Killing vectors with null bivectors

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Abstract: A counter-example to a result of Fayos and Sopuerta is given.

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In [1] Fayos and Sopuerta outlined a formalism for treating space-times that admit an isometry (Killing vector). In one of the applications of this formalism, they claimed to have proved that no vacuum metric of Petrov type III and non-zero expansion (Newman-Penrose coefficient ρ) admits a Killing vector whose covariant differential (Killing or Papapatrou bivector) is null.

However, consider the u-independent Robinson-Trautman type III metric, see [2]:

$$ds^2 = 2\frac{r^2}{P^2}\,d\zeta\,d\overline{\zeta} - 2H(\zeta,\overline{\zeta})\,du^2 - 2\,dr\,du, \tag{1}$$

where $P = P(\zeta,\overline{\zeta})$, and $H(\zeta,\overline{\zeta}) = P^2\,(\log P)_{\zeta\overline{\zeta}}$. Defining $\Delta = 2P^2\partial_\zeta\partial_{\overline{\zeta}}$, this is vacuum if $\Delta\Delta\log P = 0$ and type III if $(\Delta\log P)_\zeta \neq 0$. The repeated Debever-Penrose direction is ∂_r, which is diverging but non-twisting, as required in the definition of Robinson-Trautman solutions.

Metric (1) admits the obvious Killing vector $X^a = \partial_u$, and a simple calculation shows that (taking coordinates in the order $(u, r, \zeta, \overline{\zeta})$)

$$X_{a;b} = \begin{pmatrix} 0 & 0 & -H_\zeta & -H_{\overline{\zeta}} \\ 0 & 0 & 0 & 0 \\ H_\zeta & 0 & 0 & 0 \\ H_{\overline{\zeta}} & 0 & 0 & 0 \end{pmatrix} = -2k_{[a}\omega_{b]},$$

where $k_a = du$ is null and $\omega_a = dH = H_\zeta\,d\zeta + H_{\overline{\zeta}}\,d\overline{\zeta}$, is space-like. Hence the Killing bivector is null in this case.

It would appear that the error in the proof in [1] is the equation $\delta(\alpha+\overline{\beta}) = 0$ on p. 366, which it is claimed follows from equation 7.41 of [2]. We can see that this is not true in general by considering the special case with $P(\zeta,\overline{\zeta}) = (\zeta+\overline{\zeta})^{3/2}$, when $H = -\frac{3}{2}(\zeta+\overline{\zeta})$. The obvious complex null tetrad built around the vector k^a above,

$$k^a = \partial_r; \quad \ell^a = \partial_u - H\partial_r; \qquad m^a = \frac{P}{r}\partial_\zeta,$$

is a Weyl canonical tetrad for metric (1), as $\kappa = \sigma = \Psi_0 = \Psi_1 = \Psi_2 = 0$, and also has $\epsilon = \gamma = \tau = \pi = \lambda = 0$. To put this into the tetrad used in [1] we need to perform a boost so that the components of X^a along k^a and ℓ^a are equal. We modify the tetrad by changing to

$$k^a = (-H)^{1/2}\partial_r; \quad \ell^a = (-H)^{-1/2}\partial_u - (-H)^{1/2}\partial_r.$$

In the boosted tetrad we find using $\alpha+\overline{\beta} = -dk_a(k^a,\overline{m}^a)$, which follows as $\tau = 0$ from [3] equation 4.13.44, that

$$\alpha+\overline{\beta} = -\frac{1}{2r}\left(\zeta+\overline{\zeta}\right)^{1/2},$$

and $\delta(\alpha+\overline{\beta})$ is clearly not zero.

Curiously, metric (1) also admits a proper homothety $Y^a = u\partial_u + r\partial_r$, as is easily seen. Its associated homothetic bivector $F_{ab} = 2Y_{[a;b]}$ is also null, in fact u times the bivector for X^a above — the results of [1] guarantee that if the homothetic bivector is null, then its principal null direction has to be a repeated Debever-Penrose direction.

A proof that metric (1) gives the only Robinson-Trautman vacuum solutions admitting null Killing or homothetic bivectors will be given elsewhere [4].

References

[1] Fayos F and Sopuerta C F *Class. Quant. Grav.* **18** (2001) p. 353

[2] Kramer D, Stephani H, MacCallum, M and Herlt E *"Exact Solutions of Einstein's Field equations"* **VEB Deutscher Verlag der Wissenschaften** (1980)

[3] Penrose R and Rindler W *"Spinors and Space-Time Vol 1"* **Cambridge Universty Press** (1986)

[4] Steele, J.D. *"A formalism for space-times with a homothety"* Preprint, **University of New South Wales** (2001) – in preparation.