TOWARDS SOME EXACT RESULTS FOR THE (VANISHING) ALGEBRAIC ENTROPY OF (INTEGRABLE) LATTICE EQUATIONS

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Abstract. We study growth of degrees of lattice equations, some of which are known to be integrable. We present a conjecture that helps us to prove polynomial growth of a certain class of equations including Q_V. In addition, we also study growth of degree of several non-integrable equations. Exponential growth of degrees of these equations is also proved subject to a conjecture. Our technique is to determine the ambient degree growth of the equations and a conjectured growth of their common factors at each vertex, allowing the true degree growth to be found.

1. Introduction

In recent years, there has been growing interest in discrete integrable systems. Similar to continuous integrable systems, discrete integrable systems have rich properties – however they are rare. Roughly speaking, discrete integrable systems refer to systems that behave 'nicely'. It is known that for mappings, integrability often associates with 'low complexity' of the maps. For examples, when we iterate rational maps we will expect some cancelations and resulting slow growth of degree of the composed maps. This integrability detectors goes by the name of algebraic entropy and Diophantine integrability [5, 4, 6, 15].

There has been also some work developed by Viallet to study algebraic entropy of lattice equations. Similar to the mapping cases, it is claimed that integrable lattice equations also give us a vanishing algebraic entropy[14, 17]. Using the fact that discrete integrable equations have "low complexity", Hietarinta and Viallet used factorizations to obtain some integrable equations (in certain forms) on the square [7]. Diophantine integrability for lattice equations has been also mentioned by Halburd but has not gained a lot of attention [5].

In this paper, we study growth of degrees of integrable lattice equations in more detail. Given a multi-affine equation on a square, we write each vertex in term of projective coordinates. By looking at the factorization at the point $(2, 2)$ of some certain equations, we give a conjecture that shows a common divisor of the numerator and denominator of each vertex when we use the rule. This helps us to prove polynomial growth of a certain class of integrable equations, for example equations in the Adler-Bobenko-Suris (ABS) classification [1], Q_V [18] and others in the paper written by Hietarinta and Viallet [7].

This paper is organized as follows. In section 2, we give a setting to measure the complexity of a certain lattice equation. Using the equation, one can easily write down each vertex in projective coordinates and obtain the corresponding rules in projective coordinates. In this section, we also give a list of integrable lattice equations that we consider in this paper and a list of corresponding rules in a projective space. In the next section, we explore growth of degrees over $\mathbb{Q}$. Initial values are given as polynomials in w on the horizontal and vertical axes of the $p \times p$ square in the first quadrant. In section 3, we calculate an upper bound for degrees of multi-linear lattice equations. We present a conjecture that seems to hold for equations in the ABS list and Q_V and other equations in the Hietarinta-Viallet list. Based on the factorization at the point $(2,2)$, we give a recursive formula to build a common divisor of the the numerator and denominator of each vertex. This gives an upper bound for the actual degree of each vertex. By considering several types of initial values, we are able to show the polynomial growth of these equations. In particular, using initial values as ratios of degree-one-polynomials, we can explain the quadratic growth of the equations which was given numerically by Viallet [17]. We also study growth of degrees of the discrete Korteweg-De Vries (KdV) equation [11]. We will show that even this equation does not have any common factor at the point $(2,2)$, one can bring it to the previous cases by shifting the starting point. In section 4, we present a similar conjecture for some non-integrable equations in the paper [7]. This conjecture then helps us to prove that these equations have exponential growth.

2. The setting and some preliminary results

In this section, we set up a procedure to measure the complexity of certain lattice equations upon iteration, following aspects of the foundational papers [14, 17, 7]. We are given a square lattice (see Figure 1) whose sites have coordinates $(l,m) \in \mathbb{Z}\times\mathbb{Z}$. Field variables u are defined on each lattice site and are related via an equation which is multi-affine in these variables, i.e. linear in each variable:

$$Q(u_{l,m}, u_{l+1,m}, u_{l,m+1}, u_{l+1,m+1}) = 0. \tag{1}$$

The function Q in (1) most generally has a total of 16 terms, made up of $\binom{4}{i}$ terms that involve products of i of the field variables, $i = 0,1,2,3,4$. [1]

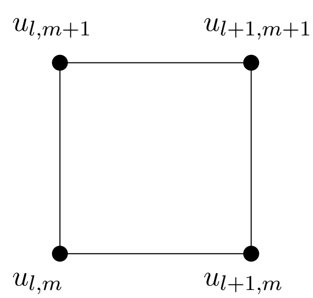

Figure 1. An elementary square of the integer lattice with field variables on the vertices.

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Generically, one can solve (1) for $u_{l+1,m+1} = P(u_{l,m}, u_{l+1,m}, u_{l,m+1})$, where P is a rational function in the arguments. Explicitly, we will assume (1) is equivalent to

$$u_{l+1,m+1}\, h_1\left(u_{l,m}, u_{l+1,m}, u_{l,m+1}\right) - h_2\left(u_{l,m}, u_{l+1,m}, u_{l,m+1}\right) = 0, \tag{2}$$

where h_1 and h_2 are multi-affine (i.e. maximally degree one) in each of their three arguments with no non-constant common factor. We **assume** that each of the three arguments appears

[1] In the connection of lattice maps to partial differential equations, the coefficients of Q are actually given in terms of lattice parameters in each direction but this will not enter into the present consideration of growth properties.

in at least one of h_1 or h_2 (so, in particular, both h_1 and h_2 are non-zero and not both constant). Then

$$u_{l+1,m+1} = \frac{h_2\left(u_{l,m}, u_{l+1,m}, u_{l,m+1}\right)}{h_1\left(u_{l,m}, u_{l+1,m}, u_{l,m+1}\right)}. \tag{3}$$

where each term in h_1 and h_2 is proportional to $u_{l,m}^{i_1}\, u_{l+1,m}^{i_2}\, u_{l,m+1}^{i_3}$, where $0 \leq i_1, i_2, i_3 \leq 1$. By assumption, for each $j \in \{1, 2, 3\}$, there is at least one term between h_1 and h_2 where $i_j = 1$. Now we write each field variable in projective coordinates, that is we introduce $u_{l,m} = \frac{x_{l,m}}{z_{l,m}}$ etc. This gives for each non-constant term in h_1 and h_2:

$$u_{l,m}^{i_1}\, u_{l+1,m}^{i_2}\, u_{l,m+1}^{i_3} = \frac{x_{l,m}^{i_1}\, z_{l,m}^{1-i_1}\, x_{l+1,m}^{i_2}\, z_{l+1,m}^{1-i_2}\, x_{l,m+1}^{i_3}\, z_{l,m+1}^{1-i_3}}{z_{l,m}\, z_{l+1,m}\, z_{l,m+1}}. \tag{4}$$

Multiplying through top and bottom of (3) by $z_{l,m}z_{l+1,m}z_{l,m+1}$ and taking $u_{l+1,m+1} = \frac{x_{l+1,m+1}}{z_{l+1,m+1}}$, we obtain the projective version of (3)

$$x_{l+1,m+1} = f\left(x_{l,m}, x_{l+1,m}, x_{l,m+1}, z_{l,m}, z_{l+1,m}, z_{l,m+1}\right), \tag{5}$$

$$z_{l+1,m+1} = g\left(x_{l,m}, x_{l+1,m}, x_{l,m+1}, z_{l,m}, z_{l+1,m}, z_{l,m+1}\right), \tag{6}$$

where f and g are homogeneous polynomials of degree 3. Our derivation illustrates:

Observation 1. *Given a multi-affine equation* (2) *defined on the square of Figure 1, with* h_1 *and* h_2 *satisfying the assumptions above, the projective coordinates* $x_{l+1,m+1}$ *and* $z_{l+1,m+1}$ *of the top right corner are homogeneous polynomials (5) and (6) of degree* 3 *where each term of these polynomials takes the form of the numerator of* (4)*. Thus each term includes exactly one projective coordinate from each of the remaining* 3 *vertices of the square.*

Table 1 is the list of lattice equations that we will consider here. For ease of notation in listing them, we denote $u := u_{l,m}, u_1 := u_{l+1,m}, u_2 := u_{l,m+1}$ and $u_{12} := u_{l+1,m+1}$. The equations satisfy the assumptions of the form (2) above and free parameters can be taken to be integer-valued.

The lattice rules Q_i and H_i, $(i = 1, 2, 3)$ belong to the Adler-Bobenko-Suris (ABS) list satisfying *consistency around a cube* [1]. The remaining rules are not in the ABS classification but have some nice properties. We start with equations in the Adler-Bobenko-Suris classification and other famous discrete integrable systems such as sine-Gordon (sG), modified Korteweg- De Vries (mKdV), KdV, Lotka-Volterra and Tzitzeica (Tz) equations [1, 8, 9, 2]. These equations are integrable in the sense that they have Lax pairs. In particular, the KdV equation can be obtained from two copies of H_1 [11]. The Lotka-Volterra equation can be obtained from the KdV equation via Muira transformation [12]. The rest of the list were introduced by Hietarinta and Viallet [7]. These equations have polynomial growth in the north-east direction. Among these equations, equations E21 and E22 can reduce to equations $H1$ and H_3 in the ABS list.

Table 2 presents the corresponding projective versions of Table 1. They take the form of (5)-(6) and illustrate Observation 1.

Name	*Lattice equation*	*Reference*
Q_1	$\alpha(u-u_2)(u_1-u_{12})-\beta(u-u_1)(u_2-u_{12})+\delta^2\alpha\beta(\alpha-\beta)=0$	[1]
Q_2	$\alpha(u-u_2)(u_1-u_{12})-\beta(u-u_1)(u_2-u_{12})+\alpha\beta(\alpha-\beta)(u+u_1+u_2+u_{12})-\alpha\beta(\alpha-\beta)(\alpha^2-\alpha\beta+\beta^2)=0$	[1]
Q_3	$(\beta^2-\alpha^2)(uu_{12}+u_1u_2)+\beta(\alpha^2-1)(uu_1+u_2u_{12})-\alpha(\beta^2-1)(uu_2+u_1u_{12})-\delta^2(\alpha^2-\beta^2)(\alpha^2-1)(\beta^2-1)/(4\alpha\beta)=0$	[1]
H_1	$(u-u_{12})(u_1-u_2)+\beta-\alpha=0$	[1, 11]
H_2	$(u-u_{12})(u_1-u_2)+(\alpha-\beta)(u+u_1+u_2+u_{12})+\beta^2-\alpha^2=0$	[1]
H_3	$\alpha(uu_1+u_2u_{12})-\beta(uu_2+u_1u_{12})+\delta(\alpha^2-\beta^2)=0$	[1]
Q_5	$p_1uu_1u_2u_{12}+p_2\big(uu_1u_2+uu_1u_{12}+uu_2u_{12}+u_1u_2u_{12}\big)+p_3\big(uu_1+u_2u_{12}\big)+p_4\big(uu_2+u_1u_{12}\big)+p_5\big(uu_{12}+u_1u_2\big)+p_6\big(u+u_1+u_2+u_{12}\big)+p_7=0$	[1]
mKdV	$uu_2-u_1u_{12}+\alpha uu_1-\beta u_2u_{12}=0$	[16]
sG	$uu_1u_2u_{12}+\alpha\,(uu_{12}-u_1u_2)-\beta=0$	[16]
E16	$uu_1p_1+uu_2p_5(p_1p_3+p_2)+(uu_{12}+u_1u_2)p_2+u_1u_{12}p_6+u_2u_{12}p_3(p_5p_6-p_2)=0$	[7]
E21	$(uu_2+u_1u_{12})\,p_5+(u_1u+u_2u_{12})\,p_3+s=0$	[7]
E22	$(u-u_{12})\,(u_1-u_2)+r_1\,(u+u_1+u_2+u_{12})+s=0$	[7]
E24	$(uu_2+u_1u_{12})(p_2+p_3)+(u_1u+u_2u_{12})p_3+uu_{12}+u_1u_2+p_3p_2(p_2+p_3)s^2=0$	[7]
E25	$uu_{12}+u_1u_2+(u_1u+u_2u_{12})p_3-(uu_2+u_1u_{12})(p_3+1)+(u_{12}-u)r_4+(u_1-u_2)r_2-(s(p_3+1)+r_4)\,(sp_3+r_4)+sr_2=0$	[7]
KdV	${u_1}^{-1}-{u_2}^{-1}-u+u_{12}=0$	[11]
E20	$u_{12}u_1p_6+u_{12}u_2p_3+u_1up_1+u_2up_5+u_{12}p_3p_6r_4+u_1p_6r_2+u_2p_3r_3+u\big(-p_1p_5r_4+p_1r_3+p_5r_2\big)+s=0$	[7]
E26	$u_{12}u_1{p_3}^2+u_2u{p_6}^2+u_{12}u_2p_1p_3+u_1up_3{p_6}^2p_1^{-1}+\big(u_{12}p_3+u_1p_3+u_2p_6+up_6\big)r_1+{r_1}^2=0$	[7]
E27	$u_{12}u_1{p_6}^2+u_2u{p_3}^2+u_2u_{12}p_6(p_3-1)p_1^{-1}+u_1up_1p_6(p_3-1)+(uu_{12}+u_1u_{l2})p_6+(u_{12}p_6+u_1p_6+u_2p_3+up_3)r_4+{r_4}^2=0$	[7]
E28	$uu_{12}+u_1u_2+p_3(u_2u_{12}+uu_1)+u_{12}u_1p_6+uu_2(p_3-1)^2p_6^{-1}+r_3(p_6-p_3+1)(u_1+u)+p_6(p_6+1){r_3}^2=0$	[7]
E30	$uu_1+u_2u_{12}+(uu_{12}+u_1u_2)p_3+(p_3-1)(uu_2+u_1u_{12})+(u-u_2+u_1-u_{12})r_4+{r_4}^2=0$	[7]
E17	$p_4uu_{12}+p_5uu_2+p_2u_1u_2+p_6u_1u_{12}+r_1u+r_2u_1+r_3u_2+r_4u_{12}+s=0$	[7]

TABLE 1. A list of the lattice rules considered in this paper. For most cases, we retain the parametrisations of the original references. Equations above the double-line division are integrable and those below it are non-integrable.

Eq.	$x_{l+1,m+1}$	$z_{l+1,m+1}$
Q_1	$\alpha\ x_{l+1,m}x_{l,m}z_{l,m+1} - \alpha\ x_{l+1,m}x_{l,m+1}z_{l,m} - \beta\ x_{l,m+1}x_{l,m}z_{l+1,m} + \beta\ x_{l,m+1}x_{l+1,m}z_{l,m} + (\delta\alpha^2\beta - \delta\alpha\beta^2)\ z_{l+1,m}z_{l,m}z_{l,m+1}$	$(\alpha - \beta)\ x_{l,m}z_{l+1,m}z_{l,m+1} - \alpha\ x_{l,m+1}z_{l,m}z_{l+1,m} + \beta\ x_{l+1,m}z_{l,m}z_{l,m+1}$
Q_2	$(\beta\ x_{l,m+1}z_{l+1,m} - \alpha\ x_{l+1,m}z_{l,m+1} + (\alpha\beta^2 - \alpha^2\beta)\ z_{l+1,m}z_{l,m+1})x_{l,m} + ((\alpha - \beta)\ z_{l,m}x_{l+1,m} + (\alpha\beta^2 - \alpha^2\beta)\ z_{l+1,m}z_{l,m})x_{l,m+1} + (\alpha\beta^2 - \alpha^2\beta)\ z_{l,m+1}z_{l,m}x_{l+1,m} + (-\alpha\beta^4 + 2\,\alpha^2\beta^3 + \alpha^4\beta - 2\alpha^3\beta^2)\ z_{l+1,m}z_{l,m+1}z_{l,m}$	$(-\alpha + \beta)\ z_{l+1,m}z_{l,m+1}x_{l,m} + \alpha\, x_{l,m+1}z_{l,m}z_{l+1,m} - \beta\, x_{l+1,m}z_{l,m}z_{l,m+1} + (\alpha^2\beta - \alpha\,\beta^2)\ z_{l+1,m}z_{l,m+1}z_{l,m}$
Q_3	$(4\,\alpha^2\beta^3 - 4\,\alpha^2\beta)\ x_{l,m}z_{l+1,m}x_{l,m+1} + (-4\,\alpha^3\beta^2 + 4\alpha\beta^2)\ z_{l,m+1}x_{l+1,m}x_{l,m} + (4\,\delta\,\alpha^3\beta - 4\delta\alpha\beta^3)\ z_{l,m}x_{l+1,m}x_{l,m+1} + (-\alpha^4 + \beta^4 + \alpha^4\beta^2 - \alpha^2\beta^4 + \alpha^2 - \beta^2)\ z_{l+1,m}z_{l,m+1}z_{l,m}$	$4(-\delta\,\alpha^2 + \delta\beta^2)\beta\,\alpha\ z_{l+1,m}z_{l,m+1}x_{l,m} + 4(\alpha^2\beta - \beta)\beta\alpha\, z_{l+1,m}z_{l,m}x_{l,m+1} + 4\,(-\alpha\beta^2 + \alpha)\beta\alpha\ z_{l,m+1}z_{l,m}x_{l+1,m}$
H_1	$-x_{l,m}x_{l+1,m}z_{l,m+1} + x_{l,m}x_{l,m+1}z_{l+1,m} + (\alpha - \beta)\ z_{l,m}z_{l+1,m}z_{l,m+1}$	$(-x_{l+1,m}z_{l,m+1} + x_{l,m+1}z_{l+1,m})\ z_{l,m}$
H_2	$x_{l,m}x_{l+1,m}z_{l,m+1} - x_{l,m}x_{l,m+1}z_{l+1,m} + (\beta - \alpha)\ x_{l,m}z_{l+1,m}z_{l,m+1} + (\beta - \alpha)\ x_{l+1,m}z_{l,m}z_{l,m+1} + (\beta - \alpha)\ x_{l,m+1}z_{l,m}z_{l+1,m} + (\beta^2 - \alpha^2)\ z_{l,m}z_{l+1,m}z_{l,m+1}$	$(x_{l+1,m}z_{l,m+1} - x_{l,m+1}z_{l+1,m} + (\alpha - \beta)\ z_{l+1,m}z_{l,m+1})z_{l,m}$
H_3	$-\alpha\ x_{l+1,m}x_{l,m}z_{l,m+1} + \beta\ x_{l,m+1}x_{l,m}z_{l+1,m} + (\delta\beta^2 - \delta\alpha^2)\ z_{l+1,m}z_{l,m}z_{l,m+1}+$	$(\alpha\ x_{l,m+1}z_{l+1,m} - \beta\ x_{l+1,m}z_{l,m+1})z_{l,m}$
mKdV	$(x_{l,m+1}z_{l+1,m} + \alpha\ x_{l+1,m}z_{l,m+1})x_{l,m}$	$(x_{l+1,m}z_{l,m+1} + \beta\ x_{l,m+1}z_{l+1,m})z_{l,m}$
sG	$z_{l,m}(\alpha\ x_{l+1,m}x_{l,m+1} + \beta\ z_{l+1,m}z_{l,m+1})$	$(x_{l+1,m}x_{l,m+1} + \alpha\ z_{l+1,m}z_{l,m+1})\ x_{l,m}$
E16	$-p_1\ x_{l,m}x_{l+1,m}z_{l,m+1} - p_1p_3p_5\ x_{l,m}x_{l,m+1}z_{l+1,m} - p_2p_5\ x_{l,m}x_{l,m+1}z_{l+1,m} - p_2\ x_{l+1,m}x_{l,m+1}z_{l,m}$	$-p_1\ x_{l,m}x_{l+1,m}z_{l,m+1} - p_1p_3p_5\ x_{l,m}x_{l,m+1}z_{l+1,m} - p_2p_5\ x_{l,m}x_{l,m+1}z_{l+1,m} - p_2\ x_{l+1,m}x_{l,m+1}z_{l,m}$
E21	$-p_5\ x_{l,m}x_{l,m+1}z_{l+1,m} - p_3\ x_{l+1,m}x_{l,m}z_{l,m+1} - sz_{l+1,m}z_{l,m}z_{l,m+1}$	$z_{l,m}(p_5\ x_{l+1,m}z_{l,m+1} + p_3\ x_{l,m+1}z_{l+1,m})$
E22	$x_{l+1,m}x_{l,m}z_{l,m+1} - x_{l,m}x_{l,m+1}z_{l+1,m} + r_1\ x_{l,m}z_{l+1,m}z_{l,m+1} + r_1\ x_{l+1,m}z_{l,m+1}z_{l,m} + r_1\ x_{l,m+1}z_{l+1,m}z_{l,m} + s\ z_{l+1,m}z_{l,m}z_{l,m+1}$	$z_{l,m}(x_{l+1,m}z_{l,m+1} - x_{l,m+1}z_{l+1,m} - r_1\ z_{l+1,m}z_{l,m+1})$
E24	$-p_2\ x_{l,m}x_{l,m+1}z_{l+1,m} - p_3\ x_{l,m}x_{l,m+1}z_{l+1,m} - x_{l+1,m}x_{l,m+1}z_{l,m} - p_3\ x_{l+1,m}x_{l,m}z_{l,m+1} - p_3{p_2}^2s^2\ z_{l+1,m}z_{l,m+1}z_{l,m} - {p_3}^2p_2s^2\ z_{l+1,m}z_{l,m+1}z_{l,m}$	$p_2\ x_{l+1,m}z_{l,m+1}z_{l,m} + p_3\ x_{l+1,m}z_{l,m+1}z_{l,m} + p_3\ x_{l,m+1}z_{l+1,m}z_{l,m} + x_{l,m}z_{l+1,m}z_{l,m+1}$
E25	$p_3\ x_{l,m}x_{l,m+1}z_{l+1,m} + x_{l,m}x_{l,m+1}z_{l+1,m} - x_{l+1,m}x_{l,m+1}z_{l,m} - p_3\ x_{l+1,m}x_{l,m}z_{l,m+1} - x_{l+1,m}r_2z_{l,m}z_{l,m+1} + r_2\ x_{l,m+1}z_{l,m}z_{l+1,m} + s^2{p_3}^2\ z_{l,m}z_{l,m+1}z_{l+1,m} + 2\,sp_3r_4\ z_{l,m}z_{l,m+1}z_{l+1,m} + s^2p_3\ z_{l,m}z_{l,m+1}z_{l+1,m} + sr_4\ z_{l,m}z_{l,m+1}z_{l+1,m} + {r_4}^2\ z_{l,m}z_{l,m+1}z_{l+1,m} + r_4\ x_{l,m}z_{l,m+1}z_{l+1,m} - sr_2\ z_{l,m}z_{l,m+1}z_{l+1,m}$	$x_{l,m}z_{l+1,m}z_{l,m+1} + p_3\ x_{l,m+1}z_{l+1,m}z_{l,m} - p_3\ x_{l+1,m}z_{l,m+1}z_{l,m} - x_{l+1,m}z_{l,m}z_{l,m+1} + r_4\ z_{l,m}z_{l,m+1}z_{l+1,m}$
KdV	$-x_{l,m+1}z_{l,m}z_{l+1,m} + x_{l+1,m}z_{l,m}z_{l,m+1} + x_{l,m}x_{l+1,m}x_{l,m+1}$	$z_{l,m}x_{l+1,m}x_{l,m+1}$

TABLE 2. List of integrable lattice rules of Table 1 when written in projective coordinates (5)-(6).

We can iteratively use these lattice rules to evaluate $x_{l,m}$ and $z_{l,m}$ throughout a region of the square lattice. In this paper, we prescribe *corner initial conditions* as previously considered in e.g. [14, 7]. That is, we prescribe the initial values $x_{l,0}$ and $z_{l,0}$ and $x_{0,m}$ and $z_{0,m}$, with $l, m \in \mathbb{N}$, on the borders of the first quadrant of the lattice. Working out from the origin using these initial conditions and the lattice rule to generate the top right entry in each square, one can evaluate $x_{l+1,m+1}$ and $z_{l+1,m+1}$ given by (5)-(6). It is clear that $x_{i,j}$ and $z_{i,j}$, with $i, j \in \mathbb{N}$, are both multinomial expressions in the $2(i + j + 1)$ variables given by the initial conditions $x_{l,0}$ and $z_{l,0}$, $0 \leq l \leq i$, and $x_{0,m}$ and $z_{0,m}$, $0 < m \leq j$, i.e.

$$x_{l,m} = F\left(x_{0,0}, x_{1,0}, \ldots, x_{l,0}, x_{0,1}, \ldots, x_{0,m}; z_{1,0}, \ldots, z_{l,0}, z_{0,1}, \ldots, z_{0,m}\right), \tag{7}$$

$$z_{l,m} = G\left(x_{0,0}, x_{1,0}, \ldots, x_{l,0}, x_{0,1}, \ldots, x_{0,m}; z_{1,0}, \ldots, z_{l,0}, z_{0,1}, \ldots, z_{0,m}\right). \tag{8}$$

Determining $x_{i,j}$ and $z_{i,j}$ requires having built $x_{l,m}$ and $z_{l,m}$ at all other lattice sites in the $i \times j$ square extending out from the origin in the first quadrant.

As observed previously [14, 7], some lattice rules distinguish themselves by a systematic factorization of the multinomial expressions for $x_{i,j}$ and $z_{i,j}$ and the appearance of common factors in them. Given their projective nature, these common factors can be ignored or *cancelled*, leading to a lower degree in the reduced multinomial expressions compared to the case where no common factor arose. One manifestation of this factorization process (leading to a diagnostic for detecting it) is that if the corner initial conditions are taken to be integers, the resulting integers $x_{i,j}$ and $z_{i,j}$ after cancellation of common integer factors would be smaller in magnitude than generically expected. This is equivalent to saying in the non-projective setting of (3), and with corner initial conditions taken in $\mathbb{Q}$, that some lattice rules give a lower *height* for the rational number $u_{l+1,m+1}$ than generically expected. This is closely connected to the concept of Diophantine integrability for lattice maps, explored in the last part of [5].

One approach to studying and observing the phenomenon of cancellation that has clear computational advantages for both speed and memory is to take the initial values $x_{l,0}(w)$ and $z_{l,0}(w)$ and $x_{0,m}(w)$ and $z_{0,m}(w)$, where $l, m \in \mathbb{N}$, as polynomials in an indeterminate w. [2] Using the lattice rule, one can calculate $x_{l,m}(w)$ and $z_{l,m}(w)$ with $l, m > 0$. We factor $x_{l,m}(w)$ and $z_{l,m}(w)$ (over the integers?) and define $\gcd_{l,m}(w)$ to be their greatest common divisor so we can write

$$x_{l,m}(w) = \gcd_{l,m}(w)\, \bar{x}_{l,m}(w), \tag{9}$$

$$z_{l,m}(w) = \gcd_{l,m}(w)\, \bar{z}_{l,m}(w). \tag{10}$$

As usual, $\gcd_{l,m}(w)$ is taken to be a monic polynomial and $x_{l,m}(w)$ and $z_{l,m}(w)$ are relatively prime if and only if $\gcd_{l,m}(w) \equiv 1$. Suppressing the w-dependence for ease of presentation, define the non-negative integers

$$d_{l,m} = \max(\deg(x_{l,m}), \deg(z_{l,m})) \geq 0 \tag{11}$$

$$g_{l.m} = \deg(\gcd_{l,m}) \geq 0. \tag{12}$$

$$\bar{d}_{l,m} = \max(\deg(\bar{x}_{l,m}), \deg(\bar{z}_{l,m})) = d_{l,m} - g_{l.m} \geq 0 \tag{13}$$

The key issue (for algebraic entropy) relates to the growth of the degree of $g_{l,m}$. Because $x_{l,m}(w)$ and $z_{l,m}(w)$ are considered projectively, when $g_{l,m} \geq 1$, they can be replaced by their barred versions. This corresponds to the cancellation of $\gcd_{l,m}(w)$ from the numerator and

[2]The specialisation to univariate polynomials also enables concepts like *greatest common divisor*, which are not always defined in the multinomial case.

denominator of the rational function $u_{l,m}(w)$. Correspondingly, $d_{l,m}$ is replaced by the lesser $\bar{d}_{l,m}$ as the (true) degree at the vertex.

In typical entropy calculations, the gcd is discarded and only its degree is noted. For what follows in the next section, we find it useful to also observe some of its internal structure as we proceed iteratively away from the origin. We have the following properties:

Proposition 2. *For the projective lattice rules of Observation 1, we have the following:*

1. $0 \leq d_{l+1,m+1} \ \leq \ d_{l,m} + d_{l+1,m} + d_{l,m+1}$
2. *If* $x_{l,m}(w) = z_{l,m}(w) \equiv 0$, *then* $x_{l+i,m+j}(w) = z_{l+i,m+j}(w) \equiv 0$, *for all integers* $i, j \geq 1$.
3. $\mathrm{gcd}_{l,m}(w)\ \mathrm{gcd}_{l+1,m}(w)\ \mathrm{gcd}_{l,m+1}(w) \ \big|\ \mathrm{gcd}_{l+1,m+1}(w)$ *which implies when* $x_{l+1,m+1}(w)$ *and* $z_{l+1,m+1}(w)$ *are not both* 0 *that*
   3a. $g_{l+1,m+1} \ \geq\ g_{l,m} + g_{l+1,m} + g_{l,m+1}$
   3b. $\mathrm{gcd}_{l,m}(w) \ \big|\ \mathrm{gcd}_{l,m+1}(w) \implies g(l, m+1) \ \geq\ g(l,m)$
   3c. $\mathrm{gcd}_{l,m}(w) \ \big|\ \mathrm{gcd}_{l+1,m}(w) \implies g(l+1, m) \ \geq\ g(l,m)$
   3d. $\mathrm{gcd}_{l,m}(w)^3 \ \big|\ \mathrm{gcd}_{l+1,m+1}(w) \implies g(l+1, m+1) \ \geq\ 3\,g(l,m)$

Proof. Observation 1 tells us that each term of $x_{l+1,m+1}(w)$ and $z_{l+1,m+1}(w)$ is a product involving one of $x(w)$ or $z(w)$ from each of the other 3 vertices. So the degree of each term is bounded above by $d_{l,m} + d_{l+1,m} + d_{l,m+1}$, as is a sum of such terms, giving statement 1. Furthermore, if both of $x(w)$ and $z(w)$ vanish on a vertex, all terms vanish in the expressions for $x(w)$ and $z(w)$ on vertices to the right and upwards, giving 2. For 3., from (9)-(10), it is clear that $\mathrm{gcd}_{l,m}(w)\ \mathrm{gcd}_{l+1,m}(w)\ \mathrm{gcd}_{l,m+1}(w)$ divides each term of $x_{l+1,m+1}(w)$ and $z_{l+1,m+1}(w)$, and hence divides $x_{l+1,m+1}(w)$ and $x_{l+1,m+1}(w)$ and their gcd (including the case $x_{l+1,m+1}(w) \equiv 0$ and $z_{l+1,m+1}(w) \equiv 0$, whence $\mathrm{gcd}_{l,m+1}(w) \equiv 0$). If $x_{l+1,m+1}(w) \equiv 0$ and $z_{l+1,m+1}(w)$ is not, then $\mathrm{gcd}_{l+1,m+1}(w) = z_{l+1,m+1}(w)$, by definition, and the statement of 3. is trivially true. Likewise, in the reverse case. The statements of (3a)-(3d) follow easily from the first statement in 3. □

Remark 3. *If* $x_{l,m}(w)$ *and* $z_{l,m}(w)$ *share the same degree,* (11)-(13) *become:*

$$d_{l,m} \ = \ \deg(x_{l,m}) = \deg(z_{l,m})) \geq 0 \tag{14}$$

$$\bar{d}_{l,m} \ = \ \deg(\bar{x}_{l,m}) = \deg(\bar{z}_{l,m})) \geq 0. \tag{15}$$

If this degree equality of the two components holds at each of the 3 vertices used in (9)-(10), then generically $x_{l+1,m+1}(w)$ *and* $z_{l+1,m+1}(w)$ *share the same degree, since each of their terms has degree* $d_{l,m} + d_{l+1,m} + d_{l,m+1}$ *and a reduction of degree in adding these terms would constitute a condition relating the coefficients of the polynomials* $x_{l,m}(w)$,$x_{l+1,m}(w)$,$x_{l,m+1}(w)$, $z_{l,m}(w)$, $z_{l+1,m}(w)$ *and* $z_{l,m+1}(w)$. *So, for generic coefficients, the degree equality of the two components is retained and*

$$d_{l+1,m+1} = \deg(\bar{x}_{l+1,m+1}) = \deg(\bar{z}_{l,m})) = d_{l,m} + d_{l+1,m} + d_{l,m+1}. \tag{16}$$

Part 3. of Proposition 2 illustrates the nesting and propagation of the greatest common divisors that occurs for growing l and growing m when there starts to be a non-trivial gcd at some point. At any lattice point in the first quadrant, the gcd of $x_(w)$ and $z(w)$ includes all the greatest common divisors of the polynomials at the $lm + l + m$ lattice points to its left and/or below it. This leads to great multiplicities of factors in the gcd. We remark that [7] used cancellation (i.e. non-trivial gcd) after 2 steps on the diagonal as a criterion to detect integrable rules and Part 3d. of Proposition 2 shows that once this occurs, it propagates at an exponential rate.

With respect to statement 3. of Proposition 2, we remark that once we factor out the product $\gcd_{l,m}(w)\ \gcd_{l+1,m}(w)\ \gcd_{l,m+1}(w)$ from $x_{l+1,m+1}(w)$ and $z_{l+1,m+1}(w)$, we are left with f of (5) and g of (6), but now with the barred arguments as defined by (9) and (10). These polynomials in the barred variables may have a further non-trivial gcd. We define

$$\begin{aligned}\overline{\gcd}_{l+1,m+1}(w) = \gcd\{&f\left(\bar{x}_{l,m}, \bar{x}_{l+1,m}, \bar{x}_{l,m+1}, \bar{z}_{l,m}, \bar{z}_{l+1,m}, \bar{z}_{l,m+1}\right), \\ &g\left(\bar{x}_{l,m}, \bar{x}_{l+1,m}, \bar{x}_{l,m+1}, \bar{z}_{l,m}, \bar{z}_{l+1,m}, \bar{z}_{l,m+1}\right)\}\end{aligned} \tag{17}$$

so that

$$\bar{x}_{l+1,m+1}(w) = f\left(\bar{x}_{l,m}, \bar{x}_{l+1,m}, \bar{x}_{l,m+1}, \bar{z}_{l,m}, \bar{z}_{l+1,m}, \bar{z}_{l,m+1}\right) / \overline{\gcd}_{l+1,m+1}(w) \tag{18}$$

$$\bar{z}_{l+1,m+1}(w) = g\left(\bar{x}_{l,m}, \bar{x}_{l+1,m}, \bar{x}_{l,m+1}, \bar{z}_{l,m}, \bar{z}_{l+1,m}, \bar{z}_{l,m+1}\right) / \overline{\gcd}_{l+1,m+1}(w) \tag{19}$$

Remark 4. *Computationally, it follows that at each vertex, it is most efficient to record the triple* $\{\bar{x}_{l,m}(w), \bar{z}_{l,m}(w), \gcd_{l,m}(w)\}$*, from which one can reconstruct* $x_{l,m}(w)$ *and* $z_{l,m}(w)$ *if necessary. One takes barred variables as the arguments in the right-hand side of (5) and (6) and checks the greatest common divisor,* $\overline{\gcd}_{l+1,m+1}(w)$ *of the resulting* f *and* g*. This gives* $\bar{x}_{l+1,m+1}(w)$ *of* (18) *and* $\bar{z}_{l+1,m+1}(w)$ *of* (19) *whereas* $\gcd_{l+1,m+1}(w)$ *is updated via*

$$\gcd_{l+1,m+1}(w) = \overline{\gcd}_{l+1,m+1}(w)\ \gcd_{l,m}(w)\ \gcd_{l+1,m}(w)\ \gcd_{l,m+1}(w). \tag{20}$$

3. Growth of ambient degrees before cancellation

Our approach in this section is to identify a linear partial difference equation satisfied by the upper bound on the degree at each vertex, which also represents the generic degree at each vertex before common factors and cancellations are considered. This enables the exponential growth of these degrees to be verified.

3.1. Solution of recurrence for the upper bound on degrees.

Part 1. of Proposition 2 shows that an upper bound for the degree $d_{l+1,m+1}$ is provided by the non-negative integer sequence $(a_{l+1,m+1})$ satisfying

$$a_{l+1,m+1} = a_{l,m} + a_{l+1,m} + a_{l,m+1}. \tag{21}$$

The value $d_{l+1,m+1}$ satisfies this same recurrence at any vertex if either $x_{l+1,m+1}$ or $z_{l+1,m+1}$ contains the term comprising a maximal degree polynomial in w from each of the other 3 vertices. The argument of Remark 3 above suggests that, generically, $d_{l+1,m+1} = a_{l+1,m+1}$ when the boundary values are chosen with both components at each vertex having the same degree in w.

The two cases of such boundary values we consider are:

$$\text{Case I: }\begin{cases} x_{0,0} = aw + b \text{ and } z_{0,0} = cw + d, \text{where } a, b, c, d \in \mathbb{Z}, \\ x_{0,m} = c_m \in \mathbb{Z} \text{ and } z_{0,m} = d_m \in \mathbb{Z}, \quad m = 1, 2, \ldots \\ x_{l,0} = e_l \in \mathbb{Z} \text{ and } z_{l,0} = f_l \in \mathbb{Z}, \quad l = 1, 2, \ldots. \end{cases} \tag{22}$$

$$\text{Case II: }\begin{cases} x_{0,m} = c_m w + b_m, \quad c_m \neq 0, b_m \in \mathbb{Z}, \quad m = 0, 1, 2, \ldots, \\ z_{0,m} = c_m w + b_m, \quad c_m, b_m \in \mathbb{Z}, \quad m = 0, 1, 2, \ldots, \\ x_{l,0} = c_m w + b_m, \quad c_m, b_m \in \mathbb{Z}, \quad l = 1, 2, \ldots, \\ z_{l,0} = c_m w + b_m, \quad c_m, b_m \in \mathbb{Z}, \quad l = 1, 2, \ldots. \end{cases} \tag{23}$$

For Case I boundary values, we have $d_{0,0} = 1$ and $d_{0,m} = d_{l,0} = 0$ for $l, m > 0$. For Case II boundary values, we have $d_{l,0} = d_{0,m} = 1$ for $l, m \geq 0$.

Our numerical experiments on the equations of Table 2 use Maple to calculate the polynomials $x_{l,m}(w)$ and $z_{l,m}(w)$ for $0 \le l, m \le 11$ (i.e. a square of 144 lattice sites based at the origin) and Case I or Case II boundary values with random integer coefficients chosen in the interval $[0, 400]$. They confirm the agreement $d_{l+1,m+1} = a_{l+1,m+1}$ where $d_{l+1,m+1}$ is calculated from (11) with (5) and (6) and $a_{l+1,m+1}$ from (21), with the corresponding same boundary values in each case. Consequently, we call $a_{l+1,m+1}$ the *ambient* degree at the vertex (i.e. before any analysis of a possible common factor and cancellation of this factor). If, furthermore, there is actually no non-trivial gcd for the two polynomials $x_{l+1,m+1}$ and $z_{l+1,m+1}$ (so *no* cancellations possible), then $d_{l+1,m+1} = a_{l+1,m+1} = \bar{d}_{l+1,m+1}$ gives the true degree throughout the lattice.

The solution of (21) with Case I boundary values can be represented as (part of) a semi-infinite array $A_{0,0} = A_{0,0}[l, m]$, with the indices $l \ge 0$ and $m \ge 0$ measured vertically, respectively to the right, with respect to the origin at the bottom left corner so $A_{0,0}[0, 0] = 1$. The solution for Case II boundary values corresponds to removing the first column and the last row of this array.

$$A_{0,0}[l,m] = \begin{bmatrix} 0 & 1 & 17 & 145 & 833 & 3649 & 13073 & 40081 & 108545 & 265729 \\ 0 & 1 & 15 & 113 & 575 & 2241 & 7183 & 19825 & 48639 & 108545 \\ 0 & 1 & 13 & 85 & 377 & 1289 & 3653 & 8989 & 19825 & 40081 \\ 0 & 1 & 11 & 61 & 231 & 681 & 1683 & 3653 & 7183 & 13073 \\ 0 & 1 & 9 & 41 & 129 & 321 & 681 & 1289 & 2241 & 3649 \\ 0 & 1 & 7 & 25 & 63 & 129 & 231 & 377 & 575 & 833 \\ 0 & 1 & 5 & 13 & 25 & 41 & 61 & 85 & 113 & 145 \\ 0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

A formula for the double-indexed sequence $A_{0,0}[l, m]$ is given by the following theorem.

Theorem 5. *Let $A_{0,0}[l, m]$ denote the integer sequence solution $a_{l,m}$ satisfying* (21) *for all $l, m \ge 0$ with boundary values $a_{0,0} = 1$ and $a_{0,m} = a_{l,0} = 0$ for $l, m > 0$. For $l, m > 0$, let $c_{l,m}$ be the coefficient of x^{m-1} in the Taylor expansion of the following function $g_l(x)$ around* 0*, i.e,*

$$g_l(x) = \frac{(1+x)^{l-1}}{(1-x)^l} = \sum_{m=1}^{\infty} c_{l,m}\, x^{m-1}. \tag{25}$$

We have for all $l, m > 0$,

$$A_{0,0}[l, m] = c_{l,m} = \sum_{i+j=m-1} \binom{l-1}{i} \binom{j+l-1}{j}. \tag{26}$$

Proof. It is easy to see from (21) that $a_{1,m} = a_{l,1} = 1$ is what results from applying the recurrence with prescribed initial conditions given on the axes as in the statement of the theorem. On the other hand, (25) also shows $c_{1,m} = 1$ as the Taylor coefficients of $g_1(x) = 1/(1-x)$ for all $m \ge 1$ and $c_{l,1} = g_l(0) = 1$ for all $l \ge 1$. It is adequate to prove that $c_{l,m}$ satisfies the same recurrence (21) as $a_{l,m}$. We have

$$g_{l+1}(x) = \sum_{m=1}^{\infty} c_{l+1,m}\, x^{m-1}.$$

On the other hand, we also have

$$\begin{aligned} g_{l+1}(x) &= g_l(x)\,\frac{1+x}{1-x} \\ &= \left(\sum_{i=1}^{\infty} c_{l,i}\,x^{i-1}\right)\left(1+\sum_{j=1}^{\infty} 2\,x^j\right). \end{aligned}$$

Equating coefficients of x^{m-1}, we obtain

$$c_{l+1,m} = c_{l,m} + \sum_{i=1}^{m-1} 2\,c_{l,i}.$$

This implies

$$\begin{aligned} c_{l+1,m+1} &= \sum_{i=1}^{m} 2c_{l,i} + c_{l,m+1} \\ &= \left(\sum_{i=1}^{m-1} 2c_{l,i} + c_{l,m}\right) + c_{l,m} + c_{l,m+1} \\ &= c_{l+1,m} + c_{l,m} + c_{l,m+1}, \end{aligned}$$

which is the same recurrence given by (21).

We now calculate $c_{l,m}$ using the following formula for $l \geq 1$, cf.

$$\frac{1}{(1-x)^l} = \sum_{j=0}^{\infty} \binom{j+l-1}{j} x^j.$$

Expanding $(1+x)^{l-1}$, we get

$$g_l(x) = \left(\sum_{i=0}^{\infty} \binom{l-1}{i} x^i\right)\left(\sum_{j=0}^{\infty} \binom{j+l-1}{j} x^j\right).$$

This gives us the formula given by (26). □

From the particular solution $A_{0,0}[l,m]$ of (26), we can generate the general solution to (21) with arbitrary boundary conditions $a_{0,0}$, $a_{0,m}$ and $a_{l,0}$. In this case, we can write the boundary conditions in the form

$$a_{l,m} = a_{0,0}\,\delta(l)\,\delta(m) + \sum_{i=1}^{\infty} [\,a_{i,0}\,\delta(i-l) + a_{0,i}\,\delta(i-m)\,], \quad l\cdot m = 0. \tag{27}$$

Here $\delta(r)$ is the Dirac delta function satisfying $\delta(0) = 1$ and $\delta(r) = 0$, otherwise. Equation (27) is to be regarded as a superposition of elemental boundary conditions, each of which has value 0 on all sites of the corner axes except for one site where $a_{i,0} = 1$. Let $A_{i,0}[l,m]$ refers to the corresponding solution of (21) with this boundary condition, represented as an array similar to $A_{0,0}$ above. Since (21) is a linear difference equation, the solution to a linear combination of boundary conditions is the linear combination of their corresponding solutions:

$$a_{l,m} = a_{0,0}\,A_{0,0}\,[l,m] + \sum_{i=1}^{l} a_{i,0}\,A_{i,0}\,[l,m] + \sum_{i=1}^{m} a_{0,i}\,A_{0,i}\,[l,m]\,, \quad l,m \geq 0. \tag{28}$$

Note that only a finite sum occurs in (28), as only a finite number of boundary values are needed to determine the value at a vertex within the quadrant.

The solution $A_{i,0}$, $i \geq 2$, can be deduced from $A_{1,0}$ via

$$A_{i,0}\,[l,m] = \begin{cases} A_{1,0}\,[l+1-i,m] & \text{if } l \geq i \\ 0 & \text{if } 0 \leq l < i, \end{cases} \tag{29}$$

i.e. a version of $A_{1,0}$ shifted to the right. We can rewrite this using the discrete Heaviside function defined on $n \in \mathbb{Z}$:

$$H[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{if } n < 0, \end{cases} \tag{30}$$

so that

$$A_{i,0}\,[l,m] = H[l-i]\,A_{1,0}\,[l+1-i,m]. \tag{31}$$

Also, since (21) is symmetric in the interchange of indices, we have for $i \geq 1$

$$A_{0,i} = A_{i,0}^T, \tag{32}$$

meaning the array $A_{0,i}$ is the transpose of that of $A_{i,0}$. Furthermore, the solution $A_{1,0}$ can actually be deduced from a certain linear combination of (shifted) versions of $A_{0,0}$. It has $A_{1,0}[0,m] = 0$ and $A_{1,0}[1,m] = 1$ for all $m \geq 0$ and $A_{1,0}[l,0] = \delta(l-1)$. The corner boundary conditions formed by column $l = 1$ and row $m = 0$ can replicated by taking column $l = 1$ of $A_{0,0}$ and adding column $l = 0$ of $A_{0,0}$ to it. Continuing to propagate this superposition of boundary conditions shows

$$A_{1,0}\,[l,m] = H[l-1]\,\{A_{0,0}\,[l-1,m] + A_{0,0}\,[l,m]\}, \tag{33}$$

whence from (34)

$$A_{i,0}\,[l,m] = H[l-i]\,\{A_{0,0}\,[l-i,m] + A_{0,0}\,[l+1-i,m]\}. \tag{34}$$

Taken altogether, we have shown

Proposition 6. *The general solution to the recurrence* (21) *with corner boundary values* $a_{0,0}$*,* $a_{0,m}$ *and* $a_{l,0}$ *is given by*

(35)

$$a_{l,m} = a_{0,0}\,A_{0,0}\,[l,m] + \sum_{i=1}^{l} a_{i,0}\,H[l-i]\,A_{1,0}\,[l+1-i,m] + \sum_{i=1}^{m} a_{0,i}\,H[m-i]\,A_{1,0}\,[m+1-i,l]\,, \quad l,m \geq 0,$$

where $A_{0,0}$ *is the array defined by* (24) *and* (26) *and* $A_{1,0}$ *is the array defined by* (33)*. Note that symmetric boundary conditions along the axes produce a symmetric solution* $a_{l,m} = a_{m,l}$*, as expected.*

3.2. **Exponential growth of the ambient degree.** We remark that the sequence $a_{l,m}$ is well known as the Delannoy numbers. Apart from the *ad hoc* approach to our solutions above, an alternative way to study the recurrence (21) is as a linear partial difference equation with constant coefficients, for which the method of bivariate generating functions applies [**?**, Chapter 6.4],[?]. Generating functions for the full double sequence and diagonal (central) sequence with Case I boundary conditions are known:

$$\begin{aligned} A(x,y) &= \sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a_{l,m}\, x^l y^m = \frac{1}{1-x-y-xy} \\ D(x) &= \sum_{m=0}^{\infty} a_{m,m}\, x^m = \frac{1}{\sqrt{1-6x+x^2}}. \end{aligned}$$

These lead to the asymptotics

$$a_{m,m} \sim \frac{\cosh(\frac{\log 2}{4})}{\sqrt{\pi}}\,(3+2\sqrt{2})^m\,\frac{1}{\sqrt{m}}$$

whence the diagonal entropy

$$\epsilon = \lim_{m\to\infty}\frac{1}{m}\log(a_{m,m}) = \log(3+2\sqrt{(2)}) = 1.76.. \tag{36}$$

4. Integrable lattice rules-vanishing entropy

In this section, we identify linear recurrences for the degree of the greatest common divisor at each vertex for the lattice equations of Table 2. Together with the recurrence (21), this helps to furnish us with a linear partial difference equation for $\bar{d}_{l,m}$ of (11), whose growth is ultimately our interest. Sometimes, we can solve this recurrence and conclude polynomial growth of degree in the vertex coordinates l and m, or exponential growth.

4.1. **Exponentially growing gcd: the inherited gcd and the spontaneous gcd.** As mentioned in the previous section, we use Maple to calculate the polynomials $x_{l,m}(w)$ and $z_{l,m}(w)$ for $0 \le l, m \le 9$ and Case I or Case II boundary values (22)-(23) with random integer coefficients chosen in the interval $[0, 400]$. Now, however, we are interested in the common factor of these polynomials so we follow the procedure outlined in Remark 4 of Section 2. For all equations except KdV of Table 1, the degree array $g_{l,m}$ of the gcd (see (12) for Case I boundary values (22) is given by

$$g^{I}_{l,m} = \begin{bmatrix} 0 & 0 & 14 & 140 & 826 & 3640 & 13062 & 40068 & 108530 & 265712 \\ 0 & 0 & 12 & 108 & 568 & 2232 & 7172 & 19812 & 48624 & 108530 \\ 0 & 0 & 10 & 80 & 370 & 1280 & 3642 & 8976 & 19812 & 40068 \\ 0 & 0 & 8 & 56 & 224 & 672 & 1672 & 3642 & 7172 & 13062 \\ 0 & 0 & 6 & 36 & 122 & 312 & 672 & 1280 & 2232 & 3640 \\ 0 & 0 & 4 & 20 & 56 & 122 & 224 & 370 & 568 & 826 \\ 0 & 0 & 2 & 8 & 20 & 36 & 56 & 80 & 108 & 140 \\ 0 & 0 & 0 & 2 & 4 & 6 & 8 & 10 & 12 & 14 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{37}$$

and for Case II boundary values (23) is

$$(38) \qquad g^{II}_{l,m} = \begin{bmatrix} 0 & 0 & 144 & 1104 & 5568 & 22272 & 75408 & 224016 & 598272 & 1462400 \\ 0 & 0 & 112 & 784 & 3584 & 12992 & 39984 & 108432 & 265600 & 598272 \\ 0 & 0 & 84 & 532 & 2184 & 7112 & 19740 & 48540 & 108432 & 224016 \\ 0 & 0 & 60 & 340 & 1240 & 3592 & 8916 & 19740 & 39984 & 75408 \\ 0 & 0 & 40 & 200 & 640 & 1632 & 3592 & 7112 & 12992 & 22272 \\ 0 & 0 & 24 & 104 & 288 & 640 & 1240 & 2184 & 3584 & 5568 \\ 0 & 0 & 12 & 44 & 104 & 200 & 340 & 532 & 784 & 1104 \\ 0 & 0 & 4 & 12 & 24 & 40 & 60 & 84 & 112 & 144 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The zero entries in the first row and column of $g^I_{l,m}$ and $g^{II}_{l,m}$ reflect the generic case of no common factors in the boundary values, i.e.

$$(39) \qquad g_{l,0} = g_{0,m} = 0, \quad l, m \geq 0.$$

The entries of $g^I_{l,m}$ and $g^{II}_{l,m}$ illustrate the gcd properties Part 3. (a-d) of Proposition 2. The numbers on the diagonal in both cases appear to be growing exponentially with an exponent $\simeq 1.7$. This is close to the rate of exponential growth of the maximal degree on the diagonal (36) and exceeds the minimum exponential growth of $\log(3) \simeq 1.099$ expected by Part 3.(d). This is because of spontaneous new additions to $\mathrm{gcd}_{l+1,m+1}(w)$ that augment the product $\mathrm{gcd}_{l,m}(w)\ \mathrm{gcd}_{l+1,m}(w)\ \mathrm{gcd}_{l,m+1}(w)$ inherited from the 3 other vertices of the lattice square. We call the latter product the *inherited* gcd and we use the term *spontaneous* gcd for $\overline{\mathrm{gcd}}_{l+1,m+1}(w)$ of (17) and (20) for $l, m \geq 0$. Defining

$$(40) \qquad \overline{g}_{l+1,m+1} = \deg(\overline{\mathrm{gcd}}_{l+1,m+1}) \geq 0,$$

we have from (20):

$$(41) \qquad \overline{g}_{l+1,m+1} = g_{l+1,m+1} - g_{l,m} - g_{l+1,m} - g_{l,m+1}, \quad l, m \geq 0.$$

Remark 7. *Given the generic boundary conditions* (39)*, it follows that there must occur some spontaneous* gcd *at some vertex* (l, m) *– like the quadratic* $\mathrm{gcd}_{3,2}(w)$ *in* $g^I_{l,m}$ *– in order to get a non-trivial* gcd *on the lattice (equivalently, to get a positive* $g_{l,m}$*). Once this occurs once at some vertex* (l, m)*, Part 3 of Proposition 2 shows that it propagates. In fact, Part 3.(d) shows that* $g_{l+i,m+i} = 3^i\, g_{l,m}$*, i.e. the* gcd *propagates exponentially with exponent* $\ln 3 \simeq 1.099\ldots$*. This is a lower exponential rate than* (**??**) *so that a finite number of occurrences of a spontaneous* gcd *throughout the lattice are not enough to give polynomial growth for* $\bar{d}_{l,m}$*. Continual and sustained occurrences of a spontaneous* gcd *are guaranteed if there is a factorization over a small square of the lattice - see below.*

Using (41), we can calculate the array of spontaneous gcd degrees corresponding to (37) and (38) – see the left hand side of, respectively, Figures 3 and 4.1. Using (24) and the gcd values (37) and (38), we can also calculate the reduced degree values (13) in each case - see the right hand side of Figures 3 and 4.1. These arrays are found to be the same for all rules of Table 2, excepting KdV.

The spontaneous gcd degrees reveal more apparent structure than their unbarred counterparts. In (42), $\overline{g}^I_{l+1,m+1} = \overline{g}^I_{l,m} + 4$, once a non-zero entry appears on a diagonal $m - l \in \mathbb{N}$. In (43), $\overline{g}^{II}_{l+1,m} = \overline{g}^{II}_{l,m} + 4(m-1)$ for $l, m \geq 2$. We also observe from our numerical experiments:

$$\bar{g}^{I}_{l,m} = \begin{bmatrix} 0 & 0 & 2 & 6 & 10 & 14 & 18 & 22 & 26 & 28 \\ 0 & 0 & 2 & 6 & 10 & 14 & 18 & 22 & 24 & 26 \\ 0 & 0 & 2 & 6 & 10 & 14 & 18 & 20 & 22 & 22 \\ 0 & 0 & 2 & 6 & 10 & 14 & 16 & 18 & 18 & 18 \\ 0 & 0 & 2 & 6 & 10 & 12 & 14 & 14 & 14 & 14 \\ 0 & 0 & 2 & 6 & \boxed{\mathbf{8}} & 10 & 10 & 10 & 10 & 10 \\ 0 & 0 & 2 & \boxed{\mathbf{4}} & 6 & 6 & 6 & 6 & 6 & 6 \\ 0 & 0 & 0 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \bar{d}^{I}_{l,m} = \begin{bmatrix} 0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 \\ 0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 15 \\ 0 & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 13 & 13 \\ 0 & 1 & 3 & 5 & 7 & 9 & 11 & 11 & \underline{\mathit{11}} & \mathbf{11} \\ 0 & 1 & 3 & 5 & 7 & 9 & 9 & \mathbf{9} & 9 & \underline{\mathit{9}} \\ 0 & 1 & 3 & 5 & 7 & 7 & 7 & \underline{\mathit{7}} & \mathbf{7} & 7 \\ 0 & 1 & \textcircled{3} & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\ 0 & 1 & 3 & \textcircled{3} & 3 & 3 & 3 & 3 & 3 & 3 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{42}$$

FIGURE 2. Array of spontaneous gcd degrees defined by (41) (left) and reduced degrees defined by (13) (right) for the equation H_1 of Table 2 and boundary values Case I. The two numbers enclosed by boxes in the left array and by circles in the right array illustrate the relationship (44). In the right array, the three numbers in bold and the three numbers in underlining/italic in $d^{I}_{l,m}$ illustrate the recurrence (51)

$$\bar{g}^{II}_{l,m} = \begin{bmatrix} 0 & 0 & 32 & 64 & 96 & 128 & 160 & 192 & 224 & 256 \\ 0 & 0 & 28 & 56 & 84 & 112 & 140 & 168 & 196 & 224 \\ 0 & 0 & 24 & 48 & 72 & 96 & 120 & 144 & 168 & 192 \\ 0 & 0 & 20 & 40 & 60 & 80 & 100 & 120 & 140 & 160 \\ 0 & 0 & 16 & 32 & 48 & 64 & 80 & 96 & 112 & 128 \\ 0 & 0 & 12 & 24 & 36 & \boxed{\mathbf{48}} & 60 & 72 & 84 & 96 \\ 0 & 0 & 8 & 16 & \boxed{\mathbf{24}} & 32 & 40 & 48 & 56 & 64 \\ 0 & 0 & 4 & 8 & 12 & 16 & 20 & 24 & 28 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \bar{d}^{II}_{l,m} = \begin{bmatrix} 1 & 19 & 37 & 55 & 73 & 91 & 109 & 127 & 145 & 163 \\ 1 & 17 & 33 & 49 & 65 & 81 & 97 & 113 & 129 & 145 \\ 1 & 15 & 29 & 43 & 57 & 71 & 85 & 99 & 113 & 127 \\ 1 & 13 & 25 & 37 & 49 & 61 & 73 & 85 & 97 & 109 \\ 1 & 11 & 21 & 31 & 41 & 51 & 61 & 71 & 81 & 91 \\ 1 & 9 & 17 & 25 & 33 & 41 & 49 & 57 & \underline{\mathit{65}} & \mathbf{73} \\ 1 & 7 & 13 & \textcircled{19} & 25 & 31 & 37 & \mathbf{43} & 49 & \underline{\mathit{55}} \\ 1 & 5 & 9 & 13 & \textcircled{17} & 21 & 25 & \underline{\mathit{29}} & \mathbf{33} & 37 \\ 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 & 17 & 19 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{43}$$

FIGURE 3. Array of spontaneous gcd degrees defined by (41) (left) and reduced degrees defined by (13) (right) for the equation H_1 of Table 2 and boundary values Case II. The two numbers enclosed by boxes in the left array and by circles in the right array illustrate the relationship (44). In the right array, the three numbers in bold and the three numbers in underlining/italic in $d^{II}_{l,m}$ illustrate the recurrence (51)

Observation 8. *For the lattice equations of Table 2 and the boundary conditions Case I and Case II, we have found that*

$$\bar{g}_{l,m} + \bar{g}_{l+1,m+1} = 2(\bar{d}_{l,m-1} + \bar{d}_{l-1,m}) \tag{44}$$

holds for all equations excepting KdV for $l, m \geq 2$. For KdV, it holds for $l, m \geq 3$.

It will turn out that (44) is a consequence of Theorem 10 of the next section.

4.2. **Polynomial growth of integrable lattice rules.** In remark 7 above, we made the point that sustained occurrences of a spontaneous gcd as we iterate the rule in the first quadrant was a necessary condition for subexponential growth of $\bar{d}_{l,m}$. An important observation of [7] provides a mechanism for this and one that can be verified quickly for a rule. From [7], it is known that for equations in the ABS list and for any 2×2 lattice square $[l-1,l+1]\times[m-1,m+1]$, we obtain a common factor $A_{l+1,m+1}$ of $x_{l+1,m+1}$ and $z_{l+1,m+1}$ of (5)-(6) for *arbitrary* initial values at the 5 corner sites $\{(l-1,m-1),(l-1,m),(l-1,m+1),(l,m-1),(l+1,m-1)\}$, see figure 4. For most equations, as indicated in the figure, the common factor $A_{l+1,m+1}$ can actually be written in terms of coordinates x and z at just the 2 sites $(l-1,m)$ and $(l,m-1)$. This factorization property over any 2×2 lattice square was used to search for new integrable equations in [7]. Table 3 gives the details of $A_{l,m}$ for equations in the ABS list and some other integrable equations given by Hietarinta and Viallet [7] (retaining their notation for these equations).

Q_1	$\beta^2\delta z_{l,m-1}^2 z_{l-1,m}^2 + \alpha^2\delta z_{l,m-1}^2 z_{l-1,m}^2 - 2\beta\alpha\delta z_{l,m-1}^2 z_{l-1,m}^2 + 2 z_{l-1,m} z_{l,m-1} x_{l-1,m} x_{l,m-1} - z_{l,m-1}^2 x_{l-1,m}^2 - z_{l-1,m}^2 x_{l,m-1}^2$
Q_2	$z_{l-1,m}^2 x_{l,m-1}^2 - 2\alpha^2 z_{l-1,m}^2 z_{l,m-1} x_{l,m-1} + \alpha^4 z_{l-1,m}^2 z_{l,m-1}^2 - 4\beta\alpha^3 z_{l-1,m}^2 z_{l,m-1}^2 + 4\beta\alpha z_{l,m-1} z_{l-1,m}^2 x_{l,m-1} + 6\beta^2\alpha^2 z_{l-1,m}^2 z_{l,m-1}^2 - 4\beta^3\alpha z_{l,m-1}^2 z_{l-1,m}^2 - 2\beta^2 z_{l,m-1} z_{l-1,m}^2 x_{l,m-1} + \beta^4 z_{l-1,m}^2 z_{l,m-1}^2 - 2 z_{l-1,m} z_{l,m-1} x_{l-1,m} x_{l,m-1} + z_{l,m-1}^2 x_{l-1,m}^2 + 4\beta\alpha z_{l,m-1}^2 z_{l-1,m} x_{l-1,m} - 2\alpha^2 z_{l,m-1}^2 z_{l-1,m} x_{l-1,m} - 2\beta^2 z_{l,m-1}^2 z_{l-1,m} x_{l-1,m}$
Q_3	$\alpha^4\delta^2 z_{l-1,m}^2 z_{l,m-1}^2 - 4\beta\alpha^3 z_{l-1,m} z_{l,m-1} x_{l-1,m} x_{l,m-1} - 2\beta^2\alpha^2\delta^2 z_{l-1,m}^2 z_{l,m-1}^2 + 4\beta^2\alpha^2 z_{l,m-1}^2 x_{l-1,m}^2 + 4\beta^2\alpha^2 z_{l-1,m}^2 x_{l,m-1}^2 - 4\beta^3\alpha z_{l-1,m} z_{l,m-1} x_{l-1,m} x_{l,m-1} + \beta^4\delta^2 z_{l-1,m}^2 z_{l,m-1}^2$
H_1	$(x_{l-1,m} z_{l,m-1} - z_{l-1,m} x_{l,m-1})^2$
H_2	$(x_{l,m-1} z_{l-1,m} - z_{l,m-1} x_{l-1,m} + (\alpha-\beta) z_{l,m-1} z_{l-1,m})(-x_{l,m-1} z_{l-1,m} + z_{l,m-1} x_{l-1,m} + (\alpha-\beta) z_{l,m-1} z_{l-1,m})$
H_3	$(\alpha x_{l-1,m} z_{l,m-1} - \beta x_{l,m-1} z_{l-1,m})(\alpha z_{l-1,m} x_{l,m-1} - \beta z_{l,m-1} x_{l-1,m})$
mKdV	$(x_{l,m-1} z_{l-1,m} + z_{l,m-1}\beta x_{l-1,m})(x_{l-1,m} z_{l,m-1} + \alpha z_{l-1,m} x_{l,m-1})$
sG	$(x_{l,m-1} x_{l-1,m} + \alpha z_{l,m-1} z_{l-1,m})(\alpha x_{l,m-1} x_{l-1,m} + \beta z_{l,m-1} z_{l-1,m})$
E16	$(z_{l-1,m} x_{l,m-1} + p_5 p_3 z_{l,m-1} x_{l-1,m})(p_1 p_3 p_5 p_6 x_{l-1,m} z_{l,m-1} - p_1 p_2 p_3 x_{l-1,m} z_{l,m-1} + p_2 p_5 p_6 x_{l-1,m} z_{l,m-1} - p_2^2 z_{l,m-1} x_{l-1,m} + p_1 p_6 z_{l-1,m} x_{l,m-1})$
E21	$(p_3 x_{l,m-1} z_{l-1,m} + p_5 z_{l,m-1} x_{l-1,m})(p_5 x_{l,m-1} z_{l-1,m} + p_3 z_{l,m-1} x_{l-1,m})$
E22	$(-z_{l,m-1} x_{l-1,m} + r_1 z_{l,m-1} z_{l-1,m} + x_{l,m-1} z_{l-1,m})(-x_{l,m-1} z_{l-1,m} + z_{l,m-1} x_{l-1,m} + r_1 z_{l,m-1} z_{l-1,m})$
E24	$-2 p_2 p_3 z_{l-1,m} z_{l,m-1} x_{l-1,m} x_{l,m-1} + p_2^2 p_3 s^2 z_{l,m-1}^2 z_{l-1,m}^2 + p_2 p_3^2 z_{l,m-1}^2 z_{l-1,m}^2 s^2 + x_{l-1,m} z_{l,m-1} x_{l,m-1} z_{l-1,m} - p_2 p_3 z_{l,m-1}^2 x_{l-1,m}^2 - p_2 p_3 z_{l-1,m}^2 x_{l,m-1}^2 - p_3^2 x_{l,m-1}^2 z_{l-1,m}^2 - 2 p_3^2 z_{l-1,m} z_{l,m-1} x_{l-1,m} x_{l,m-1} - p_2^2 z_{l-1,m} z_{l,m-1} x_{l-1,m} x_{l,m-1} - p_3^2 z_{l,m-1}^2 x_{l-1,m}^2$
E25	$(s z_{l-1,m} z_{l,m-1} - z_{l,m-1} x_{l-1,m} + z_{l-1,m} x_{l,m-1})(p_3^2 s z_{l,m-1} z_{l-1,m} + p_3 s z_{l-1,m} z_{l,m-1} - z_{l-1,m} z_{l,m-1} r_2 + 2 r_4 p_3 z_{l-1,m} z_{l,m-1} + r_4 z_{l-1,m} z_{l,m-1} - p_3 z_{l-1,m} x_{l,m-1} - p_3^2 z_{l-1,m} x_{l,m-1} + p_3 z_{l,m-1} x_{l-1,m} + p_3^2 z_{l,m-1} x_{l-1,m})$

TABLE 3. List of common factors $A_{l+1,m+1}$ of $x_{l+1,m+1}$ and $z_{l+1,m+1}$ of the rules of Table 2 for *arbitrary* initial values at the 5 corner sites $\{(l-1,m-1),(l-1,m),(l-1,m+1),(l,m-1),(l+1,m-1)\}$ of the 2×2 lattice square – see also figure 4. It is noted that KdV is not included in this list as the first common factors appear over a 2×3 lattice square and a 3×2 lattice square.

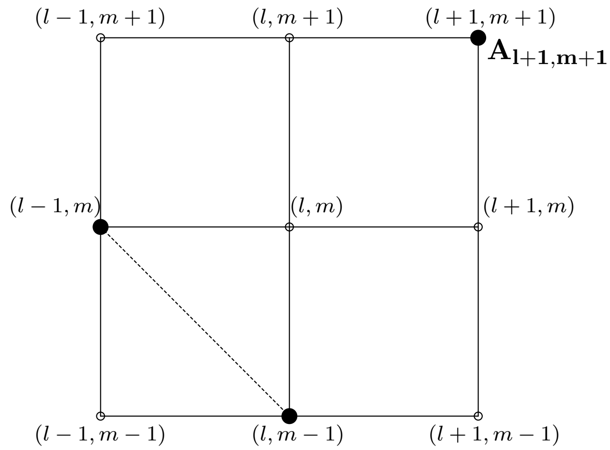

FIGURE 4. Some lattice equations produce a common factor $A_{l+1,m+1}$ of $x_{l+1,m+1}$ and $z_{l+1,m+1}$ over any 2×2 square that often only depends on x and z at the 2 sites $(l-1, m)$ and $(l, m-1)$.
.

It should be noted that for all equations in the ABS list, the common factor A is 'quartic' in terms of the off diagonal variables of the first square. It is also known that Q_V of Table 1 has vanishing entropy [18] and that all the equations in the ABS list can be obtained from this equation by choosing the parameters appropriately. Therefore, one should check whether Q_V fits in this framework. It is shown that Q_V has a factor at the point $(2, 2)$ [18] and in general the common factor is

$$\begin{aligned} A := &\left(p_1p_5 - {p_2}^2\right) {x_{l,m-1}}^2 {x_{l-1,m}}^2 + (p_1p_6 - p_2p_3 - p_2p_4 + p_2p_5)\, {x_{l,m-1}}^2 x_{l-1,m} z_{l-1,m} \\ &+ (p_2p_6 - p_3p_4)\, {x_{l,m-1}}^2 {z_{l-1,m}}^2 + (p_1p_6 - p_2p_3 - p_2p_4 + p_2p_5)\, x_{l,m-1} {x_{l-1,m}}^2 z_{l,m-1} \\ &+ \left(p_1p_7 - {p_3}^2 - {p_4}^2 + {p_5}^2\right) x_{l,m-1} x_{l-1,m} z_{l,m-1} z_{l-1,m} \\ &+ (p_2p_7 - p_3p_6 - p_4p_6 + p_5p_6)\, x_{l,m-1} z_{l,m-1} {z_{l-1,m}}^2 + (p_2p_6 - p_3p_4)\, {x_{l-1,m}}^2 {z_{l,m-1}}^2 \\ &+ (p_2p_7 - p_3p_6 - p_4p_6 + p_5p_6)\, x_{l-1,m} {z_{l,m-1}}^2 z_{l-1,m} + \left(p_5p_7 - {p_6}^2\right) {z_{l,m-1}}^2 {z_{l-1,m}}^2 \end{aligned}$$

Assuming corner boundary values as polynomials in an indeterminate w, such as (22) and (23), we assume that at the point $(l+1, m+1)$:

- $\mathrm{gcd}_{l,m}(w)\, \mathrm{gcd}_{l+1,m}(w)\, \mathrm{gcd}_{l,m+1}(w) \mid \mathrm{gcd}_{l+1,m+1}(w)$,
- $A_{l,m}(w) \mid \mathrm{gcd}_{l+1,m+1}(w)$

where all quantities belong to the ring of integer polynomials $\mathbb{Z}[w]$. The first statement is part 3 of Proposition 2. From standard divisibility results, suppressing the w dependence for brevity, we have :

$$\mathrm{lcm}(\mathrm{gcd_{l,m}}\, \mathrm{gcd_{l+1,m}}\, \mathrm{gcd_{l,m+1}},\ \mathrm{A_{l,m}}) = \frac{\mathrm{gcd_{l,m}}\, \mathrm{gcd_{l+1,m}}\, \mathrm{gcd_{l,m+1}}\ \ \mathrm{A_{l,m}}}{\mathrm{gcd}(\mathrm{gcd_{l,m}}\, \mathrm{gcd_{l+1,m}}\, \mathrm{gcd_{l,m+1}},\ \mathrm{A_{l,m}})} \mid \mathrm{gcd_{l+1,m+1}}. \tag{45}$$

Motivated by the form of the divisor on the rhs of (45), we define the sequence $G_{l,m}$ by the recurrence:

$$G_{l+1,m+1} = \frac{G_{l,m}\ G_{l+1,m}\ G_{l,m+1}\ A_{l,m}}{\gcd(G_{l,m}\ G_{l+1,m}\ G_{l,m+1},\ A_{l,m})}, \quad l,m \geq 2. \tag{46}$$

We take Case II boundary values of (23) for $l,m \leq 9$ such that $\gcd_{l,m} = 1$ if $l \leq 1$ or $m \leq 1$, and we define $G_{l,m} = \gcd_{l,m} = 1$ if $l \leq 1$ or $m \leq 1$. We compare the results of iterating the lattice equations of Table 3 plus Q_V, to find $\gcd_{l+1,m+1}$ (which involves factoring $x_{l,m}$ and $z_{l,m}$ at every site) to $G_{l,m}$ calculated by iterating (46) (which avoids needing to factor at each site). We find

$$G_{l,m}(w) = \gcd_{l,m}(w) \cdot C_{l,m}, \quad 2 \leq l,m \leq 9, \tag{47}$$

where $C_{l,m}$ is a lattice-dependent constant that depends on the equation. This continued to be the case even when we took initial values to be non-homogenous, i.e., initial values $x_{0,m}, z_{0,m}$ and $x_{l,0}, z_{l,0}$ were generated randomly in $\mathbb{Z}[w]$ with degrees between 1 and 4.

We now have a closer look at the rhs of (45). From Proposition 2 and the form of $A_{l+1,m+1}$ in Table 3 and figure 4, we can write

$$\gcd(\gcd_{l,m}\gcd_{l+1,m}\gcd_{l,m+1},\ A_{l+1,m+1}) = \gcd^2_{l-1,m}\gcd^2_{l,m-1}\ B_{l,m}. \tag{48}$$

Recalling also the definition of $\overline{\gcd}_{l,m}$ of (20), we have

$$\frac{\gcd_{l,m}\gcd_{l+1,m}\gcd_{l,m+1}\ A_{l,m}}{\gcd(\gcd_{l,m}\gcd_{l+1,m}\gcd_{l,m+1},\ A_{l,m})} = \frac{\gcd_{l-1,m-1}\ \gcd_{l+1,m}\ \gcd_{l,m+1}\ A_{l,m}}{\gcd_{l-1,m}\gcd_{l,m-1}}\ \frac{\overline{\gcd}_{l,m}}{B_{l,m}}. \tag{49}$$

Motivated by the first term on the rhs of (49), we consider the following recurrence

$$G'_{l+1,m+1} = \frac{G'_{l-1,m-1}\ G'_{l+1,m}\ G'_{l,m+1}\ A_{l,m}}{G'_{l-1,m}\ G'_{l,m-1}} \tag{50}$$

Taking boundary values $G'_{l,m} = \gcd_{l,m} = 1$ if $l \leq 1$ or $m \leq 1$ and evaluating the values $G'_{l+1,m+1}$ for $2 \leq l,m \leq 9$. Using the recursive formula of $G_{l,m}$, we predict that $\overline{G}_{l,m}\overline{G}_{l+1,m+1} = \overline{A}_{l,m}$, where $\overline{A}_{l,m} = A_{l,m}/(G^2_{l-1,m}\ G^2_{l,m-1})$. Or in other words, we have the following:

Conjecture 9 (Enabling Conjecture)**.** *Given arbitrary Type I boundary values in the first quadrant and the recurrence* (50)*, we conjecture*

$$G'_{l,m}(w) = \gcd_{l,m}(w) \cdot D_{l,m} \implies \deg(G'_{l,m}(w)) = \deg(\gcd_{l,m}(w)) = g_{l,m},$$

where $D_{l,m}$ is a lattice-dependent constant that depends on the equation.

The conjecture leads to:

Theorem 10. *If the enabling conjecture 9 holds, the reduced degrees $\overline{d}_{l,m} = \deg(\overline{x}_{l,m}(w)) = \deg(\overline{z}_{l,m}(w))$ of the lattice equations of Table 3 satisfy the following linear partial difference equation with constant coefficients:*

$$\overline{d}_{l+1,m+1} = \overline{d}_{l+1,m} + \overline{d}_{l,m+1} + \overline{d}_{l-1,m-1} - \overline{d}_{l,m-1} - \overline{d}_{l-1,m}, \tag{51}$$

and the observed relationship (44) *between the spontaneous* gcd *and the reduced degrees holds. All the equations given in Table 1 have vanishing entropy when the boundary values are Case I or II, since $\overline{d}_{l,m}$ grow linearly, respectively quadratically.*

Proof. From conjecture 9 and (50), $g_{l,m}$ satisfies the following recurrence for $l, m \geq 1$

$$g_{l+1,m+1} = 2(d_{l,m-1} + d_{l-1,m}) + g_{l-1,m-1} + g_{l+1,m} + g_{l,m+1} - g_{l-1,m} - g_{l,m-1}. \tag{52}$$

Adding this to (16) and its downshifted version gives (51). And replacing $g_{l,m}$ and $d_{l,m}$ using (41) and (13) gives (44).

Let $v_{l,m} = \overline{d}_{l+1,m+1} - \overline{d}_{l,m}$. Using equation (51), we have

$$v_{l,m} + v_{l-1,m-1} = v_{l-1,m} + v_{l,m-1}. \tag{53}$$

This is equivalent to $v_{l,m} - v_{l-1,m} = v_{l,m-1} - v_{l-1,m-1}$. Thus, we obtain $v_{l,m} - v_{l-1,m} = v_{l,0} - v_{l-1,0}$. Using this identity, we have

$$\begin{aligned} v_{l-1,m} - v_{l-2,m} &= v_{l-1,0} - v_{l-2,0}, \\ v_{l-2,m} - v_{l-3,m} &= v_{l-2,0} - v_{l-3,0}, \\ &\vdots \qquad\qquad \vdots \\ v_{1,m} - v_{0,m} &= v_{1,0} - v_{0,0}. \end{aligned}$$

It yields $v_{l,m} = v_{l,0} + v_{0,m} - v_{0,0}$. For $l > 0$, we have

$$\begin{aligned} v_{l,0} &= \overline{d}_{l+1,1} - \overline{d}_{l,0} = d_{l,1} + d_{l+1,0} \\ &= d_{l-1,1} + d_{l-1,0} + d_{l,0} + d_{l+1,0} \\ &= d_{0,1} + d_{0,0} + 2d_{1,0} + 2d_{2,0} \ldots + d_{l-1,0} + d_{l,0} + d_{l+1,0}. \end{aligned}$$

Similarly, for $m > 0$ we get

$$v_{0,m} = d_{1,0} + d_{0,0} + 2d_{0,1} + 2d_{0,2} + \ldots + 2d_{0,m-1} + d_{0,m} + d_{0,m+1},$$

and $v_{0,0} = d_{1,0} + d_{0,1}$.

- For the first case where initial values are constant except for the origin, we have $\overline{d}_{l,1} = \overline{d}_{1,m} = 1$ and $\overline{d}_{d,0} = \overline{d}_{0,m} = 0$ for all $l, m > 0$ and $d_{0,0} = 1$. It implies that $v_{0,0} = 0$ and $v_{l,0} = v_{0,m} = 1$ for all $l, m > 0$. Therefore $v_{l,m} = 2$ for all $l, m > 0$. It suggests that $\overline{d}_{l,m}$ grows linearly along the diagonal, i.e. along the direction from (l, m) to $(l + 1, m + 1)$.
- For the second case, we will show that $\overline{d}_{l,m}$ has quadratic growth along the diagonal by calculate the first difference of $v_{l,m}$ along the diagonal. We have for $l, m > 0$

  $$\begin{aligned} \Delta_{l,m} &= v_{l+1,m+1} - v_{l,m} \\ &= v_{l+1,0} - v_{l,0} + v_{0,m+1} - v_{0,m} = d_{l+2,0} + d_{l,0} + d_{0,m+2} + d_{0,m} \\ &= 4 \end{aligned}$$

  It implies that $\overline{d}_{l,m}$ has quadratic growth along the diagonal.

□

Remark 11. *It is noted that equation* (53) *satisfies consistency around the cube cf.* [3]

The coefficient 2 in (52) proves crucial in the proof of Theorem 10. It is a consequence of the fact that $A_{l+1,m+1}$ is quartic in its arguments. It will be shown in the next section that when instead $A_{l+1,m+1}$ is quadratic in its arguments, the growth of the reduced degree can be exponential.

4.3. **Polynomial growth of KdV and Mikhailov-Xenitidis equations.** Consider the KdV equation of Table 1 and the Mikhailov-Xenitidis (MX) equation [10] given by

$$(u + u_{12})u_1u_2 + 1 = 0. \tag{54}$$

The latter equation is known to be integrable as it has a Lax pair and can be obtained from the Tzitzeica equation. Although the KdV and MX equations have a different factorisation pattern compared to the other integrable equations mentioned in the previous subsection, Conjecture 9 and Theorem 10 still apply.

By using direct calculation, with arbitrary corner boundary values, we find that the first non-trivial gcd of $x_{l,m}$ and $z_{l,m}$ for KdV appears at the vertices $(2, 3)$ and $(3, 2)$ where $\text{gcd}_{2,3} = x_{1,1}$ and $\text{gcd}_{3,2} = x_{1,1}$. At the vertex $(3, 3)$, the gcd is given as follows

$$\text{gcd}_{3,3} = x^2_{1,2}\, x^2_{2,1}\, \text{gcd}_{2,3}\, \text{gcd}_{3,2}. \tag{55}$$

It is noted that this formula is quite large if we write it in terms of the boundary values.

For the MX equation, at the point $(2, 2)$ the common factor is $\text{gcd}_{2,2} = x_{1,0}\, x_{0,1}$, a quadratic rather than a quartic factor. However at the vertex $(3, 3)$ we also obtain (55). This shows that Therefore, for both equations one can try to consider $A_{3,3} = x^2_{1,2}x^2_{2,1}$ as a factor which plays the similar roll as the A given in the previous section. In general, for $l, m \geq 2$, $x^2_{l-1,m}\, x^2_{l,m-1}$ is a common factor of $x_{l+1,m+1}$ and $z_{l+1,m+1}$. We recover the setup of the previous subsection by taking for the KdV and MX equations:

$$A_{l+1,m+1} = x^2_{l-1,m}\, x^2_{l,m-1}, \quad l, m \geq 2.$$

That is, we replicate the situation of figure 4, except for the fact that the 2×2 squares over which the factorization emerges start one column in and one row up from the previous subsection. Numerical experiments support that conjecture 9 holds for $l, m \geq 2$ and the fact that $A_{l+1,m+1}$ is again quartic here implies Theorem 10, yielding polynomial growth for the KdV and MX equations.

5. Non integrable equations with factorization

In this section, we consider the equations of Table 1 below the double-line division. These equations also have some factorizations at the top right corner of any $(2, 2)$ lattice square, as in figure 4, but the degree of the factor is insufficient to prevent exponential growth of the reduced degrees as found heuristically in [7].

5.1. **Equations with "quadratic" factorization.** We consider the equations E20, E26, E27,E28 and E30 of Table 1 and present their factors $A_{l+1,m+1}$ in table 4. Replicating the approach of section 4.2, we have again that (45) holds and, as before, we find numerically that (46) reproduces $\text{gcd}_{l,m}(w)$ up to a lattice-dependent constant, analogous to (47). Because the $A_{l+1,m+1}$ of table 4 are quadratic in their arguments rather than quartic, we have (48) but without the exponent 2 on each *gcd*. Consequently, we were led to consider the following recurrence

$$G'_{l+1,m+1} = \frac{A_{l,m}G'_{l+1,m}G'_{l,m+1}G'_{l,m}}{G'_{l,m-1}G'_{l-1,m}}, \tag{56}$$

for $l, m \geq 2$ and $G'_{l,m} = \text{gcd}_{l,m}$ if $l < 2$ or $m < 2$.

Numerical experiments for different trials of boundary values on 7×7 lattice squares lead us to

$E20$	$p_6 x_{l,m-1} z_{l-1,m} + p_3 z_{l,m-1} x_{l-1,m} + p_3 p_6 r_4 z_{l,m-1} z_{l-1,m}$
$E26$	$p_1 r_1 z_{l,m-1} z_{l-1,m} + p_1 p_6 z_{l,m-1} x_{l-1,m} + p_3 p_6 x_{l,m-1} z_{l-1,m}$
$E27$	${p_1}^2 {p_6}^2 (p_3 - 1)^2 (p_3 z_{l,m-1} x_{l-1,m} + p_1 p_6 z_{l-1,m} x_{l,m-1} + r_4 z_{l,m-1} z_{l-1,m})$
$E28$	${p_6}^2 (-p_6 x_{l,m-1} z_{l-1,m} + p_6 r_3 z_{l-1,m} z_{l,m-1} + x_{l-1,m} z_{l,m-1} - p_3 x_{l-1,m} z_{l,m-1})$
$E30$	$x_{l-1,m} z_{l,m-1} + x_{l,m-1} z_{l-1,m} + r_4 z_{l,m-1} z_{l-1,m}$

TABLE 4. List of common factors $A_{l+1,m+1}$ of $x_{l+1,m+1}$ and $z_{l+1,m+1}$ of the indicated rules of Table 1 for *arbitrary* initial values at the 5 corner sites $\{(l-1,m-1),(l-1,m),(l-1,m+1),(l,m-1),(l+1,m-1)\}$ of the 2×2 lattice square – see also figure 4.

Conjecture 12 (Enabling Conjecture)**.** *Suppose Type I or Type II boundary values are given in the first quadrant such that $gcd_{l,m}(w) = 1$ for all $l < 2$ or $m < 2$. Then for $l, m \geq 2$, we find* $\gcd_{l,m}(w)$ *is a divisor of* $G'_{l,m}(w)$ *given by the recurrence* (56)*. For lattice rules E20, E26 and E27, we actually find* $\gcd_{l,m}(w)$ *agrees with* $G'_{l,m}(w)$ *up to a lattice-dependent constant.*

The conjecture leads to:

Theorem 13. *If the enabling conjecture 12 holds, the reduced degrees* $\overline{d}_{l,m} = \deg(\overline{x}_{l,m}(w)) = \deg(\overline{z}_{l,m}(w))$ *of the lattice equations of Table 4 are bounded below by the sequence* $\overline{d}'_{l,m}$ *satisfying the following linear partial difference equation with constant coefficients:*

$$\overline{d}'_{l+1,m+1} = \overline{d}'_{l+1,m} + \overline{d}'_{l,m+1} + \overline{d}'_{l,m} - \overline{d}'_{l,m-1} - \overline{d}'_{l-1,m}. \tag{57}$$

Consequently, the equations of Table 4 have non-vanishing entropy.

Proof. It follows directly from the formula (56) that the degree of $G'_{l,m}$ satisfies the recurrence

$$g'_{l+1,m+1} = (d_{l-1,m} + d_{l,m-1}) + g'_{l,m} + g'_{l+1,m} + g'_{l,m+1} - (g'_{l,m-1} + g'_{l-1,m}). \tag{58}$$

where $\overline{d}'_{l,m} = d_{l,m} - g'_{l,m}$. Denote $v_{l,m} = \overline{d}'_{l+1,m+1} - \overline{d}'_{l,m}$. Using (57) we obtain for $l, m > 0$

$$v_{l,m} = v_{l,m-1} + v_{l-1,m}, \tag{59}$$

which forms a Pascal's triangle. It implies that $v_{l,m} \geq v_{l,m-1}$ and $v_{l,m} \geq v_{l,m-1}$. Thus, we also have $v_{l,m-1} \geq v_{l-1,m-1}$ and $v_{l-1,m} \geq v_{l-1,m-1}$. Therefore, we get $v_{l,m} \geq v_{l-1,m-1}$. In particular, along the diagonal we obtain $v_{l,l} \geq 2^l v_{0,0}$. Since $v_{0,0} = \overline{d}'_{1,1} - \overline{d}'_{0,0} = d_{1,0} + d_{0,1} > 0$ for non-constant initial values at $(0,1)$ and $(1,0)$. Note that in the case of constant initial values, we have $v_{l,l} \geq 2^{l-1} v_{1,1}$ where $v_{1,1} > 0$. It shows that $v_{l,l} = \overline{d}'_{l+1,l+1} - \overline{d}'_{l,l}$ grows exponentially. Hence, $d'_{l,l}$ grows exponentially. □

We note that in the case where initial values are linear, we have found the following.

- We can write(57) as follows

$$\overline{d}'_{l+1,m+1} - \overline{d}'_{l,m+1} - \overline{d}'_{l+1,m} = \overline{d}'_{l,m} - \overline{d}'_{l-1,m} - \overline{d}'_{l,m-1}.$$

- It leads to $\overline{d}'_{l,m} = \overline{d}'_{l-1,m} + \overline{d}'_{l,m-1} + 1$

- For $l, m > 0$, we have

$$v_{l,m} = 2\binom{l+m+2}{m+1} - 2\binom{l+m}{m},$$
$$\overline{d}'_{l,m} = 2\binom{l+m}{m} - 1.$$

This can be proved by induction.

5.2. **Equation** $E17$. Finally, we consider $E17$ given in Table 1. This equation has "bigger factorization" than the non-integrable equations mentioned in the previous subsection. However, we will show that the factorization is not big enough to allow this equation to have vanishing entropy. It can be checked, with reference to figure 4, that the common factor of $x_{l+1,m+1}$ and $z_{l+1,m+1}$ is

(60)
$$A17_{l+1,m+1} = \left(p_4 x_{l-1,m-1} z_{l,m-1} + x_{l,m-1} p_6 z_{l-1,m-1} + r_4 z_{l-1,m-1} z_{l,m-1}\right) z_{l-1,m}{}^2 = z_{l-1,m} z_{l,m}.$$

It is important to note that factorization appears first at the point $(1,2)$, where $gcd_{1,2} = z_{0,1}$. We know that $gcd_{l+1,m}, gcd_{l,m+1}, z_{l,m}$ are devisors of $gcd_{l+1,m+1}$. We first used the test for $G_{l,m}$ as described in subsection 4.2 and we found that $G_{l,m}$ is a divisor of the actual $gcd_{l,m}$. However, we found that

$$\overline{d}_{l+1,m+1} = \overline{d}_{l+1,m} + \overline{d}_{l,m+1}. \quad (61)$$

It implies that

$$g_{l+1,m+1} = d_{l,m} + g_{l+1,m} + g_{l,m+1}.$$

Therefore, we predict that

$$gcd_{l+1,m+1} = z_{l,m} gcd_{l+1,m} gcd_{l,m+1}, \quad (62)$$

up to a constant. We have tested this recurrence and obtain the following conjecture.

Conjecture 14. *Given Case II boundary values, let $G_{l,m} = gcd_{l,m}$ for $l < 2$ or $m < 2$ and let $G_{l+1,m+1}$ be defined by the following recursive formula*

$$G_{l+1,m+1} = z_{l,m} G_{l+1,m} G_{l,m+1}. \quad (63)$$

Then $G_{l,m} = gcd_{l,m}$ up to a constant factor.

Using this conjecture, we have the following corollary

Corollary 15. *Given Case II boundary values, then equation $E17$ has exponential growth along diagonals, i.e. going from (l,m) to $(l+1,m+1)$.*

Proof. Using the recursive formula (61), we have $\overline{d}_{l+1,m+1} \geq 2\overline{d}_{l,m}$. This shows that $\overline{d}_{l,m}$ grows exponentially along the diagonal. □

We note that in the case where initial values are linear, for $l, m > 0$ one can prove

$$\overline{d}_{l,m} = \frac{2l+m}{l+m}\binom{l+m}{l}.$$

The proof was done by using induction. We note that the Lotka-Volterra equation which is given as follows

$$u_2\left(1+u\right) - u_1\left(1+u_{12}\right) = 0 \quad (64)$$

is a special case of $E17$, however the Lotka-Volterra equation has the vanishing entropy.

6. Discussion

We have presented two conjectures that have been used to obtain some results of growth of degrees of some integrable and non-integrable lattice equations including Q_V-the rational version of Q_4. Given an equation on the square, by looking at the factorization locally at the top right corner of any $(2, 2)$ square, one might be able to predict the integrability in the sense of having vanishing entropy. However, the question here is "how big is that factor in order to have vanishing entropy". We have given some examples in which the algebraic entropy is vanishing. It seems that for these equations, the common factor needs to be "quartic" in terms of the off diagonal variables of the first square. We also gave some examples where the common factor is "quadratic" in terms of the off diagonal variables of the first square. These equations turn out to be non-integrable as they have non-vanishing entropy. Therefore, it is worth studying this problem in the future.

In addition, there are some integrable equations where the factorization behaves in a more complicated way such as Tzitzeica and Lotka-Volterra equations. With these equations we need to go beyond the 2×2 square. We have not been able to provide any recursive formula of the gcd for these equations. Thus, it might be interesting to investigate this problem further. Furthermore, there are non-linear affine equations which are integrable in the sense of possessing a Lax pair [13]. It would be worth to study growth of degrees of these equations as well.

Acknowledgements

This work was supported by the Australian Research Council. We would like to thank Joshua Capel for useful discussions.

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