On Generalised p.p. waves

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Abstract

This paper collects and extends mathematical results on generalised p.p.waves, taken here to be those space-times admitting a covariantly constant null bivector. Several equivalent definitions of generalised p.p.waves are given, and the possibilities for motion isotropies in such space-times are dealt with. A study is made of the subclass of generalised plane waves, with a complete description of their possible algebras of Killing, homothetic and conformal vectors given.

On Generalised p.p. waves

1: Introduction

One of the best-known and mathematically simplest class of solutions to Einstein's equations in classical general relativity is the class of generalised p.p.waves [5,12,19], also called Robinson solutions [15]. The term p.p.wave is due to Ehlers and Kundt [5], who investigated the properties of the vacuum p.p.waves. The abbreviation "p.p." stands for "plane-fronted gravitational waves with parallel rays". In vacuum, "plane-fronted gravitational waves" is taken to mean that there is a geodesic null vector field whose twist, expansion and shear are zero. This definition is chosen by comparison with the plane-fronted waves of electromagnetic theory — see [5] for further details. The "parallel rays" part of the definition is the requirement that the rotation of the vector field also vanishes. For vacuum type N, this implies the existence of a (necessarily null) covariantly constant vector field, which is parallel to the vector field mentioned above. Since Ehlers and Kundt's article many results have been published on generalisations of p.p.waves, although there has been some difference on what actually constitutes a generalised p.p.wave.

We begin section 2 with the definition of generalised p.p.wave that we will use, and after some preliminary results of interest in their own right, we discuss the different definitions that have been given, showing that some of them are equivalent to the one given here. We also give the form of the metric as derived by Ehlers and Kundt in the so-called "harmonic coordinates", which shows that the metric involves one arbitrary function of three of these coordinates.

In section 3 we turn to the subclass of generalised plane waves, defining them by a coordinate requirement on the arbitrary function in the metric of generalised p.p.waves, and showing that this is equivalent to a coordinate-independent requirement on the Weyl tensor of a generalised p.p.wave.

The next section considers non-discrete motion isotropies in generalised p.p.waves, and we give an example to show that although in vacuum any such isotropy implies the generalised p.p.wave is a generalised plane wave, the situation is different in the non-vacuum case. We give theorems about the possible groups of motions and motion isotropy that non-plane generalised p.p.waves can have.

Section 5 is devoted to further characterisations of generalised plane waves in terms of the groups of motions and homotheties that a space-time possesses.

The final section considers conformal symmetries in generalised p.p.waves. We give the general form of a conformal vector in a type N generalised p.p.wave, and discuss the situation for homothetic and Killing vectors in the conformally flat case. We then consider symmetries in generalised plane waves, and give the general form of the metric for homogeneous plane waves. We also use the general form of conformal vectors in plane waves to strengthen a result on conformally symmetric spaces given in [17] and finish with a discussion on special conformal vectors in generalised plane waves.

Our conventions throughout will largely follow those of Kramer *et al*, [12]. We recall some results and terminology that will be used: for any vector field X, $\mathcal{L}_X g_{ab} = X_{a;b} + X_{b;a}$. We call X a **conformal vector** if $\mathcal{L}_X g = 2\psi g$ for a real valued function ψ called the **divergence** of X. A conformal vector with a constant non-zero divergence is a **homothetic vector**, and with an identically zero divergence is a **Killing vector**. We can write the covariant differential of a conformal (homothetic, Killing) vector as $X_{a;b} = \psi g_{ab} + F_{ab}$, where the bivector F_{ab} is called the **conformal** (or **homothetic, Killing**) **bivector**. We will also make frequent use of the Géhéniau-Debever decomposition of the Riemann tensor:

$$R_{abcd} = C_{abcd} + E_{abcd} + \frac{R}{6} g_{a[c}g_{d]b} \tag{1.1}$$

2: Definition and Basic Properties

A **generalised p.p.wave** is a non-flat spacetime (M, g) admitting a covariantly constant real null bivector. This definition is more restrictive than the one given in Kramer *et al* [12], which just required the existence of a covariantly constant null vector, but preserves most of the properties, in particular the form of the line element (q.v. section 3.2) of the vacuum p.p. waves that were studied by Ehlers and Kundt [5]. As we shall see, our definition is equivalent to the one in Kramer *et al* in the vacuum, Einstein-Maxwell or pure radiation cases, see theorem 2.3. Note that since the Levi-Civita tensor is covariantly constant, the dual and hence the self-dual of any covariantly constant real bivector is covariantly constant, and also the real and imaginary parts of a covariantly constant complex bivector are covariantly constant. Thus a space-time is a generalised p.p.wave if and only if it admits a self-dual, covariantly constant (complex) null bivector.

Now let (x, U) be a contractable chart of a space-time M, and let $\widetilde{C}_{abcd} = C_{abcd} + iC^*_{abcd}$ be the complex self-dual of the Weyl tensor. Then if in U the Weyl tensor is non-zero and $\widetilde{C}_{abcd;e} = \widetilde{C}_{abcd}p_e$, for some non-zero complex 1-form p_e — the **recurrence vector** — we say that the Weyl tensor is **complex recurrent** on U [14]. If $p \equiv 0$ on U we call the Weyl tensor **conformally symmetric** [14]. If the Weyl tensor is conformally symmetric or complex recurrent on U then its Debever-Penrose directions are recurrent and repeated, and, as the following lemma shows, the Petrov type is N or D according as to whether the Ricci scalar R is zero or not respectively, see Hall [7].

Lemma 2.1 [10]

If a space-time admits a recurrent null vector field ℓ, the Weyl tensor is algebraically special, the Petrov type being II or D if and only if the Ricci scalar is non-zero. Furthermore, ℓ is a (repeated) Debever-Penrose vector if the Weyl tensor is non-zero and a Ricci eigenvector if the Ricci tensor is non-zero.

We also recall that if a tensor, T, is recurrent and the recurrence vector is a gradient, then T is proportional to a covariantly constant tensor. For recurrent null vectors, this is equivalent to requiring that $\ell^a R_{abcd} = 0$, [12].

Using results of [7] on type N conformally symmetric or complex recurrent space-times leads to the following result

Theorem 2.2 [19], cf. [7]

Let U be a connected, smoothly contractable chart of a space-time (M, g). The following are equivalent:

(a) The Weyl tensor is non-zero on U and U admits a covariantly constant null bivector.

(b) The Weyl tensor is conformally symmetric or complex recurrent with gradient recurrence vector and is Petrov type N on U.

(c) The Weyl tensor is conformally symmetric or complex recurrent on U and the Ricci tensor is either zero or of Segre type [(2,1,1)] eigenvalues zero.

(d) The Petrov type is N on U, and there is a covariantly constant null vector field ℓ on U.

(e) The Weyl tensor is non-zero and the Ricci tensor is either zero or of Segre type [(2,1,1)] with zero eigenvalues on U and there is a covariantly constant null vector on U.

The vector ℓ in (d) or (e) spans the (unique) Debever-Penrose direction, is a null Ricci eigenvector, and spans the principal null direction of the bivector in (a).

Proof :

Hall [7] showed that (a)$\Leftrightarrow$(b). Suppose that (a) holds. Then (a)$\Leftrightarrow$(b) implies that the Petrov type is N. If $V_{ab} = 2\ell_{[a}p_{b]}$ is the covariantly constant null bivector, where $\ell^a p_a = 0$, then the principal null direction ℓ is recurrent with gradient recurrence vector, and so there is a covariantly constant null vector parallel to ℓ, hence (a)$\Rightarrow$(d). Conversely, if (d) holds then from the Ricci identity, $R_{abcd}\ell^d = 0$ and lemma 2.1 implies that the Ricci scalar R is zero and $C_{abcd}\ell^d = 0$. Thus we find using (1.1) that $E_{abcd}\ell^d = 0$, hence $R_{ab} = A\ell_a\ell_b$ for some A, possibly zero, [4]. Let $\{\ell^a, n^a, m^a, \overline{m}^a\}$ be a Weyl canonical tetrad, and define $V_{ab} = 2\ell_{[a}\overline{m}_{b]}$. Then the above results imply that V_{ab} is recurrent (cf. [5] p. 88) and has vanishing skew derivative [7]. Hence by the Poincaré

Lemma the recurrence vector is a gradient, so there is a covariantly constant (complex) null bivector proportional to V, and (d)$\Rightarrow$(a).

Now from (d)$\Leftrightarrow$(b)$\Leftrightarrow$(a) and the above, it follows that (d)$\Rightarrow$(c). Conversely, if (c) holds then from the remarks before lemma 1.1, lemma 1.1 itself, and since $R_{ab} = A\ell_a\ell_b$ (so that $R = 0$) the Petrov type must be N and ℓ is a recurrent, repeated Debever-Penrose vector. But then $R_{abcd}\ell^d = 0$ follows from equation (1.1), and thus (c)$\Rightarrow$(d).

Finally, it is clear from the above that (d)$\Rightarrow$(e). Conversely, if (e) holds, then from equation (1.1) $C_{abcd}\ell^d = 0$, so (e)$\Rightarrow$(d).

Q.E.D.

In vacuum, a non-flat space-time admitting a covariantly constant vector is automatically type N and the vector is null. For type O space-times with a covariantly constant null vector, lemma 2.1 shows that $R = 0$ and the rest of the proof (e)$\Leftrightarrow$(d)$\Leftrightarrow$(a) then goes through. Thus

Theorem 2.3 [19], cf. [7]

A non-flat space-time is a generalised p.p. wave if and only if it admits a covariantly constant null vector field and the Petrov type is N or O, or equivalently if and only if it has a covariantly constant null vector field and is vacuum or has Ricci tensor of Segre type [(2,1,1)] with zero eigenvalues. In vacuum a non-flat space-time with a covariantly constant vector is necessarily a p.p.wave and the vector is null. The covariantly constant vector is a Debever-Penrose vector if the Weyl tensor is non-zero, and a Ricci eigenvector if the Ricci tensor is non-zero. If type N, the Weyl tensor is necessarily complex recurrent with gradient recurrence vector or conformally symmetric, and if non-zero the Ricci tensor is recurrent.

Now consider a space-time satisfying condition (a) of theorem 2.2. If we denote the self-dual of the covariantly constant null bivector by $V_{ab} = 2\ell_{[a}\overline{m}_{b]}$ for some complex vector $\overline{m}^a$ such that $\overline{m}_a\overline{m}^a = \ell_a\overline{m}^a = 0$, then we find from the proof (d)$\Rightarrow$(a) above and the canonical form of type N Weyl tensors that we can write, for a complex function Φ,

$$\widetilde{C}_{abcd} = \Phi V_{ab}V_{cd}. \tag{2.1}$$

Furthermore, from theorem 2.2, there is a real valued function ϕ such that

$$R_{ab} = 2\phi\ell_a\ell_b. \tag{2.2}$$

Writing the Bianchi identities as [12]

$$C^a{}_{bcd;a} = \frac{1}{2}\left(R_{bc;d} - R_{bd;c} + \frac{1}{6}(g_{bd}R_{,c} - g_{bc}R_{,d})\right), \tag{2.3}$$

we have for generalised p.p. waves

$$C^a{}_{bcd;a} = 2\phi_{,[d}\ell_{c]}\ell_b. \tag{2.4}$$

This equation will prove useful later.

If we now consider the metric of a generalised p.p. wave, we note that the argument of [5] pp.89–91 uses only the existence of a covariantly constant null bivector to find the metric for vacuum p.p. waves, and therefore this argument will go through for generalised p.p. waves to give the metric

$$ds^2 = dx^2 + dy^2 + 2\,du dv + 2H du^2. \tag{2.5}$$

Here $u_{,a} = \ell_a$ is covariantly constant, v is an affine parameter along the (geodesic) vector field represented by ℓ and $x_{,a}$ and $y_{,a}$ span a wave surface of ℓ and are therefore spacelike. The function H is independent of v (i.e. $H_{,a}\ell^a = 0$) and in fact $H(u, x, y) = -\frac{1}{2}\,v_{,a}v^{,a}$. We will always assume that the coordinates of a generalised p.p. wave are $\{u, v, x, y\}$, in that order. These coordinates satisfy the "harmonic gauge" condition, $g^{ab}\Gamma^c_{ab} = 0$. Clearly, generalised p.p. waves are Kerr-Schild metrics, and from equation 28.13 of [12], see also [5], the Ricci tensor of a generalised p.p. wave is

$$\begin{aligned} R_{ab} &= -H^{;c}{}_c\,\ell_a\ell_b \\ &= -\left(H_{xx} + H_{yy}\right)\,\ell_a\ell_b. \end{aligned} \tag{2.6}$$

The sign difference between equation (2.6) and [5] p.91 arises since Ehlers and Kundt use a different convention in the definition of the Riemann tensor to the one used in [12]. We find (cf. [5] theorem 2–5.6) for Φ as in (2.1),

$$\Phi = -\left(H_{xx} - H_{yy} + 2iH_{xy}\right) \tag{2.7}$$

If $z = x + iy$, we find that $\Phi = -H_{\bar{z}\bar{z}}$ and $\phi = -\frac{1}{2}H_{z\bar{z}}$. Thus Φ and ϕ are independent of v so $\Phi_{,a}\ell^a = \phi_{,a}\ell^a = 0$. We note that generalised p.p. waves satisfy the energy conditions if and only if $\phi \geqslant 0$, i.e. if and only if $H_{z\bar{z}} \leqslant 0$.

For our final characterisations of generalised p.p. waves, we consider the infinitesimal holonomy group, see [6,10] for a discussion of these groups. Now for a generalised p.p.wave the existence of a covariantly constant null vector and the fact that the Petrov type is N or O implies that the infinitesimal holonomy group is (in Schell's notation [16]) either an R_8, that is, is the two-parameter group of proper null rotations, [12] equation 3.15, or is an R_3: a one-parameter group of proper null rotations.

Conversely, in both R_3 and R_8 cases the Petrov type is N or O and there is a covariantly constant null vector. Hence, generalising the result of Goldberg and Kerr [6], we have

Theorem 2.4

A space-time is a generalised p.p. wave if and only if its infinitesimal holonomy group is R_3 or R_8. In vacuum the infinitesimal holonomy group must be R_8 perfect.

Note that from this theorem it is simple to prove that the definition of a generalised p.p. wave given by Sippel and Goenner [18] is the same as the one given above, as the following theorem shows.

Theorem 2.5 [19]

A space-time is a generalised p.p. wave if and only if it admits a covariantly constant vector and the Riemann tensor satisfies

$$R^a{}_{bcd}R^b{}_{aef} = 0.$$

3: Generalised Plane Waves

A generalised p.p. wave is called a **generalised plane wave** if the function H in equation (2.1) is a quadratic in the coordinates x and y with coefficients functions of u, i.e. can be written

$$\begin{aligned} H(x, y, u) &= Ax^2 + By^2 + Cxy + Dx + Ey + F \\ &= Mz^2 + \overline{M}\overline{z}^2 + Pz\overline{z} + Qz + \overline{Q}\overline{z} + S, \end{aligned} \tag{3.1}$$

where $z = x + iy$, with $A, \ldots, F$, P and S real-valued, M and Q complex-valued functions of u. From the forms of Φ and ϕ given before, we see that for generalised plane waves $\Phi = -2\overline{M}$ and $\phi = -\frac{1}{2}P$ in (2.1) and (2.2), and

Lemma 3.1

A generalised p.p. wave is a generalised plane wave if and only if Φ and ϕ in (2.1) and (2.2) are functions of u only.

Thus the above definition of generalised plane waves is the same as that given in [12]. Furthermore, the above lemma leads to, [19], cf. [7],

Theorem 3.2

Let (M, g) be a generalised p.p. wave. The following are equivalent:

(a) (M, g) is a generalised plane wave.

(b) The Weyl tensor is either Petrov type O, conformally symmetric, or its recurrence vector is parallel to a real null vector.

Also

(c) The Ricci tensor of a generalised plane wave is either recurrent with recurrence vector parallel to a real null vector, covariantly constant or zero.

Proof :

For a, respectively, non conformally flat or non vacuum generalised p.p. wave we have, from equations (2.1) and (2.2),

$$\widetilde{C}_{abcd;e} = \widetilde{C}_{abcd} \, (\log \Phi)_{,e}. \tag{3.2}$$

$$R_{ab;c} = R_{ab} \, (\log 2\phi)_{,c}. \tag{3.3}$$

Then (a)⇒(b) and (a)⇒(c) follow from lemma 3.1. Conversely, (b)⇒(a) follows from the Bianchi identity in the form (2.3).

Q.E.D.

Corollary 3.3

A generalised p.p. wave is a generalised plane wave if and only if $\Phi = \Phi(u)$ *in (2.1).*

In fact it is always possible to perform a coordinate change for generalised plane waves so that the metric function H takes the form (3.1), but with $D = E = F = 0$, equivalently $Q = S = 0$ [5,12]. We will therefore always assume this done. For references to the earlier history and discovery of certain classes of plane waves, see [5,12].

We also note that it is possible to perform a coordinate change on a generalised plane wave to put the metric into the Rosen form

$$ds^2 = g_{AB}(u) dx^A dx^B + 2du' dv \qquad A, B = 1, 2. \tag{3.4}$$

See [12] for the requisite coordinate change. The Rosen form is the form found by Kruchkovich [13], and is used in the study of colliding plane waves, as the coordinates u' and v are both null, see e.g. [20].

4: Isotropy and Generalised p.p. waves

In vacuum, the Weyl tensor (equivalently the Riemann tensor) is conformally symmetric or complex recurrent if and only if we have a generalised p.p. wave [5], and the Petrov type of a vacuum p.p. wave is N. The following characterisation of vacuum plane waves follows from theorem 3.2 and the Bianchi identities (2.3):

Theorem 4.1 [5]

A vacuum p.p. wave is a vacuum plane wave if and only if the recurrence vector of the self-dual Riemann tensor is zero or parallel to a real vector, which is necessarily null.

Now let (M, g) be a vacuum p.p. wave that is not conformally symmetric and X be any Killing vector of it. Then using equation (4.9) from [21] p.16,

$$\mathcal{L}_X \widetilde{C}_{abcd;e} = 0 \qquad (\text{equivalently} \quad \mathcal{L}_X p_e = 0) \tag{4.1}$$

where p_e is the (non-zero) recurrence vector of $\widetilde{C}_{abcd}$. This equation also holds in the non-vacuum case, as does the equation $p_e \ell^e = 0$, which follows from the form of the recurrence vector given in the proof of theorem 3.2.

Suppose that Y is a Killing vector of (M, g) that vanishes at some point m, so that $\nabla Y(m)$ generates a motion isotropy. Then since the Petrov type is N, there is a space-like vector, q^a say, orthogonal to ℓ^a such that $Y_{a;b}(m) = 2\ell_{[a}q_{b]}$, see e.g. [5,12]. Thus at m, $\mathcal{L}_Y p^a = p^a_{;b} Y^b - Y^a_{;b} p^b = -\ell^a (q_b p^b)$, which is zero from (4.1). Thus $p_a q^a = 0$.

Write the covariantly constant bivector V as $V_{ab} = 2\ell_{[a}\overline{m}_{b]}$ for some complex vector $\overline{m}^a$, where $\overline{m}^a = x^a - iy^a$ for space-like vectors x^a, y^a orthogonal to ℓ^a. Then the Bianchi identities in the form (2.3) give $p_a V^a{}_b = 0$ so that $p_a \overline{m}^a = 0$. Now we write $p^a = r^a + is^a$ and $q^a = \lambda\ell^a + \mu x^a + \nu y^a$ for real vectors r^a, s^a and real constants λ, μ, ν where $\mu^2 + \nu^2 \neq 0$. Then the equations $p_a q^a = p_a \overline{m}^a = p_a \ell^a = 0$ show that r^a and s^a are both orthogonal to the same three linearly independent vectors (ℓ^a, x^a and y^a) and are hence parallel. However r^a and s^a being parallel implies that p^a is parallel to a real vector, and thus from theorem 4.1:

Theorem 4.2 cf. [5] p. 94

A (vacuum) p.p. wave that admits a motion isotropy is a plane wave.

This theorem is not true for non-vacuum p.p. waves. The following metric, due to Sippel and Goenner [18], provides the counterexample.

$$ds^2 = dx^2 + dy^2 + 2\,dudv + 2\,ke^{2(\rho x - \sigma y)} du^2. \tag{4.2}$$

Here k, ρ and σ are constants such that $k \neq 0$ and $\rho^2 + \sigma^2 \neq 0$. Thus at least one of ρ and σ is non-zero, and we assume, without loss of generality, that $\rho \neq 0$.

This metric is not a generalised plane wave, satisfies the energy conditions if and only if $k < 0$, is neither vacuum nor an electromagnetic field [18], and admits the following five independent Killing vectors, cf. [18]:

$$\begin{aligned} &X_1 = \partial_v \qquad X_2 = \partial_u \qquad X_3 = \rho\partial_y + \sigma\partial_x \\ &X_4 = -\partial_x + \rho(u\partial_u - v\partial_v) \\ &X_5 = u\,(\sigma\partial_x + \rho\partial_y) - (\sigma x + \rho y)\partial_v. \end{aligned} \tag{4.3}$$

The orbit of these five Killing vectors is four dimensional. At a point $m = (u_0, v_0, x_0, y_0)$ the isotropy group is generated by the vector

$$X_0 = (u - u_0)\,(\sigma\partial_x + \rho\partial_y) - (\sigma(x - x_0) + \rho(y - y_0))\,\partial_v, \tag{4.4}$$

which has a null 2-space of fixed points given by $u = u_0, (\sigma(x - x_0) + \rho(y - y_0)) = 0$. We also find that, at m, $X_{0\,a;b}$ is a null bivector, as expected from the Petrov type.

We do however have the following three theorems for generalised p.p. waves:

Theorem 4.3 [2]

A spacetime with a three-parameter group of motions with two dimensional null orbits that is vacuum or an invariant Einstein-Maxwell field [1] *is a generalised plane wave.*

Theorem 4.4 [12]

Vacuum, Einstein-Maxwell or pure radiation fields admitting a group of motions acting transitively on null hypersurfaces are generalised plane waves.

Theorem 4.5 [19]

A non-flat generalised p.p. wave admitting a two parameter (or greater) motion isotropy group is a generalised plane wave.

The metric (4.2) shows that this is the best we can do. The Killing algebra of (4.2) has two four-parameter subalgebras, spanned by $\{X_1, X_2, X_3, X_4\}$ and $\{X_1, X_3, X_4, X_5\}$. The former has a four dimensional orbit (all four Killings are everywhere linearly independent), the latter a three-dimensional orbit that is timelike if $u \neq 0$ and null if $u = 0$. The three dimensional subalgebra generated by $\{X_1, X_3, X_5\}$ has a two dimensional null orbit. If we inspect all possible cases of motion isotropy in generalised p.p. waves, we have, for a G_r of motions;

Orbit V_4: If $r \geqslant 6$ the p.p. wave is a plane wave from theorem 4.5, if $r = 5$ there is metric (4.2).

[1] $\mathcal{L}_X \widetilde{F} = 0$ for X any Killing vector, $\widetilde{F}$ the self-dual Maxwell bivector

Orbit T_3: If $r = 6$ (the maximum), theorem 4.5 implies a plane wave. Fubini's theorem forbids a G_5, metric (4.2) admits a G_4.

Orbit T_2: Here the isotropy group must consist of boosts, which is incompatible with the Petrov type.

Orbit N_3: A G_r for $r \geqslant 5$ is a plane wave (theorem 4.3, compare theorem 5.1 in the next section). Metric (4.2) admits a G_4 on an N_3, and this shows that the requirement that all the group orbits are null hypersurfaces in theorem 4.4 cannot be dropped.

Orbit N_2: The maximum is $r = 3$, and metric (4.2) admits this.

Note that as there is always a null (covariantly constant) Killing vector in any generalised p.p. wave, the Killing orbit cannot be spacelike. It follows from [18] that, apart from (4.2), all cases of p.p. waves with a motion isotropy are generalised plane waves. We can also show that that metric (4.2), which is always type N, does not admit any proper conformal or homothetic vectors, [19].

5: Characterisations of Generalised Plane Waves

Theorem 4.5 helps establish the following important characterisation of generalised plane waves, originally due to Kruchkovich [13].

Theorem 5.1

A non-flat space-time is a generalised plane wave if and only if it has a five-parameter group of motions acting on three dimensional orbits. These orbits are necessarily null.

Proof :

Suppose that (M, g) has a five-parameter group of motions acting on three dimensional orbits. By Fubini's Theorem the orbits must be null. Let u be a coordinate such that the orbits are the hypersurfaces $u =$constant, and let $\ell_a = u_{,a}$ be normal to the orbits. Let X be any Killing vector in the hypersurface, so that $\ell_a X^a = 0$. Covariantly differentiating this identity and using the fact that ℓ is a gradient gives

$$\mathcal{L}_X \ell_a = \ell_{a;b} X^b + X^b{}_{;a} \ell_b = 0. \tag{5.1}$$

Furthermore, as X is an affine vector field, we find that $\mathcal{L}_X \ell_{a;b} = 0$, i.e.

$$\ell_{a;bc} X^c + \ell_{c;b} X^c{}_{;a} + \ell_{a;c} X^c{}_{;b} = 0. \tag{5.2}$$

We can write $\ell_{a;b}$ in terms of a null triad $\{\ell, x, y\}$ in the hypersurfaces as

$$\ell_{a;b} = A\ell_a\ell_b + Bx_a x_b + C y_a y_b + 2D\ell_{(a}x_{b)} + 2E\ell_{(a}y_{b)} + 2Fx_{(a}y_{b)}, \tag{5.3}$$

for functions A, B, C, D, E, F. Now since we have a two parameter motion isotropy we can arrange the generators of the Killing algebra so that at any point p two of the generators, X_1 and X_2 say, vanish so from (5.1) $X^b{}_{A;a}\ell_b = 0$ at p for $A = 1, 2$. Thus the Killing bivectors of X_A must be null or spacelike at p. However, consideration of the possible algebras of bivectors shows that for these two Killing vectors both Killing bivectors, which must be non-zero at p, are null at p. Hence we must have Petrov type N or O and a Segre type [(2,1,1)] or [(1;1,1,1)] Ricci tensor at each point, cf. [5] and [12] table 5.1. Thus we can arrange that at p

$$X_{1a;b} = 2\ell_{[a}x_{b]} \qquad X_{2a;b} = 2\ell_{[a}y_{b]}.$$

If we use these expressions, and equation (5.3), in (5.2) we find after a short calculation that $B = C = D = E = F = 0$ at each point p, so ℓ is recurrent.

Now since the Petrov type is N or O, from lemma 2.1 the Ricci scalar must vanish and ℓ is both a Debever-Penrose vector and a Ricci eigenvector. Equation (1.1) now shows that $R_{abcd}\ell^a = 0$, and hence there is a covariantly constant null vector parallel to ℓ. Thus from theorem 2.3 we have a generalised p.p. wave which by theorem 4.5 is a generalised plane wave.

Conversely, we shall see in section 6.1 that all generalised plane waves admit a five-parameter group of motions acting on three dimensional (null) orbits.

Q.E.D.

If a conformal vector field has a fixed point at which the divergence is non-zero, it is said to have a **homothetic fixed point** at that point. It is possible to characterise generalised plane waves by the existence of homothetic fixed points:

Lemma 5.2

A generalised p.p. wave that has a homothetic fixed point at every point is a generalised plane wave.

Proof :

The conformally flat and conformally symmetric cases are obvious (theorem 3.2). Otherwise, for a conformal vector X such that $\mathcal{L}_X g = 2\,\psi g$, since $\psi_{,a}$ is proportional to the covariantly constant vector ℓ_a (see lemma 6.1, next section), we can use equation (4.9) of chapter 1 of [21] and the possibilities for the conformal bivector at fixed points of conformals, see [9,19], to show that p_a is proportional to a real null vector [19], whence the result follows from theorem 3.2.

Q.E.D.

This lemma enables us to prove the 'if' part of the following using techniques similar to those of the proof of theorem 5.1. The 'only if' part of this theorem will be proved in section 6.1.

Theorem 5.3 [19], see also [11]

A non-flat space-time is a generalised plane wave if and only if it has a group of motions acting transitively on three dimensional null orbits and also a homothety in those orbits.

More generally we have

Theorem 5.4 [1,8]

A space-time is a generalised plane wave if and only if it admits a homothety with a non-isolated fixed point.

The next result generalises a theorem of Kundt, see [5].

Theorem 5.5

A space-time that has a three parameter Abelian group of motions containing a null Killing vector and acting on three dimensional null orbits admits a covariantly constant null vector. If the space-time is non-flat and either Petrov type N, Petrov type O, vacuum or has Ricci tensor of Segre type [(2,1,1)] with zero eigenvalues it is a generalised plane wave. Conversely, all generalised plane waves admit a three parameter Abelian group of motions containing a null Killing vector and acting on three dimensional null orbits.

Proof :

Suppose a space-time admits a group of motions as in the statement of the theorem. Let ℓ, X_1, X_2 be Killing vectors that generate this Abelian group, where ℓ is null, X_1 and X_2 spacelike. The null vector must be orthogonal to the hypersurface, so that $\ell_a X^a_{\mathsf{A}} = 0$, for A =1,2. Since the group is Abelian $[X_{\mathsf{A}}, \ell] = 0$, and this together with the derivative of $\ell_a X^a_{\mathsf{A}} = 0$ implies that $\ell_{a;b} X^b_{\mathsf{A}} = 0$. But these equations, together with $\ell_{a;b}\ell^a = 0$ and the fact that $\ell_{a;b}$ is a bivector, imply that $\ell_{a;b} = 0$. This proves the first part.

Any of the Petrov type or Ricci tensor conditions implies by theorem 2.3 that we have a generalised p.p. wave. The conformally flat case is then obvious. Otherwise, since ℓ is covariantly constant, $[X_{\mathsf{A}}, \ell] = 0$ is equivalent to

$$X^a_{\mathsf{A};b}\ell^b = 0. \tag{5.4}$$

This equation implies that the Killing bivectors of both X_1 and X_2 are everywhere simple and either zero, null or spacelike. If we assume the space-time is non-flat at least one of these bivectors must be non-zero, else we would have three independent covariantly constant vectors, ℓ^a, X^a_1 and X^a_2, and the space-time would be flat. Since the group is Abelian, $[X_1, X_2] = 0$ or in coordinates

$$X^a_{1\,;b}X^b_2 - X^a_{2\,;b}X^b_1 = 0. \tag{5.5}$$

It follows from (5.4) and (5.5) that we can find a null tetrad $\{\ell^a, n^a, x^a, y^a\}$ such that $X_{1\,a;b} = 2\ell_{[a}x_{b]}$, $X_{2\,a;b} = 2\varepsilon\ell_{[a}y_{b]}$, with $\varepsilon = 0$ or 1.

We now assume that $V_{ab} = 2\ell_{[a}\overline{m}_{b]}$ is the self dual of the covariantly constant bivector, and write $\widetilde{C}_{abcd}$ as in equation (2.1) Then as $\ell^a\overline{m}_a = 0$, $\overline{m}^a = \alpha x^a + \beta y^a$ for complex scalars α, β. It now follows from $\mathcal{L}_{X_{\mathsf{A}}}\widetilde{C}_{abcd} = 0$ and the forms of $X_{\mathsf{A}a;b}$ given above that

$$\Phi_{,a}X^a_1 = \Phi_{,a}X^a_2 = 0.$$

And so as $\Phi_{,a}\ell^a = 0$, $\Phi_{,a}$ is proportional to ℓ_a, hence by corollary 3.3 we have a generalised plane wave.

The converse follows from the Rosen form of the metric given in equation (3.4), but will also be proved in section 6.1.

Q.E.D.

6: Conformal Symmetries of Generalised p.p. Waves

The paper by Sippel and Goenner [18] lists all the possible isometries of generalised p.p. waves, for different functions H. However, note that there are some errors in their table II. [2]

The relatively simple form of the metric for a generalised p.p. wave enables one to find possible *conformal* vectors, especially for vacuum p.p. waves (see [15], where their coordinates $\{t, r, \xi\}$ correspond to our $\{u, v, x + iy\}$) and type N generalised plane waves. This is possible because of the following result:

Lemma 6.1 [9]

The divergence ψ of a conformal vector in a type N generalised p.p. wave is a function of u only, where $\ell_a = u_{,a}$ is the covariantly constant null vector.

So, let $X_a = (\alpha, \beta, \gamma, \delta)$ (so that $X^a = \beta\partial_u + (\alpha - 2H\beta)\partial_v + \gamma\partial_x + \delta\partial_y$) be a conformal vector field of a type N generalised p.p. wave (M, g), where $\mathcal{L}_X g = 2\psi g$. Integration of Killing's equations and the equations given by Geroch [22] (see also [19]) as integrability conditions for conformal vectors then gives the general form of conformal vectors in these generalised p.p. waves as, cf. [15],

$$\alpha = 2\beta H - \psi'(x^2 + y^2)/2 - f'x - g'y + k_1 v + \alpha_1(u). \tag{6.2}$$

$$\beta = 2\Psi - k_1 u + k_2. \tag{6.3}$$

$$\gamma = \psi x + N y + f(u). \tag{6.4}$$

$$\delta = \psi y - N x + g(u). \tag{6.5}$$

Here k_1, k_2 and N are constants, f, g and α_1 are functions of u only, $\,' \equiv d/du$ and $\Psi' = \psi$. Let the Ricci tensor R_{ab} be written as in equation (2.2). Then in addition to the above equations we have the following as integrability conditions:

$$\alpha_{,u} - \beta H_{,u} + \gamma H_{,x} + \delta H_{,y} = 2\psi H. \tag{6.6}$$

$$\alpha_{,xu} - (\beta H_{,x})_{,u} = N H_{,y} + (k_1 - \psi) H_{,x} - \gamma H_{,xx} - \delta H_{,xy}. \tag{6.7a}$$

[2] For instance for their no. 9 the fourth Killing vector should be $-\partial_y + \rho(u\partial_u - v\partial_v)$ and the fifth $u(\rho\partial_z + \sigma\partial_y) - (\sigma y + \rho z)\partial_v$.

$$\alpha_{,yu} - (\beta H_{,y})_{,u} = -NH_{,x} + (k_1 - \psi)H_{,y} - \gamma H_{,xy} - \delta H_{,yy}. \tag{6.7b}$$

$$-\psi'' = \mathcal{L}_X \phi + 2\phi(2\psi - k_1). \tag{6.8}$$

Setting ψ =constant will give the homothetic vectors, and $\psi = 0$ the Killing vectors. In vacuum $\phi = -\frac{1}{2}\left(H_{xx} + H_{yy}\right) = 0$ and thus $\psi'' = 0$ for all conformals in vacuum p.p. waves.

If the Petrov type is O, in which case we have a generalised plane wave of course, lemma 6.1 does not necessarily hold. However, the equations (6.2)–(6.8) are still valid for those conformals whose divergences are functions of u only, and in particular for Killing and homothetic vectors. This will prove to suffice for our purposes.

The above equations confirm that all generalised p.p. waves admit the Killing vector $X_a = (1, 0, 0, 0)$ (i.e. ∂_v) formed by setting

$$\Psi = k_1 = k_2 = N = f = g = \alpha_1' = 0.$$

This is the covariantly constant vector. The above form for a Killing vector in a generalised p.p. wave can be compared to that given in [12].

6.1: Symmetries in Plane Waves

Consider a generalised plane wave with $H = Ax^2 + By^2 + Cxy$ where A, B and C are functions of u only which we will call the **metric functions**.

The integrability conditions (6.6) and (6.7) for a conformal whose divergence is a function of u only become, in the notation of (6.2)–(6.5):

$$f'' = 2fA + gC, \tag{6.9a}$$

$$g'' = 2gB + fC, \tag{6.9b}$$

$$\alpha_1' = 0, \tag{6.9c}$$

$$2\,(2\psi - k_1)A + \beta A' = \ NC + \psi''/2, \tag{6.10a}$$

$$2\,(2\psi - k_1)B + \beta B' = -NC + \psi''/2, \tag{6.10b}$$

$$2\,(2\psi - k_1)C + \beta C' = \ 2N(B - A). \tag{6.10c}$$

Note that in vacuum $A = -B$ and $\psi'' = 0$ from (6.8), so that (6.10a) and (6.10b) are then identical.

Let $m = (u_0, v_0, x_0, y_0)$ be any event of a generalised plane wave, and consider the system (6.9). For a given set of initial conditions, $\mathbf{v} = (f(u_0), g(u_0), f'(u_0), g'(u_0))$, this system of ordinary differential equations for f and g admits a unique solution in a neighbourhood of m. Then let (f_1, g_1) be the solution to the system with initial conditions $\mathbf{v} = (1, 0, 0, 0)$, (f_2, g_2) be the solution with initial conditions $\mathbf{v} = (0,1,0,0)$, (f_3, g_3) the solution with initial conditions $\mathbf{v} = (0,0,1,0)$ and (f_4, g_4) the solution with initial conditions $\mathbf{v} = (0,0,0,1)$. Now define four vector fields X_a for $\mathsf{a} = 1,\ldots,4$ in a neighbourhood of m by

$$X_\mathsf{a} = -(f'_\mathsf{a} x + g'_\mathsf{a} y)\partial_v + f_\mathsf{a}\partial_x + g_\mathsf{a}\partial_y, \tag{6.11}$$

and let $X_5 = \partial_v$. Then these five vector fields are linearly independent in a neighbourhood of M and are Killing vectors of the generalised plane wave. If we calculate the commutators of these Killing vectors we get $[X_5, X_\mathsf{a}] = 0$ for all a and

$$[X_\mathsf{a}, X_\mathsf{c}] = \{f'_\mathsf{c} f_\mathsf{a} - f'_\mathsf{a} f_\mathsf{c} + g'_\mathsf{c} g_\mathsf{a} - g'_\mathsf{a} g_\mathsf{c}\} X_5.$$

The expression in braces must of course be constant — in fact we can prove this by differentiating the expression with respect to u and using the conditions (6.9). Thus $\{X_1, X_2, X_3, X_4, X_5\}$ generates a five-parameter group of motions G_5 whose orbits are the (null) hypersurfaces u =constant. This proves the converse of theorem 5.1.

Also, we see from the choices made for the f_a and g_a that $[X_1, X_2] = 0$, as $f'_1 = g'_1 = f'_2 = g'_2 = 0$ at m. But this implies that $\{X_1, X_2, X_5\}$, which are linearly independent at m, generate a three-parameter Abelian group of motions containing a null Killing vector and having three dimensional null orbits. This proves the converse of theorem 5.5.

Furthermore, setting $\psi' = 0$, $N = k_2 = 0$, $k_1 = 2\psi$ we see that in addition to the above G_5 of motions we have for all generalised plane waves a homothety H with

$$\alpha = 2\psi v \quad \beta = 0 \quad \gamma = \psi x \quad \delta = \psi y, \tag{6.12}$$

where ψ is the divergence, so that H is a multiple of $2v\partial_v + x\partial_x + y\partial_y$. Thus H lies in the null hypersurfaces u =constant, and we have proved the converse of theorem 5.3.

Using equations (6.9) and (6.10) we can find the possibilities for generalised plane waves that admit Killing vectors over and above those that generate the G_5. These further Killing vectors must be of the form:

$$\alpha = 2\beta H + k_1 v, \tag{6.13a}$$

$$\beta = -k_1 u + k_2, \tag{6.13b}$$

$$\gamma = Ny, \qquad \delta = -Nx, \tag{6.13c}$$

and the integrability conditions (6.10) for these Killing vectors become:

$$-2k_1 A + (k_2 - k_1 u)A' = \quad NC. \tag{6.14a}$$

$$-2k_1 B + (k_2 - k_1 u)B' = -NC. \tag{6.14b}$$

$$-2k_1 C + (k_2 - k_1 u)C' = 2N(B - A). \tag{6.14c}$$

Now if the generalised plane wave is Petrov type O, the metric functions satisfy $A = B$, $C = 0$ (section 3.2). Thus from (6.14), we see that in such space-times there is always a Killing vector of the form (6.13) with $k_1 = k_2 = 0$. That is, all type O generalised plane waves admit the Killing vector $y\partial_x - x\partial_y$ over and above the Killing vectors that generate the G_5, and this sixth Killing vector also lies in the orbits of the G_5. This gives us the maximum (three parameter) motion isotropy for a space-time whose Ricci tensor has Segre type [(2,1,1)] — the isotropy group is an R_{11} in Schell's notation. Clearly, no Petrov type N generalised plane wave can admit this Killing vector, or indeed any Killing vector tangential to the null hypersurfaces, by isotropy considerations. Thus in searching for extra Killing vectors we can rule out $y\partial_x - x\partial_y$ and assume that in (6.13) and (6.14) k_1 and k_2 are not both zero, and in fact assume that only one of them is non-zero.

Suppose there is an extra Killing vector, given by (6.13). Then the system (6.14) is a linear, non-autonomous system of ordinary differential equations in u for A, B and C. Its general solution is unique and is:

$$A = G^{-2}\Big(k - l\cos(2NF + \theta)\Big), \tag{6.15a}$$

$$B = G^{-2}\Big(k + l\cos(2NF + \theta)\Big), \tag{6.15b}$$

$$C = 2G^{-2} l \sin(2NF + \theta). \tag{6.15c}$$

Where k, l and θ are constants, $G = k_2 - k_1 u \neq 0$ and $F = \int (du/G)$. Thus above the Killing vectors that generate the ever present G_5 (G_6 for type O) of motions in null hypersurfaces there is at most *one* extra Killing vector, which is not tangential to the null hypersurfaces. The most general form of this Killing vector is given by equations (6.13).

From the above remarks there are four subcases of generalised plane waves admitting extra Killing vectors. These cases are:

I	$N = k_2 = 0,\ k_1 = 1 \Rightarrow A = a\,u^{-2},\ B = b\,u^{-2},\ C = c\,u^{-2}$, a, b, c constants.
II	$N = k_1 = 0,\ k_2 = 1 \Rightarrow A,\ B,\ C$ are constant.
III	$k_2 = 0,\ k_1 = 1 \Rightarrow G = -u,\ F = -\log\lvert u\rvert + c$, c a constant.
IV	$k_1 = 0,\ k_2 = 1 \Rightarrow G = 1,\ F = u + c$, c a constant.

If we compare these cases with table II of Sippel and Goenner [18] we find that I corresponds to their number 11, II to their number 13, III to their number 12 and IV to their number 14.

We also note that for vacuum the constant k in equations (6.15a) and (6.15b) must be zero and we recover the last two cases in table 2–5.1 of Ehlers and Kundt [5]. See also table III of Salazar *et al*, [15].

Next we note that for conformally symmetric generalised plane waves we have $A - B$ and C constant. In this case (6.10c) becomes $(2\psi - k_1)C = N(B - A)$ and thus ψ is constant unless C is zero. If C is zero then (6.10a) – (6.10b) gives that $(2\psi - k_1)(A - B) = 0$, so that if the space-time is not conformally flat ψ is constant. This implies that *conformally symmetric Petrov type N space-times* (which by theorems 2.2 and 3.2 must be generalised plane waves) *admit no non-homothetic conformal vectors*. This strengthens the result of Sharma [16].

Finally, we turn to special conformal vectors, which are conformal vectors with $\psi_{,a;b} = 0$. Clearly, for type N this is equivalent to $\psi'' = 0$, which is always true in vacuum. If the Petrov type is O, then since a non-flat type O space-time can only admit at most one covariantly constant vector, see e.g. [10], $\psi_{,a;b} = 0$ implies that $\psi_{,a}$ is parallel to ℓ_a, i.e. $\psi = \psi(u)$. So a conformal in a non-flat generalised plane wave is a special conformal vector if and only if $\psi = \lambda u + \mu$ for constants λ and μ. Since there is always a proper homothety H and the G_5 of motions in generalised plane waves, if we look for generalised plane waves admitting special conformal vectors, we can assume the conformals are of the form of (6.2)–(6.5) with $\psi = \lambda u$, $f \equiv g \equiv \alpha_1 \equiv 0$. Then the system of equations (6.10) for special conformal vectors again admits a unique solution of the form (6.15), but with $G = \lambda u^2 - k_1 u + k_2$, and $F = \int(du/G)$. Thus no homogeneous generalised plane wave (with metric functions of the form (6.15) with G *linear* in u) can admit a special conformal vector field (as it would have to have metric functions of the form (6.15) with G *quadratic* in u). Since in vacuum every conformal is a special conformal, conformals in vacuum space-times can only exist in p.p. waves. This means that no homogeneous vacuum plane wave can admit a non-homothetic conformal vector field. From this we can deduce that there cannot be more than a seven-parameter group of conformals in vacuum, which agrees with the result of Collinson and French [3].

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