STANDARD KLESHCHEV MULTIPARTITIONS AND DRINFELD POLYNOMIALS OF INTEGRAL TYPE

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Abstract. By assuming the parameter $q \in \mathbb{C}^*$ is not a root of unity and introducing the notion of standard Kleshchev multipartitions, we establish a one-to-one correspondence between standard Kleshchev multipartitions and irreducible representations with integral weights of the affine Hecke algebra of type A. Then, on the one hand, we extend the correspondence to all Kleshchev multipartitions for a given Ariki-Koike algebra of integral type. On the other hand, with the affine quantum Schur–Weyl duality, we further extend this to a correspondence between standard Kleshchev multipartitions and Drinfeld multipolynomials of integral type whose associated irreducible representations determine all irreducible polynomial representations for the quantum loop algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$. As an application of this, we identify skew shape representations of the affine Hecke algebra in terms of multisegments and compute the Drinfeld polynomials of their induced skew shape representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$.

1. Introduction

Representations of affine Hecke algebras $\mathcal{H}_\vartriangle(r) = \mathcal{H}_\vartriangle(r)_\mathbb{C}$ of type A with a parameter $q \in \mathbb{C}$ link, on the one hand, representations of Ariki-Koike algebra $\mathcal{H}_{\underline{u}}(r)$ with extra parameters $\underline{u} = (u_1, \dots, u_m)$ and link, on the other hand, representations of quantum loop algebras $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ of $\mathfrak{gl}_n$, when q is not a root of unity. In the latter case, irreducible $\mathcal{H}_\vartriangle(r)$-modules $V_\mathfrak{s}$ have been classified by Rogawski [25], built on Zelevinsky [29], in terms of multisegments $\mathfrak{s}$. Moreover, by the affine quantum Schur–Weyl duality developed in [9], this classification determines in turn the isomorphism types of irreducible modules of the affine q-Schur algebras $\mathcal{S}_\vartriangle(n, r)$, which are inflated to irreducible $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-modules. Thus, by [16], we may index these irreducible modules by Drinfeld multipolynomials.

However, irreducible $\mathcal{H}_\vartriangle(r)$-modules are also inflations of irreducible $\mathcal{H}_{\underline{u}}(r)$-modules which are labelled by Kleshchev multipartitions [2]. Thus, a Kleshchev multipartition defines an irreducible $\mathcal{H}_\vartriangle(r)$-modules and, hence, an irreducible $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-module and then a Drinfeld multipolynomial. So it is natural to ask how a Kleshchev multipartition of r determines an n-tuple of Drinfeld polynomials. Since this determination relation is a many-to-one relation, a second natural question is how to single out certain Kleshchev multipartitions to obtain a one-to-one relation. The purpose of this paper is to answer these two questions.

Our approach was motivated by the work of Vazirani [28] and Fu and the first author [13]. First, it suffices to consider the Ariki–Koike algebras whose extra parameters u_i are powers of q^2. These algebras are called Ariki–Koike algebras of integral type. When q is

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not a root of unity, Kleshchev multipartitions relative to such an Ariki–Koike algebra have an easy characterization following [28, 6]. Second, it is enough to consider the irreducible $\mathcal{H}_\Delta(r)$-modules indexed by integral multisegments. Following [25], we order an integral multisegment in a unique way so that it defines a certain so-called standard words. This motivated us to introduce *standard* Kleshchev multipartitions. In this way, we immediately solved the one-to-one relation problem between standard Kleshchev multipartitions and integral multisegments.

We further extend the construction for the one-to-one relation to a many-to-one relation. Thus, we start with an arbitrary multipartition $\underline{\lambda}$ which is dual to a Kleshchev multipartition and defines an irreducible representation $D^{\underline{\lambda}}$ of an Ariki-Koike algebra of integral type. We form an integral multisegment $\mathfrak{s}$ by reading the column residual segments and establish an $\mathcal{H}_\Delta(r)$-module isomorphism $V_{\mathfrak{s}} \cong D^{\underline{\lambda}}$ by taking advantage of the connection between Kleshchev multipartitions and multisegments given in [28]. Our key observation is that, for an arbitrary integral segment $\mathfrak{s}$, twisting the irreducible $\mathcal{H}_\Delta(r)$-module $M_{\mathfrak{s}}$ considered in [28] by Zelevinsky's involution results in, up to isomorphism, the irreducible module $V_{\mathfrak{s}}$ constructed in [25]; see Lemma 5.2.3.

Next, we extend the construction, by the affine quantum Schur–Weyl duality developed in [9], to irreducible representations of the quantum loop algebra $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$. By introducing Drinfeld polynomials of integral type, we prove that tensoring $D^{\underline{\lambda}}$ with the tensor space gives an irreducible $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-module whose Drinfeld polynomials having roots associated with the row residual segments of $\underline{\lambda}$. Like the Hecke algebra case, we show that irreducible polynomial $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-modules are completely determined by those associated with Drinfeld polynomials of integral type. This is another interesting application of affine q-Schur algebras and the Schur–Weyl duality. Moreover, the symmetry between column-reading for multisegments and row-reading for Drinfeld polynomials reveals a key role played by Kleshchev multipartitions in the affine quantum Schur–Weyl duality.

We organise the paper as follows. We first briefly review in §2 affine Hecke algebras and the classification of their irreducible modules, following Rogawski. We then introduce integral multisegments and their associated standard words. In §3, we introduce Specht modules of Ariki–Koike algebras and explicitly work out the action of Jucys–Murphy operators on Specht modules. We further interpret the inflation of a Specht module as an induced $\mathcal{H}_\Delta(r)$-module. Then, we introduce Ariki–Koike algebras of integral type and their associated Kleshchev multipartitions. With these preparations, we are ready to get the main results of the paper in the next three sections. Standard Kleshchev multipartitions are introduced in §4 and are proved to be in one-to-one correspondence with integral multisegments. In §5, we construct, for every irreducible module of an Ariki–Koike algebra of integral type, the corresponding multisegment when regarded as an irreducible $\mathcal{H}_\Delta(r)$-module. Drinfeld polynomials of integral type are introduced in §6 where we will extend the one-to-one relation to irreducible representations of affine q-Schur algebras. In particular, we prove that the irreducible polynomial representations with Drinfeld polynomials of integral type determine all irreducible polynomial representations. Finally, in the last section, we provide an application to the skew shape representations.

Throughout the paper, all algebras and modules considered are defined over $\mathbb{C}$. *Let* $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ *and assume that* $q \in \mathbb{C}^*$ *is not a root of unity.*

2. Irreducible representations of affine Hecke algebras

In this section, we shall give a review on the classification of irreducible representations of affine Hecke algebras, following Rogawski [25] and Zelevinsky [29]. We will focus on the subcategory $\mathcal{H}_{\vartriangle}(r)\text{-}\mathsf{mod}^{\mathbb{Z}}$ of $\mathcal{H}_{\vartriangle}(r)$-modules with integral weights.

2.1. The affine Hecke algebra $\mathcal{H}_{\vartriangle}(r)$. For $r \geq 1$, the affine Hecke algebra $\mathcal{H}_{\vartriangle}(r)$ is the associative algebra over $\mathbb{C}$ generated by $T_1, \dots, T_{r-1}$, $X_1, \dots, X_r$ and $X_1^{-1}, \dots, X_r^{-1}$ subject to the following relations:

$$\begin{aligned} (T_i - q^2)(T_i + 1) = 1, \quad 1 \leq i \leq r-1, \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i, \quad |i-j| > 1, \\ X_iX_i^{-1} = 1 = X_i^{-1}X_i, \quad X_iX_j = X_jX_i, \quad 1 \leq i, j \leq r, \\ T_iX_iT_i = q^2X_{i+1}, X_jT_i = T_iX_j, \quad j \neq i, i+1. \end{aligned}$$

Let $\mathfrak{S}_r$ be the symmetric group on $\{1, 2, \dots, r\}$ which is generated by the simple transpositions $s_i = (i, i+1)$ with $1 \leq i \leq r-1$. Let $\mathcal{H}(r)$ be the subalgebra of $\mathcal{H}_{\vartriangle}(r)$ generated by $T_1, \dots, T_{r-1}$. Then $\mathcal{H}(r) = \mathcal{H}(\mathfrak{S}_r)$ is the (Iwahori-)Hecke algebra associated to the symmetric group $\mathfrak{S}_r$. For $w \in \mathfrak{S}_r$ with a reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_k}$, let $T_w = T_{i_1}T_{i_2}\cdots T_{i_k}$. It is known that T_w is independent of the choice of the reduced expression of w and moreover $\{T_w | w \in \mathfrak{S}_r\}$ is a basis of the Hecke algebra $\mathcal{H}(r)$. Generally, given a composition $\mu = (\mu_1, \mu_2, \dots, \mu_\ell)$ of r, that is, a finite sequence of non-negative integers $\mu = (\mu_1, \dots, \mu_\ell)$ such that $|\mu| = \mu_1 + \cdots + \mu_\ell = r$, denote by $\mathfrak{S}_\mu$ the associated Young subgroup and let $\mathcal{H}(\mu)$ be the subalgebra of $\mathcal{H}(r)$ corresponding to $\mathfrak{S}_\mu$. Furthermore, let $\mathcal{H}_{\vartriangle}(\mu)$ be the corresponding subalgebra of $\mathcal{H}_{\vartriangle}(r)$, that is, the subalgebra generated by $X_1^{\pm 1}, \dots, X_r^{\pm 1}$ and T_i for $1 \leq i \leq r-1$ such that $s_i \in \mathfrak{S}_\mu$. In other words, $\mathcal{H}_{\vartriangle}(\mu) = \langle \mathcal{H}(\mu), X_1^{\pm 1}, \dots, X_r^{\pm 1} \rangle \cong \mathcal{H}_{\vartriangle}(\mu_1) \otimes \cdots \otimes \mathcal{H}_{\vartriangle}(\mu_\ell)$.

2.2. Classification of irreducible $\mathcal{H}_{\vartriangle}(r)$-modules. Take $\underline{a} = (a_1, a_2, \dots, a_r) \in (\mathbb{C}^*)^r$ and set

$$M(\underline{a}) = \mathcal{H}_{\vartriangle}(r)/J(\underline{a}),$$

the quotient of $\mathcal{H}_{\vartriangle}(r)$ by the left ideal $J(\underline{a})$ generated by $X_1 - a_1, \dots, X_r - a_r$. Suppose M is a finite dimensional left $\mathcal{H}_{\vartriangle}(r)$-module and $\underline{a} = (a_1, a_2, \dots, a_r) \in (\mathbb{C}^*)^r$, define the $\underline{a}$-weight space

$$M_{\underline{a}} = \{m \in M \mid (X_k - a_k)m = 0, 1 \leq k \leq r\}.$$

Lemma 2.2.1 ([25])**.** *The following holds:*

1. *Every (finite dimensional) irreducible $\mathcal{H}_{\vartriangle}(r)$-module is isomorphic to a quotient of a certain $M(\underline{a})$.*
2. *For $\underline{a} = (a_1, a_2, \dots, a_r) \in (\mathbb{C}^*)^r$, the restriction $M(\underline{a})|_{\mathcal{H}(r)}$ is isomorphic to the left regular representation of $\mathcal{H}(r)$ via the linear map*

$$\begin{aligned} \psi : \mathcal{H}(r) &\longrightarrow M(\underline{a}) \\ T_w &\longmapsto \overline{T}_w, \quad \textit{for } w \in \mathfrak{S}_r, \end{aligned}$$

where $\overline{T}_w$ denotes the image of T_w in the quotient $M(\underline{a})$.

Let $l : \mathfrak{S}_r \to \mathbb{N}$ be the length function relative to the s_i. Let $\leq$ be the Bruhat order on $\mathfrak{S}_r$, and for $w' \leq w$, let $P_{w',w}(q)$ be the Kazhdan-Lusztig polynomial [20]. Define the element $C_w \in \mathcal{H}(r)$ by

$$C_w = q^{l(w)} \sum_{w' \leq w} (-1)^{l(w)-l(w')} q^{-2l(w')} P_{w',w}(q^{-2}) T_{w'}.$$

It is known from [20] that $\{C_w | w \in \mathfrak{S}_r\}$ is a basis of $\mathcal{H}(r)$, and moreover

$$T_i C_w = -C_w \text{ if } s_i w < w. \tag{2.2.1}$$

For a composition μ of r, let w_μ^0 be the longest element in the Young subgroup $\mathfrak{S}_\mu$. Then by (2.2.1) we obtain

$$T_\sigma C_{w_\mu^0} = (-1)^{l(\sigma)} C_{w_\mu^0} \tag{2.2.2}$$

for any $\sigma \in \mathfrak{S}_\mu$. Let

$$I(\mu) = \mathcal{H}(r) C_{w_\mu^0}$$

be the left ideal of $\mathcal{H}(r)$ generated by the element $C_{w_\mu^0}$. It is known that $I(\mu)$ has dimension given by

$$\dim I(\mu) = \frac{r!}{\prod_i \mu_i!}. \tag{2.2.3}$$

The cyclic subgroup $\langle q^2 \rangle$ of $\mathbb{C}^*$ is isomorphic to $\mathbb{Z}$. We will call the elements $L_a := a\langle q^2 \rangle$ of the quotient group $\mathbb{C}^*/\langle q^2 \rangle$ "lines". A finite *consecutive* subset of a line is called a segment. Thus, a *segment* s that lies on line L_a is an ordered sequence of the form $\mathsf{s} = (aq^{2i}, aq^{2(i+1)}, \ldots, aq^{2j})$ for some $i, j \in \mathbb{Z}$ and $i \leq j$. Given a segment $\mathsf{s} = (aq^{2i}, \ldots, aq^{2j})$, set

$$\widetilde{\mathsf{s}} = (aq^{2j}, aq^{2j-2}, \ldots, aq^{2i})$$

and $|\mathsf{s}| = j - i + 1$ called the length of s. Denote by $\mathscr{C}$ the set of segments. Given a sequence of segments $\mathbf{s} = (\mathsf{s}_1, \mathsf{s}_2, \ldots, \mathsf{s}_t) \in \mathscr{C}^t$ for $t \geq 1$, let

$$\mathbf{s}^\vee = \mathsf{s}_1 \vee \cdots \vee \mathsf{s}_t \quad (\text{resp., } \widetilde{\mathbf{s}}^\vee = \widetilde{\mathsf{s}}_1 \vee \cdots \vee \widetilde{\mathsf{s}}_t) \tag{2.2.4}$$

be the r-tuple obtained by juxtaposing the sequences[1] $\mathsf{s}_1, \ldots, \mathsf{s}_t$ (resp. $\widetilde{\mathsf{s}}_1, \ldots, \widetilde{\mathsf{s}}_t$) and set

$$\mu(\mathbf{s}) = (|\mathsf{s}_1|, |\mathsf{s}_2|, \ldots, |\mathsf{s}_t|).$$

As an $\mathcal{H}(r)$-module, $M(\mathbf{s}^\vee)$ can be identified with the left regular representation by Lemma 2.2.1(2), via which any subspace of $\mathcal{H}(r)$ and hence $I(\mu(\mathbf{s}))$ can be identified with a subspace of $M(\mathbf{s}^\vee)$. To simplify notations, we write

$$I_{\mathbf{s}} = I(\mu(\mathbf{s})).$$

Let $E_{\mu(\mathbf{s})}$ be the left cell module of $\mathcal{H}(r)$ containing $w^0_{\mu(\mathbf{s})}$. It is known that as an $\mathcal{H}(r)$-module, $E_{\mu(\mathbf{s})}$ occurs with multiplicity one in $I_{\mathbf{s}}$.

Given $\mathbf{s} = (\mathsf{s}_1, \ldots, \mathsf{s}_t), \mathbf{s}' = (\mathsf{s}'_1, \ldots, \mathsf{s}'_t) \in \mathscr{C}^t$ for some $t \geq 1$, we define an equivalence relation $\mathbf{s} \sim \mathbf{s}'$ if $\mathbf{s}$ and $\mathbf{s}'$ are equal up to a rearrangement, that is, there exists $\sigma \in \mathfrak{S}_t$ such that $\mathsf{s}'_k = \mathsf{s}_{\sigma(k)}$ for $1 \leq k \leq t$. The equivalent class of $\mathbf{s} = (\mathsf{s}_1, \ldots, \mathsf{s}_t)$, denoted by $\bar{\mathbf{s}} = \{\mathsf{s}_1, \ldots, \mathsf{s}_t\}$, is called a *multisegment*. For $t \geq 1$, set

$$\mathscr{C}_r^t = \{\mathbf{s} = (\mathsf{s}_1, \ldots, \mathsf{s}_t) \in \mathscr{C}^t \mid \sum_i |\mathsf{s}_i| = r\}$$

[1]Here we changed the notation $\chi(\mathbf{s})$ in [25] to $\mathbf{s}^\vee$ to be consistent with the notation $\lambda^\vee$ in §3.1.

and let

$$\mathscr{S}_r = \bigcup_{t\geq 1}(\mathscr{C}_r^t/\sim)$$

be the set of multisegments of total length r. Then we have the following results due to Rogawski [25] (cf. Zelevinsky [29]).

Proposition 2.2.2 ([25, Proposition 4.4, Theorem 5.2])**.** *Suppose* $\mathbf{s}, \mathbf{s}' \in \mathscr{C}_r^t$ *for* $t \geq 1$. *Then*

1. $I_{\mathbf{s}}$ *is an* $\mathcal{H}_\vartriangle(r)$*-submodule of* $M(\mathbf{s}^\vee)$.
2. $C_{w^0_{\mu(\mathbf{s})}} \in (I_{\mathbf{s}})_{\widetilde{\mathbf{s}}^\vee}$.
3. *As an* $\mathcal{H}_\vartriangle(r)$*-module,* $I_{\mathbf{s}}$ *has a unique composition factor* $V_{\mathbf{s}}$ *such that as an* $\mathcal{H}(r)$*-module,* $E_{\mu(\mathbf{s})}$ *appears as a constituent of* $V_{\mathbf{s}}$.
4. $V_{\mathbf{s}} \cong V_{\mathbf{s}'}$ *if and only if* $\mathbf{s} \sim \mathbf{s}'$.
5. *Every (finite dimensional) irreducible* $\mathcal{H}_\vartriangle(r)$*-module is isomorphic to* $V_{\mathbf{s}}$ *for some* $\mathbf{s} \in \mathscr{C}_r^t$ *with* $t \geq 1$. *Hence the set* $\mathscr{S}_r$ *parametrizes all irreducible* $\mathcal{H}_\vartriangle(r)$*-modules.*

By Proposition 2.2.2(4), isomorphism types of irreducible $\mathcal{H}_\vartriangle(r)$-modules are indexed by the set $\mathscr{S}_r$. Thus, we will use the notation $V_{\mathfrak{s}}$ to denote a member in the isomorphism class labelled by $\mathfrak{s} \in \mathscr{S}_r$. We also note that, for $\mathbf{s} \sim \mathbf{s}'$, $I_{\mathbf{s}}$ is not necessarily isomorphic to $I_{\mathbf{s}'}$.

2.3. The structure of $I_{\mathbf{s}}$. It is well known that, for any $u \in \mathbb{C}^*$, there is an evaluation homomorphism

$$\begin{aligned} \mathrm{ev}_u : \mathcal{H}_\vartriangle(r) &\twoheadrightarrow \mathcal{H}(r) \\ X_1 \mapsto u, &\quad T_i \mapsto T_i \end{aligned} \tag{2.3.1}$$

for $1 \leq i \leq r-1$. Given an $\mathcal{H}(r)$-module N, let $\mathrm{ev}_u^*(N)$ denote the inflation of N to an $\mathcal{H}_\vartriangle(r)$-module via this homomorphism ev_u. In particular, the "trivial" and "sign" modules $\mathbf{1}$ and $-\mathbf{1}$ of $\mathcal{H}(r)$ give two 1-dimensional $\mathcal{H}_\vartriangle(r)$-modules $\mathrm{ev}_u^*(\mathbf{1})$ and $\mathrm{ev}_u^*(-\mathbf{1})$.

For a segment $\mathsf{s} = (aq^{2i}, \dots, aq^{2(i+r-1)})$, introduce the notation

$$\mathbb{C}_{\mathsf{s}} := \mathrm{ev}^*_{aq^{2i}}(\mathbf{1}) \quad \text{and} \quad \mathbb{C}_{\widetilde{\mathsf{s}}} := \mathrm{ev}^*_{aq^{2(i+r-1)}}(-\mathbf{1}). \tag{2.3.2}$$

Then, for $1 \leq p \leq r-1$ and $1 \leq k \leq r$, T_p acts as q^2 and X_k acts as $aq^{2(i+k-1)}$ on $\mathbb{C}_{\mathsf{s}}$, while on $\mathbb{C}_{\widetilde{\mathsf{s}}}$ the elements T_p and X_k act as -1 and $aq^{2(i+r-k)}$, respectively.

Lemma 2.3.3. *Suppose* $\mathbf{s} = (\mathsf{s}_1, \mathsf{s}_2, \dots, \mathsf{s}_t) \in \mathscr{C}_r^t$ *for some* $t \geq 1$. *Then*

1. *The one-dimensional space* $\mathbb{C}C_{w^0_{\mu(\mathbf{s})}}$ *affords an* $\mathcal{H}_\vartriangle(\mu(\mathbf{s}))$*-submodule of* $I_{\mathbf{s}}$ *and moreover*
$$\mathbb{C}C_{w^0_{\mu(\mathbf{s})}} \cong \mathbb{C}_{\widetilde{\mathsf{s}}_1} \otimes \cdots \otimes \mathbb{C}_{\widetilde{\mathsf{s}}_t}.$$
2. *As* $\mathcal{H}_\vartriangle(r)$*-modules, we have*
$$\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu(\mathbf{s}))}^{\mathcal{H}_\vartriangle(r)}(\mathbb{C}_{\widetilde{\mathsf{s}}_1} \otimes \cdots \otimes \mathbb{C}_{\widetilde{\mathsf{s}}_t}) \cong I_{\mathbf{s}}.$$

Proof. The first part follows from (2.2.2), (2.3.2) and Proposition 2.2.2(2). Then by Frobenius reciprocity, there exists an $\mathcal{H}_\vartriangle(r)$-homomorphism

$$\begin{aligned} \psi : \mathrm{ind}_{\mathcal{H}_\vartriangle(\mu(\mathbf{s}))}^{\mathcal{H}_\vartriangle(r)} \mathbb{C}C_{w^0_{\mu(\mathbf{s})}} &\longrightarrow I_{\mathbf{s}} \\ h \otimes C_{w^0_{\mu(\mathbf{s})}} &\longmapsto hC_{w^0_{\mu(\mathbf{s})}}, \quad \forall h \in \mathcal{H}_\vartriangle(r). \end{aligned}$$

On the other hand, $I_{\mathbf{s}}$ is generated by $C_{w^0_{\mu(\mathbf{s})}}$ and this implies that ψ is surjective. Observe that $\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu(\mathbf{s}))}^{\mathcal{H}_\vartriangle(r)}\mathbb{C}C_{w^0_{\mu(\mathbf{s})}}$ and $I_{\mathbf{s}}$ have the same dimension given by (2.2.3), therefore the second part of the lemma is proved. $\square$

It is known from [29] that if $\bar{\mathbf{s}} = \overline{\mathbf{s}'} = \mathfrak{s}$ then $I_{\mathbf{s}}$ and $I_{\mathbf{s}'}$ have the same composition factors including the unique $V_{\mathfrak{s}}$. We now look at a sufficient condition on $\mathbf{s}$ for $V_{\mathfrak{s}}$ appearing in the head of $I_{\mathbf{s}}$.

Definition 2.3.4. (1) Let s_1 and s_2 be on the same line L_a, say $\mathsf{s}_1 = (aq^{2i_1}, \dots, aq^{2j_1})$, and $\mathsf{s}_2 = (aq^{2i_2}, \dots, aq^{2j_2})$. We say that s_1 *precedes* s_2 and write $\mathsf{s}_1 \preccurlyeq \mathsf{s}_2$ if either $j_1 < j_2$ or $j_1 = j_2$ and $i_1 \geq i_2$ (cf. [25, Theorem 5.2]).

(2) Given $\mathbf{s} = (\mathsf{s}_1, \mathsf{s}_2, \dots, \mathsf{s}_t) \in \mathscr{C}^t$ with $t \geq 1$, we say that $\mathbf{s}$ is *standard* (relative to $\preccurlyeq$) if there exist $0 = k_0 < k_1 < \dots < k_p = t$ and disjoint lines $L_{a_1}, L_{a_2}, \dots, L_{a_p}$ with $a_1, \dots, a_p \in \mathbb{C}^*$ such that, for each $1 \leq i \leq p$, the segments s_j for $k_{i-1}+1 \leq j \leq k_i$ lie on the line L_{a_i} and $\mathsf{s}_{k_{i-1}+1} \preccurlyeq \mathsf{s}_{k_{i-1}+2} \preccurlyeq \dots \preccurlyeq \mathsf{s}_{k_i}$. In this case, we set $\mathbf{s}^{(i)} = (\mathsf{s}_{k_{i-1}+1}, \dots, \mathsf{s}_{k_i})$ for $1 \leq i \leq p$ and write

$$\mathbf{s} = \mathbf{s}^{(1)} \cup \mathbf{s}^{(2)} \cup \dots \cup \mathbf{s}^{(p)}.$$

(3) For a segment $\mathsf{s} = (aq^{2i}, \dots, aq^{2j})$ lying on line L_a, define

$$\mathsf{s}^{-1} = (a^{-1}q^{-2j}, \dots, a^{-1}q^{-2i})$$

a segment lying on line $L_{a^{-1}}$. A sequence $\mathbf{s} = (\mathsf{s}_1, \mathsf{s}_2, \dots, \mathsf{s}_t)$ of segments is said to be *anti-standard* if $\mathbf{s}^{-1} := (\mathsf{s}_1^{-1}, \mathsf{s}_2^{-1}, \dots, \mathsf{s}_t^{-1})$ is standard.

Following [26, Definition 3.1], we can define another order $\mathsf{s}_1 \preccurlyeq' \mathsf{s}_2$ if either $j_1 < j_2$ or $j_1 = j_2$ and $i_1 \leq i_2$ for two segments $\mathsf{s}_1 = (aq^{2i_1}, \dots, aq^{2j_1})$ and $\mathsf{s}_2 = (aq^{2i_2}, \dots, aq^{2j_2})$ on the same line L_a. Analogous to Definition 2.3.4(2), one can define the standard sequence of segments with respect to the order $\preccurlyeq'$.

Proposition 2.3.5 ([26, Theorem 3.3])**.** *For $\mathfrak{s} \in \mathscr{S}_r$, if $\mathbf{s} \in \mathfrak{s}$ is standard with respect to the order $\preccurlyeq'$, then $I_{\mathbf{s}}$ has simple head* $\mathrm{hd}(I_{\mathbf{s}})$ *and moreover*

$$V_{\mathfrak{s}} \cong \mathrm{hd}(I_{\mathbf{s}}).$$

Lemma 2.3.6 ([26, Proposition 5.2])**.** *Suppose that s_1 and s_2 are on the same line L_a with $\mathsf{s}_1 = (aq^{2i_1}, \dots, aq^{2j_1})$, $\mathsf{s}_2 = (aq^{2i_2}, \dots, aq^{2j_2})$. If $j_1 = j_2$, then*

$$\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)}(\mathbb{C}_{\widetilde{\mathsf{s}}_1} \otimes \mathbb{C}_{\widetilde{\mathsf{s}}_2}) \cong \mathrm{ind}_{\mathcal{H}_\vartriangle(\mu^o)}^{\mathcal{H}_\vartriangle(r)}(\mathbb{C}_{\widetilde{\mathsf{s}}_2} \otimes \mathbb{C}_{\widetilde{\mathsf{s}}_1}),$$

that is, $I_{(\mathsf{s}_1,\mathsf{s}_2)} \cong I_{(\mathsf{s}_2,\mathsf{s}_1)}$, where $\mu = (|\mathsf{s}_1|, |\mathsf{s}_2|)$, $\mu^o = (|\mathsf{s}_2|, |\mathsf{s}_1|)$ and $r = |\mathsf{s}_1| + |\mathsf{s}_2|$.

Lemma 2.3.7. *Let $\mathbf{s} = (\mathsf{s}_1, \mathsf{s}_2, \dots, \mathsf{s}_t) \in \mathscr{C}_r^t$ be standard relative to $\preccurlyeq$ with $\mathsf{s}_1, \dots, \mathsf{s}_t$ being on the same line. Reorder the segments in $\mathbf{s}$ such that $\mathbf{s}' = (\mathsf{s}_{i_1}, \mathsf{s}_{i_2}, \dots, \mathsf{s}_{i_t})$ is standard with respect to $\preccurlyeq'$. Then*

$$I_{\mathbf{s}} \cong I_{\mathbf{s}'}.$$

Proof. Suppose $\mathsf{s}_1, \dots, \mathsf{s}_t$ are on the line L_a and $\mathsf{s}_k = (aq^{2i_k}, \dots, aq^{2j_k})$ for $1 \leq k \leq t$. Observe that there exist unique $0 = k_0 < k_1 < k_2 < \dots < k_d = t$ such that $j_{k_{b-1}+1} = j_{k_{b-1}+2} = \dots = j_{k_b}$ and $i_{k_{b-1}+1} \geq i_{k_{b-1}+2} \geq \dots \geq i_{k_b}$ for each $1 \leq b \leq d$ and $j_{k_e} < j_{k_e+1}$ for $1 \leq e \leq d-1$. Then $\mathbf{s}'$ has the form given by

$$\mathbf{s}' = (\mathsf{s}_{k_1}, \mathsf{s}_{k_1-1}, \dots, \mathsf{s}_1, \mathsf{s}_{k_2}, \mathsf{s}_{k_2-1} \cdots, \mathsf{s}_{k_1+1}, \dots, \mathsf{s}_{k_d}, \mathsf{s}_{k_d-1}, \dots, \mathsf{s}_{k_{d-1}+1}).$$

For $1 \le b \le d$, set

$$I_b = \mathrm{ind}_{\mathcal{H}_\vartriangle(\mu^{(b)})}^{\mathcal{H}_\vartriangle(r_b)} \mathbb{C}_{\widetilde{\mathsf{s}}_{k_{b-1}+1}} \otimes \cdots \otimes \mathbb{C}_{\widetilde{\mathsf{s}}_{k_b-1}} \otimes \mathbb{C}_{\widetilde{\mathsf{s}}_{k_b}},$$
$$I'_b = \mathrm{ind}_{\mathcal{H}_\vartriangle((\mu^{(b)})^o)}^{\mathcal{H}_\vartriangle(r_b)} \mathbb{C}_{\widetilde{\mathsf{s}}_{k_b}} \otimes \mathbb{C}_{\widetilde{\mathsf{s}}_{k_b-1}} \otimes \cdots \otimes \mathbb{C}_{\widetilde{\mathsf{s}}_{k_{b-1}+1}},$$

where $\mu^{(b)} = (|\mathsf{s}_{k_{b-1}+1}|, \dots, |\mathsf{s}_{k_b}|)$, $(\mu^{(b)})^o = (|\mathsf{s}_{k_b}|, \dots, |\mathsf{s}_{k_{b-1}+1}|)$ and $r_b = |\mu^{(b)}|$. By Lemma 2.3.6, we obtain

$$I_b \cong I'_b$$

for $1 \le b \le d$. Hence, by Lemma 2.3.3(2) and associativity of induction, one can deduce that

$$I_{\mathbf{s}} \cong \mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)} I_1 \otimes \cdots \otimes I_d \cong \mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)} I'_1 \otimes \cdots \otimes I'_d \cong I_{\mathbf{s}'},$$

where $\mu = (|\mu^{(1)}|, \dots, |\mu^{(d)}|)$. □

Proposition 2.3.8. *For $\mathfrak{s} \in \mathscr{S}_r$, if $\mathbf{s} \in \mathfrak{s}$ is standard (with respect to $\preccurlyeq$), then $I_{\mathbf{s}}$ has simple head* $\mathrm{hd}(I_{\mathbf{s}})$ *and*

$$V_{\mathfrak{s}} \cong \mathrm{hd}(I_{\mathbf{s}}).$$

Proof. Suppose $\mathbf{s} = (\mathsf{s}_1, \mathsf{s}_2, \dots, \mathsf{s}_t)$ is standard with the decomposition $\mathbf{s} = \mathbf{s}^{(1)} \cup \mathbf{s}^{(2)} \cup \cdots \cup \mathbf{s}^{(p)}$ in the sense of Definition 2.3.4(2). Reorder the segments in $\mathbf{s}^{(i)}$ such that $(\mathbf{s}^{(i)})'$ is standard with respect to the order $\preccurlyeq'$ for $1 \le i \le p$. Then $\mathbf{s}' := (\mathbf{s}^{(1)})' \cup (\mathbf{s}^{(2)})' \cup \cdots \cup (\mathbf{s}^{(p)})'$ is standard with respect to the order $\preccurlyeq'$. Meanwhile by Lemma 2.3.3(2) we have

$$I_{\mathbf{s}} \cong \mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)} I_{\mathbf{s}^{(1)}} \otimes \cdots \otimes I_{\mathbf{s}^{(p)}} \quad \text{and} \quad I_{\mathbf{s}'} \cong \mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)} I_{(\mathbf{s}^{(1)})'} \otimes \cdots \otimes I_{(\mathbf{s}^{(p)})'}.$$

Thus, Lemma 2.3.7 implies $I_{\mathbf{s}} \cong I_{\mathbf{s}'}$ and so the result follows from Proposition 2.3.5. □

Note that this result is not true in general for non-standard $\mathbf{s}$. From now on, by standard sequence of segments we mean those satisfying the property given in Definition 2.3.4(2) where the order $\preccurlyeq$ is used.

2.4. The category $\mathcal{H}_\vartriangle(r)$-$\mathsf{mod}^{\mathbb{Z}}$, integral multisegments and standard words. Let $\mathcal{H}_\vartriangle(r)$-$\mathsf{mod}^{\mathbb{Z}}$ denote the full subcategory of finite dimensional left $\mathcal{H}_\vartriangle(r)$-modules on which the eigenvalues of $X_1, \dots, X_r$ are powers of q^2 (i.e., the weights are integral). It is known (cf. [21, 18]) that to understand irreducible $\mathcal{H}_\vartriangle(r)$-modules, it suffices to study those irreducible modules in $\mathcal{H}_\vartriangle(r)$-$\mathsf{mod}^{\mathbb{Z}}$. More precisely, given standard $\mathbf{s} = (\mathsf{s}_1, \mathsf{s}_2, \dots, \mathsf{s}_t) \in \mathscr{C}_r^t$ with $t \ge 1$, and assume $\mathbf{s} = \mathbf{s}^{(1)} \cup \mathsf{s}^{(2)} \cup \cdots \cup \mathsf{s}^{(p)}$ in the sense of Definition 2.3.4(2). Then it is known [21, Lemma 6.1.2] that $\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)}(V_{\mathbf{s}^{(1)}} \otimes \cdots \otimes V_{\mathbf{s}^{(p)}})$ is irreducible and hence by Proposition 2.3.8 the following holds

$$V_{\mathbf{s}} \cong \mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)}(V_{\mathbf{s}^{(1)}} \otimes \cdots \otimes V_{\mathbf{s}^{(p)}}), \tag{2.4.1}$$

where $\mu = (\mu_1, \dots, \mu_p)$ with $\mu_i = \sum_j |\mathsf{s}_j^{(i)}|$. Therefore, it is reduced to study the irreducible modules $V_{\mathbf{s}}$ for $\mathbf{s} = (\mathsf{s}_1, \mathsf{s}_2, \dots, \mathsf{s}_t)$ with $\mathsf{s}_1, \dots, \mathsf{s}_t$ being on the same line.

For $a \in \mathbb{C}^*$, let σ_a be the automorphism of $\mathcal{H}_\vartriangle(r)$ given by

$$\sigma_a(T_i) = T_i, \quad \sigma_a(X_k) = aX_k \tag{2.4.2}$$

for $1 \le i \le r-1$ and $1 \le k \le r$. For an $\mathcal{H}_\vartriangle(r)$-module V, we can twist the action with σ_a to get a new module denoted by V^{σ_a}. Given an arbitrary segment $\mathsf{s} = (zq^{2i}, \dots, zq^{2j})$ and $a \in \mathbb{C}^*$, set

$$\mathsf{s}^{\sigma_a} = (azq^{2i}, \dots, azq^{2j}). \tag{2.4.3}$$

Similarly we can define $\mathbf{s}^{\sigma_a}$ and $\mathfrak{s}^{\sigma_a}$ for $\mathbf{s} \in \mathscr{C}_r^t$ and $\mathfrak{s} \in \mathscr{S}_r$.

Lemma 2.4.9. *Let $a \in \mathbb{C}^*$ and $\mathfrak{s} \in \mathscr{S}_r$. Then*

$$V_{\mathfrak{s}}^{\sigma_a} \cong V_{\mathfrak{s}^{\sigma_a}}.$$

Proof. Take $\mathbf{s} = (\mathsf{s}_1, \ldots, \mathsf{s}_t) \in \mathfrak{s}$. Note that $\mathbb{C}_{\widetilde{\mathsf{s}}_k}^{\sigma_a} \cong \mathbb{C}_{\widetilde{\mathsf{s}_k^{\sigma_a}}}$ for $1 \le k \le t$ by (2.3.2) and hence by Lemma 2.3.3 we obtain

$$\begin{aligned} I_{\mathbf{s}}^{\sigma_a} &\cong \big(\mathrm{ind}_{\mathcal{H}_\Delta(\mu(\mathbf{s}))}^{\mathcal{H}_\Delta(r)}(\mathbb{C}_{\widetilde{\mathsf{s}}_1} \otimes \cdots \otimes \mathbb{C}_{\widetilde{\mathsf{s}}_t})\big)^{\sigma_a} \\ &\cong \mathrm{ind}_{\mathcal{H}_\Delta(\mu(\mathbf{s}))}^{\mathcal{H}_\Delta(r)}(\mathbb{C}_{\widetilde{\mathsf{s}}_1}^{\sigma_a} \otimes \cdots \otimes \mathbb{C}_{\widetilde{\mathsf{s}}_t}^{\sigma_a}) \text{ (by Frobenius reciprocity)} \\ &\cong \mathrm{ind}_{\mathcal{H}_\Delta(\mu(\mathbf{s}))}^{\mathcal{H}_\Delta(r)}(\mathbb{C}_{\widetilde{\mathsf{s}_1^{\sigma_a}}} \otimes \cdots \otimes \mathbb{C}_{\widetilde{\mathsf{s}_t^{\sigma_a}}}) \\ &\cong I_{\mathbf{s}^{\sigma_a}}. \end{aligned}$$

Therefore, $V_{\mathbf{s}}^{\sigma_a}$ is a composition factor of $I_{\mathbf{s}^{\sigma_a}}$. Meanwhile, since $V_{\mathbf{s}}^{\sigma_a} \cong V_{\mathbf{s}}$ as $\mathcal{H}(r)$-modules and hence $E_{\mu(\mathbf{s}^{\sigma_a})} = E_{\mu(\mathbf{s})}$ appears as a constituent of $V_{\mathbf{s}}^{\sigma_a}$. By Proposition 2.2.2, we get $V_{\mathbf{s}}^{\sigma_a} \cong V_{\mathbf{s}^{\sigma_a}}$ and this implies $V_{\mathfrak{s}}^{\sigma_a} \cong V_{\mathfrak{s}^{\sigma_a}}$. □

Now assume that $\mathbf{s} = (\mathsf{s}_1, \ldots, \mathsf{s}_t)$ with s_k being on the same line L_a for some $a \in \mathbb{C}^*$. Then it is easy to check that $\mathsf{s}_k^{\sigma_{a^{-1}}}$ is integral for $1 \le k \le n$ and moreover by Lemma 2.4.9

$$V_{\mathbf{s}} \cong (V_{\mathbf{s}^{\sigma_{a^{-1}}}})^{\sigma_a}.$$

Hence, to understand irreducible modules $V_{\mathbf{s}}$, it is enough to consider those $\mathbf{s} \in \mathscr{C}_r^t$ with $t \ge 1$ such that $\bar{\mathbf{s}} \in \mathscr{S}_r^{\mathbb{Z}}$, where

$$\mathscr{S}_r^{\mathbb{Z}} = \{\bar{\mathbf{s}} \in \mathscr{S}_r \mid \mathbf{s} \text{ lies on } L_1\}$$

is the set of all *integral multisegments* of total length r.

Define $\mathscr{C}_r^{\mathbb{Z}}$ similarly to be the set of *standard sequences of integral segments* of total length r. For any multisegment $\mathfrak{s} \in \mathscr{S}_r^{\mathbb{Z}}$, there exists a unique standard sequence $\mathbf{s}$ such that $\mathbf{s} \in \mathfrak{s}$. Thus, the map

$$\begin{aligned} \flat_1 :\ mathscrC_r^{\mathbb{Z}} &\longrightarrow \mathscr{S}_r^{\mathbb{Z}} \\ \mathbf{s} &\longmapsto \bar{\mathbf{s}} \end{aligned} \tag{2.4.4}$$

is a bijection. We now use this fact to describe integral multisegments in terms of standard words.

Consider the set $\mathcal{W}$ of nonempty words in the alphabet $\mathbb{Z}^2 = \{\binom{j}{i} \mid i, j \in \mathbb{Z}\}$. We always identify such a word $\binom{j_1}{i_1} \cdots \binom{j_t}{i_t}$ as $\binom{j_1 \ldots j_t}{i_1 \ldots i_t}$ and define $\left|\binom{j_1 \ldots j_t}{i_1 \ldots i_t}\right| := \sum_{k=1}^t (j_k - i_k + 1)$. Let

$$\mathcal{W}(r) = \{w = \begin{pmatrix} j_1 \ldots j_t \\ i_1 \ldots i_t \end{pmatrix} \in \mathcal{W} \mid i_k \le j_k, 1 \le k \le t, |w| = r\}.$$

Clearly, each word in $\mathcal{W}(r)$ defines a segment sequence on the line L_1. A word $\binom{j_1 \ldots j_t}{i_1 \ldots i_t}$ in $\mathcal{W}(r)$ is called a *standard word* if $j_1 \le j_2 \le \ldots \le j_t$ and $i_k \ge i_{k+1}$ whenever $j_k = j_{k+1}$. Let $\mathcal{W}^s(r)$ be the set of standard words in $\mathcal{W}(r)$.

Clearly, by definition, there is a bijection

$$\flat_2 : \mathcal{W}^s(r) \longrightarrow \mathscr{C}_r^{\mathbb{Z}} \tag{2.4.5}$$

sending a standard word $\binom{j_1 j_2 \dots j_t}{i_1 i_2 \dots i_t}$ to a standard sequence $\mathsf{s} = (\mathsf{s}_1, \mathsf{s}_2, \dots, \mathsf{s}_t)$, where, for $1 \le k \le t$, $\mathsf{s}_k = (q^{2i_k}, q^{2(i_k+1)}, \dots, q^{2j_k})$. This induces a bijection

$$\begin{aligned} \flat = \flat_1 \circ \flat_2 : \mathcal{W}^s(r) &\longrightarrow \mathscr{S}_r^{\mathbb{Z}} \\ w &\longmapsto \overline{\flat_2(w)}. \end{aligned} \tag{2.4.6}$$

We remark that the reduction to the integral case allows us to link irreducible modules $V_{\mathfrak{s}}$ for $\mathfrak{s} \in \mathscr{S}_r^{\mathbb{Z}}$ with irreducible modules over the Ariki-Koike algebras of integral type considered in §3.5.

3. Ariki-Koike algebras and Specht modules

In this section, we shall recall the construction of Specht modules for Ariki-Koike algebras (cf. [12, 15]), compute the action of Jucys–Murphy elements on Specht modules, and introduce Kleshchev multipartitions for Ariki–koike algebras of integral type.

3.1. Basics on multipartitions. Recall that a composition μ of r is a finite sequence of non-negative integers $\mu = (\mu_1, \dots, \mu_\ell)$ such that $|\mu| = \mu_1 + \cdots + \mu_\ell = r$. If, in addition, $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_\ell$, then μ is called a partition of r and write $\mu \vdash r$. Denote by $\mathcal{P}(r)$ the set of partitions of r. It is known that a partition λ can be identified with its Young diagram which is formally defined to be set of points $\{(i,j) | 1 \le i \le \ell(\lambda), 1 \le j \le \lambda_i\} \subseteq \mathbb{N}^2$ and the elements (i,j) are called *nodes*. Here $\ell(\lambda)$ denotes the number of nonzero parts of λ. Given a composition μ, its conjugate $\mu' = (\mu'_1, \mu'_2, \dots)$ is the partition defined via $\mu'_i = \sharp\{k | \mu_k \ge i\}$ for $i \ge 1$.

Suppose $\lambda \in \mathcal{P}(r)$. A λ-tableau is a labeling of the nodes in the Young diagram λ with integers $1, 2, \dots, r$. Let t^λ (resp. t_λ) be the λ-tableau in which the numbers $1, 2, \dots, r$ appear in order along successive rows (resp. column). For example, if $\lambda = (3, 2)$, then

$$t^\lambda = \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \cline{1-2} \end{array}, \quad t_\lambda = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \cline{1-2} \end{array}.$$

Observe that the symmetric group $\mathfrak{S}_r$ naturally acts on λ-tableaux for $\lambda \in \mathcal{P}(r)$. Given $\lambda \in \mathcal{P}(r)$, let w_λ be the permutation satisfying

$$w_\lambda t^\lambda = t_\lambda.$$

Observe that

$$w_\lambda^{-1} = w_{\lambda'}. \tag{3.1.1}$$

An m-tuple of partitions $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)})$ with $|\lambda^{(1)}| + \cdots + |\lambda^{(m)}| = r$ is called a multipartition of r. Call $\underline{\lambda}$ a *sincere* multipartition if $|\lambda^{(i)}| > 0$ for all i. Denote by $\mathcal{P}_m(r)$ the set of multipartitions $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)})$ of r. Given $\underline{\lambda} \in \mathcal{P}_m(r)$, set

$$\underline{\lambda}' = (\lambda^{(m)'}, \dots, \lambda^{(2)'}, \lambda^{(1)'}).$$

By concatenating the components of $\underline{\lambda}$, the resulting composition of r will be denoted by

$$\underline{\lambda}^\vee = \lambda^{(1)} \vee \lambda^{(2)} \vee \cdots \vee \lambda^{(m)}.$$

Define

$$[\underline{\lambda}] = [a_0, a_1, \dots, a_m]$$

such that $a_0 = 0, a_k = \sum_{j=1}^k |\lambda^{(j)}|$ for $1 \le k \le m$. Then all $[\underline{\lambda}]$ form the set

$$\Lambda[m,r] := \{\mathbf{a} = [a_0, a_1, \dots, a_m] \mid 0 = a_0 \le a_1 \le \cdots \le a_m = r, a_i \in \mathbb{Z}, 0 \le i \le m\}.$$

Note that this set is often associated with the set of compositions of r into m parts:

$$\Lambda(m,r) := \{(a_1-a_0, a_2-a_1, \ldots, a_m-a_{m-1}) \mid [a_0, a_1, \ldots, a_m] \in \Lambda[m,r]\}. \tag{3.1.2}$$

For $\mathbf{a} = [a_0, a_2, \ldots, a_m] \in \Lambda[m,r]$, let

$$\mathbf{a}' = [r-a_m, r-a_{m-1}, \ldots, r-a_1, r-a_0]$$

and define $w_{\mathbf{a}} \in \mathfrak{S}_r$ to be the element satisfying

$$w_{\mathbf{a}}(a_{j-1}+k) = r - a_j + k$$

for $1 \le k \le a_j - a_{j-1}$ and $1 \le j \le m$ with $a_{j-1} < a_j$. For example, if $\mathbf{a} = [0,4,8,9]$, then $w_{\mathbf{a}}$ has the form

$$w_{\mathbf{a}} = \begin{pmatrix} 1\,2\,3\,4 & 5\,6\,7\,8 & 9 \\ \underline{6\,7\,8\,9} & \underline{2\,3\,4\,5} & \underline{1} \end{pmatrix}.$$

In particular, if $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(m)})$ is a multipartition of r, then

$$w_{[\underline{\lambda}]}(|\lambda^{(1)}| + \cdots + |\lambda^{(j-1)}| + b) = |\lambda^{(j+1)}| + \cdots + |\lambda^{(m)}| + b \tag{3.1.3}$$

for $1 \le b \le |\lambda^{(j)}|$ and $1 \le j \le m$ with $|\lambda^{(j)}| \ge 1$.

A multipartition $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ can also be identified with its Young diagram which is formally defined to be set

$$\{(i,j,k) | 1 \le i \le \ell(\lambda^{(k)}), 1 \le j \le \lambda_i^{(k)}, 1 \le k \le m\} \subseteq \mathbb{N}^3.$$

The elements (i,j,k) are called nodes. For example, the 3-partition $\underline{\lambda} = ((3,1),(2,2),(1))$ is identified with

$$\underline{\lambda} = \left(\begin{array}{|c|c|c|} \hline & & \\ \hline & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|} \hline & \\ \hline & \\ \hline \end{array}, \quad \begin{array}{|c|} \hline \\ \hline \end{array} \right).$$

Similarly, a $\underline{\lambda}$-tableau is a labeling of the nodes in the diagrams with numbers $1, 2, \ldots, r$. Let $t^{\underline{\lambda}}$ be the $\underline{\lambda}$-tableau in which the numbers $1, 2, \ldots, r$ appear in order along successive rows in the first diagram of $\underline{\lambda}$, then in the second diagram and so on. Meanwhile let $t_{\underline{\lambda}}$ be the $\underline{\lambda}$-tableau in which the numbers $1, 2, \ldots, r$ appear in order along successive columns in the last diagram of $\underline{\lambda}$, then in the second last diagram and so on. For example, in the case $\underline{\lambda} = ((3,1),(2,2),(1))$, we have

$$\begin{aligned} t^{\underline{\lambda}} &= \left(\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|} \hline 5 & 6 \\ \hline 7 & 8 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 9 \\ \hline \end{array} \right), \\ t_{\underline{\lambda}} &= \left(\begin{array}{|c|c|c|} \hline 6 & 8 & 9 \\ \hline 7 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}, \quad \begin{array}{|c|c|} \hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array}, \quad \begin{array}{|c|} \hline 1 \\ \hline \end{array} \right). \end{aligned} \tag{3.1.4}$$

For a multipartition $\underline{\lambda} \in \mathcal{P}_m(r)$, let $w_{\underline{\lambda}} \in \mathfrak{S}_r$ be the element satisfying

$$w_{\underline{\lambda}} t^{\underline{\lambda}} = t_{\underline{\lambda}}.$$

Similarly, we have

$$w_{\underline{\lambda}}^{-1} = w_{\underline{\lambda}'}. \tag{3.1.5}$$

3.2. The Ariki-Koike algebra $\mathcal{H}_{\underline{u}}(r)$ and Jucys-Murphy elements. Recall $q \in \mathbb{C}^*$. Let $\underline{u} = (u_1, u_2, \ldots, u_m) \in \mathbb{C}^m$ with $m \ge 1$. The *Ariki-Koike* algebra $\mathcal{H}_{\underline{u}}(r)$ is the associative algebra over $\mathbb{C}$ generated by $T_0, T_1, \ldots, T_{r-1}$ subject to the relations:

$$\begin{aligned}(T_i - q^2)(T_i + 1) &= 0, \quad 1 \le i \le r-1,\\ T_iT_{i+1}T_i &= T_{i+1}T_iT_{i+1}, \quad 1 \le i \le r-2,\\ T_iT_j &= T_jT_i, \quad 1 \le i, j \le r-1, |i-j| > 1,\\ (T_0 - u_1)(T_0 - u_2)\cdots(T_0 - u_m) &= 0,\\ T_0T_1T_0T_1 &= T_1T_0T_1T_0.\end{aligned}$$

Observe that the Hecke algebra $\mathcal{H}(r)$ can be viewed as a special case of Ariki-Koike algebras obtained by take $m = 1$. The *Jucys-Murphy* elements J_k $(1 \le k \le r)$ of $\mathcal{H}(r)$ are defined to be $J_1 = 1$ and

$$J_k = q^{2(1-k)} T_{k-1} \cdots T_1 T_1 \cdots T_{k-1} \tag{3.2.1}$$

for $k \ge 2$. It is known that these elements commute with each other and they play an important role in the representation theory of $\mathcal{H}(r)$ (cf. [11]). These elements have been generalized to the Ariki-Koike algebra $\mathcal{H}_{\underline{u}}(r)$ given by:

$$L_1 = T_0, \quad L_k = q^{2(1-k)} T_{k-1} \cdots T_1 T_0 T_1 \cdots T_{k-1} = q^{-2} T_{k-1} L_{k-1} T_{k-1} \tag{3.2.2}$$

for $2 \le k \le r$. Again, it is easy to check that these elements commute with each other. By taking advantage of these elements, Ariki-Koike algebras can be regarded as quotients of affine Hecke algebras as follows, generalising the evaluation map (2.3.1).

If $u_1, \ldots, u_m$ are all non-zero,[2] there exists a surjective homomorphism

$$\begin{aligned}\varpi_{\underline{u}} : \mathcal{H}_{\vartriangle}(r) &\twoheadrightarrow \mathcal{H}_{\underline{u}}(r)\\ X_k \mapsto L_k, &\quad T_i \mapsto T_i\end{aligned} \tag{3.2.3}$$

for $1 \le k \le r$ and $1 \le i \le r-1$ and $\varpi_{\underline{u}}$ induces the algebra isomorphism

$$\mathcal{H}_{\underline{u}}(r) \cong \mathcal{H}_{\vartriangle}(r)/\langle (X_1 - u_1)(X_1 - u_2)\cdots(X_1 - u_m)\rangle.$$

This means any $\mathcal{H}_{\underline{u}}(r)$-module M can be *inflated* to an $\mathcal{H}_{\vartriangle}(r)$-module in this case. We will use the same letter to denote its inflation.

A direct check of relations shows that there exists an anti-automorphism τ on $\mathcal{H}_{\underline{u}}(r)$ defined by

$$\begin{aligned}\tau : \mathcal{H}_{\underline{u}}(r) &\longrightarrow \mathcal{H}_{\underline{u}}(r),\\ T_k &\longmapsto T_k\end{aligned} \tag{3.2.4}$$

for $0 \le k \le r-1$.

3.3. Specht modules for $\mathcal{H}_{\underline{u}}(r)$. We shall mainly follow the constructions of Specht modules for the Hecke algebra $\mathcal{H}(r)$ and the Ariki-Koike algebras $\mathcal{H}_{\underline{u}}(r)$ established in [10, 11, 14, 15], where right modules are considered. Observe that by applying the anti-automorphism τ, the definitions and statements therein can be reformulated for left modules as follows.

Given a composition $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ of r, set

$$x_\mu := \sum_{\sigma \in \mathfrak{S}_\mu} T_\sigma, \quad y_\mu := \sum_{\sigma \in \mathfrak{S}_\mu} (-q^2)^{l(\sigma)} T_\sigma.$$

[2]This guarantees that all L_k are invertible since all X_k are invertible in $\mathcal{H}_{\vartriangle}(r)$.

It is easy to check that

$$T_\sigma x_\mu = x_\mu T_\sigma = q^{2l(\sigma)} x_\mu, \quad T_\sigma y_\mu = y_\mu T_\sigma = (-1)^{l(\sigma)} y_\mu \tag{3.3.1}$$

for $\sigma \in \mathfrak{S}_\mu$. For $\lambda \in \mathcal{P}(r)$, define

$$z_\lambda := y_{\lambda'} T_{w_{\lambda'}} x_\lambda.$$

The *Specht* module [10] associated to a partition λ of r is the left ideal S^λ of $\mathcal{H}(r)$ generated by z_λ, that is,

$$S^\lambda = \mathcal{H}(r) z_\lambda. \tag{3.3.2}$$

We remark that by (3.1.1) our z_λ is exactly the element obtained by apply τ to the elements also denoted by z_λ in [10, P. 34], where right modules are studied.

Following [15], we shall introduce the notion of Specht modules for the Ariki-Koike algebras $\mathcal{H}_{\underline{u}}(r)$ in the rest of this section. For $u \in \mathbb{C}$ and a positive integer k, let

$$\pi_0(u) = 1, \quad \pi_k(u) = (L_1 - u)(L_2 - u) \cdots (L_k - u).$$

Then, for $\mathbf{a} = [a_0, a_1, \ldots, a_m] \in \Lambda[m, r]$, we define (cf. [12, 14, 17])

$$\begin{aligned} \pi_{\mathbf{a}} =& \pi_{a_1}(u_2) \cdots \pi_{a_{m-1}}(u_m), \quad \widetilde{\pi}_{\mathbf{a}} = \pi_{a_1}(u_{m-1}) \cdots \pi_{a_{m-1}}(u_1), \\ v_{\mathbf{a}} =& \widetilde{\pi}_{\mathbf{a}'} T_{w_{\mathbf{a}'}} \pi_{\mathbf{a}}. \end{aligned}$$

For a multipartition $\underline{\lambda}$ of r, let

$$\begin{aligned} x_{\underline{\lambda}} =& \pi_{[\underline{\lambda}]} x_{\underline{\lambda}^\vee} = x_{\underline{\lambda}^\vee} \pi_{[\underline{\lambda}]}, \quad y_{\underline{\lambda}} = \widetilde{\pi}_{[\underline{\lambda}]} y_{\underline{\lambda}^\vee} = y_{\underline{\lambda}^\vee} \widetilde{\pi}_{[\underline{\lambda}]}, \\ z_{\underline{\lambda}} =& y_{\underline{\lambda}'} T_{w_{\underline{\lambda}'}} x_{\underline{\lambda}}. \end{aligned}$$

Again, the element $z_{\underline{\lambda}}$ defined here is the one obtained by applying the anti-automorphism τ to the one also denoted by $z_{\underline{\lambda}}$ constructed in [15, Definition 2.1] since τ satisfies

$$\tau(T_{w_{\underline{\lambda}}}) = T_{w_{\underline{\lambda}'}}$$

due to (3.1.5). By (3.3.1), we get

$$T_\sigma z_{\underline{\lambda}} = (-1)^{l(\sigma)} z_{\underline{\lambda}} \tag{3.3.3}$$

for $\sigma \in \mathfrak{S}_{(\underline{\lambda}')^\vee}$.

Definition 3.3.1. The Specht module $S^{\underline{\lambda}}$ associated to a multipartition $\underline{\lambda} \in \mathcal{P}_m(r)$ is defined to be

$$S^{\underline{\lambda}} = \mathcal{H}_{\underline{u}}(r) z_{\underline{\lambda}},$$

that is, the left ideal of $\mathcal{H}_{\underline{u}}(r)$ generated by $z_{\underline{\lambda}}$.

Lemma 3.3.2 ([14, Proposition 3.1])**.** *Suppose* $\mathbf{a} \in \Lambda[m, r]$ *and let* $\mathfrak{S}_{\mathbf{a}}$ *be the Young subgroup of* $\mathfrak{S}_r$ *associated to the composition* $(a_1, a_2 - a_1, \ldots, a_m - a_{m-1})$ *of* r*. Then*

(1) $T_k v_{\mathbf{a}} = v_{\mathbf{a}} T_{w_{\mathbf{a}}^{-1}(k)}$ *for* $1 \le k \le r$ *such that* $s_k \in \mathfrak{S}_{\mathbf{a}}$*.*
(2) $L_k v_{\mathbf{a}} = u_j v_{\mathbf{a}}$ *for* $1 \le k \le r$ *satisfying* $k = r - a_j + 1$ *for some* j *with* $a_{j-1} < a_j$*.*

Observe that, if $\mathbf{a} = [\underline{\lambda}]$, then Lemma 3.3.2 implies (cf. (3.1.3))

$$L_{r-a_j+1} v_{[\underline{\lambda}]} = u_j v_{[\underline{\lambda}]}, \tag{3.3.4}$$

$$T_{r-a_j+b} v_{[\underline{\lambda}]} = v_{[\underline{\lambda}]} T_{a_{j-1}+b} \tag{3.3.5}$$

for $1 \le b \le |\lambda^{(j)}|$ and $1 \le j \le m$ with $|\lambda^{(j)}| \ge 1$.

We remark that the Specht module $S^{\underline{\lambda}}$ defined above is known [15, Theorem 2.9] to be isomorphic to those cell modules associated with $\underline{\lambda}'$ defined in [12] up to a twist by an automorphism of $\mathcal{H}_{\underline{u}}(r)$. The following lemma will be used later on.

Lemma 3.3.3 ([15, Corollary 2.3])**.** *Suppose that $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ is a multipartition of r with $|\lambda^{(k)}| = r_k$ for $1 \le k \le m$. Then*

$$z_{\underline{\lambda}} = v_{[\underline{\lambda}]}(z_{\lambda^{(1)}} \otimes \cdots \otimes z_{\lambda^{(m)}}) = (z_{\lambda^{(m)}} \otimes \cdots \otimes z_{\lambda^{(1)}})v_{[\underline{\lambda}]}, \tag{3.3.6}$$

where $z_{\lambda^{(1)}} \otimes \cdots \otimes z_{\lambda^{(m)}} \in \mathcal{H}(r_1) \otimes \cdots \otimes \mathcal{H}(r_m) \subseteq \mathcal{H}(r)$. Moreover, as $\mathcal{H}(r)$-modules, the following holds

$$S^{\underline{\lambda}}|_{\mathcal{H}(r)} \cong \mathrm{ind}_{\mathcal{H}(r_m)\otimes\cdots\otimes\mathcal{H}(r_1)}^{\mathcal{H}(r)}(S^{\lambda^{(m)}} \otimes \cdots \otimes S^{\lambda^{(1)}}). \tag{3.3.7}$$

3.4. The action of Jucys-Murphy elements on $S^{\underline{\lambda}}$. For later use, we now study the action of Jucys-Murphy elements on Specht modules for Ariki-Koike algebras.

In the Hecke algebra $\mathcal{H}(r)$, there is another set of interesting elements (also called Jucys-Murphy elements in some contexts (cf. [10])) defined by

$$\begin{aligned} J'_1 &= 0, \\ J'_k &= q^{-2}T_{(k-1,k)} + q^{-4}T_{(k-2,k)} + \cdots + q^{-2(k-1)}T_{(1,k)} \end{aligned}$$

for $2 \le k \le r$, where (i,j) denotes the transposition switching i and j for $1 \le i,j \le r$. It is easy to check that J'_k is related to the elements J_k defined in (3.2.1) via:

$$J_k = (q^2 - 1)J'_k + 1. \tag{3.4.1}$$

Lemma 3.4.4 ([11, Theorem 3.14])**.** *Let $\lambda \in \mathcal{P}(r)$ and $1 \le k \le r$. Assume that the node occupied by k in t_λ is (i,j) with $1 \le i \le \ell(\lambda)$ and $1 \le j \le \lambda_i$. Then*

$$J'_k z_\lambda = (1 + q^2 + \cdots + q^{2(j-i-1)})z_\lambda.$$

Given $\lambda \in \mathcal{P}(r)$ and a node $\rho = (i,j) \in \lambda$, define the residue of ρ relative to q to be

$$\mathrm{res}(\rho) = q^{2(j-i)}.$$

Accordingly, for a λ-tableau t and $1 \le k \le r$, let $\mathrm{res}_t(k)$ be the residue of the node occupied by k in t. For example, if $t = \begin{array}{|c|c|c|}\hline 3 & 4 & 1 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}$, then $\mathrm{res}_t(4) = q^2$. Then by Lemma 3.4.4 and (3.4.1), we obtain the following.

Corollary 3.4.5. *Let $\lambda \in \mathcal{P}(r)$ and $1 \le k \le r$. Then*

$$J_k z_\lambda = \mathrm{res}_{t_\lambda}(k) z_\lambda.$$

We now generalise the formula to Ariki–Koike algebras.

For $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}) \in \mathcal{P}_m(r)$ and $\rho = (i,j,k) \in \underline{\lambda}$, define its residue relative to the given parameters $\{q, u_1, \dots, u_m\}$ to be

$$\mathrm{res}(\rho) = u_k q^{2(j-i)}.$$

For a $\underline{\lambda}$-tableau $t = (t^{(1)}, t^{(2)}, \dots, t^{(m)})$ and $1 \le k \le r$, let $\mathrm{res}_t(k)$ be the residue of the node occupied by k in t. For example, for the tableau given in (3.1.4), we have $\mathrm{res}_t(4) = u_2 q^2$.

Proposition 3.4.6. *The following holds for any multipartition $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ of r:*

$$L_k z_{\underline{\lambda}} = \mathrm{res}_{t_{\underline{\lambda}}}(k) z_{\underline{\lambda}}, \quad 1 \le k \le r.$$

Proof. Fix $1 \le k \le r$ and assume $[\underline{\lambda}] = \mathbf{a} = [a_0, a_1, \dots, a_m]$. Suppose that $k = r - a_j + b = |\lambda^{(j+1)}| + \cdots + |\lambda^{(m)}| + b$ for $1 \le j \le m$ with $|\lambda^{(j)}| \ge 1$ and $1 \le b \le |\lambda^{(j)}|$. Then note that the node in $t_{\underline{\lambda}}$ occupied by k coincides with the node occupied by b in $t_{\lambda^{(j)}}$ and hence

$$\mathrm{res}_{t_{\underline{\lambda}}}(k) = u_j \mathrm{res}_{t_{\lambda^{(j)}}}(b). \tag{3.4.2}$$

Observe that by (3.2.2) the following holds

$$L_k = L_{r-a_j+b} = q^{2(1-b)} T_{r-a_j+b-1} \cdots T_{r-a_j+1} L_{r-a_j+1} T_{r-a_j+1} \cdots T_{r-a_j+b-1}.$$

This together with (3.3.6),(3.3.4) and (3.3.5) implies that

$$\begin{aligned} L_k z_{\underline{\lambda}} =& q^{2(1-b)} T_{r-a_j+b-1} \cdots T_{r-a_j+1} L_{r-a_j+1} T_{r-a_j+1} \cdots T_{r-a_j+b-1} z_{\underline{\lambda}} \\ =& q^{2(1-b)} T_{r-a_j+b-1} \cdots T_{r-a_j+1} L_{r-a_j+1} T_{r-a_j+1} \cdots T_{r-a_j+b-1} v_{[\underline{\lambda}]} (z_{\lambda^{(1)}} \otimes \cdots \otimes z_{\lambda^{(m)}}) \\ =& q^{2(1-b)} T_{r-a_j+b-1} \cdots T_{r-a_j+1} L_{r-a_j+1} v_{[\underline{\lambda}]} T_{a_{j-1}+1} \cdots T_{a_{j-1}+b-1} (z_{\lambda^{(1)}} \otimes \cdots \otimes z_{\lambda^{(m)}}) \\ =& u_j q^{2(1-b)} T_{r-a_j+b-1} \cdots T_{r-a_j+1} v_{[\underline{\lambda}]} T_{a_{j-1}+1} \cdots T_{a_{j-1}+b-1} (z_{\lambda^{(1)}} \otimes \cdots \otimes z_{\lambda^{(m)}}) \\ =& u_j q^{2(1-b)} v_{[\underline{\lambda}]} T_{a_{j-1}+b-1} \cdots T_{a_{j-1}+1} T_{a_{j-1}+1} \cdots T_{a_{j-1}+b-1} (z_{\lambda^{(1)}} \otimes \cdots \otimes z_{\lambda^{(m)}}). \end{aligned} \tag{3.4.3}$$

Meanwhile, observe that the element $q^{1-b} T_{a_{j-1}+b-1} \cdots T_{a_{j-1}+1} T_{a_{j-1}+1} \cdots T_{a_{j-1}+b-1}$ belongs to the subalgebra $\mathcal{H}(\mu)$ of the Hecke algebra $\mathcal{H}(r)$ and moreover it can be identified with $1^{\otimes j-1} \otimes J_b \otimes 1^{\otimes m-j}$ by (3.2.1), where $\mu = (|\lambda^{(1)}|, \dots, |\lambda^m|)$. Then by Corollary 3.4.5, we obtain

$$\begin{aligned} & q^{2(1-b)} T_{a_{j-1}+b-1} \cdots T_{a_{j-1}+1} T_{a_{j-1}+1} \cdots T_{a_{j-1}+b-1} (z_{\lambda^{(1)}} \otimes \cdots \otimes z_{\lambda^{(m)}}) \\ & = z_{\lambda^{(1)}} \otimes \cdots \otimes J_b z_{\lambda^{(j)}} \otimes \cdots \otimes z_{\lambda^{(m)}} \\ & = \mathrm{res}_{t_{\lambda^{(j)}}}(b) z_{\lambda^{(1)}} \otimes \cdots \otimes z_{\lambda^{(m)}}. \end{aligned}$$

Thus by (3.4.3) and (3.4.2) one can deduce that

$$L_k z_{\underline{\lambda}} = u_j \mathrm{res}_{t_{\lambda^{(j)}}}(b) v_{[\underline{\lambda}]} (z_{\lambda^{(1)}} \otimes \cdots \otimes z_{\lambda^{(m)}}) = \mathrm{res}_{t_{\underline{\lambda}}}(k) z_{\underline{\lambda}},$$

as desired. □

Recall that any $\mathcal{H}_{\underline{u}}(r)$-module M can be regarded as an $\mathcal{H}_{\vartriangle}(r)$-module by inflation in the case that $u_1, \dots, u_m$ are all non-zero. The following result will be used later on.

Proposition 3.4.7. *Suppose that $\underline{u} = (u_1, \dots, u_m) \in (\mathbb{C}^*)^m$ and $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)}) \in \mathcal{P}_m(r)$. Then the inflation of the $\mathcal{H}_{\underline{u}}(r)$-module $S^{\underline{\lambda}}$ gives an $\mathcal{H}_{\vartriangle}(r)$-module isomorphism:*

$$S^{\underline{\lambda}} \cong \mathrm{ind}_{\mathcal{H}_{\vartriangle}(\mu)}^{\mathcal{H}_{\vartriangle}(r)} (\mathrm{ev}^*_{u_m}(S^{\lambda^{(m)}}) \otimes \cdots \otimes \mathrm{ev}^*_{u_1}(S^{\lambda^{(1)}})),$$

where $\mu = (|\lambda^{(m)}|, \dots, |\lambda^{(2)}|, |\lambda^{(1)}|)$.

Proof. By (3.2.3), Lemma 3.3.2(1) and Proposition 3.4.6, it is straightforward to check that the following map

$$\begin{aligned} \zeta : \mathrm{ev}^*_{u_m}(S^{\lambda^{(m)}}) \otimes \cdots \otimes \mathrm{ev}^*_{u_1}(S^{\lambda^{(1)}}) &\longrightarrow S^{\underline{\lambda}} \\ z_{\lambda^{(m)}} \otimes \cdots \otimes z_{\lambda^{(1)}} &\mapsto z_{\underline{\lambda}} \end{aligned}$$

is an $\mathcal{H}_{\vartriangle}(\mu)$-homomorphism. Then by Frobenius reciprocity, there exists a nonzero homomorphism $\bar{\zeta} : \mathrm{ind}_{\mathcal{H}_{\vartriangle}(\mu)}^{\mathcal{H}_{\vartriangle}(r)} \mathrm{ev}^*_{u_m}(S^{\lambda^{(m)}}) \otimes \cdots \otimes \mathrm{ev}^*_{u_1}(S^{\lambda^{(1)}}) \to S^{\underline{\lambda}}$ which is also surjective since $S^{\underline{\lambda}}$ is generated by $z_{\underline{\lambda}}$. By (3.3.7), a dimension comparison proves the proposition. □

Remark 3.4.8. By Proposition 3.4.7, the $\mathcal{H}_\vartriangle(r)$-module $\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)}(\mathrm{ev}_{u_m}^*(S^{\lambda^{(m)}}) \otimes \cdots \otimes \mathrm{ev}_{u_1}^*(S^{\lambda^{(1)}}))$ affords an $\mathcal{H}_{\underline{u}}(r)$-module. Actually, it is not hard to check directly that the polynomial $(X_1 - u_1)(X_1 - u_2) \cdots (X_1 - u_m)$ acts as zero and, hence, the $\mathcal{H}_\vartriangle(r)$-module structure factors through the surjective homomorphism $\varpi_{\underline{u}}$ followed by the $\mathcal{H}_{\underline{u}}(r)$-module structure on $\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)}(\mathrm{ev}_{u_m}^*(S^{\lambda^{(m)}}) \otimes \cdots \otimes \mathrm{ev}_{u_1}^*(S^{\lambda^{(1)}}))$.

3.5. Ariki–Koike algebras of integral type and Kleshchev multipartitions. The classification of simple $\mathcal{H}_{\underline{u}}(r)$-modules has been completed by Ariki in terms of Kleshchev multipartitions; see [2, Theorem 4.2] and [4, conjecture 2.12]. It is also known from [4, §1] that the classification of irreducible $\mathcal{H}_{\underline{u}}(r)$-modules is reduced to the cases where the parameters $u_1, \ldots, u_m$ are either all zero or all powers of q^2. For our purpose, we are mainly concerned about the latter one (see footnote 2).

For $m \geq 1$, let

$$\mathfrak{F}_m = \{(f_1, \ldots, f_m) \in \mathbb{Z}^m \mid f_1 \geq \cdots \geq f_m\},$$

and let

$$\mathfrak{F} = \bigcup_{m \geq 1} \mathfrak{F}_m. \tag{3.5.1}$$

For $f = (f_1, \ldots, f_m) \in \mathfrak{F}$, let $f^* = (-f_m, \ldots, -f_1)$. Clearly, this gives a bijection

$$(\ )^* : \mathfrak{F} \longrightarrow \mathfrak{F}.$$

We continue to assume that q is *not a root of unity* and consider the parameter vectors of the form

$$\underline{u}_f = (q^{2f_1}, \ldots, q^{2f_m}) \quad (f \in \mathfrak{F}_m). \tag{3.5.2}$$

In other words, $u_1 = (q^2)^{f_1}, \ldots, u_m = (q^2)^{f_m}$. An Ariki–Koike algebra $\mathcal{H}_{\underline{u}}(r)$ is of *integral type* if $\underline{u} = \underline{u}_f$ for some $f \in \mathfrak{F}$. Thus, Kleshchev multipartions for an Ariki–Koike algebra $\mathcal{H}_{\underline{u}}(r)$ of integral type (and a parameter which is not a root of unity) can be described as follows.

Definition 3.5.9. Given a multipartition $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(m)})$ of r, we will say that $\underline{\lambda}$ is a *Kleshchev multipartition* with respect to $f \in \mathfrak{F}_m$ (or to $\underline{u}_f$) if it satisfies

$$\lambda^{(k)}_{j+f_k-f_{k+1}} \leq \lambda^{(k+1)}_j \tag{3.5.3}$$

for $j \geq 1$ and $1 \leq k \leq m-1$.

Note that, when q is a root of unity, the definition of Kleshchev multipartitions is much more complicated.

Remark 3.5.10. Note that if $\ell(\lambda^{(k)}) \leq f_k - f_{k+1} + 1$ then (3.5.3) is equivalent to $\lambda^{(k)}_{1+f_k-f_{k+1}} \leq \lambda^{(k+1)}_1$.

Denote by $\mathcal{K}_{\underline{u}_f}(r)$ or $\mathcal{K}_f(r)$ the set of Kleshchev multipartitions of r with respect to $\underline{u}_f$. For $\underline{u} = (u_1, \ldots, u_m)$, set

$$\underline{u}^{-1} = (u_m^{-1}, \ldots, u_1^{-1}).$$

Then by (3.5.3), a multipartition $\underline{\gamma} = (\gamma^{(1)}, \ldots, \gamma^{(m)})$ belongs to $\mathcal{K}_{\underline{u}_f^{-1}}(r) = \mathcal{K}_{f^*}(r)$ if and only if

$$\gamma^{(k)}_{j+f_{m-k}-f_{m-k+1}} \leq \gamma^{(k+1)}_j \tag{3.5.4}$$

for $j \geq 1$ and $1 \leq k \leq m-1$.

It is known [5, Corollary 2] that the Ariki-Koike algebra $\mathcal{H}_{\underline{u}_f}(r)$ for $f \in \mathfrak{F}$ and q not root of unity is isomorphic to the corresponding degenerate cyclotomic Hecke algebra over $\mathbb{C}$. Then we have the following.

Proposition 3.5.11 ([6, Theorem 3.18, Lemma 3.13] (cf. [4, 2, 28])). *Assume that $f = (f_1, \dots, f_m) \in \mathfrak{F}$ with $m \geq 1$. The following holds for all $r \geq 1$:*

(1) *For multipartition $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)}) \in \mathcal{P}_m(r)$ such that $\underline{\lambda}' \in \mathcal{K}_{f^*}(r)$, the $\mathcal{H}_\vartriangle(r)$-module $\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)}(\mathrm{ev}^*_{q^{2f_m}}(S^{\lambda^{(m)}}) \otimes \cdots \otimes \mathrm{ev}^*_{q^{2f_1}}(S^{\lambda^{(1)}}))$ has irreducible head denoted by $D^{\underline{\lambda}}$, where $\mu = (|\lambda^{(m)}|, \dots, |\lambda^{(1)}|)$. In particular, $D^{\underline{\lambda}}$ affords a simple $\mathcal{H}_{\underline{u}_f}(r)$-module.*

(2) *The set $\{D^{\underline{\lambda}'} \mid \underline{\lambda} \in \mathcal{K}_{f^*}(r)\}$ is a complete set of non-isomorphic irreducible $\mathcal{H}_{\underline{u}_f}(r)$-modules.*

By Proposition 3.4.7 and Proposition 3.5.11, we have the following.

Corollary 3.5.12. *For $\underline{\lambda} \in \mathcal{P}_m(r)$ satisfying $\underline{\lambda}' \in \mathcal{K}_{f^*}(r)$ with $f \in \mathfrak{F}_m$, the $\mathcal{H}_{\underline{u}_f}(r)$-module $S^{\underline{\lambda}}$ has irreducible head isomorphic to $D^{\underline{\lambda}}$.*

We will construct a subset of $\cup_{f\in\mathfrak{F}}\mathcal{K}_f(r)$ which labels precisely the irreducible $\mathcal{H}_\vartriangle(r)$-module with "integral weights", i.e., the irreducible objects in $\mathcal{H}_\vartriangle(r)\text{-}\mathsf{mod}^{\mathbb{Z}}$.

4. Standard Kleshchev multipartitions and integral multisegments

At the end of §2, we saw that simple objects in the category $\mathcal{H}_\vartriangle(r)\text{-}\mathsf{mod}^{\mathbb{Z}}$ are indexed by the set $\mathscr{S}_r^{\mathbb{Z}}$. The set can be further described in terms of standard sequences of integral segments or standard words. In this section, we will describe the set in terms of standard Kleshchev multipartitions. In particular, we will characterise the simple objects in the category $\mathcal{H}_\vartriangle(r)\text{-}\mathsf{mod}^{\mathbb{Z}}$ by these standard Kleshchev multipartitions and their associated irreducible modules of Ariki–Koike algebras.

4.1. Column residual segments of a multipartition.

We first construct a sequence $\mathbf{s}$ of segments via the column residual segment of a multipartition $\underline{\lambda}$ and show that $I_{\mathbf{s}}$ maps onto $S^{\underline{\lambda}}$.

For $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}) \in \mathcal{P}_m(r)$ and parameter vector $\underline{u} = (u_1, \dots, u_m)$, define the jth-*column residual segment* of $\lambda^{(i)}$

$$\mathsf{s}_j^{(i)} = (u_i q^{2(j-(\lambda^{(i)'})_j)}, \dots, u_i q^{2(j-2)}, u_i q^{2(j-1)}) \quad (1 \leq i \leq m, 1 \leq j \leq \lambda_1^{(i)})$$

by reading the residues of the nodes in the j-th column of $\lambda^{(i)}$ from bottom to top, and form the sequence of column residual segments from column 1, then column 2, and so on, starting from the last partition of $\underline{\lambda}$, then the second last partition, and so on,

$$\mathbf{s}^{\mathsf{c}}_{\underline{\lambda};\underline{u}} = \big(\mathsf{s}_1^{(m)}, \dots, \mathsf{s}^{(m)}_{\lambda_1^{(m)}}, \dots, \mathsf{s}_1^{(1)}, \dots, \mathsf{s}^{(1)}_{\lambda_1^{(1)}}\big); \tag{4.1.1}$$

cf. the order of the Young diagram $t_{\underline{\lambda}}$ in (3.1.4). We will simply write $\mathbf{s}^{\mathsf{c}}_{\underline{\lambda};f}$ in the sequel, if $\underline{u} = \underline{u}_f$ for some $f \in \mathfrak{F}_m$.

Lemma 4.1.1. *Assume $\underline{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(m)}) \in \mathcal{P}_m(r)$ and $\underline{u} = (u_1 \dots, u_m) \in (\mathbb{C}^*)^m$. Write $\mathbf{s} = \mathbf{s}^{\mathsf{c}}_{\underline{\lambda};\underline{u}}$. Then*

(1) *The subspace $\mathbb{C}z_{\underline{\lambda}}$ of $\mathcal{H}_{\underline{u}}(r)$ affords an $\mathcal{H}_\vartriangle((\underline{\lambda}')^\vee)$-submodule of the inflation of the $\mathcal{H}_{\underline{u}}(r)$-module $S^{\underline{\lambda}}$.*

(2) *There exists a surjective $\mathcal{H}_\vartriangle(r)$-homomorphism $\varphi : \mathrm{Ind}^{\mathcal{H}_\vartriangle(r)}_{\mathcal{H}_\vartriangle((\underline{\lambda}')^\vee)}\mathbb{C}z_{\underline{\lambda}} \twoheadrightarrow S^{\underline{\lambda}}$ such that $\varphi(1\otimes z_{\underline{\lambda}}) = z_{\underline{\lambda}}$.*

(3) *There exists a surjective $\mathcal{H}_\vartriangle(r)$-homomorphism $\widetilde{\varphi} : I_{\mathbf{s}} \twoheadrightarrow S^{\underline{\lambda}}$ with $\widetilde{\varphi}\big(C_{w^0_{\mu(\mathbf{s})}}\big) = z_{\underline{\lambda}}$.*

Proof. By (3.3.3), one can deduce that

$$T_i z_{\underline{\lambda}} = -z_{\underline{\lambda}}$$

for any $T_i \in \mathcal{H}((\underline{\lambda}')^\vee)$. On the other hand, by Proposition 3.4.6 we have

$$X_k z_{\underline{\lambda}} = L_k z_{\underline{\lambda}} = \mathrm{res}_{t_{\underline{\lambda}}}(k) z_{\underline{\lambda}}$$

for $1 \le k \le r$. Since $\mathcal{H}_\vartriangle((\underline{\lambda}')^\vee)$ is generated by $\mathcal{H}((\underline{\lambda}')^\vee)$ and $X_1^{\pm1}, \ldots, X_r^{\pm1}$, Part (1) is verified.

Now, by Frobenius reciprocity, there exists an $\mathcal{H}_\vartriangle(r)$-homomorphism

$$\varphi : \mathrm{Ind}^{\mathcal{H}_\vartriangle(r)}_{\mathcal{H}_\vartriangle((\underline{\lambda}')^\vee)}\mathbb{C}z_{\underline{\lambda}} \to S^{\underline{\lambda}}$$

which sends $h\otimes z_{\underline{\lambda}}$ to $hz_{\underline{\lambda}}$ for all $h \in \mathcal{H}_\vartriangle(r)$. Since $S^{\underline{\lambda}}$ is generated by $z_{\underline{\lambda}}$, Part (2) holds.

Observe from (4.1.1), (2.2.4), and the standard tableau $t_{\underline{\lambda}}$ (see (3.1.4)) that the following holds

$$\begin{aligned} \widetilde{\mathbf{s}}^\vee &= (\mathrm{res}_{t_{\underline{\lambda}}}(1), \mathrm{res}_{t_{\underline{\lambda}}}(2), \ldots, \mathrm{res}_{t_{\underline{\lambda}}}(r)), \\ \mu(\mathbf{s}) &= (\underline{\lambda}')^\vee. \end{aligned} \tag{4.1.2}$$

Then by (2.2.2) and Proposition 2.2.2(2) we have

$$\begin{aligned} T_i C_{w^0_{\mu(\mathbf{s})}} &= -\,C_{w^0_{\mu(\mathbf{s})}}, \\ X_k C_{w^0_{\mu(\mathbf{s})}} &= \mathrm{res}_{t_{\underline{\lambda}}}(k) C_{w^0_{\mu(\mathbf{s})}} \end{aligned}$$

for $T_i \in \mathcal{H}((\underline{\lambda}')^\vee)$ and $1 \le k \le r$. This means

$$\mathbb{C}C_{w^0_{\mu(\mathbf{s})}} \cong \mathbb{C}z_{\underline{\lambda}}$$

as $\mathcal{H}_\vartriangle((\underline{\lambda}')^\vee)$-modules by the proof of Part (1). Hence, by Lemma 2.3.3, there exists an isomorphism

$$\phi : I_{\mathbf{s}} \xrightarrow{\cong} \mathrm{Ind}^{\mathcal{H}_\vartriangle(r)}_{\mathcal{H}_\vartriangle((\underline{\lambda}')^\vee)}\mathbb{C}z_{\underline{\lambda}}.$$

This together with Part (2) gives Part (3) if we set $\widetilde{\varphi} = \varphi\circ\phi$. □

4.2. Kleshchev multipartitions associated with integral multisegments. We shall construct a Kleshchev multipartition associated to an integral multisegment as follows.

Given $\binom{j_1 j_2 \ldots j_t}{i_1 i_2 \ldots i_t} \in \mathcal{W}^s(r)$ so that $\mathfrak{s} = \flat\binom{j_1 j_2 \ldots j_t}{i_1 i_2 \ldots i_t} \in \mathscr{S}^{\mathbb{Z}}_r$ (see (2.4.6)), let $1 \le k_1 \le t$ be the right-most column index satisfying $j_a = j_{a-1}+1$, for $2 \le a \le k_1$, and $i_1 < i_2 < \cdots < i_{k_1}$. Define $\gamma^{(1)} = (\gamma^{(1)}_1, \ldots, \gamma^{(1)}_{k_1})$ by setting

$$\gamma^{(1)}_b = |j_b - i_b + 1|$$

for $1 \le b \le k_1$. Note that $k_1 = j_{k_1} - j_1 + 1$ and $\gamma^{(1)}$ is a partition.

Let

$$\mathfrak{s}^1 = \flat\begin{pmatrix} j_{k_1+1} j_{k_1+2} \cdots j_t \\ i_{k_1+1} i_{k_1+2} \ldots i_t \end{pmatrix} \in \mathscr{S}^{\mathbb{Z}}_{r-|\gamma^{(1)}|}.$$

Now applying the same procedure to $\mathfrak{s}^1$, we obtain a positive number $1 \leq k_2 \leq t - k_1$, a partition $\gamma^{(2)}$ and $\mathfrak{s}^2 \in \mathscr{S}^{\mathbb{Z}}_{r-|\gamma^{(1)}|-|\gamma^{(2)}|}$. Continuing in this way, we will end up with $1 \leq k_1, \dots, k_m \leq t$ and a multipartition

$$\underline{\gamma}_{\mathfrak{s}} = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(m)}), \tag{4.2.1}$$

where $\gamma^{(a)} = (\gamma^{(a)}_1, \gamma^{(a)}_2, \dots, \gamma^{(a)}_{k_a})$ with

$$\gamma^{(a)}_b = |j_{k_1+\cdots+k_{a-1}+b} - i_{k_1+\cdots+k_{a-1}+b} + 1|$$

for all $1 \leq a \leq m$ and $1 \leq b \leq k_a$. Note that

$$k_a = j_{k_1+\cdots+k_a} - j_{k_1+\cdots+k_{a-1}+1} + 1. \tag{4.2.2}$$

Meanwhile, define

$$\underline{u}_{\mathfrak{s}} = (u_1, \dots, u_m) \tag{4.2.3}$$

by setting

$$u_1 = q^{2(j_{k_1+\cdots+k_{m-1}+1})},\ u_2 = q^{2(j_{k_1+\cdots+k_{m-2}+1})},\ \dots, u_m = q^{2j_1}.$$

(So $f_i = j_{k_1+\cdots+k_{m-i}+1}$ in the notation of (3.5.2).) Observe that

$$\mathfrak{s} = \mathfrak{s}' \text{ if and only if } \underline{\gamma}_{\mathfrak{s}} = \underline{\gamma}_{\mathfrak{s}'}, \underline{u}_{\mathfrak{s}} = \underline{u}_{\mathfrak{s}'}. \tag{4.2.4}$$

Example 4.2.2. Suppose

$$\mathfrak{s} = \flat(w),\ \text{where } w = \begin{pmatrix} -1 & 0 & 1 & 2 & 2 \\ -4 & -5 & -2 & -1 & -2 \end{pmatrix}.$$

Then by the procedure described above, one can deduce that $k_1 = 1, k_2 = 3, k_3 = 1$, and $\underline{\gamma}_{\mathfrak{s}} = (\gamma^{(1)}, \gamma^{(2)}, \gamma^{(3)})$, where

$$\gamma^{(1)} = (4), \gamma^{(2)} = (6, 4, 4),\ \text{and } \gamma^{(3)} = (5).$$

Moreover, $\underline{u}_{\mathfrak{s}} = (u_1, u_2, u_3)$ with

$$u_1 = q^4, u_2 = q^0, u_3 = q^{-2}.$$

We also observe from this example that $\underline{\gamma}_{\mathfrak{s}} \in \mathcal{K}_{\underline{u}_{\mathfrak{s}}^{-1}}(r)$ is Kleshchev. Moreover, for multipartition

$$\underline{\lambda}_{\mathfrak{s}} = \underline{\gamma}'_{\mathfrak{s}} = ((1^5), (3^4 1^2), (1^4))$$

and the $u_{\mathfrak{s}}$ above, the sequence $\mathbf{s}^{\mathsf{c}}_{\underline{\lambda}_{\mathfrak{s}};\underline{u}_{\mathfrak{s}}}$ constructed in (4.1.1) is the standard element in $\mathfrak{s}$.

In fact, the last observation holds in general.

Proposition 4.2.3. *For $\mathfrak{s} \in \mathscr{S}^{\mathbb{Z}}_r$, let $\gamma_{\mathfrak{s}}$ and $u_{\mathfrak{s}}$ be defined as in (4.2.1) and (4.2.3). Then $\underline{\gamma}_{\mathfrak{s}}$ is a sincere Kleshchev multipartition with respect to $u_{\mathfrak{s}}^{-1}$, and moreover $\mathbf{s}^{\mathsf{c}}_{\underline{\gamma}'_{\mathfrak{s}};\underline{u}_{\mathfrak{s}}}$ is the standard element in $\mathfrak{s}$.*

Proof. Suppose $w = \binom{j_1 j_2 \dots j_t}{i_1 i_2 \dots i_t} \in \mathcal{W}^s(r)$ with $\mathfrak{s} = \flat(w)$. Write $\underline{\gamma}_{\mathfrak{s}} = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(m)})$ with $\ell(\gamma^{(a)}) = k_a = j_{k_1+\cdots+k_a} - j_{k_1+\cdots+k_{a-1}+1} + 1$ for $1 \leq a \leq m$. So $\underline{\gamma}_{\mathfrak{s}}$ is sincere. By (3.5.4) and (4.2.3), it suffices to show that the following holds for $1 \leq a \leq m - 1$ and $b \geq 1$:

$$\gamma^{(a)}_{b+j_{k_1+\cdots+k_a+1} - j_{k_1+\cdots+k_{a-1}+1}} \leq \gamma^{(a+1)}_b \tag{4.2.5}$$

Fix $1 \leq a \leq m-1$ and $b \geq 1$. If $\gamma^{(a)}_{b+j_{k_1+\cdots+k_a+1} - j_{k_1+\cdots+k_{a-1}+1}} = 0$, then (4.2.5) automatically holds.

Suppose now $\gamma^{(a)}_{b+j_{k_1+\cdots+k_a+1}-j_{k_1+\cdots+k_{a-1}+1}} > 0$. Since w is standard, the sequence $j_1,\dots,j_t$ is weakly increasing. By (4.2.2),

$$j_{k_1+\cdots+k_a+1} - j_{k_1+\cdots+k_{a-1}+1} \geq j_{k_1+\cdots+k_a} - j_{k_1+\cdots+k_{a-1}+1} = k_a - 1. \tag{4.2.6}$$

Hence, we obtain from the definition of the partition $\gamma^{(a)}$ that

$$0 < \gamma^{(a)}_{b+j_{k_1+\cdots+k_a+1}-j_{k_1+\cdots+k_{a-1}+1}} \leq \gamma^{(a)}_{b+k_a-1},$$

which forces $b = 1$ and $1 + j_{k_1+\cdots+k_a+1} - j_{k_1+\cdots+k_{a-1}+1} = k_a$ since $\ell(\gamma^{(a)}) = k_a$ (cf. Remark 3.5.10). Thus, (4.2.6) must be an equality and consequently,

$$j_{k_1+\cdots+k_a+1} = j_{k_1+\cdots+k_a}. \tag{4.2.7}$$

Hence, the fact that w is standard implies

$$i_{k_1+\cdots+k_a} \geq i_{k_1+\cdots+k_a+1}. \tag{4.2.8}$$

Finally, by (4.2.7) and (4.2.8),

$$\begin{aligned}\gamma^{(a)}_{b+j_{k_1+\cdots+k_a+1}-j_{k_1+\cdots+k_{a-1}+1}} &= \gamma^{(a)}_{k_a}\\ &= j_{k_1+\cdots+k_a} - i_{k_1+\cdots+k_a} + 1\\ &= j_{k_1+\cdots+k_a+1} - i_{k_1+\cdots+k_a} + 1\\ &\leq j_{k_1+\cdots+k_a+1} - i_{k_1+\cdots+k_a+1} + 1\\ &= \gamma^{(a+1)}_1 = \gamma^{(a+1)}_b\end{aligned}$$

proving (4.2.5).

The last assertion follows from the definition. □

Theorem 4.2.4. *For any given multisegment $\mathfrak{s} \in \mathscr{S}_r^{\mathbb{Z}}$, let $\underline{\gamma}_{\mathfrak{s}}$ and $\underline{u}_{\mathfrak{s}}$ be defined as in (4.2.1) and (4.2.3) and let $\underline{\lambda}_{\mathfrak{s}} = \underline{\gamma}'_{\mathfrak{s}}$. Then the irreducible $\mathcal{H}_\vartriangle(r)$-module $V_{\mathfrak{s}}$ is isomorphic to the inflated $\mathcal{H}_\vartriangle(r)$-module $D^{\underline{\lambda}_{\mathfrak{s}}}$. In particular, $V_{\mathfrak{s}}$ affords an $\mathcal{H}_{\underline{u}_{\mathfrak{s}}}(r)$-module.*

Proof. Let $\mathbf{s} \in \mathfrak{s}$ be standard. By Proposition 4.2.3, $\underline{\lambda}'_{\mathfrak{s}} = \underline{\gamma}_{\mathfrak{s}} \in \mathcal{K}_{\underline{u}_{\mathfrak{s}}^{-1}}(r)$ and $\mathbf{s} = \mathbf{s}^{\mathfrak{c}}_{\underline{\lambda}_{\mathfrak{s}};\underline{u}_{\mathfrak{s}}}$. Thus, by Corollary 3.5.12, the Specht module $S^{\underline{\lambda}_{\mathfrak{s}}}$ has the simple head $D^{\underline{\lambda}_{\mathfrak{s}}}$. On the other hand, by Lemma 4.1.1, there exists a surjective $\mathcal{H}_\vartriangle(r)$-homomorphism $I_{\mathbf{s}} \twoheadrightarrow S^{\underline{\lambda}_{\mathfrak{s}}}$, which extends to a surjective $\mathcal{H}_\vartriangle(r)$-homomorphism $I_{\mathbf{s}} \twoheadrightarrow D^{\underline{\lambda}_{\mathfrak{s}}}$. This together with Proposition 2.3.8 gives the required isomorphism.

The last assertion is clear since $D^{\underline{\lambda}_{\mathfrak{s}}}$ is an $\mathcal{H}_{\underline{u}_{\mathfrak{s}}}(r)$-module. □

In the case $m = 1, u_1 = 1$, Theorem 4.2.4 recovers [13, Proposition 8.2].

Remark 4.2.5. Fix $\mathfrak{s} \in \mathscr{S}_r^{\mathbb{Z}}$. It is known [1] that the decomposition numbers for the Ariki-Koike algebras $\mathcal{H}_{\underline{u}_{\mathfrak{s}}}(r)$ over $\mathbb{C}$ can be computed in terms of the canonical basis of an associated integral highest weight module of the Lie algebra $\mathfrak{gl}_\infty(\mathbb{C})$. Using Theorem 4.2.4 and Lemma 3.3.3, one can derive a dimension formula for $V_{\mathfrak{s}}$ and the decomposition of the restriction $V_{\mathfrak{s}}$ to $\mathcal{H}(r)$ in terms of Spech modules S^λ for $\lambda \in \mathcal{P}(r)$.

4.3. Standard Kleshchev multipartitions and simple objects in $\mathcal{H}_\vartriangle(r)$-$\mathsf{mod}^{\mathbb{Z}}$. We now characterise the Kleshchev multipartition $\underline{\gamma}_{\mathfrak{s}}$ constructed from a standard word w with $\mathfrak{s} = \flat(w)$.

Recall the sets $\mathfrak{F}_m, \mathfrak{F}$ in (3.5.1) and the elements $\underline{u}_f = (q^{2f_1}, q^{2f_2}, \ldots, q^{2f_m})$ for $f = (f_1, \ldots, f_m) \in \mathfrak{F}_m$ in (3.5.2). For $f \in \mathfrak{F}_m$, let

$$\|f\| = \min\{f_i - f_{i+1} \mid 1 \le i < m\}.$$

Definition 4.3.6. A multipartitions $\underline{\gamma} = (\gamma^{(1)}, \ldots, \gamma^{(m)})$ of r is called a *standard Kleshchev multipartition* relative to $f = (f_1, \ldots, f_m) \in \mathfrak{F}$, if it satisfies

(SK1) $\underline{\gamma}$ is sincere (or equivalently, $\ell(\gamma^{(a)}) \ge 1$ for each $1 \le a \le m$).

(SK2) $\ell(\gamma^{(a)}) \le f_a - f_{a+1} + 1$ and $\gamma^{(a)}_{f_a - f_{a+1}+1} \le \gamma^{(a+1)}_1$ for each $1 \le a \le m-1$.

(SK3) If $\ell(\gamma^{(a)}) = f_a - f_{a+1}$ for some $1 \le a \le m-1$, then $\gamma^{(a)}_{f_a - f_{a+1}} \le \gamma^{(a+1)}_1 - 1$.

Observe from the condition (SK2) that a standard Kleshchev multipartition is a Kleshchev multipartition; see Remark 3.5.10.

Let $\mathcal{K}^s_f(r)$ be the set of all standard Kleshchev multipartitions of r relative to $f \in \mathfrak{F}$. Then

$$\mathcal{K}^s_f(r) \subseteq \mathcal{K}_f(r) = \mathcal{K}_{\underline{u}_f}(r).$$

Lemma 4.3.7. *Suppose $f = (f_1, \ldots, f_m) \in \mathfrak{F}$. Then*

(1) *For any $c \in \mathbb{Z}$, we have*

$$\mathcal{K}^s_f(r) = \mathcal{K}^s_{f+c}(r)$$

where $f + c = (f_1 + c, \ldots, f_m + c)$.

(2) *If $m = 1$, then $\mathcal{K}^s_f(r) = \mathcal{P}(r)$, the set of partitions of r.*

(3) *If $m > 1$ and $\|f\| \ge r$, then $m \le r$ and $\mathcal{K}^s_f(r)$ is the set of all sincere multipartitions in $\mathcal{P}_m(r)$.*

(4) *If $f_1 = f_2 = \cdots = f_m$, then $m \le r$ and*

$$\mathcal{K}^s_f(r) \equiv \{(a_1, \ldots, a_m) \in \Lambda(m, r) \mid 1 \le a_1 \le \cdots \le a_m\}.$$

Proof. Observe that the conditions (SK1)-(SK3) depend only on $f_a - f_{a+1}$ for $1 \le a \le m-1$ and hence Part (1) holds. Clearly Part (2) is true. Now suppose $f_a - f_{a+1} \ge r$ for all $1 \le a \le m-1$ and $\underline{\gamma} = (\gamma^{(1)}, \ldots, \gamma^{(m)}) \in \mathcal{P}_m(r)$ is sincere. Then $|\gamma^{(a)}| \le r - m + 1 \le r - 1$ and hence (SK2) and (SK3) automatically hold for $\underline{\gamma}$ and thus $\underline{\gamma} \in \mathcal{K}^s_f(r)$, proving Part (3). Finally, if $f_1 = f_2 = \cdots = f_m$, the $\underline{\gamma} = (\gamma^{(1)}, \ldots, \gamma^{(m)}) \in \mathcal{K}^s_f(r)$ if and only if $\gamma^{(i)} = (\gamma^{(i)}_1)$ for $1 \le i \le m$ and $\gamma^{(1)}_1 \le \cdots \le \gamma^{(m)}_1$ by (SK2). This proves Part (4). $\square$

Recall from §4.2 that associated to each $\mathfrak{s} \in \mathscr{S}^{\mathbb{Z}}_r$, there exist $\underline{\gamma}_{\mathfrak{s}}$ and $\underline{u}_{\mathfrak{s}}$ such that $\underline{\gamma}_{\mathfrak{s}} \in \mathcal{K}_{\underline{u}_{\mathfrak{s}}^{-1}}(r)$. Thus, every $f \in \mathfrak{F}$ defines a subset

$$\mathscr{S}^{\mathbb{Z}}_{r,f} := \{\mathfrak{s} \in \mathscr{S}^{\mathbb{Z}}_r \mid \underline{u}_{\mathfrak{s}} = \underline{u}_f\},$$

and a map

$$\begin{aligned} \eta_f : \mathscr{S}^{\mathbb{Z}}_{r,f} &\longrightarrow \mathcal{K}_{\underline{u}_f^{-1}}(r) = \mathcal{K}_{f^*}(r) \\ \mathfrak{s} &\longmapsto \underline{\gamma}_{\mathfrak{s}} \end{aligned}$$

which is injective by (4.2.4).

Proposition 4.3.8. *With the notation above we have $\mathrm{im}(\eta_f) = \mathcal{K}^s_{f^*}(r)$ and hence the map*

$$\eta_f : \mathscr{S}^{\mathbb{Z}}_{r,f} \longrightarrow \mathcal{K}^s_{f^*}(r)$$

is a bijection.

Proof. Suppose $\mathfrak{s} \in \mathscr{S}_{r,f}^{\mathbb{Z}}$ and assume that $\mathfrak{s} = \flat(w)$ with $w = \binom{j_1 j_2 \dots j_t}{i_1 i_2 \dots i_t} \in \mathcal{W}^s(r)$. Then, $\underline{u}_{\mathfrak{s}} = \underline{u}_f$ and so $f_i = j_{k_1+\cdots+k_{m-i}+1}$ with k_a as given in (4.2.2). In particular,

$$j_{k_1+\cdots+k_{a-1}+1} = f_{m-a+1}, \quad j_{k_1+\cdots+k_a} = f_{m-a+1} + k_a - 1$$

for all $1 \le a \le m$. We want to show that $\underline{\gamma}_{\mathfrak{s}} = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(m)})$ defined in (4.2.1) satisfies the conditions (SK1)–(SK3).

First,

$$\ell(\gamma^{(a)}) = k_a \ge 1, \text{ for all } 1 \le a \le m, \tag{4.3.1}$$

and $f_a^* = -f_{m-a+1} = -j_{k_1+\cdots+k_{a-1}+1}$ for $1 \le a \le m$. Also, the hypothesis that the word w is standard implies that

$$k_a - 1 + f_{m-a+1} = j_{k_1+\cdots+k_a} \le j_{k_1+\cdots+k_a+1} = f_{m-a}.$$

Hence,

$$k_a \le f_{m-a} - f_{m-a+1} + 1 = f_a^* - f_{a+1}^* + 1 \tag{4.3.2}$$

for all $1 \le a \le m-1$. Thus, $\underline{\gamma}_{\mathfrak{s}}$ satisfies (SK1) and (SK2) by (4.3.1) and (4.3.2) and Proposition 4.2.3. It remains to prove (SK3). If $k_a = f_a^* - f_{a+1}^* = f_{m-a} - f_{m-a+1}$ for some $1 \le a \le m-1$, then we have

$$j_{k_1+\cdots+k_a} = f_{m-a+1} + k_a - 1 = f_{m-a} - 1 = j_{k_1+\cdots+k_a+1} - 1.$$

This implies, by the choice of k_a in the construction of $\underline{\gamma}_{\mathfrak{s}}$, that

$$i_{k_1+\cdots+k_a} \ge i_{k_1+\cdots+k_a+1}.$$

Hence,

$$\begin{aligned} \gamma_{k_a}^{(a)} &= j_{k_1+\cdots+k_a} - i_{k_1+\cdots+k_a} + 1 \\ &= j_{k_1+\cdots+k_a+1} - i_{k_1+\cdots+k_a} \\ &\le j_{k_1+\cdots+k_a+1} - i_{k_1+\cdots+k_a+1} = \gamma_1^{(a+1)} - 1, \end{aligned}$$

proving (SK3) and so $\underline{\gamma}_{\mathfrak{s}} \in \mathcal{K}_{f^*}^s(r)$. In other words, we have proved $\mathrm{im}(\eta_f) \subseteq \mathcal{K}_{f^*}^s(r)$.

Conversely, let $\underline{\gamma} = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(m)}) \in \mathcal{K}_{f^*}^s(r)$ be a standard Kleshchev multipartition (or satisfy (SK1)–(SK3)). We now construct an integral multisegment $\mathfrak{s} \in \mathscr{S}_r^{\mathbb{Z}}$ such that $\underline{u}_{\mathfrak{s}} = \underline{u}_f$ and $\underline{\gamma}_{\mathfrak{s}} = \underline{\gamma}$.

Set $k_a = \ell(\gamma^{(a)})$ for $1 \le a \le m$ and let $t = k_1 + \cdots + k_m$. Given $1 \le c \le t$, assume that $c = k_1 + k_2 + \cdots + k_{a-1} + b$ with $1 \le b \le \ell(\gamma^{(a)}) = k_a$ for some $1 \le a \le m$ and define

$$j_c = f_{m-a+1} + b - 1, \quad i_c = f_{m-a+1} - \gamma_b^{(a)} + b. \tag{4.3.3}$$

We claim that

$$\begin{pmatrix} j_1 j_2 \cdots j_t \\ i_1 i_2 \dots i_t \end{pmatrix} \in \mathcal{W}^s(r). \tag{4.3.4}$$

Indeed, firstly, since $\underline{\gamma} = (\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(m)}) \in \mathcal{K}_{f^*}^s(r)$, we have $k_a \le f_{m-a} - f_{m-a+1} + 1$ by (SK2). In other words,

$$f_{m-a+1} + k_a - 1 \le f_{m-a},$$

which together with (4.3.3) implies that

$$j_{k_1+\cdots+k_a} \le j_{k_1+\cdots+k_a+1} \tag{4.3.5}$$

for $1 \le a \le m-1$. If $j_{k_1+\cdots+k_a} = j_{k_1+\cdots+k_a+1}$ for some $1 \le a \le m-1$, then $f_{m-a+1}+k_a-1 = f_{m-a}$ and so $\ell(\gamma^{(a)}) = k_a = f_{m-a} - f_{m-a+1} + 1$. Then, by (SK2),

$$i_{k_1+\cdots+k_a} = f_{m-a+1} - \gamma^{(a)}_{k_a} + k_a = f_{m-a} - \gamma^{(a)}_{k_a} + 1 \ge f_{m-a} - \gamma^{(a+1)}_1 + 1 = i_{k_1+\cdots+k_a+1}.$$

Thus, this together with (4.3.3) and (4.3.5) shows that $j_1 \le \ldots \le j_t$ and $i_c \ge i_{c+1}$ whenever $j_c = j_{c+1}$, proving (4.3.4).

Now, if $j_{k_1+\cdots+k_a} = j_{k_1+\cdots+k_a+1} - 1$ for some $1 \le a \le m-1$, then by (4.3.3) we have $f_{m-a} = f_{m-a+1} + k_a$ and, by (SK3)

$$i_{k_1+\cdots+k_a} = f_{m-a+1} - \gamma^{(a)}_{k_a} + k_a = f_{m-a} - \gamma^{(a)}_{k_a} \ge f_{m-a} - \gamma^{(a+1)}_1 + 1 = i_{k_1+\cdots+k_a+1}.$$

This and (4.3.3) show that $\underline{u}_{\mathfrak{s}} = \underline{u}_f$ and $\underline{\gamma}_{\mathfrak{s}} = \underline{\gamma}$ if we put $\mathfrak{s} = \flat\binom{j_1 j_2 \ldots j_n}{i_1 i_2 \ldots i_n}$. Hence, $\mathrm{im}(\eta_f) \supseteq \mathcal{K}^s_{f^*}(r)$. $\square$

For $r \ge 1$, set

$$\mathcal{K}^s(r) = \dot{\bigcup}_{f \in \mathfrak{F}} \mathcal{K}^s_f(r) = \dot{\bigcup}_{f \in \mathfrak{F}} \mathcal{K}^s_{f^*}(r),$$

where $\dot{\cup}$ means a disjoint union. Observe that

$$\mathscr{S}^{\mathbb{Z}}_r = \dot{\bigcup}_{f \in \mathfrak{F}} \mathscr{S}^{\mathbb{Z}}_{r,f}.$$

For $\underline{\gamma} \in \mathcal{K}^s_{\underline{u}}(r)$, recall that $D^{\underline{\gamma}'}$ is the irreducible $\mathcal{H}_{\underline{u}^{-1}}(r)$-module associated to $\underline{\gamma}'$ defined in Proposition 3.5.11.

Theorem 4.3.9. *The bijective maps η_f $(f \in \mathfrak{F})$ give rise to a bijection*

$$\begin{aligned} \eta : \mathscr{S}^{\mathbb{Z}}_r &\longrightarrow \mathcal{K}^s(r) \\ \mathfrak{s} &\longmapsto \underline{\gamma}_{\mathfrak{s}}. \end{aligned}$$

Moreover, the set $\{D^{\underline{\gamma}'} \mid \underline{\gamma} \in \mathcal{K}^s(r)\}$ after inflation forms a complete set of non-isomorphic irreducible objects in the category $\mathcal{H}_{\vartriangle}(r)$-$\mathsf{mod}^{\mathbb{Z}}$.

Proof. By Proposition 4.3.8, the first assertion is clear. For $\underline{\gamma} \in \mathcal{K}^s_f(r)$, suppose $\eta^{-1}(\underline{\gamma}) = \mathfrak{s}$. Then $\underline{\gamma} = \underline{\gamma}_{\mathfrak{s}}, \underline{u}_f = \underline{u}_{\mathfrak{s}}^{-1}$ and by Theorem 4.2.4 we have

$$D^{\underline{\gamma}'_{\mathfrak{s}}} \cong V_{\mathfrak{s}}.$$

Hence, the set of modules $D^{\underline{\gamma}'}$ for $\underline{\gamma} \in \mathcal{K}^s(r)$ coincides with the set of modules $V_{\mathfrak{s}}$ with $\mathfrak{s} \in \mathscr{S}^{\mathbb{Z}}_r$ up to isomorphism. Then the last assertion follows from Proposition 2.2.2(5) by restricting to $\mathscr{S}^{\mathbb{Z}}_r$. $\square$

5. Identification of irreducible representations of an Ariki-Koike algera of integral type

In this section, we shall consider the inflation of irreducible representations of Ariki-Koike algeras of integral type via the canonical surjective homomorphism $\varpi_{\underline{u}} : \mathcal{H}_{\vartriangle}(r) \to \mathcal{H}_{\underline{u}}(r)$ given in (3.2.3) and identify the corresponding integral multisegments. We will extend the isomorphisms $D^{\underline{\lambda}'} \cong V_{\mathfrak{s}}$ for $\underline{\lambda} \in \mathcal{K}^s_f(r)$, where $\mathfrak{s} = \overline{\mathbf{s}^{\mathsf{c}}_{\underline{\lambda}';f}}$ (see Proposition 4.2.3 and Theorem 4.2.4), to all $\underline{\lambda} \in \mathcal{K}_f(r)$.

5.1. Row residual segments of a multipartition. Given a sequence of integral segments $\mathbf{s} = (\mathsf{s}_1, \mathsf{s}_2, \ldots, \mathsf{s}_t)$ $(t \geq 1)$ with $\mathsf{s}_k = (q^{2i_k}, \ldots, q^{2j_k})$, by Definition 2.3.4, $\mathbf{s}$ is anti-standard if it satisfies

$$i_1 \geq i_2 \geq \cdots \geq i_t, \text{ and } j_k \leq j_{k+1} \text{ whenever } i_k = i_{k+1}. \tag{5.1.1}$$

This order is exactly the so-called *right order* introduced in [28, §2.3]. Observe that any multisegment $\mathfrak{s} \in \mathscr{S}_r^{\mathbb{Z}}$ contains a unique anti-standard sequence $\mathbf{s} = (\mathsf{s}_1, \ldots, \mathsf{s}_t) \in \mathfrak{s}$.

For a multisegment $\mathfrak{s} = \bar{\mathbf{s}} \in \mathscr{S}_r$, set $\mathfrak{s}^{-1} = \overline{\mathbf{s}^{-1}} = \overline{(\mathsf{s}_1^{-1}, \ldots, \mathsf{s}_t^{-1})}$. The function

$$(\ )^{-1} : \mathscr{S}_r^{\mathbb{Z}} \longrightarrow \mathscr{S}_r^{\mathbb{Z}}$$

is a bijection.

For $\mathfrak{s} \in \mathscr{S}_r^{\mathbb{Z}}$, assume that $\mathbf{s} = (\mathsf{s}_1, \ldots, \mathsf{s}_t) \in \mathfrak{s}$ is anti-standard and define

$$M_{\mathfrak{s}} = \mathrm{hd}(\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)} \mathbb{C}_{\mathsf{s}_1} \otimes \cdots \otimes \mathbb{C}_{\mathsf{s}_t}), \tag{5.1.2}$$

where $\mu = (|\mathsf{s}_1|, \ldots, |\mathsf{s}_t|)$. By [28, Theorem 2.2], $M_{\mathfrak{s}}$ is irreducible. We will reproduce the fact in Lemma 5.2.3.

Given $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)}) \in \mathcal{P}_m(r)$ and $f \in \mathfrak{F}_m$, define the *i-th row residual segment* of $\lambda^{(k)}$

$$\mathsf{s}_i^{(k)} = (q^{2(f_k+1-i)}, q^{2(f_k+2-i)}, \ldots, q^{2(f_k+\lambda_i^{(k)}-i)}) \quad (1 \leq k \leq m, 1 \leq i \leq \ell(\lambda^{(k)}))$$

by reading the residues of the nodes in the ith row of $\lambda^{(k)}$ from left to right, and then form the sequence of segments in the order of row 1, row 2, ... of the first partition of $\underline{\lambda}$, of the second partition, and so on:

$$\mathbf{s}_{\underline{\lambda};f}^{\mathsf{r}} = (\mathsf{s}_1^{(1)}, \ldots, \mathsf{s}_{\ell(\lambda^{(1)})}^{(1)}, \ldots, \mathsf{s}_1^{(m)}, \ldots, \mathsf{s}_{\ell(\lambda^{(m)})}^{(m)}). \tag{5.1.3}$$

By (4.1.1) and (5.1.3), a direct calculation shows that the following holds

$$(\mathbf{s}_{\underline{\lambda};f}^{\mathsf{c}})^{-1} = \mathbf{s}_{\underline{\lambda}';f^*}^{\mathsf{r}}. \tag{5.1.4}$$

for any $\underline{\lambda} \in \mathcal{P}_m(r)$ and $f \in \mathfrak{F}_m$.

Proposition 5.1.1 ([28, Theorem 3.4])**.** *Let $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)}) \in \mathcal{K}_{\underline{u}}(r)$ with $\underline{u} = \underline{u}_f$ and $f \in \mathfrak{F}_m$. Assume that $\mathfrak{s}$ is the multisegment containing $\mathbf{s}_{\underline{\lambda};f}^{\mathsf{r}}$. Then the $\mathcal{H}_\vartriangle(r)$-module* $\mathrm{hd}\,\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)}(\mathrm{ev}_{u_1}^*(S^{\lambda^{(1)}}) \otimes \cdots \otimes \mathrm{ev}_{u_m}^*(S^{\lambda^{(m)}}))$ *is irreducible, and moreover*

$$M_{\mathfrak{s}} \cong \mathrm{hd}\,\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)}(\mathrm{ev}_{u_1}^*(S^{\lambda^{(1)}}) \otimes \cdots \otimes \mathrm{ev}_{u_m}^*(S^{\lambda^{(m)}})),$$

where $\mu = (|\lambda^{(1)}|, \ldots, |\lambda^{(m)}|)$.

Remark 5.1.2. (1) Observe that $\mathcal{H}_\vartriangle(r)$ possesses an anti-automorphism $*$ with $T_i^* = T_i$ and $X_k^* = X_k$. Using this we can introduce a natural duality $\circledast$ on finite dimensional left $\mathcal{H}_\vartriangle(r)$-modules: $M^\circledast$ denotes the dual space M^* with the right action shifted to a left action by the anti-automorphism $*$. It is known [21, Corollary 3.7.5] that

$$(\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)} \mathrm{ev}_{u_1}^*(S^{\lambda^{(1)}}) \otimes \cdots \otimes \mathrm{ev}_{u_m}^*(S^{\lambda^{(m)}}))^\circledast \cong \mathrm{ind}_{\mathcal{H}_\vartriangle(\mu^o)}^{\mathcal{H}_\vartriangle(r)} \mathrm{ev}_{u_m}^*(S^{\lambda^{(m)}}) \otimes \cdots \otimes \mathrm{ev}_{u_1}^*(S^{\lambda^{(1)}})$$

for any $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)}) \in \mathcal{P}_m(r)$ and $\underline{u} = (u_1, \ldots, u_m) = \underline{u}_f$ with $f \in \mathfrak{F}_m$, where $\mu = (|\lambda^{(1)}|, \ldots, |\lambda^{(m)}|)$ and $\mu^o = (|\lambda^{(m)}|, \ldots, |\lambda^{(1)}|)$.

(2) It is straightforward to check that there exists an automorphism $\sharp$ on $\mathcal{H}_\vartriangle(r)$ defined by

$$\begin{aligned} \sharp : \mathcal{H}_\vartriangle(r) &\longrightarrow \mathcal{H}_\vartriangle(r) \\ T_i \longmapsto -q^2 T_i^{-1}, &\quad X_k \longmapsto X_k^{-1} \end{aligned} \tag{5.1.5}$$

for $1 \le i \le r-1$ and $1 \le k \le r$. Given an $\mathcal{H}_\vartriangle(r)$-module M, we denote by $M^\sharp$ the $\mathcal{H}_\vartriangle(r)$-module obtained from M by twisting the action via the automorphism $\sharp$. It is easy to deduce that for an $\mathcal{H}_\vartriangle(a)$-module M and an $\mathcal{H}_\vartriangle(b)$-module N with $1 \le a, b \le r$ and $a+b=r$, we have

$$(\mathrm{ind}_{\mathcal{H}_\vartriangle(a)\otimes\mathcal{H}_\vartriangle(b)}^{\mathcal{H}_\vartriangle(r)} M \otimes N)^\sharp \cong \mathrm{ind}_{\mathcal{H}_\vartriangle(a)\otimes\mathcal{H}_\vartriangle(b)}^{\mathcal{H}_\vartriangle(r)} M^\sharp \otimes N^\sharp. \tag{5.1.6}$$

5.2. Identification of irreducible representations. We now prove the main result of this section.

Lemma 5.2.3. *For any $\mathfrak{s} \in \mathscr{S}_r^{\mathbb{Z}}$,*

$$M^\sharp_{\mathfrak{s}^{-1}} \cong V_{\mathfrak{s}}$$

and hence $M_{\mathfrak{s}}$ is irreducible.

Proof. Clearly, the second assertion follows from the first assertion.

Let $\mathsf{s} = (\mathsf{s}_1, \dots, \mathsf{s}_t) \in \mathfrak{s}$ be standard. By Definition 2.3.4, $\mathsf{s}^{-1} = (\mathsf{s}_1^{-1}, \dots, \mathsf{s}_t^{-1})$ is the unique anti-standard element in $\mathfrak{s}^{-1}$ and hence by (5.1.2)

$$M_{\mathfrak{s}^{-1}} \cong \mathrm{hd}(\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)} \mathbb{C}_{\mathsf{s}_1^{-1}} \otimes \cdots \otimes \mathbb{C}_{\mathsf{s}_t^{-1}}).$$

This together with (5.1.6) implies

$$M^\sharp_{\mathfrak{s}^{-1}} \cong \mathrm{hd}(\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)} \mathbb{C}^\sharp_{\mathsf{s}_1^{-1}} \otimes \cdots \otimes \mathbb{C}^\sharp_{\mathsf{s}_t^{-1}}).$$

Observe that $\mathbb{C}^\sharp_{\mathsf{s}^{-1}} \cong \mathbb{C}_{\widetilde{\mathsf{s}}}$ for any segment s and hence we obtain

$$M^\sharp_{\mathfrak{s}^{-1}} \cong \mathrm{hd}(\mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)} \mathbb{C}_{\widetilde{\mathsf{s}}_1} \otimes \cdots \otimes \mathbb{C}_{\widetilde{\mathsf{s}}_t}) \cong \mathrm{hd}(I_{\mathsf{s}}) \cong V_{\mathsf{s}} \cong V_{\mathfrak{s}},$$

where the second isomorphism is due to Lemma 2.3.3 and the third isomorphism follows from Proposition 2.3.8 since s is standard. □

Remark 5.2.4. It is easy to check that $\mathcal{H}_\vartriangle(r)$ affords another automorphism $\diamond$ which sends T_i, X_k to T_{r-i}, X_{r+1-k}^{-1} for $1 \le i \le r-1$ and $1 \le k \le r$, respectively, and moreover the composition $\sharp \circ \diamond$ is exactly the Zelevinsky's involution [29] on $\mathcal{H}_\vartriangle(r)$. By (5.1.2) and [28, Corollary 6.1] one can directly compute that $M^\diamond_{\mathfrak{s}} \cong M_{\mathfrak{s}^{-1}}$. Hence, Lemma 5.2.3 implies that $V_{\mathfrak{s}}$ is isomorphic to the twist of $M_{\mathfrak{s}}$ via Zelevinsky's involution.

Observe also that the automorphism $\sharp$ on $\mathcal{H}_\vartriangle(r)$ also defines an automorphism on $\mathcal{H}(r)$ still denoted by $\sharp$ which maps T_w to $(-q^2)^{l(w)} T_{w^{-1}}^{-1}$ for $w \in \mathfrak{S}_r$. Since q is not a root of unity, each Specht module S^λ is self-dual for $\lambda \in \mathcal{P}(r)$. Furthermore it is known [11, Theorem 3.5] that

$$(S^\lambda)^\sharp \cong S^{\lambda'} \tag{5.2.1}$$

for $\lambda \in \mathcal{P}(r)$.

Recall that for $\underline{\lambda} \in \mathcal{K}_{f^*}(r)$ with $f \in \mathfrak{F}$, $D^{\underline{\lambda}'}$ is an irreducible $\mathcal{H}_{\underline{u}_f}(r)$-module, which can be also regarded as an $\mathcal{H}_\vartriangle(r)$-module by inflation.

We are now ready to generalise Theorem 4.2.4 to arbitrary Kleshchev multipartitions.

Theorem 5.2.5. *Suppose $\underline{\lambda} \in \mathcal{P}_m(r)$ such that $\underline{\lambda}' \in \mathcal{K}_{f^*}(r)$ for some $f \in \mathfrak{F}_m$. Then there is an isomorphism of $\mathcal{H}_\vartriangle(r)$-modules:*

$$D^{\underline{\lambda}} \cong V_{\mathbf{s}^{\mathsf{c}}_{\underline{\lambda};f}}.$$

Proof. By Lemma 5.2.3 and (5.1.4), the following holds

$$V_{\mathbf{s}^{\mathsf{c}}_{\underline{\lambda};f}} \cong M^{\sharp}_{\mathbf{s}^{\mathsf{r}}_{\underline{\lambda}';f^*}}. \tag{5.2.2}$$

Since $\underline{\lambda}' \in \mathcal{K}_{f^*}(r) = \mathcal{K}_{\underline{u}_f^{-1}}(r)$, by Proposition 5.1.1 we obtain

$$M_{\mathbf{s}^{\mathsf{r}}_{\underline{\lambda}';f^*}} \cong \mathrm{hd}\big(\mathrm{ind}^{\mathcal{H}_\vartriangle(r)}_{\mathcal{H}_\vartriangle(\mu)} \mathrm{ev}^*_{u_m^{-1}}(S^{\lambda^{(m)'}}) \otimes \cdots \otimes \mathrm{ev}^*_{u_1^{-1}}(S^{\lambda^{(1)'}})\big),$$

where $\mu = (|\lambda^{(m)}|, \ldots, |\lambda^{(1)}|)$ and $u_k = q^{2f_k}$ for $1 \le k \le m$. This together with (5.2.2) and (5.1.6) gives rise to

$$\begin{aligned} V_{\mathbf{s}^{\mathsf{c}}_{\underline{\lambda};f}} &\cong \mathrm{hd}(\mathrm{ind}^{\mathcal{H}_\vartriangle(r)}_{\mathcal{H}_\vartriangle(\mu)} \mathrm{ev}^*_{u_m^{-1}}(S^{\lambda^{(m)'}}) \otimes \cdots \otimes \mathrm{ev}^*_{u_1^{-1}}(S^{\lambda^{(1)'}}))^{\sharp} \\ &\cong \mathrm{hd}\ \mathrm{ind}^{\mathcal{H}_\vartriangle(r)}_{\mathcal{H}_\vartriangle(\mu)} \mathrm{ev}^*_{u_m^{-1}}(S^{\lambda^{(m)'}})^{\sharp} \otimes \cdots \otimes \mathrm{ev}^*_{u_1^{-1}}(S^{\lambda^{(1)'}})^{\sharp}. \end{aligned} \tag{5.2.3}$$

By (2.3.1) and (5.1.5), one can easily check that

$$\mathrm{ev}^*_u(M)^{\sharp} \cong \mathrm{ev}^*_{u^{-1}}(M^{\sharp})$$

for any $\mathcal{H}(r)$-module M and hence by (5.2.1)

$$\mathrm{ev}^*_u(S^{\lambda})^{\sharp} \cong \mathrm{ev}^*_{u^{-1}}((S^{\lambda})^{\sharp}) \cong \mathrm{ev}^*_{u^{-1}}(S^{\lambda'}).$$

This means

$$\mathrm{ind}^{\mathcal{H}_\vartriangle(r)}_{\mathcal{H}_\vartriangle(\mu)}(\mathrm{ev}^*_{u_m^{-1}}(S^{\lambda^{(m)'}})^{\sharp} \otimes \cdots \otimes \mathrm{ev}^*_{u_1^{-1}}(S^{\lambda^{(1)'}})^{\sharp}) \cong \mathrm{ind}^{\mathcal{H}_\vartriangle(r)}_{\mathcal{H}_\vartriangle(\mu)}(\mathrm{ev}^*_{u_m}(S^{\lambda^{(m)}}) \otimes \cdots \otimes \mathrm{ev}^*_{u_1}(S^{\lambda^{(1)}})).$$

Then by (5.2.3) we obtain

$$V_{\mathbf{s}^{\mathsf{c}}_{\underline{\lambda};f}} \cong \mathrm{hd}\ \mathrm{ind}^{\mathcal{H}_\vartriangle(r)}_{\mathcal{H}_\vartriangle(\mu)}(\mathrm{ev}^*_{u_m}(S^{\lambda^{(m)}}) \otimes \cdots \otimes \mathrm{ev}^*_{u_1}(S^{\lambda^{(1)}})) \cong D^{\underline{\lambda}},$$

where the second isomorphism is due to Corollary 3.5.12 since $\underline{\lambda}' \in \mathcal{K}_{\underline{u}_f^{-1}}(r) = \mathcal{K}_{f^*}(r)$. This proves the theorem. ∎

5.3. A branching property. When the parameter q is not a root of unity, the Hecke algebra $\mathcal{H}(r)$ is semisimple. Thus, in this case, the restriction to $\mathcal{H}(r)$ of an irreducible $\mathcal{H}_\vartriangle(r)$-module $V_{\mathfrak{s}}$ is a direct sum of Specht modules S^λ, $\lambda \vdash r$. The determination of S^λ which appears in $V_{\mathfrak{s}}$ is called the affine branching rule in [9, Problem 4.3.6].

We now look at a relatively simple problem. Given $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)}) \in \mathcal{P}_m(r)$, let $\underline{\lambda}^+ = (\underline{\lambda}_1^+, \underline{\lambda}_2^+, \ldots) \in \mathcal{P}(r)$ with

$$\underline{\lambda}_i^+ = \lambda_i^{(1)} + \lambda_i^{(2)} + \cdots + \lambda_i^{(m)}, \text{ for } i \ge 1.$$

In other words, $\underline{\lambda}^+ = \lambda^{(1)} + \cdots + \lambda^{(m)}$.

Generally speaking, by (3.3.7) and the Littlewood-Richardson rule, one can deduce that the Specht module $S^{\underline{\lambda}^+}$ over $\mathcal{H}(r)$ occurs in the restriction $S^{\underline{\lambda}}|_{\mathcal{H}(r)}$ with multiplicity one. (One can also prove this by an induced cell module argument; see [24, Theorem 2.8].)

If, in addition, $\underline{\lambda}'$ is a Kleshchev multipartition, does $S^{\underline{\lambda}^+}$ appear in $D^{\underline{\lambda}}|_{\mathcal{H}(r)}$? This sounds like a simple question but we didn't see a straightforward proof in literature.

Corollary 5.3.6. *Let $\underline{\lambda} \in \mathcal{P}_m(r)$ such that $\underline{\lambda}' \in \mathcal{K}_{f^*}(r)$ with $f \in \mathfrak{F}_m$. Then*

(1) $V_{\mathbf{s}^{\mathfrak{c}}_{\underline{\lambda};f}}$ *is a quotient of* $I_{\mathbf{s}^{\mathfrak{c}}_{\underline{\lambda};f}}$.

(2) *The Specht* $\mathcal{H}(r)$*-module* $S^{\underline{\lambda}^+}$ *occurs with multiplicity one in the restriction* $D^{\underline{\lambda}}|_{\mathcal{H}(r)}$.

Proof. Part (1) follows from Lemma 4.1.1 and Theorem 5.2.5. By Proposition 2.2.2(3) and Theorem 5.2.5, the cell module $E_{\mu(\mathbf{s}^{\mathfrak{c}}_{\underline{\lambda};f})}$ occurs with multiplicity one in $D^{\underline{\lambda}}$ as $\mathcal{H}(r)$-modules. Observe that by [22, I, (1.8)] and (4.1.2) we have

$$\underline{\lambda}^+ = ((\underline{\lambda}')^\vee)' = \mu(\mathbf{s}^{\mathfrak{c}}_{\underline{\lambda};f})'$$

and, hence, $S^{\underline{\lambda}^+} \cong E_{\mu(\mathbf{s}^{\mathfrak{c}}_{\underline{\lambda};f})}$ occurs with multiplicity one in $D^{\underline{\lambda}}|_{\mathcal{H}(r)}$. □

Remark 5.3.7. It would be interesting to find a direct approach to verifying that the Specht module $S^{\underline{\lambda}^+}$ must occur in $D^{\underline{\lambda}}|_{\mathcal{H}(r)}$ whenever $D^{\underline{\lambda}}$ is nonzero. Then combining this with Lemma 4.1.1 gives an elementary way to determine the multisegments corresponding to the irreducible modules associated to Kleshchev multipartitions without using Proposition 5.1.1 of Vazirani.

6. Drinfeld polynomials of integral type

In this section, we shall compute the Drinfeld polynomials associated with the irreducible $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-modules which are constructed by the irreducible $\mathcal{H}_{\underline{u}}(r)$-modules associated with Kleshchev multipartitions.

6.1. The affine Schur-Weyl duality. The *quantum loop algebra* $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ (or *quantum affine* $\mathfrak{gl}_n$) is the $\mathbb{C}$-algebra generated by $\mathtt{x}^{\pm}_{i,s}$ ($1 \le i < n$, $s \in \mathbb{Z}$), $\mathtt{k}_i^{\pm 1}$ and $\mathtt{g}_{i,t}$ ($1 \le i \le n$, $t \in \mathbb{Z}\backslash\{0\}$) with the following relations:

(QLA1) $\mathtt{k}_i\mathtt{k}_i^{-1} = 1 = \mathtt{k}_i^{-1}\mathtt{k}_i,\ [\mathtt{k}_i, \mathtt{k}_j] = 0;$

(QLA2) $\mathtt{k}_i\mathtt{x}^{\pm}_{j,s} = q^{\pm(\delta_{i,j}-\delta_{i,j+1})}\mathtt{x}^{\pm}_{j,s}\mathtt{k}_i,\ [\mathtt{k}_i, \mathtt{g}_{j,s}] = 0;$

(QLA3) $[\mathtt{g}_{i,s}, \mathtt{x}^{\pm}_{j,t}] = \begin{cases} 0, & \text{if } i \ne j,\ j+1; \\ \pm q^{-js}\frac{[s]_q}{s}\mathtt{x}^{\pm}_{j,s+t}, & \text{if } i = j; \\ \mp q^{-js}\frac{[s]_q}{s}\mathtt{x}^{\pm}_{j,s+t}, & \text{if } i = j+1; \end{cases}$

(QLA4) $[\mathtt{g}_{i,s}, \mathtt{g}_{j,t}] = 0;$

(QLA5) $[\mathtt{x}^{+}_{i,s}, \mathtt{x}^{-}_{j,t}] = \delta_{i,j}\frac{\phi^+_{i,s+t}-\phi^-_{i,s+t}}{q-q^{-1}};$

(QLA6) $\mathtt{x}^{\pm}_{i,s}\mathtt{x}^{\pm}_{j,t} = \mathtt{x}^{\pm}_{j,t}\mathtt{x}^{\pm}_{i,s}$, for $|i-j| > 1$, and $[\mathtt{x}^{\pm}_{i,s+1}, \mathtt{x}^{\pm}_{j,t}]_{q^{\pm c_{i,j}}} = -[\mathtt{x}^{\pm}_{j,t+1}, \mathtt{x}^{\pm}_{i,s}]_{q^{\pm c_{i,j}}};$

(QLA7) $[\mathtt{x}^{\pm}_{i,s}, [\mathtt{x}^{\pm}_{j,t}, \mathtt{x}^{\pm}_{i,p}]_q]_q = -[\mathtt{x}^{\pm}_{i,p}, [\mathtt{x}^{\pm}_{j,t}, \mathtt{x}^{\pm}_{i,s}]_q]_q$ for $|i-j| = 1$,

where $[x,y]_a = xy - ayx$, $C = (c_{i,j})_{i,j\in I}$ denotes the generalized Cartan matrix of type $\tilde{A}_{n-1}$, and $\phi^{\pm}_{i,s}$ are defined via the generating functions in indeterminate u by

$$\Phi^{\pm}_i(u) := \tilde{\mathtt{k}}_i^{\pm 1}\exp\big(\pm(q-q^{-1})\sum_{m\ge 1}\mathtt{h}_{i,\pm m}u^{\pm m}\big) = \sum_{s\ge 0}\phi^{\pm}_{i,\pm s}u^{\pm s}$$

with $\tilde{\mathtt{k}}_i = \mathtt{k}_i/\mathtt{k}_{i+1}$ ($\mathtt{k}_{n+1} = \mathtt{k}_1$) and $\mathtt{h}_{i,\pm m} = q^{\pm(i-1)m}\mathtt{g}_{i,\pm m} - q^{\pm(i+1)m}\mathtt{g}_{i+1,\pm m}$ ($1 \le i < n$).

Consider the $\mathbb{C}$-space V_n spanned by $\{\omega_1, \ldots, \omega_n\}$ and set

$$\Omega_{(n)} = V_n \otimes_{\mathbb{C}} \mathbb{C}[X, X^{-1}] = V_n[X, X^{-1}].$$

Following [27], there is a right action of $\mathcal{H}_{\vartriangle}(r)_{\mathbb{C}}$ on $\Omega_{(n)}^{\otimes r}$. Then the endomorphism algebra

$$\mathcal{S}_{\vartriangle}(n,r) := \mathrm{End}_{\mathcal{H}_{\vartriangle}(r)}(\Omega_{(n)}^{\otimes r}) \tag{6.1.1}$$

is known as the affine q-Schur algebra. In particular, $\Omega_{(n)}^{\otimes r}$ becomes an $\mathcal{S}_\vartriangle(n,r)$-$\mathcal{H}_\vartriangle(r)$-bimodule. By a double Hall algebra interpretation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ (see [9, §3.5]), there is a left action of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-module on $\Omega_{(n)}^{\otimes r}$ which commutes with the right action of $\mathcal{H}_\vartriangle(r)$ on $\Omega_{(n)}^{\otimes r}$. Furthermore, this gives rise to an algebra homomorphism

$$\zeta_r : \mathrm{U}_q(\widehat{\mathfrak{gl}}_n) \longrightarrow \mathrm{End}_{\mathcal{H}_\vartriangle(r)}(\Omega_{(n)}^{\otimes r}) = \mathcal{S}_\vartriangle(n,r), \tag{6.1.2}$$

which is surjective [9, Corollary 3.8.4]. We omit the details of the actions of $\mathcal{H}_\vartriangle(r)$ and $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ on $\Omega_{(n)}^{\otimes r}$ and the construction for ζ_r here as it is irrelevant in this paper.

6.2. Identification of irreducible representations of $\mathcal{S}_\vartriangle(n,r)$. For $1 \le i \le n$ and $s \in \mathbb{Z}$, define the elements $\mathscr{Q}_{i,s} \in \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ through the generating functions

$$\mathscr{Q}_i^{\pm}(x) := \exp\Big(-\sum_{t\ge 1} \frac{1}{[t]_q} \mathtt{g}_{i,\pm t}(qx)^{\pm t} \Big) = \sum_{s\ge 0} \mathscr{Q}_{i,\pm s} x^{\pm s} \in \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)[[x,x^{-1}]]. \tag{6.2.1}$$

Let V be a finite dimensional $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-module and let $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{Z}^n$. A nonzero ($\lambda$-weight) vector $w \in V$ is called a *pseudo-highest weight vector* if there exist some $Q_{i,s} \in \mathbb{C}$ such that

$$\mathtt{x}_{j,s}^+ w = 0, \quad \mathscr{Q}_{i,s} w = Q_{i,s} w, \quad \text{and} \quad \mathtt{k}_i w = q^{\lambda_i} w, \tag{6.2.2}$$

for all $1 \le i \le n$, $1 \le j < n$, and $s \in \mathbb{Z}$. A $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-module V is called a *pseudo-highest weight module* if $V = \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)w$ for some pseudo-highest weight vector w.

Following [16], an n-tuple of polynomials $\mathbf{Q} = (Q_1(x),\ldots,Q_n(x))$ with constant terms 1 is called *dominant* if, for each $1 \le i \le n-1$, the ratio $Q_i(xq^{i-1})/Q_{i+1}(xq^{i+1})$ is a polynomial in x. Let $\mathcal{Q}(n)$ be the set of dominant n-tuples of polynomials.

For a polynomial $Q(x) = \prod_{1\le i\le m}(1-a_i x) \in \mathbb{C}[x]$ with $a_i \in \mathbb{C}^*$, put $Q^+(x) = Q(x)$ and define

$$Q^-(x) = \prod_{1\le i\le m} (1 - a_i^{-1}x^{-1}) \in \mathbb{C}[x^{-1}].$$

Given a $\mathbf{Q} = (Q_1(x),\ldots,Q_n(x)) \in \mathcal{Q}(n)$, define $Q_{i,s} \in \mathbb{C}$, for $1 \le i \le n$ and $s \in \mathbb{Z}$, by the following formula

$$Q_i^{\pm}(x) = \sum_{s\ge 0} Q_{i,\pm s} x^{\pm s}. \tag{6.2.3}$$

Let $I(\mathbf{Q})$ be the left ideal of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ generated by $\mathtt{x}_{j,s}^+$, $\mathscr{Q}_{i,s} - Q_{i,s}$, and $\mathtt{k}_i - q^{\lambda_i}$, for $1 \le j < n$, $1 \le i \le n$, and $s \in \mathbb{Z}$, where $\lambda_i = \deg Q_i(x)$, and define

$$M(\mathbf{Q}) = \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)/I(\mathbf{Q}). \tag{6.2.4}$$

Then $M(\mathbf{Q})$ has a unique simple quotient, denoted by $L(\mathbf{Q})$. The polynomial $Q_i(x)$ are called *Drinfeld polynomials* associated to $L(\mathbf{Q})$. Clearly, $L(\mathbf{Q})$ is a pseudo-highest weight module with pseudo-highest weight $\lambda = (\lambda_1,\ldots,\lambda_n) \in \Lambda(n,r)$, where $\Lambda(n,r)$ denotes the set of compositions of r into n parts; see (3.1.2).

Let

$$\mathcal{Q}(n)_r = \big\{ \mathbf{Q} = (Q_1(x),\ldots,Q_n(x)) \in \mathcal{Q}(n) \mid r = \sum_{1\le i\le n} \deg Q_i(x) \big\}.$$

Observe that given a left $\mathcal{H}_\vartriangle(r)$-module M, the tensor product $\Omega_{(n)}^{\otimes r} \otimes_{\mathcal{H}_\vartriangle(r)} M$ naturally affords a left $\mathcal{S}_\vartriangle(n,r)$-module. Recall that $\{V_\mathfrak{s} | \mathfrak{s} \in \mathscr{S}_r\}$ is a complete set of simple $\mathcal{H}_\vartriangle(r)$-modules. Further, set

$$\mathscr{S}_r^{(n)} = \{\mathfrak{s} = \{\mathsf{s}_1, \ldots, \mathsf{s}_t\} \in \mathscr{S}_r \mid t \geq 1, |\mathsf{s}_i| \leq n, \forall i\}.$$

Clearly, if $n \geq r$, then $\mathscr{S}_r^{(n)} = \mathscr{S}_r$.

Given $n, r \geq 1$ and $\mathfrak{s} \in \mathscr{S}_r^{(n)}$, take $\mathbf{s} = (\mathsf{s}_1, \mathsf{s}_2, \ldots, \mathsf{s}_t) \in \mathfrak{s}$ with

$$\mathsf{s}_k = (z_k q^{2i_k}, z_k q^{2(i_k+1)}, \ldots, z_k q^{2j_k}) \in (\mathbb{C}^*)^{r_k},$$

where $r_k = |\mathsf{s}_k| = j_k - i_k + 1 \in \mathbb{Z}_+$ for $1 \leq k \leq t$. We set

$$\partial(\mathfrak{s}) := \mathbf{Q}_\mathfrak{s} = (Q_1(x), \ldots, Q_n(x)) \in \mathcal{Q}(n)_r$$

by setting recursively

$$Q_k(x) = \prod_{k \leq l \leq n} P_l(xq^{l+1-2k}) = P_k(xq^{-k+1}) P_{k+1}(xq^{-k+2}) \cdots P_{n-1}(xq^{n-2k}) P_n(xq^{n+1-2k}), \tag{6.2.5}$$

where

$$P_k(x) = \prod_{\substack{1 \leq b \leq t \\ r_b = k}} (1 - z_b q^{i_b + j_b} x) \tag{6.2.6}$$

for $1 \leq k \leq n$. Here we use the convention that $P_i(x) = 1$ if there does not exist $1 \leq j \leq t$ such that $r_j = i$. This gives a map

$$\partial : \mathscr{S}_r^{(n)} \to \mathcal{Q}(n)_r.$$

Thus, we obtain the associated simple $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-module $L(\mathbf{Q}_\mathfrak{s})$. The following result is due to Deng, Du and Fu [8, 9, 13].

Proposition 6.2.1 ([8, Theorem 6.6], [13, Theorem 4.4])**.** *Assume $n, r \geq 1$. Then*

(1) *Both the set $\{L(\mathbf{Q}) \mid \mathbf{Q} \in \mathcal{Q}(n)_r\}$ and the set $\{\Omega_{(n)}^{\otimes r} \otimes_{\mathcal{H}_\vartriangle(r)} V_\mathfrak{s} \mid \mathfrak{s} \in \mathscr{S}_r^{(n)}\}$ are complete sets of non-isomorphic irreducible representations of $\mathcal{S}_\vartriangle(n,r)$.*

(2) *For each $\mathfrak{s} \in \mathscr{S}_r^{(n)}$, there is a $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-module isomorphism (or equivalently, an $\mathcal{S}_\vartriangle(n,r)$-module isomorphism)*

$$L(\mathbf{Q}_\mathfrak{s}) \cong \Omega_{(n)}^{\otimes r} \otimes_{\mathcal{H}_\vartriangle(r)} V_\mathfrak{s}.$$

Hence, the map ∂ is a bijection.

6.3. Drinfeld polynomials of integral type. Recall from §2.4 the category $\mathcal{H}_\vartriangle(r)$-$\mathsf{mod}^\mathbb{Z}$ whose simple objects are indexed by the set $\mathscr{S}_r^\mathbb{Z}$ of integral multisegment and consider the functor

$$\mathcal{F} : \mathcal{H}_\vartriangle(r)\text{-}\mathsf{mod}^\mathbb{Z} \longrightarrow \mathcal{S}_\vartriangle(n,r)\text{-}\mathsf{mod}, \quad L \longmapsto \Omega_{(n)}^{\otimes r} \otimes_{\mathcal{H}_\vartriangle(r)} L.$$

We form full subcategory $\mathcal{S}_\vartriangle(n,r)$-$\mathsf{mod}^\mathbb{Z}$ whose objects are isomorphic to $\mathcal{F}(L)$ for $L \in \mathrm{Ob}(\mathcal{H}_\vartriangle(r)$-$\mathsf{mod}^\mathbb{Z})$. We now show that simple objects in $\mathcal{S}_\vartriangle(n,r)$-$\mathsf{mod}^\mathbb{Z}$ for all $r \geq 0$ determine all irreducible polynomial representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$.

A polynomial $Q(x) \in \mathbb{C}[x]$ of constant term 1 is said to be of *integral type* if its roots are powers of q^2. Let $\mathcal{Q}(n)^{\mathbb{Z}}$ be the set of dominant n-tuples of polynomials of integral type. Set

$$\mathcal{Q}(n)_r^{\mathbb{Z}} = \mathcal{Q}(n)^{\mathbb{Z}} \cap \mathcal{Q}(n)_r$$
$$\mathscr{S}_r^{(n),\mathbb{Z}} = \mathscr{S}_r^{\mathbb{Z}} \cap \mathscr{S}_r^{(n)}.$$

Given $\mathfrak{s} \in \mathscr{S}_r^{(n)}$ with $\mathsf{s} = (\mathsf{s}_1, \mathsf{s}_2, \ldots, \mathsf{s}_t) \in \mathfrak{s}$ and

$$\mathsf{s}_k = (z_k q^{2i_k}, z_k q^{2(i_k+1)}, \ldots, z_k q^{2j_k})$$

for $1 \le k \le t$. Write $\mathbf{Q}_{\mathfrak{s}} = (Q_1(x), \ldots, Q_n(x))$. Observe that by (6.2.5) and (6.2.6) we have

$$\begin{aligned} Q_k(x) = \prod_{k \le l \le n} P_l(xq^{l+1-2k}) &= \prod_{k \le l \le n} \prod_{\substack{1 \le b \le t, \\ j_b - i_b + 1 = l}} (1 - z_b q^{i_b + j_b + l + 1 - 2k} x) \\ &= \prod_{\substack{1 \le b \le t, \\ j_b - i_b + 1 \ge k}} (1 - z_b q^{2(j_b + 1 - k)} x). \end{aligned} \tag{6.3.1}$$

This implies that $\mathbf{Q}_{\mathfrak{s}} \in \mathcal{Q}(n)_r^{\mathbb{Z}}$ if and only if $\mathfrak{s} \in \mathscr{S}_r^{(n),\mathbb{Z}}$. Hence, the restriction of the map ∂ gives a bijection

$$\partial^{\mathbb{Z}} : \mathscr{S}_r^{(n),\mathbb{Z}} \longrightarrow \mathcal{Q}(n)_r^{\mathbb{Z}}. \tag{6.3.2}$$

For $a \in \mathbb{C}^*$, analogous to (2.4.2), there is a Hopf algebra automorphism $\omega_a : \mathrm{U}_q(\widehat{\mathfrak{gl}}_n) \longrightarrow \mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ defined by

$$\mathtt{x}_{i,s}^{\pm} \mapsto a^s \mathtt{x}_{i,s}^{\pm}, \quad \mathtt{g}_{j,t} \mapsto a^t \mathtt{g}_{j,t}, \quad \mathtt{k}_j^{\pm 1} \mapsto \mathtt{k}_j^{\pm 1}. \tag{6.3.3}$$

Thus, we may twist a $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-module M by ω_a to get a new module denoted by M^{ω_a}. For $\mathbf{Q} = (Q_1(x), \ldots, Q_n(x)) \in \mathcal{Q}(n)_r$, define $\mathbf{Q}^{\omega_a} = (Q_1^{\omega_a}(x), \ldots, Q_n^{\omega_a}(x))$ by setting

$$Q_k^{\omega_a}(x) = Q_k(ax) \tag{6.3.4}$$

for $1 \le k \le n$. Analogous to Lemma 2.4.9, we have the following.

Lemma 6.3.2. *Suppose $a \in \mathbb{C}^*$ and $\mathbf{Q} = (Q_1(x), \ldots, Q_n(x)) \in \mathcal{Q}(n)_r$. Then*

$$L(\mathbf{Q})^{\omega_a} \cong L(\mathbf{Q}^{\omega_a}).$$

Proof. Let w be a pseudo-highest weight vector of $L(\mathbf{Q})$. By (6.3.3), (6.2.1) and (6.2.2), $\omega_a(\mathscr{Q}_{i,s}) = a^s \mathscr{Q}_{i,s}$ and hence

$$\omega_a(\mathtt{x}_{j,s}^{+})w = 0, \quad \omega_a(\mathscr{Q}_{i,s})w = a^s \mathscr{Q}_{i,s} w = a^s Q_{i,s} w, \quad \omega_a(\mathtt{k}_i)w = q^{\lambda_i} w,$$

where $\lambda_i = \deg Q_i(x)$ and $Q_{i,s}$ is defined in (6.2.3) for $1 \le i \le n, 1 \le j < n$ and $s \in \mathbb{Z}$. This implies that w is also a pseudo-highest weight vector of $L(\mathbf{Q})^{\omega_a}$ and moreover $L(\mathbf{Q})^{\omega_a}$ is a quotient of $M(\mathbf{Q}^{\omega_a})$ by (6.2.4) and (6.3.4). Then the lemma is proved since $L(\mathbf{Q})^{\omega_a}$ is irreducible. □

It is known from [16, §4.3] that $L(\mathbf{Q})$ for $\mathbf{Q} \in \mathcal{Q}(n)$ are all non-isomorphic finite dimensional irreducible polynomial representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$. (We refer the reader to [9, 16] for the definition of polynomial representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$.) We now use the discussion in §2.4 and Proposition 6.2.1 to prove the following.

Theorem 6.3.3. *For $\mathbf{Q} \in \mathcal{Q}(n)$, there exist $\mathbf{Q}^{(1)}, \ldots, \mathbf{Q}^{(p)} \in \mathcal{Q}(n)^{\mathbb{Z}}$ and $a_1, \ldots, a_p \in \mathbb{C}^*$ such that as $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-modules*

$$L(\mathbf{Q}) \cong L(\mathbf{Q}^{(1)})^{\omega_{a_1}} \otimes \cdots \otimes L(\mathbf{Q}^{(p)})^{\omega_{a_p}}.$$

Proof. Write $\mathbf{Q} = (Q_1(u), \ldots, Q_n(u))$ and let $r = \sum_i \mathrm{deg} Q_i(u)$. By Proposition 6.2.1, there exists $\mathfrak{s} \in \mathscr{S}_r^{(n)}$ such that $\mathbf{Q} = \mathbf{Q}_{\mathfrak{s}}$ and

$$L(\mathbf{Q}) \cong \Omega_{(n)}^{\otimes r} \otimes_{\mathcal{H}_\vartriangle(r)} V_{\mathfrak{s}}. \tag{6.3.5}$$

Suppose $\mathbf{s} \in \mathfrak{s}$ is standard in the sense of Definition 2.3.4(2) with the decomposition $\mathbf{s} = \mathbf{s}^{(1)} \cup \mathbf{s}^{(2)} \cup \cdots \cup \mathbf{s}^{(p)}$ and $\mathbf{s}^{(k)}$ lying on the line L_{a_k} for $1 \le k \le p$. Let $\mathfrak{s}^{(k)}$ be the multisegment containing $\mathbf{s}^{(k)}$ for $1 \le k \le p$. By (2.4.1), (6.3.5), and Proposition 6.2.1, we have the following isomorphisms of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-modules

$$\begin{aligned} L(\mathbf{Q}) &\cong \Omega_{(n)}^{\otimes r} \otimes_{\mathcal{H}_\vartriangle(r)} \mathrm{ind}_{\mathcal{H}_\vartriangle(\mu)}^{\mathcal{H}_\vartriangle(r)} (V_{\mathfrak{s}^{(1)}} \otimes \cdots \otimes V_{\mathfrak{s}^{(p)}}) \\ &\cong \Omega_{(n)}^{\otimes r} \otimes_{\mathcal{H}_\vartriangle(\mu)} \big(V_{\mathfrak{s}^{(1)}} \otimes \cdots \otimes V_{\mathfrak{s}^{(p)}}\big) \\ &\cong \big(\Omega_{(n)}^{\otimes r_1} \otimes_{\mathcal{H}_\vartriangle(r_1)} V_{\mathfrak{s}^{(1)}}\big) \otimes \cdots \otimes \big(\Omega_{(n)}^{\otimes r_p} \otimes_{\mathcal{H}_\vartriangle(r_p)} V_{\mathfrak{s}^{(p)}}\big) \\ &\cong L(\mathbf{Q}_{\mathfrak{s}^{(1)}}) \otimes \cdots \otimes L(\mathbf{Q}_{\mathfrak{s}^{(p)}}), \end{aligned} \tag{6.3.6}$$

where $\mu = (r_1, \ldots, r_p)$ and $r_k = \sum_i |\mathsf{s}_i^{(k)}|$ for $1 \le k \le p$.

Furthermore, set

$$\mathbf{s}^{(k)\prime} = (\mathbf{s}^{(k)})^{\sigma_{a_k^{-1}}} \tag{6.3.7}$$

for $1 \le k \le p$. Then $\mathfrak{s}^{(k)\prime} := \overline{\mathbf{s}^{(k)\prime}} \in \mathscr{S}_{r_k}^{\mathbb{Z}}$ and hence

$$\mathbf{Q}^{(k)} := \mathbf{Q}_{\mathfrak{s}^{(k)\prime}} \in \mathcal{Q}(n)_{r_k}^{\mathbb{Z}}$$

for $1 \le k \le p$. Then by (6.3.7), (6.3.1) and (2.4.3), one can easily deduce that

$$\mathbf{Q}_{\mathfrak{s}^{(k)}} = \mathbf{Q}_{(\mathfrak{s}^{(k)\prime})^{\sigma_{a_k}}} = \mathbf{Q}_{\mathfrak{s}^{(k)\prime}}^{\omega_{a_k}} = (\mathbf{Q}^{(k)})^{\omega_{a_k}}$$

for $1 \le k \le p$. This together with Lemma 6.3.2 and (6.3.6) gives rise to an isomorphism of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-modules

$$L(\mathbf{Q}) \cong L(\mathbf{Q}^{(1)})^{\omega_{a_1}} \otimes \cdots \otimes L(\mathbf{Q}^{(p)})^{\omega_{a_p}},$$

as desired. □

Remark 6.3.4. Retain the notations in Theorem 6.3.3. Observe that $\mathcal{S}_\vartriangle(n,r)$ can be naturally regarded as a subalgebra of the tensor product $\mathcal{S}_\vartriangle(n,r_1) \otimes \cdots \otimes \mathcal{S}_\vartriangle(n,r_p)$. Then, by (6.3.6), $L(\mathbf{Q})$ is isomorphic to the restriction to $\mathcal{S}_\vartriangle(n,r)$ of the irreducible $\mathcal{S}_\vartriangle(n,r_1) \otimes \cdots \otimes \mathcal{S}_\vartriangle(n,r_p)$-module $L(\mathbf{Q}_{\mathfrak{s}^{(1)}}) \otimes \cdots \otimes L(\mathbf{Q}_{\mathfrak{s}^{(p)}})$. Hence by Theorem 6.3.3, to study irreducible $\mathcal{S}_\vartriangle(n,r)$-modules or finite dimensional irreducible polynomial representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$, it is enough to consider those $L(\mathbf{Q})$ associated with Drinfeld polynomials $\mathbf{Q}$ of integral type. In other words, every irreducible polynomial representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ is determined by the simple objects in the category $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)\text{-}\mathsf{mod}^{\mathbb{Z}} := \oplus_{r \ge 0} \mathcal{S}_\vartriangle(n,r)\text{-}\mathsf{mod}^{\mathbb{Z}}$.

6.4. Kleshchev multipartitions and Drinfeld polynomials. For $f \in \mathfrak{F}$ and $r \geq 1$, let

$$\mathcal{K}^s_{f,(n)}(r) = \{\underline{\gamma} \in \mathcal{K}^s_f(r) \mid \gamma_1^{(k)} \leq n, \forall k \geq 1\}$$
$$\mathcal{K}^s_{(n)}(r) = \dot{\bigcup}_{f\in\mathfrak{F}} \mathcal{K}^s_{f,(n)}(r).$$

Recall from Theorem 4.3.9 the bijective map $\theta = \eta^{-1} : \mathcal{K}^s(r) \to \mathscr{S}_r^{\mathbb{Z}}$. Restriction induces a bijection

$$\theta^{\mathbb{Z}} : \mathcal{K}^s_{(n)}(r) \to \mathscr{S}_r^{(n),\mathbb{Z}}.$$

The two parts of the following result are the counterparts of Theorems 4.3.9 and 5.2.5, respectively.

Theorem 6.4.5. *The following holds for $n, r \geq 1$:*

(1) *There is a bijection*

$$\partial^{\mathbb{Z}} \circ \theta^{\mathbb{Z}} : \mathcal{K}^s_{(n)}(r) \longrightarrow \mathcal{Q}(n)^{\mathbb{Z}}_r$$

such that, for each $\mathbf{Q} \in \mathcal{Q}(n)^{\mathbb{Z}}_r$, if $\mathbf{Q} = \partial^{\mathbb{Z}} \circ \theta^{\mathbb{Z}}(\underline{\gamma})$ for some $\underline{\gamma} \in \mathcal{K}^s_{(n)}(r)$, then the following isomorphism of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-modules (or $\mathcal{S}_\vartriangle(n,r)$-modules) holds

$$L(\mathbf{Q}) \cong \Omega^{\otimes r}_{(n)} \otimes_{\mathcal{H}_\vartriangle(r)} D^{\underline{\gamma}'}.$$

(2) *Let $\underline{\lambda} \in \mathcal{P}_m(r)$ and $f \in \mathfrak{F}$. Assume that $\underline{\lambda}' \in \mathcal{K}_{f^*}(r)$ and $\ell(\lambda^{(k)}) \leq n$ for $1 \leq k \leq m$ and write $\mathbf{s} = \mathbf{s}^{\mathsf{c}}_{\underline{\lambda};f}$ and $\mathfrak{s} = \bar{\mathbf{s}}$. Then $\mathbf{Q}_{\mathfrak{s}} = (Q_1(x), Q_2(x), \dots, Q_n(x))$ are given by*

$$Q_i(x) = \prod_{1\leq k\leq m, 1\leq j\leq \lambda_i^{(k)}} (1 - q^{2(f_k+j-i)}x).$$

In other words, for every $1 \leq i \leq n$, $Q_i(x)$ is the polynomial whose roots are determined by the residues (with respect to $\underline{u}_f$) of the nodes in ith row of $\lambda^{(k)}$ for all $1 \leq k \leq m$. Moreover, the following isomorphism of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-modules holds:

$$\Omega^{\otimes r}_{(n)} \otimes_{\mathcal{H}_\vartriangle(r)} D^{\underline{\lambda}} \cong L(\mathbf{Q}_{\mathfrak{s}}).$$

Proof. Suppose $\mathbf{Q} \in \mathcal{Q}(n)^{\mathbb{Z}}_r$. By Proposition 6.2.1 and (6.3.2), there exists a unique $\mathfrak{s} \in \mathscr{S}_r^{(n),\mathbb{Z}}$ such that $\mathbf{Q} = \partial^{\mathbb{Z}}(\mathfrak{s})$ and $L(\mathbf{Q}) \cong \Omega^{\otimes r}_{(n)} \otimes_{\mathcal{H}_\vartriangle(r)} V_{\mathfrak{s}}$. Then, by Theorem 4.3.9, we have $\underline{\gamma} = \eta(\mathfrak{s}) \in \mathcal{K}^s_{(n)}(r)$ and, by Theorem 4.2.4, $V_{\mathfrak{s}} \cong D^{\underline{\gamma}'}$ and so

$$L(\mathbf{Q}) \cong \Omega^{\otimes r}_{(n)} \otimes_{\mathcal{H}_\vartriangle(r)} D^{\underline{\gamma}'},$$

proving Part (1).

Fix $1 \leq i \leq n$. Recall from (4.1.1) that we have

$$\mathbf{s}^{\mathsf{c}}_{\underline{\lambda};f} = (\mathsf{s}_1^{(m)}, \dots, \mathsf{s}^{(m)}_{\lambda_1^{(m)}}, \dots, \mathsf{s}_1^{(1)}, \dots, \mathsf{s}^{(1)}_{\lambda_1^{(1)}}),$$

where

$$\mathsf{s}_j^{(k)} = (q^{2(f_k+j-\lambda_j^{(k)'})}, \dots, q^{2(f_k+j-2)}, q^{2(f_k+j-1)})$$

for $1 \le k \le m$. Then by (6.3.1), we obtain

$$Q_i(x) = \prod_{1\le k\le m} \prod_{1\le j\le \lambda_1^{(k)}, \lambda_j^{(k)\prime} \ge i} (1 - q^{2(f_k+j-i)}x)$$
$$= \prod_{1\le k\le m} \prod_{1\le j\le \lambda_i^{(k)}} (1 - q^{2(f_k+j-i)}x).$$

Now, Part (2) follows from Theorem 4.2.4 and Proposition 6.2.1. □

Note that, in the case $m = 1$ and $u_1 = 1$, Theorem 6.4.5 recovers [13, Theorem 7.2].

7. Skew shape representations of $\mathcal{H}_\vartriangle(r)$ and $U_q(\widehat{\mathfrak{gl}}_n)$

We now give an application of our theory. We will identify Ram's skew shape representations of the affine Hecke algebra in terms of (column residual) multisegments and compute the Drinfeld polynomials of their induced skew shape representations of $U_q(\widehat{\mathfrak{gl}}_n)$.

Throughout the section, we repeatedly use the notation $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ for the conjugate (or dual) partition of $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash r$.

7.1. Skew shape representations of $\mathcal{H}_\vartriangle(r)$ and multisegments. If λ and ν are partitions such that $\nu_k \le \lambda_k$ for all k, we write $\nu \subseteq \lambda$. A *skew shape* λ/ν is defined to be the diagram obtained by removing the Young diagram ν from λ for some pair of partitions $\nu \subseteq \lambda$. Recall that the residue of a node $\rho = (i, j)$ in λ is defined by $\mathrm{res}(\rho) = q^{2(j-i)}$. Accordingly the residue of nodes in a skew shape is defined. A standard λ/ν-tableau is a labeling of the skew Young diagram λ/ν with the numbers $1, 2, \ldots, |\lambda| - |\nu|$ such that the numbers strictly increase from left to right along each row and down each column. If T is a λ/ν-tableau, denote by $\mathrm{res}_T(k)$ the residue of the node of λ/ν labeled by k in T for $1 \le k \le |\lambda| - |\nu|$. Let $\mathcal{F}(\lambda/\nu)$ be the set of standard tableau of shape λ/ν.

Let λ/ν be a skew shape such that $|\lambda| - |\nu| = r$ and set

$$D^{\lambda/\nu} = \mathbb{C}\text{−span}\{v_T \mid T \in \mathcal{F}(\lambda/\nu)\}.$$

It is known [23, Theorem 4.1] the $D^{\lambda/\nu}$ affords an irreducible $\mathcal{H}_\vartriangle(r)$-module via

$$X_k v_T = \mathrm{res}_T(k) v_T, \tag{7.1.1}$$
$$T_i v_T = \frac{q^2 - 1}{1 - \frac{\mathrm{res}_T(i)}{\mathrm{res}_T(i+1)}} v_T + \Big(1 + \frac{q^2 - 1}{1 - \frac{\mathrm{res}_T(i)}{\mathrm{res}_T(i+1)}}\Big) v_{s_i T}, \tag{7.1.2}$$

where $1 \le k \le r$, $s_i T$ denotes the tableau obtained by switching $i, i+1$ in T and $v_{s_i T} = 0$ if $s_i T$ is not standard for $1 \le i \le r - 1$.

Given a skew shape λ/ν, there exist $1 \le j_1 < j_2 < \cdots < j_t \le \lambda_1$ with $t \ge 1$ such that $\lambda'_{j_k} > \nu'_{j_k}$ for $1 \le k \le t$, and $\lambda'_j = \nu'_j$ for $1 \le j \le \lambda_1$ with $j \ne j_1, j_2, \ldots, j_t$. Set

$$\mathsf{s}_k = (q^{2(j_k - \lambda'_{j_k})}, q^{2(j_k - \lambda'_{j_k} + 1)}, \ldots, q^{2(j_k - \nu'_{j_k} - 1)}) \tag{7.1.3}$$

for $1 \le k \le t$, that is, s_k is the segment obtained by reading the residues of the nodes in the kth nonempty column of the skew shape λ/ν from bottom to top. Let

$$\mathsf{s}_{\lambda/\nu} = (\mathsf{s}_1, \mathsf{s}_2, \ldots, \mathsf{s}_t).$$

Theorem 7.1.1. *Suppose that λ/ν is a skew shape with $|\lambda|-|\nu|=r$. Then*

$$D^{\lambda/\nu}\cong V_{\mathbf{s}_{\lambda/\nu}}$$

as $\mathcal{H}_\vartriangle(r)$-modules.

Proof. Let $t_{\lambda/\nu}$ be the standard λ/ν-tableau in which the numbers $1,2,\ldots,r$ appear in order along successive columns from top to bottom. By (7.1.2), one can deduce that

$$T_i v_{t_{\lambda/\nu}}=-v_{t_{\lambda/\nu}}. \tag{7.1.4}$$

for $1\le i\le r-1$ such that $T_i\in\mathcal{H}_\vartriangle(\mu(\mathbf{s}_{\lambda/\nu}))$. Meanwhile it is easy to see that

$$\widetilde{\mathbf{s}}^\vee_{\lambda/\nu}=(\mathrm{res}_{t_{\lambda/\nu}}(1),\mathrm{res}_{t_{\lambda/\nu}}(2),\ldots,\mathrm{res}_{t_{\lambda/\nu}}(r))$$

Then by (7.1.1) and (7.1.4), we obtain that $\mathbb{C}v_{t_{\lambda/\nu}}$ affords an $\mathcal{H}_\vartriangle(\mu(\mathbf{s}_{\lambda/\nu}))$-submodule of $D^{\lambda/\nu}$ and moreover by (2.2.2) and Proposition 2.2.2(2) we have

$$\mathbb{C}C_{w^0_{\mu(\mathbf{s}_{\lambda/\nu})}}\cong\mathbb{C}v_{t_{\lambda/\nu}}$$

as $\mathcal{H}_\vartriangle(\mu(\mathbf{s}_{\lambda/\nu}))$-modules. Then by Frobenius reciprocity and Lemma 2.3.3, there exists an $\mathcal{H}_\vartriangle(r)$-homomorphism

$$\xi: I_{\mathbf{s}_{\lambda/\nu}}\longrightarrow D^{\lambda/\nu},\quad C_{w^0_{\mu(\mathbf{s}_{\lambda/\nu})}}\mapsto v_{t_{\lambda/\nu}}. \tag{7.1.5}$$

By (7.1.3) and Definition 2.3.4(2), we observe that $\mathbf{s}_{\lambda/\nu}$ is standard. Then by Proposition 2.3.8, the head of $I_{\mathbf{s}_{\lambda/\nu}}$ is isomorphic to $V_{\mathbf{s}_{\lambda/\nu}}$ and therefore, (7.1.5) implies the isomorphism $D^{\lambda/\nu}\cong V_{\mathbf{s}_{\lambda/\nu}}$. □

Corollary 7.1.2. *Suppose that λ/ν is a skew shape. Let $\rho=(\rho_1,\rho_2,\ldots)$ be the partition given by $\rho_i=\#\{j\mid \lambda'_j-\nu'_j\ge i\}$ for $i\ge 1$. Then the Littlewood-Richardson coefficients $c^\lambda_{\nu\gamma}$ satisfy*

$$c^\lambda_{\nu\rho}=1.$$

Proof. Assume that $|\lambda|-|\nu|=r$. It is known [23, Theorem 6.1] that as an $\mathcal{H}(r)$-module, $D^{\lambda/\nu}$ can be decomposed as

$$D^{\lambda/\nu}\cong\oplus_{\gamma\in\mathcal{P}(r)}c^\lambda_{\nu\gamma}S^\gamma, \tag{7.1.6}$$

where $c^\lambda_{\nu\gamma}$ are Littlewood-Richardson coefficients. Then by Theorem 7.1.1 and Proposition 2.2.2(3), we obtain that the left cell module $E_{\mu(s_{\lambda/\nu})}$ occurs with multiplicity one in $D^{\lambda/\nu}$. Observe that $\mu(s_{\lambda/\nu})'=\rho$ and hence $E_{\mu(s_{\lambda/\nu})}\cong S^\rho$. This together with (7.1.6) proves the corollary. □

7.2. Skew shape representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$. Recall that given a left $\mathcal{H}_\vartriangle(r)$-module M, the tensor product $\Omega^{\otimes r}_{(n)}\otimes_{\mathcal{H}_\vartriangle(r)}M$ naturally affords a left $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-module. Suppose λ/ν is a skew shape such that $|\lambda|-|\nu|=r$ and $\lambda'_j-\nu'_j\le n$ for $1\le j\le\lambda_1$. The tensor product $\Omega^{\otimes r}_{(n)}\otimes_{\mathcal{H}_\vartriangle(r)}D^{\lambda/\nu}$ will be called a skew shape representation of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$.

Theorem 7.2.3. *Suppose that λ/ν is a skew shape such that $|\lambda|-|\nu|=r$ and $\lambda'_j-\nu'_j\le n$ for $1\le j\le\lambda_1$. Write $\mathfrak{s}=\bar{\mathbf{s}}_{\lambda/\nu}$ and $\mathbf{Q}_\mathfrak{s}=(Q_1(x),Q_2(x),\ldots,Q_n(x))$. Then*

(1) *For $1 \le i \le n$, we have*

$$Q_i(x) = \prod_{1\le j\le\lambda_1,\lambda'_j-\nu'_j\ge i} (1 - q^{2(j-(i+\nu'_j))}x),$$

that is, $Q_i(x)$ is the polynomial whose roots are determined by the residues of the ith nodes in each column of the skew shape λ/ν.

(2) *The following isomorphism of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$-modules holds:*

$$\Omega_{(n)}^{\otimes r} \otimes_{\mathcal{H}_\Delta(r)} D^{\lambda/\nu} \cong L(\mathbf{Q}_{\mathfrak{s}}),$$

Proof. By (7.1.3) and (6.3.1) we obtain

$$Q_i(x) = \prod_{1\le j\le\lambda_1,\lambda'_j-\nu'_j\ge i} (1 - q^{2(j-(i+\nu'_j))}x).$$

This proves part (1).

By Theorem 7.1.1 and Proposition 6.2.1, we have

$$\Omega_{(n)}^{\otimes r} \otimes_{\mathcal{H}_\Delta(r)} D^{\lambda/\nu} \cong \Omega_{(n)}^{\otimes r} \otimes_{\mathcal{H}_\Delta(r)} V_{\mathfrak{s}} \cong L(\mathbf{Q}_{\mathfrak{s}}).$$

Hence part (2) of the theorem is proved. □

Remark 7.2.4. Skew shape (or skew) representations of $\mathrm{U}_q(\widehat{\mathfrak{gl}}_n)$ have been considered in [19] (see also [7]) and Drinfeld polynomials $P_i(x) = Q_i(xq^{i-1})/Q_{i+1}(xq^{i+1})$ $(1 \le i \le n-1)$ are computed. A comparison shows that a shift is required between the two sets of the $P_i(x)$'s. More precisely, if g denotes the bijection $\mathscr{S}_r \to \mathscr{S}_r$ sending $\mathfrak{s} = \{\mathsf{s}_1, \dots, \mathsf{s}_t\}$ to $g(\mathfrak{s}) = \{\mathsf{s}_1 q^{-|\mathsf{s}_1|}, \dots, \mathsf{s}_t q^{-|\mathsf{s}_t|}\}$, then our $P_i(x)$ are defined relative to $\mathfrak{s}$ and their $P_i(x)$ are defined relative to $g(\mathfrak{s})$.

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