

LOCAL AND GLOBAL METHODS IN REPRESENTATIONS OF HECKE ALGEBRAS

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ABSTRACT. This paper aims at developing a “local–global” approach for various types of finite dimensional algebras, especially those related to Hecke algebras. The eventual intention is to apply the methods and applications developed here to the cross-characteristic representation theory of finite groups of Lie type. The authors first review the notions of quasi-hereditary and stratified algebras over a Noetherian commutative ring. They prove that many global properties of these algebras hold if and only if they hold locally at every prime ideal. When the commutative ring is sufficiently good, it is often sufficient to check just the prime ideals of height at most one. These methods are applied to construct certain generalized q -Schur algebras, proving they are often quasi-hereditary (the “good” prime case) but always stratified. Finally, these results are used to prove a triangular decomposition matrix theorem for the modular representations of Hecke algebras at good primes. In the bad prime case, the generalized q -Schur algebras are at least stratified, and a block triangular analogue of the good prime case is proved, where the blocks correspond to Kazhdan-Lusztig cells.

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1. INTRODUCTION.

The focus of this paper is the representation theory of generic Hecke algebras which arise in the study of principal and related series of finite groups of Lie type in cross-characteristic. Actually, our preference is for the broader context of algebras generalizing the Dipper-James notion of a q -Schur algebra, on which we will (re)focus in a later paper. Nevertheless, the q -Schur algebra generalizations we found recently

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in [DPS15], [DPS17a] already have consequences for Hecke algebra theory, as the later sections of this paper show.

The q -Schur algebras of Dipper-James were originally used to study representations of $GL_n(q)$ in cross-characteristic. For some time, these q -Schur algebras have been known to be quasi-hereditary, even over the ring $\mathbb{Z}[t, t^{-1}]$ of integral Laurent polynomials (with $t^2 = q$, an indeterminate). In the case of types besides GL_n , the use of quasi-hereditary algebras in cross-characteristic theory, while a good starting point, seems too restrictive, if one is seeking a theory for all characteristics different from the defining characteristic p .

One approach involves weakening the notion of a quasi-hereditary algebra A . In general, the definition of a quasi-hereditary algebra depends on the existence of certain idempotent ideals $J = AeA$ in A and recursively defined factor algebras (replacing A with A/J , and repeating). Here, $e^2 = e$ and eAe is required to be a semisimple algebra. Then, the axioms imply the various module categories eAe -mod form "strata" in A -mod, which collectively "glue" together to give all of A -mod, at a derived category level. The notion of a standardly stratified algebra parallels that of a quasi-hereditary algebra, but the condition that eAe be semisimple is not assumed. The categories eAe -mod still glue together to give A -mod, as before. This provides a somewhat cruder picture of A -mod, but one which is still quite useful. Stratified algebras were first introduced in [CPS96], largely over fields, and then a full study of their integral versions was begun in [DPS98a]. This later paper began a long-term project by the authors to apply stratified algebras in cross-characteristic representation theory of finite groups of Lie type. See [DPS98a], [DPS98b], [DPS15], [DPS17a], [DPS17b]. See also [CPS99, §9] and [BDK01].

In particular, [DPS98a] formulated a conjecture providing the cross-characteristic representation theory of finite groups $G(q)$ of Lie type with a kind of generalized q -Schur algebra A^+ , defined directly from the generic Hecke algebra H , of the same type as G , over the ring $\mathcal{Z} := \mathbb{Z}[t, t^{-1}]$. The authors conjectured in [DPS98a] that A^+ could be constructed as an (integral) standardly stratified algebra, with strata described in terms of Kazhdan-Lusztig cell theory. The conjecture was verified in that paper for all rank 2 cases (some of which led to standardly stratified algebras which were not quasi-hereditary). In [DPS15], the conjecture, in a slightly modified form, was established, if \mathcal{Z} is replaced by its localization at the prime ideal generated by a cyclotomic polynomial $\Phi_{2e}(t)$ with $e \neq 2$. It was required also, to be able to quote work of [GGOR03], to only treat the so-called "equal parameter" case. (The original conjecture itself, as well as the modified version, allows unequal parameters. It just requires that they appear in a Hecke algebra arising from the BN-pair structure of a finite group of Lie type.)

In §2 of this paper we develop a theory for integral quasi-hereditary algebras and height 1 prime ideals (in their base rings) strong enough to deduce global results from such local results as the above. It can also be used to deduce local results at maximal ideals from results at height 1 primes. This is applied in §5 to prove a local "triangular decomposition matrix" theorem in the spirit of Geck and Jacon [GJ11, Thm. 4.4.1] as well as a global version, giving, in particular, a new way to think about their use of Lusztig's a -function. We use more general height functions on quasi-posets arising in quasi-hereditary/stratified algebra theory. See Theorem 5.4 and Remark 5.5(b).

Though this result and most of §5 focus on quasi-hereditary algebras and “good prime” results, standardly stratified algebra methods from [DPS17a] and [DPS17b] are useful.

The two latter papers deal with \mathcal{Z} -versions of A^+ , and also with standardly stratified algebras (rather than just quasi-hereditary algebras). In §6 we stick with standardly stratified algebras and address the question of what can be said regarding (analogues of) the triangularization theorem of Geck (see [GJ11, Thm 4.4.1]) for “bad” primes. We show that a similar formulation can be obtained if the role of single ordinary irreducible characters is suitably replaced by the characters of two sided Kazhdan-Lusztig cells.

Next, §3, omitted in the above discussion so far, gives a candidate parallel treatment of the results in §2, but in a stratified algebra context. Proposition 2.2 may also be regarded as a useful part of this theory. These results are largely not needed in the later sections §§4,5,6. Also, §4 provides some technical results on Morita theory needed later in §5.

We mention that Corollary 2.6 corrects (and provides a generalization of) an old local-global result [CPS90, Thm. 3.3(c)]. The use of height one primes in the statement and “proof” of [CPS90, op.cit.] was one of the inspirations for our approach here.

Finally, we make a comment on terminology. A local commutative ring is, of course, any commutative ring with a unique maximal ideal. The main base rings used in [DPS15] are primarily local. The terminology for global rings is much less standard. Typical examples are \mathbb{Z} , $\mathbb{Z}[t]$, and also $\mathcal{Z} := \mathbb{Z}[t, t^{-1}]$, the latter used as the main base ring in [DPS17a]. In the present paper, we focus on \mathcal{Z} as well as the fraction rings $\mathbb{S}^{-1}\mathcal{Z}$, where \mathbb{S} is a multiplicative monoid generated by an explicit (and small) finite set of nonzero elements in \mathcal{Z} . Here we regard these rings as global, and try to understand algebras over them in terms of localizations $\mathcal{Z}_{\mathfrak{p}} = (\mathbb{S}^{-1}\mathcal{Z})_{\mathfrak{p}}$ at prime ideals \mathfrak{p} containing no element of \mathbb{S} , and especially those \mathfrak{p} of height ≤ 1 .

2. LOCALIZATION OF INTEGRAL QUASI-HEREDITARY ALGEBRAS (QHAs).

Let k be a commutative Noetherian ring (with 1). All algebras over k are assumed to be finitely generated as k -modules (i.e., they are k -finite). For $\mathfrak{p} \in \text{Spec } k$ and a k -module X , let

$$X(\mathfrak{p}) := X_{\mathfrak{p}}/\mathfrak{p}X_{\mathfrak{p}}$$

be the resulting module for the residue field $k(\mathfrak{p})$. The functor $X \mapsto X(\mathfrak{p}) = X \otimes_k k(\mathfrak{p})$ from the category of k -modules to the category of $k(\mathfrak{p})$ -vector spaces is right exact. If X is a k -submodule of a k -algebra A , let $\overline{X(\mathfrak{p})}$ be the image of the natural $k(\mathfrak{p})$ -map $X(\mathfrak{p}) \rightarrow A(\mathfrak{p})$. In general, $A(\mathfrak{p})$ is a finite dimensional $k(\mathfrak{p})$ -algebra, and, if X is a (left, right, 2-sided) ideal in A , then $\overline{X(\mathfrak{p})}$ is a (left, right, or 2-sided, respectively) ideal in $A(\mathfrak{p})$.

Definition 2.1. Assume that the k -algebra A is projective over k . An ideal J in A is called a *heredity ideal* provided the following conditions hold.

- (0) A/J is projective over k .
- (1) J is a projective as a left A -module.
- (2) $J^2 = J$.

(3) The k -algebra $E := \text{End}_A({}_A J)$ is k -semisimple.

The heredity ideal J is of separable (resp., semisplit, split) type provided that E is separable (resp., semisplit, split) over k . Recall that a k -algebra E is separable if the (E, E) -bimodule map $E \otimes_k E \rightarrow E$ is split. One says that E is semisplit if it is a finite direct product of algebras, each of which is separable and has center k (i.e., each factor is an Azumaya algebra over k). If each factor is the endomorphism algebra of a finite projective k -module, then E is called split.¹

Semisimple algebras over commutative rings arise in the context of relative homological algebra. Alternatively, a (finitely generated) k -algebra E is k -semisimple if and only if the $k(\mathfrak{p})$ -algebras $E(\mathfrak{p})$ are k -semisimple for all $\mathfrak{p} \in \text{Spec } k$. (See [CPS90, Thm. 2.1]. This implies, in particular, that any k -algebra Morita equivalent to E is k -semisimple. We will also need the facts that every separable k -algebra is k -semisimple, and that every module projective over k for a k -semisimple algebra E is projective over E . See [CPS90, pp. 133–135].)

Most idempotent ideals J dealt with in this paper have the form $J = AeA$ for an idempotent $e \in A$. Indeed, if the idempotent ideal J is projective as a left A -module, then A is Morita equivalent to an algebra A' having the property that the corresponding idempotent ideal J' is generated by an idempotent $e' \in A'$. In fact, for some positive integer n , we can take $A' = M_n(A)$, so that $J' = M_n(J) = M_n(A)e'M_n(A)$ for some idempotent $e' \in M_n(A)$. (See [CPS90, Rem. 1.4(b)] for more details.)

The following proposition is new in an integral context. Note that properties (0), (3) in the definition of a heredity ideal are not used.

Proposition 2.2. *Suppose A is a k -algebra which is projective over k , and J is an ideal in A satisfying conditions (1) and (2) of Definition 2.1, that is, $J = J^2$ is projective as a left A -module. Then the following statements hold.*

(a) $E := \text{End}_A({}_A J)$ is projective over k .

(b) If $J = AeA$ for some idempotent $e \in A$, then E is Morita equivalent to eAe .

(c) If $J = AeA$ for some idempotent $e \in A$, then eA is a projective left eAe -module.

Also, the natural multiplication map $Ae \otimes_{eAe} eA \rightarrow J$ is bijective, even at the derived category level, i.e., $Ae \otimes_{eAe}^L eA \xrightarrow{\sim} J$.

Proof. (a) Actually, this holds provided J is any left ideal in A which is projective as a left A -module. In fact, in that case there is a left A -module X so that $Y := J \oplus X \cong ({}_A A)^{\oplus n}$ is a free A -module. Then $\text{End}_A({}_A J)$ is a direct summand of $\text{End}_A(Y)$. The latter identifies with $n \times n$ matrices over A^{op} and hence is projective over k . This proves (a).

To prove (b), note that for each $x \in A$, the A -submodule Aex of ${}_A J$ is a homomorphic image of Ae (via the identity map $Ae \rightarrow Ae$ composed with right multiplication by x). Clearly, AeA is a sum of finitely many such submodules Aex , $x \in A$, since k and A are left Noetherian. Thus, J is a homomorphic image of the left A -module $(Ae)^{\oplus n}$ for some positive integer n . Thus, since ${}_A J$ is projective, it is a direct summand of $(Ae)^{\oplus n}$. Also, Ae is a direct summand of J , viz., $J = Ae \oplus J(1 - e)$. Of

¹In particular, full matrix algebras over k are split. This often occurs for quasi-hereditary algebras using integral standard modules, see [DPS98b, Lem. 1.6] and its proof.

course, the module $N := J(1 - e)$ is also a homomorphic image and direct summand of $(Ae)^{\oplus n}$.

Thus, letting $m = n + 1$, $Ae = M$ and $N = J(1 - e)$, we fulfill the hypothesis of the following lemma.

Lemma 2.3. *Let M, N be finitely generated (left) modules for a k -algebra A , and let m be a positive integer. Suppose there is a split A -module epimorphism $\pi : M^{\oplus(m-1)} \twoheadrightarrow N$. Then $\text{End}_A(M)$ and $\text{End}_A(M^{\oplus m})$ are each Morita equivalent to $\text{End}_A(M \oplus N)$.*

Proof. Of course, $\text{End}_A(M)$ and $\text{End}_A(M^{\oplus m})$ are trivially Morita equivalent, so it suffices to show that $\text{End}_A(M^{\oplus m})$ and $\text{End}_A(M \oplus N)$ are Morita equivalent. We will use the fact that if f is an idempotent in an algebra B , then B is Morita equivalent to the algebra fBf whenever $BfB = B$ (i.e., Bf is a progenerator of $B\text{-mod}$). In our case, we let $B := \text{End}_A(M^{\oplus m})$.

Let $f' \in \text{Hom}_A(M^{\oplus m}, M \oplus N) \cong \text{Hom}_A(M, M) \oplus \text{Hom}_A(M^{\oplus(m-1)}, N)$ be given by $f' := \text{Id}_M \oplus \pi$. Here we have written $M^{\oplus m} = M \oplus M^{\oplus(m-1)}$. This distinguishes a “first summand” $M = M_1$. Then put $f = (\text{Id}_M \oplus \sigma) \circ f'$, where σ is a fixed splitting of the projection $\pi : M^{\oplus(m-1)} \rightarrow N$. It is clear that f is an idempotent in B .

Claim: $BfB = B$.

Write $M^{\oplus m} = M_1 \oplus \cdots \oplus M_m$, where each $M_i = M$ concentrated in position i . Let $b \in B$. To show $b \in BfB$, it suffices to consider the case where $b|_{M_i} = 0$ for all but one index i , call it j . Without loss, $j = 1$. (If $bu \in BfB$, where u is a unit in B , then $b \in BfB$. Choose u to be a unit interchanging M_j and M_1 , and fixing the other summands M_i .) Thus, $b = b\pi_1$, where $\pi_1 : M^{\oplus m} \rightarrow M^{\oplus m}$ is defined by $(x_1, \dots, x_m) \mapsto (x_1, 0, \dots, 0)$. Since $\pi_1 = f\pi_1$, we have that $b = b\pi_1 = bf\pi_1 \in BfB$, proving the Claim, and then the lemma. \square

Part (b) of the Proposition follows.

Finally, we prove (c). There is an evident surjection $Ae \otimes_k eA \twoheadrightarrow J$ of left A -modules. Since ${}_A J$ is projective, ${}_A J$ is a direct summand of $Ae \otimes_k eA$. Thus, eJ is a direct summand of $eAe \otimes_k eA$ in $eAe\text{-mod}$. Thus, eJ is a projective eAe -module. Next,

$$eA \subseteq (eAe)A \subseteq eJ \subseteq eA,$$

so $eJ = eA$. Thus, eA is a projective left eAe -module.

In particular, $Ae \otimes_{eAe} eA \cong A \otimes_{eAe}^{\mathbb{L}} eA$, since $-\otimes_{eAe} eA$ is exact as, say, a functor from $A \otimes_k (eAe)^{\text{op}}\text{-mod}$ to $(A \otimes_k A^{\text{op}})\text{-mod}$. To complete the proof of (c), it remains to show that the multiplication map $Ae \otimes_{eAe} eA \xrightarrow{\mu} J$ is bijective. It is clearly surjective, hence split as a map of left A -modules. Let N be the kernel of μ , and let $J' \subseteq Ae \otimes_{eAe} eA$ be a left A -submodule complementary to N (a section of μ). Then $e(Ae \otimes_{eAe} eA) \cong e((1 - e)Ae \otimes_{eAe} eA) \oplus eAe \otimes_{eAe} eA \cong eAe \otimes_{eAe} eA \cong eA$. However, $eJ' \cong eJ$ through the bijection $\mu|_{J'}$, while $eJ = eA$, as shown above. Hence, $eN = 0$. In more detail: $Ae \otimes_{eAe} eA = J' \oplus N$, so $e(Ae \otimes_{eAe} Ae) = eJ' \oplus eN \cong eA \oplus eN \cong e(Ae \otimes_{eAe} eA) \oplus eN$ as k -modules.

Thus, the set $e(Ae \otimes_{eAe} eA)$, which clearly generates the left A -module $Ae \otimes_{eAe} eA$, is contained in J' . Thus, J' is all of $Ae \otimes_{eAe} eA$, forcing $N = 0$. This finishes the proof of (c). \square

Remarks 2.4. (a) The argument for part (c) of the above proposition was inspired by a parallel argument in the field case given in [CPS96, §2.1].

(b) It is well-known [CPS88] that quasi-hereditary algebras are “left quasi-hereditary” if and only if they are “right quasi-hereditary.” Looking ahead to standardly stratified algebras in §3 below, [CPS96, p.42] provides examples where this left-right symmetry fails for standardly stratified algebras. However, (c) can be used to show that, in the stratified case, A^{op} does satisfy the full embedding condition (3.0.6) below. (In [DPS98a, p. 180], this full embedding condition was taken as [a weaker version] of the notion of a standardly stratified algebras, while the notion used here was called a standardly stratified algebra.)

In the following result, let k be a Noetherian integrally closed domain. Let A be a (finite) k -algebra which is projective over k . Let K be the fraction field of k . If J is an ideal in A , let $\sqrt{J} := J_K \cap A$. Also, recall that, if $\mathfrak{p} \in \text{Spec } k$, then $\overline{J(\mathfrak{p})}$ denotes the image of the natural map $J(\mathfrak{p}) \rightarrow A(\mathfrak{p})$.

Now we have

Lemma 2.5. *Maintain the notation introduced above. Let $J = AeA$ be an idempotent ideal of A generated by an idempotent e . Assume that A/\sqrt{J} is a projective k -module. Also, for each $\mathfrak{p} \in \text{Spec } k$ of height ≤ 1 , assume that $\overline{J(\mathfrak{p})}$ is a heredity ideal in $A(\mathfrak{p})$ such that $\text{End}_{A(\mathfrak{p})}(A(\mathfrak{p})\overline{J(\mathfrak{p})})$ is isomorphic to a direct product of central simple algebras over $k(\mathfrak{p})$. Then*

(a) *the map $J(\mathfrak{p}) \rightarrow \overline{J(\mathfrak{p})}$ is bijective (and, thus, an isomorphism of $(A(\mathfrak{p}), A(\mathfrak{p}))$ -bimodules);*

(b) *$J = \sqrt{J}$, and, furthermore, J is a heredity ideal in A . In addition, $\text{End}_A(AJ)$ is semisplit (i.e., it is a direct product of Azumaya algebras).*

Proof. Of course, (a) is a consequence of (b), but we need to prove (a) first.

By the Peirce decomposition, eAe is a direct summand of A as a k -module. Thus, eAe is itself projective as a k -module. By hypothesis, $J(0) = \overline{J(0)}$ is a heredity ideal such that $\text{End}_{A(0)}(A(0)J(0))$ is separable over K . Therefore, $eA(0)e = (eAe)(0)$ is also a separable algebra since the multiplication map $A(0)e \otimes_{eA(0)e} eA(0) \rightarrow J(0)$ is an isomorphism.

Let \mathfrak{p} be a prime ideal of k having height 1. Because $\overline{J(\mathfrak{p})}$ is a heredity ideal, $\overline{J(\mathfrak{p})} \cong A(\mathfrak{p}) \otimes_{eA(\mathfrak{p})e} eA(\mathfrak{p})$. By the universal property of tensor products, there is a natural surjective map $\overline{J(\mathfrak{p})} \rightarrow J(\mathfrak{p})$. Thus, $\dim \overline{J(\mathfrak{p})} \geq \dim J(\mathfrak{p})$. By definition, $\overline{J(\mathfrak{p})}$ is an image of $J(\mathfrak{p})$. We conclude the surjection $\overline{J(\mathfrak{p})} \rightarrow J(\mathfrak{p})$ is an isomorphism, as is the defining surjection $J(\mathfrak{p}) \rightarrow \overline{J(\mathfrak{p})}$. This proves (a).

It follows also that $A_{\mathfrak{p}}/J_{\mathfrak{p}}$ is torsion free over the discrete valuation ring (DVR) $k_{\mathfrak{p}}$ [Reiner75, Thm. 4.25] (or [Bass68, Thm. 7.13]), so that $\sqrt{J_{\mathfrak{p}}} = J_{\mathfrak{p}}$. Also, $(\sqrt{J}/J)_{\mathfrak{p}} \subseteq (A/J)_{\mathfrak{p}}$ which is torsion free, so $(\sqrt{J}/J)_{\mathfrak{p}} = 0$. Thus,

$$(\sqrt{J})_{\mathfrak{p}} = J_{\mathfrak{p}}. \quad (2.0.1)$$

Continuing with \mathfrak{p} as above, the algebra $E' := \text{End}_{A(\mathfrak{p})}(A(\mathfrak{p})\overline{J(\mathfrak{p})})$ is, by hypothesis, a direct product of central simple algebras over $k(\mathfrak{p})$. Since $\overline{J(\mathfrak{p})}$ is, also by hypothesis, a heredity ideal in $A(\mathfrak{p})$, $eA(\mathfrak{p})e$ is, by Proposition 2.2(b) above, applied over $k(\mathfrak{p})$, Morita equivalent to E' . Thus, $(eAe)(\mathfrak{p}) \cong eA(\mathfrak{p})e$ is also a direct product of central

simple algebras over $k(\mathfrak{p})$. This statement holds for all prime ideals \mathfrak{p} of height ≤ 1 . By [CPS90, Prop. 2.3] (which uses the separability results [AG60b, Prop. 4.6]), the algebra eAe is a direct product of Azumaya algebras over k , and, in particular, it is separable over k . In particular, since eA is a projective k -module, it is a projective eAe -module. It follows that $Ae \otimes_{eAe} eA \subseteq (Ae \otimes_{eAe} eA)_K$. However, we have that a surjection $Ae \otimes_{eAe} eA \rightarrow AeA$. Composing with the inclusion $AeA \subseteq (AeA)_K \cong (Ae \otimes_{eAe} eA)_K$, we obtain the previous inclusion. It follows that ${}_A J \cong {}_A Ae \otimes_{eAe} eA$, so J is projective as a left A -module. Applying Proposition 2.2(b) again, we obtain that $\text{End}_A({}_A J)$ is semisplit over k .

To complete the proof, it suffices to show that $\sqrt{J} = J$. Observe that J is projective over k , since it is a direct summand of $Ae \otimes_{eAe} (eAe)^{\oplus r}$ for some r . Thus, $J = \bigcap_{\text{ht}(\mathfrak{p})=1} J_{\mathfrak{p}}$. However, for each $\mathfrak{p} \in \text{Spec } k$ of height 1,

$$J_{\mathfrak{p}} = (\sqrt{J})_{\mathfrak{p}} \supseteq \sqrt{J} \supseteq J$$

using (2.0.1). Hence, $J = \bigcap_{\text{ht}(\mathfrak{p})=1} J_{\mathfrak{p}}$ must equal \sqrt{J} . Here we use [Reiner75, Thm. 4.25] and k -projectivity of J . (See also [Bass68, Prop. 7.13].) \square

Theorem 2.6. *Assume that A is a finite k -algebra which is projective over a commutative Noetherian ring k . Let J be an idempotent ideal in A .*

(a) *For any given $\mathfrak{p} \in \text{Spec } k$, if $\overline{J(\mathfrak{p})}$ is a heredity ideal in $A(\mathfrak{p})$, then $J(\mathfrak{p}) \cong \overline{J(\mathfrak{p})}$, so that $J(\mathfrak{p})$ identifies with an ideal in $A(\mathfrak{p})$. Moreover, J is a heredity ideal if and only if for each $\mathfrak{p} \in \text{Spec } k$, $\overline{J(\mathfrak{p})}$ is a heredity ideal in $A(\mathfrak{p})$.*

(b) *J is a heredity ideal of separable type if and only if, for every maximal ideal \mathfrak{m} in k , $\overline{J(\mathfrak{m})}$ is a heredity ideal in $A(\mathfrak{m})$ of separable type.*

(c) *Now assume that k is an integrally closed domain and A/J is projective over k . Then J is a heredity ideal of semisplit type if and only if $\overline{J(\mathfrak{p})}$ is a semisplit heredity ideal in $A(\mathfrak{p})$ for every prime ideal \mathfrak{p} of height ≤ 1 .*

(d) *Assume that k is a regular domain of dimension at most 2, and that A/J is projective over k . Then J is a heredity ideal of split type if and only if $\overline{J(\mathfrak{p})}$ is a heredity ideal of split type in $A(\mathfrak{p})$ for every prime ideal \mathfrak{p} of height ≤ 1 .*

Proof. We begin with the following

Claim: For any $\mathfrak{p} \in \text{Spec } k$ such that $\overline{J(\mathfrak{p})}$ is a heredity ideal in $A(\mathfrak{p})$, we have:

- (i) $(A/J)_{\mathfrak{p}}$ is $k_{\mathfrak{p}}$ -projective; and
- (ii) $J(\mathfrak{p}) \cong \overline{J(\mathfrak{p})}$.

We first prove (ii). For this, we may temporarily take A to be $A_{\mathfrak{p}}$ and then even pass to its completion $\widehat{A}_{\mathfrak{p}}$, a faithfully flat base change from $A_{\mathfrak{p}}$. At this point, with A local and complete, we can assume that $J = AeA$ for an idempotent e ; see the discussion in [CPS90, §1]. Note that $\overline{J(\mathfrak{p})}$ remains a heredity ideal in $A(\mathfrak{p})$. By the well-known field case of Proposition 2.2,

$$\overline{J(\mathfrak{p})} \cong A(\mathfrak{p})e \otimes_{eA(\mathfrak{p})e} eA(\mathfrak{p}) \cong (Ae \otimes_{eAe} eA) \otimes_k k(\mathfrak{p}). \quad (2.0.2)$$

The natural multiplication map $Ae \otimes_{eAe} eA \rightarrow AeA$ gives a surjection

$$(Ae \otimes_{eAe} eA) \otimes_k k(\mathfrak{p}) \twoheadrightarrow J \otimes_k k(\mathfrak{p}) = J(\mathfrak{p}) \quad (2.0.3)$$

So $J(\mathfrak{p})$ is a homomorphic image of $\overline{J(\mathfrak{p})}$ by (2.0.2). Since $\overline{J(\mathfrak{p})}$ is defined as the image of a natural surjection $J(\mathfrak{p}) \twoheadrightarrow \overline{J(\mathfrak{p})}$, the latter must be an isomorphism, proving (ii). By [CPS90, Lem. 3.3.1] (a well-known commutative algebra fact [M80, Lem. 4]), assertion (i) also follows. This completes the proof of the Claim.

We now prove (a). First, assume that J is a heredity ideal in A . By the Claim above, $(A/J)_{\mathfrak{p}}$ is $k_{\mathfrak{p}}$ -projective for every prime ideal \mathfrak{p} in k . In particular, this holds for every maximal ideal \mathfrak{m} in k , so that A/J is a projective k -module. Thus, the sequence $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$ of k -modules is k -split. So, for any prime ideal \mathfrak{p} , the sequence remains split upon applying the functor $- \otimes_k k(\mathfrak{p})$. Thus, $\overline{J(\mathfrak{p})} \cong J(\mathfrak{p})$, and we may identify $J(\mathfrak{p})$ with its image in $A(\mathfrak{p})$, for any prime ideal \mathfrak{p} . Since J is idempotent, so is each $J(\mathfrak{p})$ idempotent, and also $J(\mathfrak{p})$ is a projective left $A(\mathfrak{p})$ -module, since it is obtained from the projective left A -module ${}_A J$ by base change from k to $k(\mathfrak{p})$. Finally, since ${}_A J$ is projective, its k -semisimple endomorphism algebra $\text{End}_A({}_A J)$ has $\text{End}_{A(\mathfrak{p})}({}_{A(\mathfrak{p})} J(\mathfrak{p}))$ as its $k(\mathfrak{p})$ -base change. The latter algebra is, therefore, $k(\mathfrak{p})$ -semisimple.

Conversely, assume $\overline{J(\mathfrak{p})}$ is a heredity ideal for all prime ideals \mathfrak{p} . By the Claim, applied just for all maximal ideals, it follows that A/J is projective over k . By [CPS90, Lem. 3.3.2], ${}_A J$ is A -projective. Finally, since ${}_A J$ is A -projective, it base changes to $\text{End}_{A(\mathfrak{p})}({}_{A(\mathfrak{p})} J(\mathfrak{p}))$, which is $k(\mathfrak{p})$ -projective by hypothesis. This statement holds for all $\mathfrak{p} \in \text{Spec } k$. Thus, $\text{End}_A({}_A J)$ is k -semisimple. This completes the proof of (a).

Part (b) is proved similarly, but using [AG60b, Thm. 4.7].

Next, consider (c). First, assume J is a heredity ideal of semisplit type. We need to show that if $\mathfrak{p} \in \text{Spec } k$ has height ≤ 1 , then $\overline{J(\mathfrak{p})}$ is a heredity ideal of semisplit type in the algebra $A(\mathfrak{p})$. By (a) above, we know that $\overline{J(\mathfrak{p})}$ is a heredity ideal in $A(\mathfrak{p})$ and $J(\mathfrak{p}) \cong \overline{J(\mathfrak{p})}$. As noted in the proof of (a), the endomorphism algebra $\text{End}_A(J)$ base changes to $\text{End}_{A(\mathfrak{p})}({}_{A(\mathfrak{p})} J(\mathfrak{p}))$. Since $\text{End}_A({}_A J)$ is a direct sum (algebra-theoretic direct product) of Azumaya algebras (i.e., it is semisplit), $\text{End}_{A(\mathfrak{p})}({}_{A(\mathfrak{p})} J(\mathfrak{p}))$ is also a direct sum of Azumaya algebras over $k(\mathfrak{p})$; see [CPS90, Rem. 2.4], [AG60b, Prop. 1.4, Cor. 1.6].

Conversely, suppose that $\overline{J(\mathfrak{p})}$ is a heredity ideal in $A(\mathfrak{p})$ of semisplit type, for every prime ideal \mathfrak{p} of height ≤ 1 . We wish to show J is an heredity ideal of semisplit type in the k -algebra A . The reader may check that this statement holds if and only if it is true for the ideal $J' = M_n(J)$ in the algebra $A' = M_n(A)$ for any particular positive integer n . As noted above Proposition 2.2, we can choose n , so that J' is generated by an idempotent. By Lemma 2.5(b), J' is a heredity ideal in A' of semisplit type.

Now consider (d). We require the following

Lemma 2.7. (a) *Let D be a DVR with maximal ideal \mathfrak{m} , fraction field F and residue field $\mathfrak{f} = D/\mathfrak{m}$. Let B be an projective algebra over D with the property that B_F and $B_{\mathfrak{f}}$ are full $n \times n$ matrix algebras over F and \mathfrak{f} , respectively. Then B is an full $n \times n$ matrix algebra over D .*

(b) *Suppose that E is an Azumaya algebra projective over a regular domain k of dimension ≤ 2 . Suppose that, for any prime ideal $\mathfrak{p} \in \text{Spec } k$ of height ≤ 1 , $E_{\mathfrak{p}}$ is split. Then E is split.*

Proof. (a) is an exercise using Nakayama's lemma.

(b) If $\mathfrak{p} = 0$, put $K := E_{\mathfrak{p}}$. Then we have $E_K \cong \text{End}_K(V)$, where $V = K^n$ for some $n > 0$. Using [DPS98a, Prop. 1.1.1], V has a E -stable full k -lattice N . Thus, identifying E with its image in $\text{End}_k(N)$ we have $E \subseteq \text{End}_k(N)$. For any prime ideal \mathfrak{p} of height ≤ 1 in k , $E_{\mathfrak{p}}$ is split, by hypothesis, so that $E_{\mathfrak{p}} \cong \text{End}_{k_{\mathfrak{p}}}(P)$ for a projective $k_{\mathfrak{p}}$ -module P (which depends on \mathfrak{p}). In particular, $E_{\mathfrak{p}}$ is a maximal order in $\text{End}_K(P_K)$, as is well-known [Reiner75, Thm. 8.7]. Clearly, the $E_{\mathfrak{p}}$ -modules $P_{\mathfrak{p}}$ and $N_{\mathfrak{p}}$ have the same rank, so are isomorphic as $E_{\mathfrak{p}}$ -modules [Reiner75, Thm. 18.7i]. Thus, $E_{\mathfrak{p}} = \text{End}_{k_{\mathfrak{p}}}(N_{\mathfrak{p}})$. Intersecting over all \mathfrak{p} of height ≤ 1 , we get that $E \cong \text{End}_k(P)$ is split as required; see [Reiner75, (11.3)] and the well-known (local version of) Auslander-Goldman’s criterion for projectivity of modules over regular domains of dimension ≤ 2 [AG60a, p. 18]. This proves the Lemma. \square

Return to the proof of (d). First, suppose that $J(\mathfrak{p})$ is a heredity ideal in $A(\mathfrak{p})$ of split type, for every prime ideal \mathfrak{p} of height ≤ 1 of the regular domain k of dimension ≤ 2 . We will show that J is an heredity ideal of split type in the k -algebra A . (This will prove the \Leftarrow direction in (d). We leave the \Rightarrow direction to the reader; it is not needed later in this paper.) The righthand hypothesis of (d) implies the righthand hypothesis of (c). The proof of (c) shows that $E = \text{End}_A(AJ)$ is a direct product of Azumala algebras which may be taken as projections of E onto the central simple components of E_K . For a prime ideal \mathfrak{p} of height 1, $E(\mathfrak{p})$ is a direct product of full matrix algebras. Clearly, if B is such a projection, the same statement holds if E is replaced by B . Dimension considerations show that if $B(\mathfrak{p})$ is itself a full matrix algebra. By (a) of the above lemma, $B_{\mathfrak{p}}$ is a full matrix algebra. Now by part (a) of the lemma, the E is a full matrix algebra over k . \square

Recall that the projective k -algebra A is called a quasi-hereditary algebra (QHA) provided there exists a finite “defining sequence” $0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_t = A$ of ideals in A such that for $0 < i \leq t$, J_i/J_{i-1} is a hereditary ideal in A/J_{i-1} . In case k is a field, this definition agrees with the classical notion of a QHA given in [CPS88]. Given such a defining sequence $\{J_{\bullet}\}$, we say that it is a *defining sequence of separable type* provided that each J_i/J_{i-1} is a heredity ideal of separable type, i.e., $\text{End}_{A/J_{i-1}}(A/J_{i-1}J_i/J_{i-1})$ is of separable type, $i = 1, \dots, t$. A defining sequence of semisplit type, etc., can be defined similarly.

We end this section with the following improvement (and correction—see the remark following it) of [CPS90, Thm. 3.3]. The proof is easily obtained from Theorem 2.6, and further details are omitted.

Corollary 2.8. *Let A be as in Theorem 2.6. Assume that*

$$0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_t = A \quad (2.0.4)$$

is a sequence of idempotent ideals. The following statements hold.

(a) *The k -algebra A is quasi-hereditary with defining sequence (2.0.4) if and only if, for each $\mathfrak{p} \in \text{Spec } k$, the algebra $A(\mathfrak{p})$ is quasi-hereditary with defining sequence*

$$0 = \overline{J_0(\mathfrak{p})} \subseteq \overline{J_1(\mathfrak{p})} \subseteq \overline{J_2(\mathfrak{p})} \subseteq \cdots \subseteq \overline{J_t(\mathfrak{p})} = A(\mathfrak{p}). \quad (2.0.5)$$

(For any given $\mathfrak{p} \in \text{Spec } k$, when these conditions hold, the isomorphisms $\overline{J_i(\mathfrak{p})} \cong J_i(\mathfrak{p})$ are also valid.)

(b) The k -algebra A is quasi-hereditary of separable type with separable type defining sequence (2.0.4) if and only if, for each maximal ideal \mathfrak{p} in k , the $k(\mathfrak{p})$ -algebra $A(\mathfrak{p})$ is quasi-hereditary of separable type with separable type defining sequence (2.0.5).

(c) Assume that k is a Noetherian integrally closed domain. Assume also that for each i , $0 \leq i < t$, A/J_i is projective over k . The k -algebra A is then quasi-hereditary of semisplit type with semisplit type defining sequence (2.0.4) if and only if, for each prime ideal $\mathfrak{p} \in \text{Spec } k$ of height ≤ 1 , the $k(\mathfrak{p})$ -algebra $A(\mathfrak{p})$ is quasi-hereditary of semisplit type with semisplit type defining sequence (2.0.5).

(d) Assume that k is a regular domain of dimension ≤ 2 . Assume also that for each i , $0 \leq i < t$, A/J_i is projective over k . The k -algebra A is then quasi-hereditary of split type with split type defining sequence (2.0.4) if and only if, for each prime ideal $\mathfrak{p} \in \text{Spec } k$ of height ≤ 1 , the $k(\mathfrak{p})$ -algebra $A(\mathfrak{p})$ is quasi-hereditary of split type with split type defining sequence (2.0.5).

Remarks 2.9. (a) Parts (a) and (b) of Corollary 2.8 are essentially the same as parts (a) and (b) in [CPS90, Thm. 3.3]. We essentially adapted the arguments given in [CPS90] to obtain Theorem 2.6(a), (b) above. Part (c) of [CPS90, Thm. 3.3] is parallel to Corollary 2.8(c) above, but for a smaller class of algebras k , omitting the assumption that the A/J_i be projective over k . This omission appears to be a mistake, and in any case the proof given in [CPS90] is incorrect. For example, the assertion on [CPS90, p. 141] that the $(E(\mathfrak{m}), A(\mathfrak{m}))$ -bimodule $J(\mathfrak{m})$ has $\overline{J(\mathfrak{m})}$ as a homomorphic image appears to be wrong.

(b) Rouquier [Ro08, Thm. 4.15] gives a variation on Corollary 2.8(b) in a highest weight category setting. In the present context, this is very close (but not identical) to using heredity ideals J_i/J_{i-1} of split type, and Corollary 2.8(c) holds as written using defining ideals with this property with a proof similar to the proof above in the semisplit case. A similar remark holds for part (b) in Theorem 2.6.

3. STRATIFIED ALGEBRAS AND THEIR LOCALIZATIONS.

This section follows the outline of the previous section on integral QHAs. The idea is to weaken the notion of a heredity ideal. As we see elsewhere, the new class of algebras, called *standardly stratified algebras* (or SSAs), arise naturally in the study of the cross-characteristic representation theory of finite groups of Lie type. Stratified algebras over a field (with some discussion over DVRs) were first introduced in [CPS96]. The version we follow here, valid over general commutative rings, was first given in [DPS98a]. Essentially, condition (3) in the definition of a heredity ideal in Definition 2.1 is dropped to give the notion of a *standard stratifying ideal*. In particular, Proposition 2.2(c) of the previous section could have been used to begin this section.

As in §2, let k be a Noetherian commutative ring, and let A be a k -algebra, always assumed to be a finite k -module which projective over k . We make the following definition, analogous to the notion of a heredity ideal.

Definition 3.1. An ideal J in A is called a *standard stratifying ideal* if the following conditions hold.

- (0) A/J is projective over k ;
- (1) J is A -projective as a left A -module.

(2) $J^2 = J$.

Observe that, in particular, a heredity ideal is a standard stratifying ideal. With the above notion, we can make the following definition.

Definition 3.2. The algebra A over k is called a *standardly stratified algebra* (SSA) provided there exists a finite “defining sequence” $0 = J_0 \subseteq J_1 \subseteq J_2 \subseteq \cdots \subseteq J_t = A$ of ideals in A such that, for $0 < i \leq t$, J_i/J_{i-1} is a standard stratifying ideal in A/J_{i-1} .

Thus, if A is a QHA over k , it is also a standardly stratified algebra over k . The definition of both types of algebras are given by defining sequences, but the requirements on the sections J_i/J_{i-1} are weaker in the SSA case.

Remarks 3.3. (a) Suppose that J is an ideal in a k -algebra A . Let M, N be A/J -modules. Of course, they can by inflation be regarded as A -modules. Thus, for any integer $n \geq 0$, there is a k -module homomorphism

$$\mathrm{Ext}_{A/J}^n(M, N) \longrightarrow \mathrm{Ext}_A^n(M, N). \quad (3.0.6)$$

By [DPS17a, Appendix B], if J satisfies conditions (1), (2) of Definition 3.1, then the maps in (3.0.6) are isomorphisms for all $n \geq 0$. In particular, they are isomorphisms provided that J is a standard stratifying ideal or a heredity ideal in A .

(b) Stratified (and quasi-hereditary) algebras often arise naturally as endomorphism algebras $\mathrm{End}_R(T)$, where T is an appropriate (right) module for a k -algebra R . See [DPS98a, (1.2.9) and Thm. 1.2.10], as well as the discussion in §§4,5 below.

Let A be a finite k -algebra which is projective over k . Let J be an ideal in A , and assume (for simplicity) that A/J is projective over k . The following proposition gives a simple condition that guarantees that J is a standard stratifying ideal of A .

Proposition 3.4. *Let A and J be as immediately above. Suppose there is an idempotent $e \in J$, a k -subalgebra E of eAe , and a projective E -submodule P of the (left) E -module eA such that the natural multiplication map*

$$Ae \otimes_E P \xrightarrow{\mu} J \quad (3.0.7)$$

is bijective. Then $J^2 = J$ and ${}_A J$ is projective.

Proof. Obviously the image of the multiplication map μ in (3.0.7) is contained in $AeP \subseteq AeA \subseteq J$. So, if $J = \mathrm{Im} \mu$, then $J = AeA$ is idempotent. (This part of the proof only requires the surjectivity of μ .)

It remains to prove that ${}_A J$ is projective. If the multiplication map μ in (3.0.7) is bijective, it gives an isomorphism of left A -modules. Thus, J is isomorphic to a direct summand of a sufficiently large direct sum $(Ae)^{\oplus n}$ of copies of Ae . This follows from the projectivity of P as an E -module, which implies that P is a direct summand of $E^{\oplus n}$ for some positive integer n . \square

Remark 3.5. Proposition 2.2(c) provides a converse to Proposition 3.4 above. Specifically, suppose that J is a standard stratifying ideal in a k -algebra A (still assumed to be a finite module which is projective over k). Then by Definition 3.1, $J^2 = J$ and ${}_A J$ is a projective A -module. Assume that $J = AeA$ for an idempotent $e \in A$. By Proposition 2.2(c), there is a surjective map (3.0.7) of k -modules taking $E = eAe$ and $P = eA$ (which is a projective left $E = eAe$ -module). In fact, μ is an isomorphism.

Proposition 3.6. *Assume that k is an integrally closed Noetherian domain. Let A and J be as immediately above Proposition 3.4. Let $e \in J$ be an idempotent and let E be a k -subalgebra of eAe . Assume, for each $\mathfrak{p} \in \text{Spec } k$ of height ≤ 1 , that $E(\mathfrak{p})$ is a direct product of copies of $k(\mathfrak{p})$. Assume E is projective over k , and that P is an E -submodule, projective over k , of eA such that multiplication induces an isomorphism*

$$(Ae)(\mathfrak{p}) \otimes_{E(\mathfrak{p})} P(\mathfrak{p}) \longrightarrow J(\mathfrak{p})$$

for each $\mathfrak{p} \in \text{Spec } k$ of height ≤ 1 . Then the multiplication map (3.0.7) is an isomorphism, and P is projective over E . In particular, the conclusions of Proposition 3.4 hold.

Proof. Since E is projective over k , and $E(\mathfrak{p})$ is semisimple for each $\mathfrak{p} \in \text{Spec } k$ having height ≤ 1 , E is the product of its projections onto simple factors of $E \otimes_k K$. (See [CPS90, Prop. 2.3b], which uses also the fact that k is an integrally closed domain with fraction field K .) It follows that E is a direct product of copies of k .

In particular, E is separable as a k -algebra in the classical sense of Auslander-Goldman [AG60b], and so each E -module projective over k is projective as an E -module. (See the discussion in [CPS90, p. 133].) In particular, P is projective as an E -module.

It remains to show that the multiplication map μ in (3.0.7) is an isomorphism. By hypothesis, for $\mathfrak{p} = (0)$, μ becomes an isomorphism upon base change to K . However, the natural map $Ae \otimes_E P \longrightarrow (Ae \otimes_E P)_K$ factors as the composite of the map $Ae \otimes_E P \rightarrow (Ae)_K \otimes_E P$ (which is an injection, using the projectivity of P over E) with the natural invertible map

$$(Ae)_K \otimes_E P = (K \otimes_k Ae) \otimes_E P \longrightarrow K \otimes_k (Ae \otimes_E P) = (Ae \otimes_E P)_K.$$

Thus, the map μ in (3.0.7), when followed by the inclusion $J \subseteq J_K$, becomes an injection. (Note that J is k -torsion free.) Hence, the displayed map itself is an injection.

Note that the injectivity of μ gives an isomorphism of $Ae \otimes_E P$ with its image $AeP \subseteq J$.

It remains to show that μ is surjective. By hypothesis, the isomorphism $(Ae)(\mathfrak{p}) \otimes_{E(\mathfrak{p})} P(\mathfrak{p}) \xrightarrow{\sim} J(\mathfrak{p})$ gives a surjection

$$\begin{aligned} (Ae \otimes_E P)_{\mathfrak{p}} &\cong (Ae)_{\mathfrak{p}} \otimes_E P \cong (Ae)_{\mathfrak{p}} \otimes_{E_{\mathfrak{p}}} E_{\mathfrak{p}} \otimes_E P \\ &\cong (Ae)_{\mathfrak{p}} \otimes_{E_{\mathfrak{p}}} P_{\mathfrak{p}} \rightarrow (Ae)(\mathfrak{p}) \otimes_{E(\mathfrak{p})} P_{\mathfrak{p}} \cong J(\mathfrak{p}), \end{aligned}$$

compatible with multiplication, for each height one prime ideal \mathfrak{p} . So multiplication gives surjections $(Ae \otimes_E P)_{\mathfrak{p}} \rightarrow J_{\mathfrak{p}}$, since $J(\mathfrak{p})$ is the head of the finitely generated k -module $J_{\mathfrak{p}}$. Since P is a projective E -module, and Ae is projective over k , the A -module $AeP \cong Ae \otimes_E P$ is projective over k . Hence,

$$AeP = \bigcap_{\text{ht}(\mathfrak{p})=1} (AeP)_{\mathfrak{p}}$$

where $\text{ht}(\mathfrak{p})$ denotes the height of the prime ideal \mathfrak{p} . The intersection is taken in $(AeP)_K$. However, the inclusion $AeP \subseteq J$ induces an isomorphism $(AeP)_K \xrightarrow{\sim} J_K$, which we use to identify J with a submodule of $(AeP)_K$. We may similarly regard any $J_{\mathfrak{p}}$ as a $k_{\mathfrak{p}}$ -submodule of $(AeP)_K$ containing, and, in fact, equal to $(AeP)_{\mathfrak{p}}$. It

is still true that $J \subseteq \bigcap_{\text{ht}(\mathfrak{p})=1} J_{\mathfrak{p}}$. Thus, $J \subseteq AeP \subseteq J$ in $(AeP)_K$. The resulting equality $AeP = J$ holds as well back in J_K . We have now obtained all the hypotheses of Proposition 3.4, and, so the proof of this proposition is complete. \square

We conclude this section with a brief discussion of stratifying systems. These are analogous to highest weight category structures for a module category $A\text{-mod}$.

By a quasi-poset, we mean a (usually finite) set Λ with a transitive and reflexive relation \leq . (In other words, \leq is a pre-order on Λ .) An equivalence relation \sim is defined on Λ by putting $\lambda \sim \mu$ if and only if $\lambda \leq \mu$ and $\mu \leq \lambda$. Let $\bar{\lambda}$ be the equivalence class containing $\lambda \in \Lambda$. Of course, $\bar{\Lambda}$ inherits a poset structure.

Given a finite quasi-poset Λ , a *height function* on Λ is a mapping $\text{ht} : \Lambda \rightarrow \mathbb{Z}$ with the properties that $\lambda < \mu \implies \text{ht}(\lambda) < \text{ht}(\mu)$ and $\bar{\lambda} = \bar{\mu} \implies \text{ht}(\lambda) = \text{ht}(\mu)$. We also say that the function ht is a height function compatible with quasi-poset structure. Given $\lambda \in \Lambda$, a sequence $\lambda = \lambda_n > \lambda_{n-1} > \cdots > \lambda_0$ is called a chain of length n starting at $\lambda = \lambda_n$. Then the standard height function $\text{ht} : \Lambda \rightarrow \mathbb{N}$ is defined by setting $\text{ht}(\lambda)$ to be the maximal length of a chain beginning at λ .

We can now review the notion of a *stratifying system* for a finite k -algebra A and a quasi-poset Λ . We follow the discussion in [DPS15, §2] fairly closely. As noted there, in the original discussion of stratifying system [DPS98a], what we define below was called a “strict” stratifying system. As in [DPS17a], we drop the word “strict” in our treatment.

Definition 3.7. Let k be as in the previous section, and let A be a finite k -algebra which is projective over k . Let Λ be a quasi-poset. For $\lambda \in \Lambda$, there is given a finitely generated A -module $\Delta(\lambda)$, projective as a k -module², and a finitely generated, projective A -module $P(\lambda)$, together with an epimorphism $P(\lambda) \twoheadrightarrow \Delta(\lambda)$. The following conditions are assumed to hold:

(1) For $\lambda, \mu \in \Lambda$,

$$\text{Hom}_A(P(\lambda), \Delta(\mu)) \neq 0 \implies \lambda \leq \mu.$$

(2) Every irreducible A -module L is a homomorphic image of some $\Delta(\lambda)$.

(3) For $\lambda \in \Lambda$, the A -module $P(\lambda)$ has a finite filtration by A -submodules with top section $\Delta(\lambda)$ and other sections of the form $\Delta(\mu)$ with $\bar{\mu} > \bar{\lambda}$.

When these conditions all hold, the data consisting of the $\Delta(\lambda)$, $P(\lambda)$, etc. form (by definition) a *stratifying system* for the category $A\text{-mod}$ of finitely generated A -modules.

The modules $\Delta(\lambda)$, $\lambda \in \Lambda$, are called the *standard modules* for the stratifying system. It is also clear that $\Delta(\lambda)_{k'}, P(\lambda)_{k'}, \dots$ is a stratifying system for $A_{k'}\text{-mod}$ for any base change $k \rightarrow k'$, provided k' is a Noetherian commutative ring. (Notice that condition (2) is redundant, if it is known that the direct sum of the projective modules in (3) is a progenerator—a property preserved by base change.)

We record the following useful result.

Lemma 3.8. ([DPS17a, Lem. 2.1]) *Suppose that A has a stratifying system as above. Let $\lambda, \mu \in \Lambda$. Then*

$$\text{Ext}_A^1(\Delta(\lambda), \Delta(\mu)) \neq 0 \implies \lambda < \mu.$$

²The assumption that each $\Delta(\lambda)$ is projective over k was (incorrectly) omitted in [DPS15], though it was used in that paper; see also footnote 3 in [DPS17a].

Given A -modules X, Y , recall that the trace module $\text{trace}_X(Y)$ of Y in X is the submodule of X generated by the images of all homomorphisms $Y \rightarrow X$.

Proposition 3.9. [DPS17a, Prop. 2.2] *Suppose that A has a stratifying system as above, and let $\text{ht} : \Lambda \rightarrow \mathbb{Z}$ be a height function. Let $\lambda \in \Lambda$. Then the Δ -sections arising from the filtration of $P(\lambda)$ in Definition 3.7(3) can be reordered (constructively, see its proof in op. cit.) so that, if we set*

$$P(\lambda)_j = \text{trace}_{P(\lambda)}\left(\bigoplus_{\text{ht}(\mu) \geq j} P(\mu)\right),$$

then $P(\lambda)_{j+1} \subseteq P(\lambda)_j$, for $j \in \mathbb{Z}$, and

$$P(\lambda)_j/P(\lambda)_{j+1}$$

is a direct sum of modules $\Delta(\mu)$ satisfying $\text{ht}(\mu) = j$.

Proof. First, fix j maximal with a section $\Delta(\mu)$ appearing in $P(\lambda)$ such that $\text{ht}(\mu) = j$. Lemma 3.8 implies that, whenever M is a module with a submodule $D \cong \Delta(\nu)$ and $M/D \cong \Delta(\mu)$, with $\mu, \nu \in \Lambda$ and $\text{ht}(\nu) \leq \text{ht}(\mu)$, then M is the direct sum of D and a submodule $E \cong \Delta(\mu)$. Of course the quotient M/E is isomorphic to D . This interchange of E with D can be repeatedly applied to adjacent Δ -sections in a filtration (SS3) of $P(\lambda)$ to construct a submodule $P(\lambda)(j)$, a term in a modified filtration, which is filtered by modules $\Delta(\nu)$ with $\text{ht}(\nu) = j$, and $P(\lambda)/P(\lambda)(j)$ filtered by modules $\Delta(\nu)$ with $\text{ht}(\nu) < j$. Axiom (SS1) clearly gives $P(\lambda)(j) = P(\lambda)_j$, and $P(\lambda)_{j+1} = 0$. Clearly, $P(\lambda)_j/P(\lambda)_{j+1}$ is a direct sum as required by the proposition. We have not used projectivity of $P(\lambda)$, only its filtration properties. Induction applied to the quotient module $P(\lambda)/P(\lambda)_j$ completes the proof. \square

In [DPS98a, Thm. 1.2.8], it is shown that if an algebra A over k has a stratifying system, then A has a standard stratification involving idempotent ideals. For our purposes, the following result using a height function ht on Λ is more useful.

Theorem 3.10. *Let A be a finite projective k -algebra which has a stratifying system*

$$\{\Delta(\lambda), P(\lambda)\}_{\lambda \in \Lambda}.$$

Put $P := \bigoplus_{\lambda \in \Lambda} P(\lambda)$ and $A' := \text{End}_A(P)^{\text{op}}$. Then

(i) A' is Morita equivalent to A by means of the functor

$$\text{Hom}_A(P, -) : A\text{-mod} \longrightarrow A'\text{-mod}.$$

(ii) the category $A'\text{-mod}$ has a stratifying system $\{\Delta'(\lambda), P'(\lambda)\}_{\lambda \in \Lambda}$ corresponding to $\{\Delta(\lambda), P(\lambda)\}_{\lambda \in \Lambda}$ under the Morita equivalence of (i).

(iii) A' is standardly stratified with a defining sequence $0 = J'_0 \subseteq J'_1 \subseteq \dots \subseteq J'_n = A'$, where $n = \max_{\lambda \in \Lambda} \{\text{ht}(\lambda)\}$, and J'_i/J'_{i-1} is a direct sum of modules $\Delta'(\lambda)$, each with $\text{ht}(\lambda) = n + 1 - i$.

Proof. The proof is an easy application of Proposition 3.9: Filter each $P(\lambda)$ and P itself by standard modules $\Delta(\mu)$'s according to height as in Proposition 3.9. Note that there are no module homomorphisms $\bigoplus_{\text{ht}(\mu) \geq j} P(\mu) \rightarrow P/P_j$ by axiom (1) and the (rearranged) version of axiom (3) in Definition 3.7, where $P_j := \bigoplus_{\lambda \in \Lambda} P(\lambda)_j$. It follows that

$$\text{Hom}_A(P, P_j) \subseteq \text{Hom}_A(P, P) = A'$$

is an ideal in A' , which we set to be J'_{n-j+1} . We also have the short exact sequence

$$0 \rightarrow \mathrm{Hom}_A(P, P_j) \longrightarrow \mathrm{Hom}_A(P, P) \longrightarrow \mathrm{Hom}_A(P, P/P_j) \rightarrow 0$$

which identifies with the short exact sequence $0 \rightarrow J'_{n-j+1} \rightarrow A' \rightarrow A'/J'_{n-j+1} \rightarrow 0$. Note that P/P_j is projective over k since it is filtered by various of the $\Delta(\mu)$. It follows that

$$A'/J'_{n-j+1} \cong \mathrm{Hom}_A(P, P/P_j)$$

is projective over k (since P is projective over A).

The remaining details follow by induction and are left to the reader. Note that

$$A/J_1 \cong \mathrm{Hom}_A(P, P/P_n) \cong \mathrm{Hom}_A(P/P_n, P/P_n) \cong \mathrm{Hom}_{A/J_1}(P/P_n, P/P_n).$$

□

Remark 3.11. (a) A itself is also standardly stratified by a sequence of defining ideals J_i corresponding to the sequence J'_i under the Morita equivalence. However, we may have less control over the summands of various J_i/J_{i-1} . A remedy is to replace A with A' .

(b) Assume the above replacement has been made. There is another useful choice of E and P in Proposition 3.6 closer to Proposition 3.4: Take $e = \sum e_\lambda \in A$. (Recall that A is the now relabeled A' ; here e_λ is the projection $P \rightarrow P(\lambda)$ in the construction of A' .) Let $E = \sum ke_\lambda$. For P in Proposition 3.6, we will use an E -module Q constructed as the direct sum of E -modules $Q_\lambda \subseteq e_\lambda A \subseteq eA$, $\lambda \in \Lambda$. Each Q_λ is a free k -module spanned by elements $a_{\lambda, \mu, s} \in e_\lambda J_1$ ($\cong \mathrm{Hom}_A(Ae_\lambda, J_1)$) where $\mu \in \Lambda$ and s belongs to a set of integers indexed by the pair λ, μ , such that $J_1 = \bigoplus_{\lambda, \mu, s} Ae_{\lambda, \mu, s} a_{\lambda, \mu, s}$ and $Ae_\lambda a_{\lambda, \mu, s} \cong \Delta(\lambda)$. As a result, we get the hypothesis of Proposition 3.6 with E as above, Q ($= P$ in Proposition 3.6) a direct sum of E -modules isomorphic to various ke_λ . Thus, Q is projective over E . A particular interest of this example is that the stratifying system can be reconstructed from this description.

4. SOME MORITA EQUIVALENCES.

Let $\mathcal{Z} = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials over the ring of integers \mathbb{Z} , and let K be its fraction field. Let $\mathcal{G} = \{G(q)\}$ be a family of finite groups of Lie type, in the sense of [CR87, Section 68.22]. The groups $G(q)$ each have a BN-pair structure and there is associated a finite Coxeter system (W, S) (which is independent of q). For simplicity, we ignore the Ree and Suzuki groups in this paper. We will consider the generic Hecke algebra H over \mathcal{Z} with generators T_s , $s \in S$. It has \mathcal{Z} -basis $\{T_w\}_{w \in W}$ and is defined by relations

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } sw > w; \\ t^{2c_s} T_{sw} + (t^{2c_s} - 1)T_w, & \text{if } sw < w. \end{cases} \quad (4.0.8)$$

Recall the index parameters c_s , $s \in S$, are given by $[B(q) : {}^s B(q) \cap B(q)] = q^{c_s}$, where $B(q)$ is a Borel subgroup. For any commutative \mathcal{Z} -algebra R , let $H_R := R \otimes_{\mathcal{Z}} H$.

The distinct irreducible left (or right) H_K -modules are indexed by a finite set, which we denote by Λ . For $\lambda \in \Lambda$, let $E_K(\lambda)$ denote the associated irreducible left H_K -module.³

Recall the pre-order \leq_{LR} on W [Lus03, Ch. 8]. Its associated equivalence classes are called two-sided (Kazhdan-Lusztig) cells. From its definition, \leq_{LR} induces a pre-order, denoted again by \leq_{LR} , on the set Ξ of two-sided cells in W .

The \mathcal{Z} -spans of the Kazhdan–Lusztig basis elements over the two-sided cells are the sections in a filtration of H by two-sided ideals, and there is a unique decomposition of the (split) semisimple algebra H_K as a direct product $\mathfrak{C}_1 \times \cdots \times \mathfrak{C}_r$ of (semisimple) two-sided ideals, one for each two-sided cell section. This provides a corresponding decomposition of Λ . Namely, given $\lambda \in \Lambda$, let $[\lambda] \subseteq \Lambda$, be the set of those μ such that $E_K(\mu)$ and $E_K(\lambda)$ are modules for the same \mathfrak{C}_i . Each $\lambda \in \Lambda$ determines a unique two sided cell, which we denote by $c[\lambda]$: thus, $E_K(\lambda)$ is a left H_K -module for a unique ideal \mathfrak{C}_i determined by $c[\lambda]$. Then \leq_{LR} gives rise to a corresponding pre-order on Λ , setting $\lambda \leq_{LR} \mu$ if and only if $c[\lambda] \leq_{LR} c[\mu]$. Similarly, the opposite pre-order \leq_{LR}^{op} may be defined on Λ .

Let $\text{ht} : \Lambda \rightarrow \mathbb{Z}$ be a height function compatible with the quasi-poset structure defined by \leq_{LR}^{op} . (See the discussion above Definition 3.7.) For convenience, we can assume that ht has image in \mathbb{N} .

There is also a pre-order \leq_L on W whose associated equivalence classes are called left cells. (It is finer than the pre-order \leq_{LR} on W , i. e., $x \leq_L y \implies x \leq_{LR} y$, for $x, y \in W$.) Let Ω be the set of left cells [Lus03]. For each left cell $\omega \in \Omega$, there is a corresponding left cell module $S(\omega) \in H\text{-mod}$; see [DPS17a, below Rem. 4.8] or [Lus03, §8.3]. Observe, by definition, that ht takes a constant value on left cells occurring in the same two-sided cell. The (H, H) -bimodule decomposition above, of H_K into the direct sum of all two-sided cell modules, can be refined into a (left) H -module decomposition of H_K into the direct sum over $\omega \in \Omega$ of all left cell modules $S(\omega)$. Consequently, given two left cells ω, ω' , if $S(\omega)_K$ and $S(\omega')_K$ have a common composition factor, then ω and ω' are contained in the same two-sided cell, and so $\text{ht}(\omega) = \text{ht}(\omega')$. In particular, the function ht is well-defined on the set of left cells. Also, \leq_{LR}^{op} makes sense on Ω , just as it does on W , Λ and Ξ . In addition, $\text{ht} : \Omega \rightarrow \mathbb{Z}$ is compatible with \leq_{LR}^{op} .

As in [DPS17a, §3], we will use the “dual left cell modules” $S_\omega := \text{Hom}_{\mathcal{Z}}(S(\omega), \mathcal{Z}) \in \text{mod-}H$. Now for an integer i , let \mathcal{S}_i be the full additive subcategory of $\text{mod-}H$ whose objects are finite direct sums of various (repetitions allowed) dual left cell modules S_ω with $\text{ht}(\omega) = i$. This notation agrees with that in [DPS17a, op. cit.] (except that our X_j below would be denoted X^j there). The (full) additive category $\mathcal{A}(\mathcal{S})$ defined there consists of objects X , with a (finite) filtration $\cdots \supseteq X_j \supseteq X_{j-1} \supseteq \cdots$ by right H -modules X_j satisfying $X_j/X_{j-1} \in \mathcal{S}_j$, for each j . Of course, these filtrations depend on the height function ht .

Note that the smallest nonzero filtration term X_i of X has the property that $X_i \in \mathcal{S}_i$. In [DPS17a] we constructed finite dimensional right H -modules X_ω , $\omega \in \Omega$. See, in particular, the discussion immediately above [DPS17a, Thm. 4.9], which uses [DPS17a, Thm. 4.7]. There is an exact category $(\mathcal{A}(\mathcal{S}), \mathcal{E}(\mathcal{S}))$ constructed in

³Thus, Λ also indexes the irreducible $\mathbb{Q}W$ -modules. For 2F_4 , which is not allowed here, the algebras $\mathbb{Q}W$ and H_K are not split.

[DPS17a, Construction 3.8]. Here $\mathcal{E}(\mathcal{S})$ is defined by all short exact sequences (in $H\text{-mod}$) of objects in $\mathcal{A}(\mathcal{S})$ which remain exact, and are even split, when passing to a section defined by any \mathcal{S}_j . Now put

$$T^\dagger = \bigoplus_{\omega \in \Omega} X_\omega^{\oplus m_\omega} \quad (4.0.9)$$

for any fixed set $\{m_\omega\}_{\omega \in \Omega}$ of positive integers. The following properties hold:

(1) For all $\omega \in \Omega$, $X_\omega \in \mathcal{A}(\mathcal{S})$, and its smallest nonzero filtration (with respect to ht) term is isomorphic to S_ω ;

(2) For all $\omega \in \Omega$,

$$\text{Ext}_{\mathcal{E}(\mathcal{S})}^1(S_\omega, T^\dagger) = 0.$$

We remark that if $X \in \mathcal{A}(\mathcal{S})$, all exact sequences $X_j \hookrightarrow X \twoheadrightarrow X/X_j$ belong to $\mathcal{E}(\mathcal{S})$, $j \in \mathbb{N}$. If $Y \in \mathcal{A}(\mathcal{S})$ is such that $\text{Ext}_{\mathcal{E}(\mathcal{S})}^1(S, Y) = 0$ for all $S \in \mathcal{S}$, then $\text{Hom}_H(-, Y)$ applied to an exact sequence in $\mathcal{E}(\mathcal{S})$ yields an exact sequence; see [DPS17a, Lem. 3.10]. This will be used in the proof of the following result.

Theorem 4.1. *Let $X \in \mathcal{A}(\mathcal{S})$ satisfy $\text{Ext}_{\mathcal{E}(\mathcal{S})}^1(S_\omega, X) = 0$ for all $\omega \in \Omega$. Let T^\dagger be as in (4.0.9). Put $T^{\dagger'} = T^\dagger \oplus X$. Then the \mathcal{Z} -endomorphism algebras $A^\dagger = \text{End}_H(T^\dagger)$ and $A^{\dagger'} = \text{End}_H(T^{\dagger'})$ are Morita equivalent. A specific Morita equivalence*

$$A^{\dagger'}\text{-mod} \xrightarrow{\sim} A^\dagger\text{-mod}$$

is given on objects by $N' \mapsto eN'$, where $e : T^{\dagger'} \rightarrow T^\dagger \subseteq T^{\dagger'}$ is projection from $T^{\dagger'}$ to T^\dagger along X , and $eA^{\dagger'}e$ is identified with A^\dagger . With this identification $e(T^{\dagger'})_j = T_j^\dagger$ for each $j \in \mathbb{N}$.

Proof. Since X lies in $\mathcal{A}(\mathcal{S})$, it has a height filtration whose sections are direct sums of dual left cell modules S_ω (all having the same height). For each $\omega \in \Omega$, S_ω appears as the lowest term in the filtration of the summand X_ω of T^\dagger . In particular, there is an inflation $S_\omega \rightarrow T^\dagger$. Apply the remark made immediately above the statement of the theorem, with $Y = T^{\dagger'} = T^\dagger \oplus X$. There is a (nonzero) surjection (in $A^{\dagger'}\text{-mod}$)

$$A^{\dagger'}e = \text{Hom}_H(T^\dagger, T^{\dagger'}) \twoheadrightarrow \text{Hom}_H(S_\omega, T^{\dagger'})$$

using condition (2) above and the $\text{Ext}_{\mathcal{E}(\mathcal{S})}^1$ -vanishing condition on X in the hypothesis of the theorem. Observe $T^{\dagger'} = T^\dagger \oplus X \in \mathcal{A}(\mathcal{S})$, since $T^\dagger \in \mathcal{A}(\mathcal{S})$ by (1) and $X \in \mathcal{A}(\mathcal{S})$ by hypothesis. As an object in $\mathcal{A}(\mathcal{S})$, $T^{\dagger'}$ has a (height compatible) filtration with sections direct sums of modules S_ω , $\omega \in \Omega$. Using the $\text{Ext}_{\mathcal{E}(\mathcal{S})}^1(-, T^{\dagger'})$ -vanishing discussed above, we find that the left regular module ${}_{A^{\dagger'}}A^{\dagger'} = {}_{A^{\dagger'}}\text{End}_H(T^{\dagger'})$ has a filtration with sections which are direct sum of $A^{\dagger'}$ -modules $\text{Hom}_H(S_\omega, T^{\dagger'})$. Thus, any irreducible $A^{\dagger'}$ -module is a homomorphic image of some module $\text{Hom}_H(S_\omega, T^{\dagger'})$, and, thus, by the surjection displayed above, of $A^{\dagger'}e$. It follows $A^{\dagger'}e$ is a projective generator for $A^{\dagger'}\text{-mod}$. The rest of the argument is either obvious, or follows from standard Morita theory, as in [Jac89, §3.12]. For example, the functor $M \mapsto \text{Hom}_{A^{\dagger'}}(A^{\dagger'}e, M) \cong eM$ from $A^{\dagger'}\text{-mod}$ to $\text{End}_{A^{\dagger'}}(A^{\dagger'}e)^{\text{op}}\text{-mod} \cong eA^{\dagger'}e\text{-mod}$ is an equivalence of categories. \square

Remark 4.2. The identification $e(T^{\dagger'})_j = T_j^{\dagger}$ above also yields an identification of the right H -modules $e(T^{\dagger'}/(T^{\dagger'})_j) = T^{\dagger}/T_j^{\dagger}$. Thus, the given Morita equivalence takes the two-sided ideal

$$\mathrm{Hom}_H(T^{\dagger'}/(T^{\dagger'})_j, T^{\dagger'})$$

of $A^{\dagger'}$ to the two-sided ideal $\mathrm{Hom}_H(T^{\dagger}/T_j^{\dagger}, T^{\dagger})$ of A^{\dagger} . This will be very useful in our later discussion of stratified algebras and the ideals in their defining sequences.

In the next theorem, we work with the commutative algebra $\mathcal{Z}_{\mathfrak{q}}$ the localization of \mathcal{Z} at a height one prime ideal $\mathfrak{q} = (\Phi_{2e})$, generated by a cyclotomic polynomial Φ_{2e} , $e \neq 2$. (In [DPS15], the algebra $\mathcal{Z}_{\mathfrak{q}}$ is denoted \mathcal{Q} .) Let $\tilde{H} := H_{\mathfrak{q}}$, the Hecke algebra over $\mathcal{Z}_{\mathfrak{q}}$ with basis T_w , $w \in W$, and relations (4.0.8). (In [DPS15], our H here is denoted \mathcal{H} , and \tilde{H} here is denoted $\tilde{\mathcal{H}}$ there.)

Recall that in [DPS15, Thm. 5.6] the $\mathcal{Z}_{\mathfrak{q}}$ -algebra $\tilde{\mathcal{A}}^+$ is realized as an \tilde{H} -endomorphism algebra

$$\tilde{\mathcal{A}}^+ := \mathrm{End}_{\tilde{H}}(\tilde{\mathcal{T}}^+), \quad \text{where} \quad \tilde{\mathcal{T}}^+ := \bigoplus_{\omega \in \Omega} \tilde{\mathcal{T}}_{\omega}, \quad (4.0.10)$$

where each $\tilde{\mathcal{T}}_{\omega}$ is a right \tilde{H} -module: If ω is a left cell containing the longest word $w_{\lambda,0}$ in a parabolic subgroup W_{λ} , then $\tilde{\mathcal{T}}_{\omega}$ in (4.0.10) is a right \tilde{H} “ q -permutation” module $x_{\lambda}\tilde{H}$. (See [DDPW08, (7.6.1)].) (Actually, $\tilde{\mathcal{T}}^+$ above allows for repetition of permutation modules as in (4.0.9), whereas the original arguments in [DPS15] do not. This does not affect the argument in [DPS15], and the endomorphism algebra $\tilde{\mathcal{A}}^+$ there is Morita equivalent to $\tilde{\mathcal{A}}^+$ here in an obvious way.) Otherwise, the \tilde{H} -modules $\tilde{\mathcal{T}}_{\omega}$ are certain right $\tilde{\mathcal{H}}$ -modules inductively constructed in [DPS15, §5C; notation of §5D] and called \tilde{X}_{ω} , indexed by the remaining left cells $\omega \in \Omega$. As pointed out in [DPS15, §5C], $\tilde{\mathcal{T}}_{\omega}$ may be constructed as $(\mathcal{T}_{\omega})_{\mathcal{Q}} = (\mathcal{T}_{\omega})_{\mathfrak{q}}$, where \mathcal{T}_{ω} is either a q -permutation modules $x_{\lambda}H$ or a \mathcal{Z} -free right H -module with certain filtration properties. Accordingly, we may write

$$\mathcal{A}^+ := \mathrm{End}_H(\mathcal{T}^+), \quad \text{where} \quad \mathcal{T}^+ := \bigoplus_{\omega \in \Omega} \mathcal{T}_{\omega} \quad (4.0.11)$$

with $\tilde{\mathcal{T}}^+ = \mathcal{T}_{\mathfrak{q}}^+ := (\mathcal{T}^+)_{\mathfrak{q}}$ and $\tilde{\mathcal{A}}^+ := \mathcal{A}_{\mathfrak{q}}^+$. Each of the localizations $(-)_{\mathfrak{q}}$ can be written as $(-)_{\mathcal{Q}}$ in the notation of [DPS15]. We also mention that the expression \mathcal{T}_{ω} above is sometimes written as X_{ω} in [DPS15].

The proof of the following result requires a height function ht which is compatible with \leq_{LR}^{op} , as used in [DPS17a]. We quote below results from [DPS15, §5C,D], which used a specific compatible height function f (called a sorting function in [GGOR03]). However, this particular choice of height function is unnecessary.

In the following theorem, we assume that the parameters c_s in (4.0.8) are all equal to 1. However, we expect that the theorem holds for any choice of parameters that corresponds to a family of finite groups of Lie type.

Theorem 4.3. *Let $\mathcal{A}^+, \mathcal{T}^+, \mathfrak{q}$ be as above, and let A^{\dagger} and T^{\dagger} be as in Theorem 4.1. Then $\tilde{\mathcal{A}}^+ = \mathcal{A}_{\mathfrak{q}}^+$ is Morita equivalent to $A_{\mathfrak{q}}^{\dagger}$ via an equivalence in which the height filtrations of $T_{\mathfrak{q}}^{\dagger}$ and $\tilde{\mathcal{T}}^+ = \mathcal{T}_{\mathfrak{q}}^+$ all correspond.*

Proof. It is easily seen from the construction in [DPS15, §5C] that each H -module \mathcal{T}_ω and thus \mathcal{T}^+ belongs to the category $\mathcal{A}(\mathcal{S})$ used in [DPS17a, Thm. 4.7]. According to *op.cit.*, there is an inflation $\mathcal{T}^+ \rightarrow X$ (in the exact category sense), where $X \in \mathcal{A}(\mathcal{S})$ satisfies the vanishing conditions of Theorem 4.1. By construction, the H -module X has a height filtration with sections X_j/X_{j-1} which are direct sums of dual left cell modules S_ω of the same height j . Each such S_ω appears as a direct summand⁴ of the lowest term $\mathcal{T}_{\omega,i}$ in some summand \mathcal{T}_ω of \mathcal{T}^+ in the construction [DPS15, §5C,D] of \mathcal{T}^+ . Here i denotes a value of the height function ht on the lowest section for with $\mathcal{T}_{\omega,i} \neq 0$.

Let T^\dagger and $T^{\dagger'} = T^\dagger \oplus X$, with X as above, be as in Theorem 4.1. To prove the theorem it is enough, by Theorem 4.1, to prove Theorem 4.3 with $T^{\dagger'}$ replacing T^\dagger , and its H -endomorphism ring $A^{\dagger'}$ replacing A^\dagger . As in the proof of Theorem 4.1, we find that the module ${}_{A^{\dagger'}}A^{\dagger'}$ is filtered by modules $\text{Hom}_{A^{\dagger'}}(S_\omega, T^{\dagger'})$, $\omega \in \Omega$. The vanishing of $\text{Ext}_{\mathcal{A}(\mathcal{S})}^1(S_\nu, T^{\dagger'})$, for all $\nu \in \Omega$, gives, by the remark immediately above Theorem 4.1, a surjection

$$\text{Hom}_H(\mathcal{T}^+, T^{\dagger'}) \twoheadrightarrow \text{Hom}_H(S_\omega, T^{\dagger'}).$$

However, the exact sequence $0 \rightarrow \mathcal{T}_\mathfrak{q}^+ \rightarrow X_\mathfrak{q} \rightarrow (X/\mathcal{T}^+)_\mathfrak{q} \rightarrow 0$, arising from the inflation above, is split in $\text{mod-}H_\mathfrak{q}$ (note $H_\mathfrak{q} = \tilde{H}$), since $X/\mathcal{T}^+ \in \mathcal{A}(\mathcal{S})$ and $\text{Ext}_{\tilde{H}}^1(\tilde{S}_\omega, \tilde{\mathcal{T}}^+) = 0$ for all $\omega \in \Omega$ by [DPS15, Cor. 4.5(2), Prop. 5.5, §5C,D]. It follows that the $A_\mathfrak{q}^{\dagger'}$ -module $\text{Hom}_{\tilde{H}}(\tilde{\mathcal{T}}^+, T_\mathfrak{q}^{\dagger'})$ is a projective generator. Now argue as in Theorem 4.1. \square

Remark 4.4. It was claimed without proof in [DPS17a] that the algebra A^\dagger constructed there had localizations $A_\mathfrak{q}^\dagger$ agreeing, up to Morita equivalence, with the corresponding localizations in [DPS15], denoted $\mathcal{A}_\mathfrak{q}^+$ in our notation here, with \mathfrak{q} as above. Theorem 4.3 provides the proof of this result. This is potentially important for future decomposition number calculations, since it was shown in [DPS15] that $\mathcal{A}_\mathfrak{q}^+$ has a completion with module category equivalent to a Cherednik algebra module category \mathcal{O} .

We will give further applications of Theorem 4.3 and [DPS17a] in the final two sections.

5. THE HECKE ALGEBRAS AT GOOD PRIMES.

We begin this section in the setting of Theorem 4.1. In particular, we will use the algebra $A^\dagger = \text{End}_H(T^\dagger)$ as in Theorem 4.1. Using [DPS98a, proof of Cor. 1.2.12], A^\dagger is projective over \mathcal{Z} .

By [DPS17a, Thm. 4.9] (which does not depend on the present discussion), A^\dagger has a (strict) stratifying system, $\{\Delta(\omega) := \text{Hom}_H(S_\omega, T^\dagger), P(\omega)\}_{\omega \in \Omega}$.

Also, using Remark 4.2, there are various ideals

$$J_j := \text{Hom}_H(T^\dagger/T_{N-j}^\dagger, T^\dagger), \quad \text{where } N = \max\{\text{ht}(\lambda)\}_{\lambda \in \Lambda}. \quad (5.0.12)$$

⁴In fact, using [Lus03, P4,P9], which certainly holds in the equal parameter setup of [DPS15], it can be shown that S_ω is the full bottom section. The proof of [DPS15, Thm. 5.6] touches this point, but without a proof. The fact here that S_ω a summand follows easily from the Kazhdan-Lusztig cell structure for the (duals of) the various parabolic right H -modules $x_\lambda H$ (for other \mathcal{T}_ω this property is automatic by construction).

Obviously, we have a sequence $0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_N = A^\dagger$ of ideals.

Lemma 5.1. *For each positive integer $j \leq N$, we have*

- (1) *the ideal J_j is idempotent;*
- (2) *the quotient A^\dagger/J_j is a \mathcal{Z} -projective module;*
- (3) *each section J_j/J_{j-1} is projective as a left A^\dagger/J_{j-1} -module.*

Proof. By the proof of Theorem 4.1, each J_j/J_{j-1} is isomorphic to $\text{Hom}_H(T_{N+1-j}^\dagger/T_{N-j}^\dagger, T^\dagger)$ and thus to a direct sum of various $\text{Hom}_H(S_\omega, T^\dagger)$, with $\text{ht}(\omega) = N + 1 - j$. We will use this throughout the proof.

Claim 1: For each j and A^\dagger/J_j -module V , we have $\text{Hom}_{A^\dagger}(P(\omega), V) = 0$ whenever $\text{ht}(\omega) \geq N - j$.

Proof of Claim 1: This fact follows from the filtrations just mentioned and the axioms for a stratifying system when V is the left regular A/J_j -module. However, it reduces to this case through the projectivity of $P(\omega)$. This proves Claim 1.

We first prove (1). As a consequence of Claim 1 and the filtrations above, $\text{Hom}_{A^\dagger}(J_j, V) = 0$ whenever V is an A^\dagger/J_j -mod and $\text{ht}(\omega) \geq N - j$. This applies, in particular, to $V = J_j/J_j^2$, forcing the latter to be 0, i.e., $J_j^2 = J_j$ and J_j is idempotent. This proves (1).

Next, (2) follows from the fact that A^\dagger/J_j is filtered by various $\Delta(\omega)$, which are \mathcal{Z} -projective.

Finally, we prove (3). We prove J_j/J_{j-1} is an A^\dagger/J_{j-1} -projective module. We know J_j/J_{j-1} is filtered by $\Delta(\omega) = \text{Hom}_H(S_\omega, T^\dagger)$, with $\text{ht}(\omega) = N + 1 - i$. For each ω there is projective A^\dagger -module $P(\omega)$ with a Δ -filtration. Also, $P(\omega)$ maps onto $\Delta(\omega)$. The other sections $\Delta(\tau)$ satisfy $\text{ht}(\tau) > \text{ht}(\omega)$.

Claim 2: Suppose $\omega \in \Lambda$ has height $N - j$. Then

$$P(\omega)/J_{j-1}P(\omega) \cong \Delta(\omega). \quad (5.0.13)$$

Proof of Claim 2. Equivalently, $J_{j-1}P(\omega) = K_\omega$, the kernel of the map $P(\omega) \rightarrow \Delta(\omega)$. Clearly, $J_{j-1}P(\omega) \subseteq K_\omega$. Also, $P(\omega)/J_{j-1}P(\omega)$ is an A^\dagger/J_{j-1} -module, as is $K_\omega/J_{j-1}P(\omega)$. However, Claim 1 implies that $\text{Hom}_{A^\dagger}(\Delta(\omega), V) = 0$ if $\text{ht}(\omega) \geq N - j$ and V is an A^\dagger/J_j -module. Since K_ω is filtered by such Δ 's, it follows taking $V = K_\omega/J_{j-1}P_\omega = 0$. This proves Claim 2.

Next, note that $P(\omega)/J_{j-1}P(\omega)$ is a projective A^\dagger/J_{j-1} -module. We know that all direct summands of J_j/J_{j-1} is a direct sum of projective A/J_{j-1} -modules (the various $\Delta(\omega)$ with $\text{ht}(\omega) = N + 1 - j$). Thus, J_j/J_{j-1} is projective as an A/J_{j-1} -module. This completes the proof of (3) and thus the lemma. \square

We now define a specific multiplicative monoid $\mathbb{S} \subseteq \mathcal{Z} = \mathbb{Z}[t, t^{-1}]$ that will be used in Theorem 5.2. Namely, assume that \mathbb{S} is generated by all the bad primes for the Weyl group W together with the cyclotomic polynomial $\Phi_4(t) = t^2 + 1$. Let $\mathcal{Z}^\natural = \mathbb{S}^{-1}\mathcal{Z}$, the localization of \mathcal{Z} and \mathbb{S} . Put $H^\natural := \mathbb{S}^{-1}H = \mathcal{Z}^\natural \otimes_{\mathcal{Z}} H$. We consider the right H -module T^\dagger defined in [DPS17a], and put $T^{\natural\dagger} = \mathbb{S}^{-1}T^\dagger$. Similarly, put $A^{\natural\dagger} = \mathbb{S}^{-1}A^\dagger \cong \text{End}_{H^\natural}(T^{\natural\dagger})$.

Also, in Theorem 5.2, we presently require the equal parameter assumption of [DPS15]. That is, we assume each $c_s = 1$ in (4.0.8). (The authors expect this

requirement will be removed in a subsequent paper, and $t^2 + 1$ will be removed from \mathbb{S} . See Remark 5.3.) In the proof, we shall also use Theorem 4.3; it serves here as a bridge between the results of [DPS17a], quoted above, and [DPS15]. The module $\tilde{\mathcal{T}}^+$ in Theorem 4.3 may be viewed as the base change to \mathcal{Q} of a module $\mathcal{T}^{\dagger\mathfrak{h}}$ for $H^{\mathfrak{h}}$. See [DPS15, §5C] which even starts with modules for H . A similar remark applies to $\tilde{\mathcal{A}}^+$.

Theorem 5.2. *With the equal parameter assumption, $A^{\dagger\mathfrak{h}}$ is a split quasi-hereditary algebra over $\mathcal{Z}^{\mathfrak{h}}$. It has a split defining sequence obtained from that in Lemma 5.1 by base-change to $\mathcal{Z}^{\mathfrak{h}}$.*

Proof. Here we work over $\mathcal{Z}^{\mathfrak{h}}$ rather than the DVR \mathcal{Q} as in [DPS15]. In fact, \mathcal{Q} is a localization of $\mathcal{Z}^{\mathfrak{h}}$ at the ideal $\mathfrak{q} = (\Phi_{2e})$, $e \neq 2$. By [DPS15, Thm. 6.4], $\mathcal{A}_{\mathfrak{q}}^+$ ($= \tilde{\mathcal{A}}^+$ in [DPS15]) is quasi-hereditary, and thus $\mathcal{A}^{\dagger\mathfrak{h}}(\mathfrak{q})$ is quasi-hereditary. The proof there shows that $\mathcal{A}_{\mathfrak{q}}^+$ is split quasi-hereditary, with the height function (used in Lemma 5.1) giving the required split defining sequence. These properties pass to $A_{\mathfrak{q}}^{\dagger\mathfrak{h}} = A_{\mathfrak{q}}^{\dagger}$ by Theorem 4.3. Also, [DPS98a, Thm. 4.2.2] implies that $A^{\dagger\mathfrak{h}}(\mathfrak{p})$ is split semisimple for all choices of height one primes $\mathfrak{p} = (p)$ with p a good prime integer, or for $\mathfrak{p} = 0$. (Here we use the fact that \mathbb{S} contains all the bad primes of W , appearing in the denominator of the generic degrees.)

The argument in [DPS98a, op. cit.] actually works in characteristic 0, and shows that $A^{\dagger\mathfrak{h}}(\mathfrak{p})$ is split semisimple, if $\mathfrak{p} = (f(t))$ for $f(t) \in \mathbb{Z}[t]$ a (nonscalar) primitive irreducible polynomial, and if $f(t)$ does not divide any product of cyclotomic polynomials $\Phi_e(t^2)$, $e \in \mathbb{N}^+$. We note that

$$\Phi_e(t^2) = \begin{cases} \Phi_{2e}(t), & e \text{ even} \\ \pm \Phi_{2e}(t)\Phi_{2e}(-t), & e \text{ odd.} \end{cases} \quad (5.0.14)$$

by an elementary argument. Thus, if $f(t)$ divides any such product, it must be either $\Phi_{2e}(t)$ or $\pm\Phi_{2e}(-t)$. The latter polynomial is conjugate, of course, to $\Phi_{2e}(t)$ by an automorphism of \mathcal{Z} . The associated prime ideals \mathfrak{p} always give, if $e \neq 2$, split quasi-hereditary algebras $A^{\dagger\mathfrak{h}}(\mathfrak{p})$ with respect to the defining sequence defined using the height function, as discussed above. Now our assertion follows from Corollary 2.8. \square

Remark 5.3. We expect to show in [DPS17b] that Theorem 5.2 holds when the multiplicative monoid \mathbb{S} is replaced by the smaller multiplicative monoid \mathbb{S}' generated by the bad primes for W . (In other words, the cyclotomic polynomial $t^2 + 1$ can be omitted. Also, in this more recent setting, we should not require equal parameters, but instead can use the setting of Theorem 4.1 in the current paper.)

In the proof of the following result, we fix a height function $\text{ht} : \Lambda \rightarrow \mathbb{N}$ which is compatible with the quasi-poset structure \leq_{LR}^{op} on Λ . We assume the stronger version of Theorem 5.2 as described in Remark 5.3 above, and allow its relaxed hypothesis: thus, $\mathcal{Z}^{\mathfrak{h}}$ is obtained from \mathcal{Z} by inverting the bad primes of W , while $t^2 + 1$ plays no special role. For a version of Theorem 5.4 proved using only Theorem 5.2 as proved in the present paper, see Remark 5.5 (a).

Theorem 5.4. *There is a family $\{E(\lambda)\}_{\lambda \in \Lambda}$ of $H^{\mathfrak{h}}$ -modules with the following properties:*

(1) Each $E(\lambda)$ is projective as a \mathcal{Z}^{\natural} -module, and $E(\lambda)_K$ is the irreducible H_K -module indexed by λ .

(2) Let \mathfrak{m} be any fixed maximal ideal of \mathcal{Z}^{\natural} . Let D be an irreducible $H^{\natural}(\mathfrak{m})$ -module. There is a unique $\lambda = \lambda(D) \in \Lambda$ such that $[E(\lambda)(\mathfrak{m}) : D] \neq 0$ and such that $[E(\mu)(\mathfrak{m}) : D] = 0$ for any $\mu \in \Lambda$ with $\text{ht}(\mu) \leq \text{ht}(\lambda)$ and $\mu \neq \lambda$.

(3) Let \mathfrak{m} , $E(\lambda)$, and $\lambda = \lambda(D)$ be as in (2). Then D is in the head of $E(\lambda)(\mathfrak{m})$.

(4) Using the notation of (3), we have $[E(\lambda)(\mathfrak{m}) : D] = 1$.

Proof. The notation Λ was introduced in §4 as a set indexing the irreducible left (or right) H_K -modules. The structure of $A_K^{\dagger} = A_K^{\natural\dagger}$ as endomorphism algebra shows that Λ also indexes in a corresponding way the irreducible left $A_K^{\dagger} = A_K^{\natural\dagger}$ -modules. It is observed in [DPS17a, Rem. 4.10] that the right regular module H_H has a split embedding into T^{\dagger} . Using the idempotent projection $e : T^{\dagger} \rightarrow H_H \subseteq T^{\dagger}$, we may identify

$$H = eA^{\dagger}e \quad \text{and} \quad H^{\natural} = eA^{\natural\dagger}e.$$

Using either identification, multiplication by e sends an irreducible $A_K^{\dagger} = A_K^{\natural\dagger}$ -module to an irreducible $H_K = H_K^{\natural}$ -module. We may take multiplication by e as defining the indexing correspondence for Λ .

Multiplication by e sends irreducible $A_k^{\natural\dagger}$ -modules to irreducible or zero H_k^{\natural} -modules, whenever k is any field with a ring homomorphism $\mathcal{Z}^{\natural} \rightarrow k$. Also, $e \text{head}(V) \subseteq \text{head}(eV)$ for any left $A^{\natural\dagger}$ -module V .

Next, notice that Theorem 5.4 makes sense if H^{\natural} and H_K are replaced in its wording by $A^{\natural\dagger}$ and $A_K^{\natural\dagger}$, respectively. Moreover, this new version implies the original version using the above remarks, as the reader may check. We now prove the new version.

Recall that Theorem 5.2 gives a defining sequence $0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_n$ for $A^{\natural\dagger}$ associated to the height function; see Lemma 5.1. The splitting property gives that each $\text{End}_j := \text{End}_{A^{\natural\dagger}}(J_j/J_{j-1})$ is a direct product of split Azumaya algebras. The latter are all \mathcal{Z}^{\natural} -endomorphism algebras of projective \mathcal{Z}^{\natural} -modules. By [Sw78, p. 111, first paragraph], the latter projective modules are free. (We use here that \mathcal{Z}^{\natural} is obtained from $\mathcal{Z} = \mathbb{Z}[t, t^{-1}]$ by inverting some rational primes in \mathbb{Z} . This uses a Swan theorem version of Serre's conjecture. We remark that another way to insure projective modules over the base ring are free is to just pass to a localization at a maximal ideal, as in Remark 5.5(a).)

Thus, the direct product decomposition of End_j may be regarded as a direct sum of ideals associated to various centrally primitive idempotents in End_j . These idempotents may be obviously indexed as e_{λ} , $\lambda \in \Lambda_j$, where Λ_j corresponds to the irreducible left $(A^{\natural\dagger})_K$ -modules in $(J_j/J_{j-1})_K$. Thus, $J_j/J_{j-1} = \bigoplus_{\lambda \in \Lambda_j} e_{\lambda}(J_j/J_{j-1})$. The endomorphism algebra $\text{End}_{A^{\natural\dagger}}(e_{\lambda}J_j/J_{j-1})$ is a full matrix algebra $M_{n_{\lambda}}(\mathcal{Z}^{\natural})$ and the $A_K^{\natural\dagger}$ -module $(e_{\lambda}J_j/J_{j-1})_K$ is a direct sum $E_K(\lambda)^{\oplus n_{\lambda}}$, where $E_K(\lambda)$ is the irreducible $A_K^{\natural\dagger}$ -module indexed by λ .

Now fix any diagonal primitive idempotent $f_{\lambda} \in M_{n_{\lambda}}(\mathcal{Z}^{\natural})$, and set

$$E(\lambda) := f_{\lambda}e_{\lambda}J_j/J_{j-1}.$$

Then $E(\lambda)$ is a projective $A^{\natural\dagger}/J_{j-1}$ -module, and $E(\lambda)_K \cong E_K(\lambda)$.

We now check the properties in the new version of Theorem 5.4. Property (1) is obvious. Fix an irreducible $A^{\natural\dagger}$ -module D . Let j be the smallest value with

$[(J_j/J_{j-1}(\mathfrak{m}) : D) \neq 0$. Thus, D is killed by J_{j-1} , so is an irreducible A/J_{j-1} -module. As such it appears in the head of $A/J_{j-1}(\mathfrak{m})$. Since it does not appear $(A/J_j)(\mathfrak{m})$, it must appear in the head of $(J_j/J_{j-1})(\mathfrak{m})$. In particular, it must appear in the head of $E(\lambda)(\mathfrak{m})$ for some $\lambda \in \Lambda_j$. Such a λ is unique, since $E(\lambda)$ is a projective A^\dagger/J_{j-1} -module, and $\text{Hom}_{A^\dagger}(E(\lambda), E(\mu)) = 0$ for $\lambda \neq \mu$ in Λ_j . This proves (2) and (3). Also, $\text{Hom}_{A^\dagger}(E(\lambda), E(\lambda))$ has rank one over \mathcal{Z}^\natural . This proves (4) and the theorem is proved. \square

Remarks 5.5. (a) A local version of the theorem, using \mathfrak{m} from the start, can be proved using the setting of Theorem 5.2 in this paper. Here one must assume that $t^2 + 1$ does not lie in \mathfrak{m} , and \mathcal{Z}^\natural and H^\natural should be replaced by $\mathcal{Z}_\mathfrak{m}^\natural$ and $H_\mathfrak{m}^\natural$, respectively. There is only one choice, in this formulation for \mathfrak{m} in (2), (3), (4). Still, it is interesting that a result for the local case of $H_\mathfrak{m}^\natural$ can be deduced from properties of the various $H_\mathfrak{p}^\natural$ for \mathfrak{p} of height ≤ 1 in \mathcal{Z} , which, in effect, happens in the proof.

(b) For any fixed \mathfrak{m} , the formulation of Theorem 5.4 is (deliberately) similar to Geck’s triangularization theorem [GJ11, Thm. 4.4.1]. The latter result uses a “unipotent support” function, an extension of Lusztig’s a -function, whereas we use a general height function. The context in [GJ11] has the virtue of relevance to unipotent conjugacy classes and character families, while ours has a similar relevance to generalized q -Schur algebras.

6. BAD PRIMES AND STANDARDLY STRATIFIED ALGEBRAS.

We again use the notation of §4, as introduced above Theorem 4.1, and again let A^\dagger denote the algebra $\text{End}_H(T^\dagger)$ over $\mathcal{Z} := \mathbb{Z}[t, t^{-1}]$ (also discussed in §4). Thus, Lemma 5.1 is available. Our aim in this section, is to give cruder versions of Theorems 5.2 and 5.4 over \mathcal{Z} , rather than \mathcal{Z}^\natural , without any preference for “good primes.” Also, unequal parameters are allowed as in the setting of [DPS17a, Thm. 4.9].

For $\omega, \omega' \in \Omega$, define a pre-order

$$\omega \preceq \omega' \iff \text{ht}(\omega) < \text{ht}(\omega'), \text{ or } \text{ht}(\omega) = \text{ht}(\omega') \text{ and } \omega \sim_{LR} \omega'.$$

(The same definition is in the preamble to [DPS17a, Thm. 4.9].). Then (Ω, \preceq) becomes a quasi-poset and ht remains a height function with respect to \preceq .

The following result is stated and proved in [DPS17a, Thm. 4.9]. It parallels Theorem 5.2 above.

Theorem 6.1. *The \mathcal{Z} -algebra $A^\dagger := \text{End}_H(T^\dagger)$ is standardly stratified. In fact, it has stratifying system, relative to the quasi-poset (Ω, \preceq) , consisting of all $\Delta(\omega) := \text{Hom}_H(S_\omega, T^\dagger)$, with S_ω ranging over the dual left cell modules.*

For $\omega \in \Omega$, let $[\omega]$ denote the two-sided cell to which ω belongs. It will be convenient in the theorem below to let $S[\omega]$ denote the two-sided cell module associated with $[\omega]$, viewed as a left H -module.⁵ Part (1) of the theorem is implicit in the discussion above Theorem 4.1. The theorem parallels the first three parts of Theorem 5.4.

Theorem 6.2. (1) *All irreducible H_K -modules appear in some $S[\omega]_K$.*

⁵modulo higher cells, using \leq_{LR}^{op} .

(2) Let \mathfrak{m} be any fixed maximal ideal of \mathcal{Z} . Let D be an irreducible $H(\mathfrak{m})$ -module. There is a unique two-sided cell $[\omega] = [\omega](D)$ with $[S[\omega](\mathfrak{m}) : D] \neq 0$ and $[S[\nu](\mathfrak{m}) : D] = 0$ for any ν with $\text{ht}(\nu) \leq \text{ht}(\omega)$ and $[\nu] \neq [\omega]$.

(3) For $[\omega] = [\omega](D)$ as above, D is in the head of $S[\omega](\mathfrak{m})$.

Proof. It suffices to prove statements (2) and (3): We recall from [DPS17a, Rem. 4.10] that the right H -module H_H is a direct summand of T^\dagger . Let $e \in A^\dagger = \text{End}_H(T^\dagger)$ be the projection $T^\dagger \rightarrow H_H \subseteq T^\dagger$. Thus, $eA^\dagger e \cong H$. Also, any $eA^\dagger e$ -module $e\Delta(\omega)$ identifies with the left H -module $S(\omega)$ using the isomorphisms

$$\text{Hom}_H(S_\omega, H_H) \cong \text{Hom}_{\mathcal{Z}}(S_\omega, \mathcal{Z}) \cong S(\omega).$$

Here the left isomorphism follows from the bilinear form induced identification $H_H \cong \text{Hom}_{\mathcal{Z}}({}_H H, \mathcal{Z})$ and the general isomorphism $\text{Hom}_H(S_\omega, \text{Hom}_{\mathcal{Z}}({}_H H, \mathcal{Z})) \cong \text{Hom}_{\mathcal{Z}}(S_\omega \otimes_H H, \mathcal{Z})$.

Let D be a given irreducible $H(\mathfrak{m})$ -module; it may also be viewed an irreducible left H -module. Viewing H as $eA^\dagger e$, there is a unique irreducible A^\dagger module L such that $eL \cong D$. (This is easily argued by passing first to $A^\dagger(\mathfrak{m})$, then to a simple algebra factor of its semisimple head.) Since $\{\Delta(\omega)\}_{\omega \in \Omega}$ is a stratifying system, there is a $\omega \in \Omega$ with $\text{Hom}_{A^\dagger}(\Delta(\omega), L) \neq 0$. Also, if eL is a composition factor of $e\Delta(\nu)$ for some $\nu \in \Omega$, then $\omega \preceq \nu$, since $\text{Hom}_{A^\dagger}(P(\omega), \Delta(\nu)) \neq 0$. If the two-sided cells $[\omega]$ and $[\nu]$ are unequal, then $\text{ht}(\nu) > \text{ht}(\omega)$ by construction of \preceq . This proves (2). Also, we get (3), since $e(\text{head}(\Delta(\omega)))$ is contained in the head of $e\Delta(\omega)$. \square

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