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★**Finite dimensional algebras and quantum groups.**
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This book provides an introduction to two algebraic approaches to quantum groups. The quantum groups considered are the quantum enveloping algebras associated by Drinfel′d and Jimbo to a symmetrizable Cartan matrix. Their presentation is a q-deformation of the presentation of a Kac-Moody Lie algebra by Chevalley generators and Serre relations.

There is a well-known correspondence between the symmetrizable Cartan matrices used in the definition of Kac-Moody Lie algebras (and their quantized analogues) and finite (possibly valued) graphs. This correspondence is only a hint of a much deeper theory. Gabriel's theorem [P. Gabriel, Manuscripta Math. **6** (1972), 71–103; correction, ibid. **6** (1972), 309; MR0332887 (48 #11212)] states that if the underlying graph of a quiver (directed graph) is a (simply laced) Dynkin graph, then the indecomposable representations correspond naturally to the positive roots of the semisimple Lie algebra associated to that Dynkin graph. This result was later generalized by V. G. Kac [Invent. Math. **56** (1980), no. 1, 57–92; MR0557581 (82j:16050)] to a result involving quivers of arbitrary type. Thus there is a strong connection between the structure of Kac-Moody Lie algebras and representations of finite-dimensional algebras.

This connection was further developed by C. M. Ringel [Invent. Math. **101** (1990), no. 3, 583–591; MR1062796 (91i:16024)], who developed the Ringel-Hall realization of symmetrizable Kac-Moody algebras in finite type. Further work by G. Lusztig [J. Amer. Math. Soc. **3** (1990), no. 2, 447–498; MR1035415 (90m:17023)] and J. A. Green [Invent. Math. **120** (1995), no. 2, 361–377; MR1329046 (96c:16016)] built on this construction to obtain realizations in arbitrary type. Lusztig's construction also resulted in canonical bases in the $\pm$-parts of quantum enveloping algebras.

A parallel connection between quantum groups and finite-dimensional algebras has been developed by Beilinson, Lusztig, and MacPherson. They used quantum Schur algebras to give a realization of the entire quantum enveloping algebras in type A. This method leads to an explicit basis, called the BLM basis, for the entire quantum enveloping algebra as well as some explicit multiplication formulas. It was proved by J. Du and B. J. Parshall [Indag. Math. (N.S.) **13** (2002), no. 4, 459–481; MR2015831 (2005b:17031)] that a triangular part of the BLM basis coincides with the Ringel-Hall algebra basis.

In addition to an introductory chapter, the book consists of 14 chapters split into 5 parts, entitled as follows:

- Part 1: Quivers and their representations. 1. Representations of quivers, 2. Algebras with Frobenius morphisms, 3. Quivers with automorphisms.
- Part 2: Some quantized algebras. 4. Coxeter groups and Hecke algebras, 5. Hopf algebras and

universal enveloping algebras, 6. Quantum enveloping algebras.

- Part 3: Representations of symmetric groups. 7. Kazhdan-Lusztig combinatorics for Hecke algebras, 8. Cells and representations of symmetric groups, 9. The integral theory of quantum Schur algebras.
- Part 4: Ringel-Hall algebras: a realization for the $\pm$-parts. 10. Ringel-Hall algebras, 11. Bases of quantum enveloping algebras of finite type, 12. Green's theorem.
- Part 5: The BLM algebras: a realization for quantum $\mathfrak{gl}_n$. 13. Serre relations in quantum Schur algebras, 14. Constructing quantum $\mathfrak{gl}_n$ via quantum Schur algebras.

It also contains three appendices: A. Varieties and affine algebraic groups, B. Quantum linear groups through coordinate algebras, C. Quasi-hereditary and cellular algebras. Each chapter concludes with exercises and historical notes.

The leading Chapter 0 outlines the main features of the book. It discusses two realizations of Cartan matrices: the graph realization and the root datum realization. These are precursors to the theories of quiver representations and quantum enveloping algebras.

Part 1 (Chapters 1–3) involves the theory of finite-dimensional algebras. The focus is on representations of quivers with automorphisms. Chapter 1 introduces the theory of quiver representations and gives a proof of Gabriel's theorem relating the representation theory of quivers to the root systems of Kac-Moody algebras. The relations between quivers, Euler forms, root systems, Weyl groups, and representation varieties are discussed.

Chapter 2 presents the theory of representations of algebras with Frobenius morphisms. It discusses the equivalence between the category of modules stable under the Frobenius twist functor and the module category of the Frobenius fixed point algebra.

In Chapter 3, the theory of Frobenius morphisms is applied to the path algebra of a quiver with automorphism. The automorphism induces a Frobenius morphism of the path algebra and the fixed point algebra is a hereditary algebra over a finite field. Up to Morita equivalence, every finite hereditary algebra can be realized in this way. Applications to Auslander-Reiten theory are also discussed.

Part 2 (Chapters 4–6) discusses the presentation via generators and relations of the algebras that play a central role in the book. Chapter 4 introduces the theory of Coxeter groups (including symmetric groups and affine Weyl groups) and Hecke algebras. It also shows that the Hecke algebras for the symmetric groups are related to the endomorphism algebra of the complete flag variety of a finite general linear group.

Chapter 5 gives an introduction to Hopf algebras, paying special attention to universal enveloping algebras, especially Kac-Moody Lie algebras and their symmetries. The chapter concludes with a treatment of quantum $\mathfrak{sl}_2$.

In Chapter 6, the book treats the topic of quantum enveloping algebras associated to symmetrizable Cartan matrices via the Drinfel′d-Jimbo presentation. Their Hopf algebra structure and braid group action is discussed. This action leads naturally to the definition of root vectors and PBW-type bases in the finite type case.

Part 3 (Chapters 7–9) consists of a modern approach to the representation of the symmetric groups and associated Hecke algebras. Chapter 7 develops the Kazhdan-Lusztig theory of Hecke algebras and cells. It discusses Kazhdan-Lusztig polynomials, dual bases, inverse Kazhdan-Lusztig polynomials, and Knuth, cell, and Vogan equivalence relations. The geometric interpretation of Kazhdan-Lusztig polynomials is briefly treated.

Chapter 8 uses the Robinson-Schensted-Knuth correspondence to explicitly determine the cells for the symmetric groups and construct the simple representations of symmetric groups and their associated Hecke algebras.

In Chapter 9, the theory of Hecke algebras is extended to a treatment of the Kazhdan-Lusztig calculus for quantum Schur algebras. It is shown that these algebras are quasi-hereditary, and canonical bases are constructed.

Part 4 (Chapters 10–12) concerns Ringel's Hall algebra approach to quantum enveloping algebras. In Chapter 10, the definition of the (integral) Hall algebra of a finitely generated algebra over a finite field is given. Certain fundamental relations are discussed which become the quantum Serre relations in the Ringel-Hall algebra. There is a surjective algebra homomorphism from the triangular part of a quantum enveloping algebra to the generic composition algebra associated with a quiver with automorphism. In the case of a Dynkin quiver, this is in fact an isomorphism.

Chapter 11 takes up the Ringel-Hall algebras of Dynkin quivers with automorphisms and the corresponding quantum enveloping algebras of finite type. Algebraic constructions are given of monomial bases, PBW-type bases and Lusztig's canonical bases.

In Chapter 12, the comultiplication on Ringel-Hall algebras defined by Green is presented. Using the compatibility of multiplication and comultiplication, combined with a theorem of Lusztig, it is shown that the surjective algebra homomorphism of Chapter 10 is an isomorphism. This completes the realization of the triangular parts of all quantum enveloping algebras.

Part 5 (Chapters 13–14) deals with the Beilinson-Lusztig-MacPherson construction of the quantum enveloping algebra associated to $\mathfrak{gl}_n$. Chapter 13 describes multiplication formulas in quantum Schur algebras and the BLM basis as well as a presentation in terms of a certain monomial basis. In Chapter 14, the BLM algebra is defined and quantum $\mathfrak{gl}_n$ is realized inside its completion.

While this book is somewhat more advanced than would be appropriate for a typical introductory graduate course, it should prove to be a valuable reference to researchers working in the field. It contains and collects many results which have not appeared before in book form.

Reviewed by *Alistair Savage*