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LETTER TO THE EDITOR

Creating and relating three-dimensional integrable maps

John A G Roberts[1,2] **and G R W Quispel**[2,3]

[1] School of Mathematics and Statistics, University of New South Wales, Sydney, NSW 2052, Australia
[2] Centre of Excellence for Mathematics and Statistics of Complex Systems, La Trobe University, Melbourne VIC 3086, Australia
[3] Department of Mathematics, La Trobe University, Melbourne VIC 3086, Australia

E-mail: Jag.Roberts@unsw.edu.au and R.Quispel@latrobe.edu.au

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Abstract
We show how some integrable third-order difference equations recently given in the literature are related to one another by the process of interchanging parameters and integrals. Using the same process, we then create a 21-parameter family of integrable third-order difference equations that contains the previous examples as special cases. Our methodology illustrates that the combination of finding 2-integrals (i.e. integrals of the second iterate of the map), exploiting linear parameter dependence and using the interchange process provides a powerful way to relate and create higher-dimensional discrete integrable systems.

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1. Introduction

Integrable systems in general are studied for various reasons: for their intrinsic physical and mathematical interest, as a starting point for a perturbative approach, as tests for various numerical methods. Discrete integrable systems in particular are studied because of their fundamental mathematical nature and their applications to various areas of physics (including statistical mechanics and quantum gravity) and because sometimes they are discrete analogues of integrable systems in classical mechanics or solid state physics [4, 14, 16–18, 23].

This letter is concerned with integrable mappings, in particular those that can be written as difference equations. Integrable mappings of the plane were first introduced by McMillan [13], with some precursors in the work of Lyness [11]. The McMillan and Lyness maps are generalized by the so-called QRT map [17, 18], which contains a large number of parameters, but is still a map of $\mathbb{R}^2$ (or a second-order difference equation). In recent years, some extensions of integrable maps or difference equations to third (and higher) order have begun to appear [1, 2, 6–9, 14, 16, 22], but a comprehensive approach has been elusive.

Building on recent advances, this letter represents a first attempt at a more comprehensive approach to third-order integrable difference equations. In particular, it uses the technique of [19] extended to the concept of 2-integrals [5] to create new integrable maps that, in their most general form, are alternating [15]. Ultimately, we derive the fractional-linear third-order difference equation $\bar{L}$ of equation (52), which contains 21 parameters (6 of which alternate from one iterate to another). Its integrability derives from the possession of two integrals (42) and a measure-preservation property (62).

2. Interchange of 2-integrals and parameters to relate three existing integrable third-order difference equations

The following third-order difference equation was derived in [6]:

$$L : x_{n+3} = \frac{1}{x_n} \frac{p_3 x_{n+1} x_{n+2} + p_4(x_{n+1} + x_{n+2}) + p_5}{p_2 x_{n+1} x_{n+2} + p_1(x_{n+1} + x_{n+2}) + p_3}, \tag{1}$$

where $p_1, p_2, \ldots, p_5$ are arbitrary parameters. It has two integrals, as shown in [6], and is also (anti) measure preserving since

$$\det \mathrm{d}L = \frac{\partial x_{n+3}}{\partial x_n} = -\frac{m(x_n, x_{n+1}, x_{n+2})}{m(x_{n+1}, x_{n+2}, x_{n+3})} \tag{2}$$

with density $m(x, y, z) = (xyz)^{-1}$. Hence, it is integrable [5]. It turns out that underlying the existence of the two integrals are two 2-integrals, I_1 and I_2, given by

$$\begin{aligned} I_1(x_n, x_{n+1}, x_{n+2}) &= \frac{p_1 x_n x_{n+1} x_{n+2} + (p_3 x_{n+1} + p_4)(x_n + x_{n+2}) + p_4 x_{n+1} + p_5}{x_n x_{n+2}} \\ I_2(x_n, x_{n+1}, x_{n+2}) &= \frac{(p_2 x_{n+1} + p_1) x_n x_{n+2} + (p_1 x_{n+1} + p_3)(x_n + x_{n+2}) + p_4}{x_{n+1}}. \end{aligned} \tag{3}$$

It can be verified that

$$I_1(n+1) := I_1(x_{n+1}, x_{n+2}, x_{n+3}) = I_2(x_n, x_{n+1}, x_{n+2}) =: I_2(n), \tag{4}$$

$$I_2(n+1) := I_2(x_{n+1}, x_{n+2}, x_{n+3}) = I_1(x_n, x_{n+1}, x_{n+2}) =: I_1(n), \tag{5}$$

whence

$$I_1(n+2) = I_1(n) \qquad \text{and} \qquad I_2(n+2) = I_2(n). \tag{6}$$

The difference equation L has a time-reversal symmetry [20], being conjugate to its inverse via the involution

$$G : (x_n, x_{n+1}, x_{n+2}) \to (x_{n+2}, x_{n+1}, x_n). \tag{7}$$

A systematic method for finding third-order difference equations that possess 2-integrals, based upon assuming that the equations possess such time-reversal symmetry, will be detailed elsewhere [21]. For the moment, we note that the existence of the 2-integrals I_1 and I_2 is a more fundamental invariance property of L that implies the existence of the integrals. The 2-integrals are also substantially more compact building blocks than the integrals. Specifically, we find that the integrals H_1 and H_2 of L given in [6] can be expressed as[4]

[4] In [6], L is equation (Y1) of the appendix. We are grateful to Professor R Hirota and his collaborators for providing us with H_1 and H_2. During the final preparation of our manuscript, [12] appeared in which the 2-integrals I_1 and I_2 for (Y1) are also found, as are the 2-integrals for other integrable cases of [6]. This allows the authors of [12] to give certain reductions of the third-order equations of [6] to pairs of second-order equations.

$$H_1 = \frac{I_1 + I_2 - p_3 H_2}{p_4} \tag{8}$$

$$H_2 = \frac{I_1 I_2 - \left(4p_1 p_4 + 2p_3^2 + p_2 p_5\right)}{p_1 p_4}. \tag{9}$$

In [19], a process was described whereby new maps with integrals could be obtained from an original map with integrals. This process works by interchanging parameters with (the values of) integrals in the original map. It can be generalized to cover the case of k-integrals depending on parameters, as we now illustrate (a complete description of this will be given in [21]).

In particular, we can use the two 2-integrals to solve for any two parameters present in them. This is facilitated by the fact that the dependence of I_1 and I_2 upon their five parameters is *linear* (in stark contrast to the case of the integrals H_1 and H_2). Consider the equations[5]

$$I_1(n) = -\alpha(n) \qquad I_2(n) = -\beta(n), \tag{10}$$

where from (4)–(6)

$$\alpha(n+1) = \beta(n) \quad \text{and} \quad \beta(n+1) = \alpha(n) \Rightarrow \alpha(n+2) = \alpha(n) \quad \text{and} \quad \beta(n+2) = \beta(n). \tag{11}$$

Equations (10) define a linear system in the parameters $p_1, p_2, \ldots, p_5$. Arbitrarily distinguishing two of these, calling them K_1 and K_2, we can write

$$J(n)\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = -\begin{pmatrix} \alpha(n) + \tilde{I}_1(n) \\ \beta(n) + \tilde{I}_2(n) \end{pmatrix}. \tag{12}$$

Here,

$$J(n) := \begin{pmatrix} \frac{\partial I_1(n)}{\partial K_1} & \frac{\partial I_1(n)}{\partial K_2} \\ \frac{\partial I_2(n)}{\partial K_1} & \frac{\partial I_2(n)}{\partial K_2} \end{pmatrix} \tag{13}$$

and $\tilde{I}_i(n), i = 1, 2$ are the parts of $I_i(n)$ not involving K_1 or K_2. Solving (12) for K_1 and K_2, we obtain

$$K_1 =: k_1(x_n, x_{n+1}, x_{n+2}, \tilde{\boldsymbol{p}}, \alpha(n), \beta(n)), \tag{14}$$

$$K_2 =: k_2(x_n, x_{n+1}, x_{n+2}, \tilde{\boldsymbol{p}}, \alpha(n), \beta(n)), \tag{15}$$

where $\tilde{\boldsymbol{p}}$ represent the parameters not chosen as either K_1 or K_2. Similarly, the equations $I_1(n+1) = \alpha(n+1)$ and $I_2(n+1) = \beta(n+1)$ lead to the system

$$J(n+1)\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = -\begin{pmatrix} \beta(n) + \tilde{I}_1(n+1) \\ \alpha(n) + \tilde{I}_2(n+1) \end{pmatrix}, \tag{16}$$

noting the use of (11) to update the parameters on the right-hand side. Since K_1 and K_2 must simultaneously satisfy both (12) and (16), we conclude that the difference equation $\bar{L}$ formed from L by replacing, respectively, K_1 and K_2 with k_1 and k_2 has the integrals k_1 and k_2 (i.e. they equal their upshift). The map $\bar{L}$ and k_1 and k_2 now contain the n-dependent parameters $\alpha(n)$ and $\beta(n)$ satisfying from (11)

$$\alpha(n) = \alpha_0 + (-1)^n \alpha_1, \qquad \beta(n) = \alpha_0 - (-1)^n \alpha_1 \tag{17}$$

where α_0 and α_1 are constants.

[5] The negative sign is introduced simply for aesthetic reasons.

$$\begin{array}{ccc} & L & \\ \swarrow & & \searrow \\ L_1 & \longrightarrow & L_2 \end{array}$$

Figure 1. Summary of relations between the difference equations L, L_1 and L_2 established by interchanging parameters and 2-integrals, as described in the text. All interchanges can be inverted to go in the other direction.

In summary, the map L containing constant parameters and possessing two 2-integrals becomes the map $\bar{L}$ containing a 2-cycle of parameters and possessing two integrals. This interchange can be encoded as

$$\{(I_1, I_2); K_1, K_2\} \to \{k_1, k_2; (\alpha, \beta)\}, \tag{18}$$

where the semicolon in each bracket separates integrals to its left from parameters on its right, and the round brackets represent a 2-cycle in integrals or in parameters (in two dimensions, alternating versions of the QRT map with one integral have been recently studied in [15]).

To exemplify the above process, first consider solving (3) and (12) for $K_1 = p_5$ and $K_2 = p_2$. This yields

$$\begin{aligned} k_1 &= -\alpha(n)x_nx_{n+2} - p_1x_nx_{n+1}x_{n+2} - p_3x_{n+1}(x_n + x_{n+2}) - p_4(x_n + x_{n+1} + x_{n+2}), \\ k_2 &= -\frac{p_1(x_nx_{n+1} + x_nx_{n+2} + x_{n+1}x_{n+2}) + p_3(x_n + x_{n+2}) + p_4 + \beta(n)x_{n+1}}{x_nx_{n+1}x_{n+2}}, \end{aligned} \tag{19}$$

where $\alpha(n)$ and $\beta(n)$ are given by (17). With these replacements, $\bar{L}$ becomes

$$L_1 : x_{n+3} = x_n \frac{p_4 + p_3x_{n+1} + \alpha(n)x_{n+2} + p_1x_{n+1}x_{n+2}}{p_4 + p_3x_{n+2} + \beta(n)x_{n+1} + p_1x_{n+1}x_{n+2}}. \tag{20}$$

The third-order difference equation L_1 is a generalization of the one studied in [22]. It is measure preserving with density $m_1(x_n, x_{n+1}, x_{n+2}) = (x_nx_{n+1}x_{n+2})^{-1}$.

Another possibility is constructing (12) from (3) with $K_1 = p_5$ and $K_2 = p_4$. This yields

$$\begin{aligned} k_1 &= -\alpha(n)x_nx_{n+2} - p_1x_nx_{n+1}x_{n+2} - p_3x_{n+1}(x_n + x_{n+2}) - (x_n + x_{n+1} + x_{n+2})k_2, \\ k_2 &= -\beta(n)x_{n+1} - p_2x_nx_{n+1}x_{n+2} - p_1(x_nx_{n+2} + x_nx_{n+1} + x_{n+1}x_{n+2}) - p_3(x_n + x_{n+2}). \end{aligned} \tag{21}$$

With these replacements, $\bar{L}$ becomes

$$L_2 : x_{n+3} = x_n - \frac{p_3(x_{n+1} - x_{n+2}) - \beta(n)x_{n+1} + \alpha(n)x_{n+2}}{p_3 + p_1(x_{n+1} + x_{n+2}) + p_2x_{n+1}x_{n+2}}. \tag{22}$$

The equation L_2 is volume preserving and represents a generalization, with alternating parameters, of equation (12) of [8] (in particular, the latter is obtained by taking $p_1 = 0$ and $\alpha(n) = \beta(n) = \alpha_0$ in L_2). Note that both integrals of L_2 are now polynomials. It is also the case that L_2 can be obtained directly from L_1, an application of the interchange of parameters and integrals as originally advanced in [19]. This is achieved by solving the second equation of (19) for p_4 and renaming the value of k_2 there by p_2. Replacing for p_4 in L_1 then yields L_2. It is also seen that the expression obtained by solving for p_4 is precisely the integral k_2 in (21), whereas k_1 of (21) follows from k_1 of (19) with p_4 also replaced. Summarizing, the three integrable difference equations L, L_1 and L_2 can all be obtained from one another via the interchange process. In other words, any two can be generated from the third. This is highlighted schematically in figure 1.

3. Reparametrization and interchange of 2-integrals and parameters to create a new class of integrable third-order difference equations

In fuller generality, analogous to the case of integrals described in [19], we can *reparametrize* all parameters in L and its two 2-integrals, in terms of two parameters K_1 and K_2. That is, we let

$$p_i \to p_i^0 + p_i^1 K_1 + p_i^2 K_2, \qquad i = 1, \ldots, 5, \tag{23}$$

i.e. each parameter p_i is 'replaced' by three parameters p_i^0, p_i^1 and p_i^2 (note that the superscripts here *do not* denote powers). Similarly, we can reparametrize

$$I_1 + \alpha \to \left(I_1^0 + I_1^1 K_1 + I_1^2 K_2\right) + (\alpha^0 + \alpha^1 K_1 + \alpha^2 K_2), \tag{24}$$

$$I_2 + \beta \to \left(I_2^0 + I_2^1 K_1 + I_2^2 K_2\right) + (\beta^0 + \beta^1 K_1 + \beta^2 K_2). \tag{25}$$

In (24), I_1^0 is the 2-integral I_1 of L with $p_i \to p_i^0$ and

$$\alpha^j(n+1) = \beta^j(n) \qquad \text{and} \qquad \beta^j(n+1) = \alpha^j(n), \qquad j = 0, 1, 2. \tag{26}$$

It is seen that (24) and (25) incorporate the p_i reparametrization in terms of K_1 and K_2 as well as the 'expansion' of α and β, respectively, satisfying (26). This extra reparametrization is simply a reflection of the fact that the addition of an alternating parameter, or a linear combination of alternating parameters, to a 2-integral still gives a 2-integral of the map. At this point, the expanded p_i, α and β, together with K_1 and K_2, should all be considered parameters on an equal footing. Substituting these reparametrizations into (1) and (3), the linear dependence of L and of I_1 and I_2 on the parameters means that we can achieve the following trivial reformulation of the properties of L: the third-order difference equation

$$\hat{L} : x_{n+3} = \frac{(N_0, N_1, N_2) \cdot (1, K_1, K_2)}{(D_0, D_1, D_2) \cdot (1, K_1, K_2)} = \frac{\boldsymbol{N} \cdot \boldsymbol{K}}{\boldsymbol{D} \cdot \boldsymbol{K}} \tag{27}$$

possesses the 2-integrals

$$\hat{I}_1 = \left(I_1^0 + \alpha^0, I_1^1 + \alpha^1, I_1^2 + \alpha^2\right) \cdot (1, K_1, K_2) = \boldsymbol{I}_1 \cdot \boldsymbol{K} \tag{28}$$

$$\hat{I}_2 = \left(I_2^0 + \beta^0, I_2^1 + \beta^1, I_2^2 + \beta^2\right) \cdot (1, K_1, K_2) = \boldsymbol{I}_2 \cdot \boldsymbol{K}. \tag{29}$$

In (27), the vector $\boldsymbol{N} := (N_0, N_1, N_2)$, where N_0 is the numerator on the rhs of L with $p_i \to p_i^0$ etc, and similarly for $\boldsymbol{D}$, the corresponding vector built from copies of the denominator. The vector $\boldsymbol{K} := (1, K_1, K_2)$. For brevity, we define $\boldsymbol{I}_1 = (I_{10}, I_{11}, I_{12})$ and similarly for $\boldsymbol{I}_2$, where

$$I_{1j} := I_1^j + \alpha^j, \qquad I_{2j} := I_2^j + \beta^j, \qquad j = 0, 1, 2. \tag{30}$$

Now, as in the similar philosophy for the case of integrals as discussed in [19], we now take K_1 and K_2 as distinguished parameters chosen to satisfy $\hat{I}_1(n) = 0$ and $\hat{I}_2(n) = 0$ respectively. These two equations, the generalizations of (10), define a linear system (12) for K_1 and K_2 where now $J = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$ and the right-hand side of (12) is now $-\begin{pmatrix} I_{10} \\ I_{20} \end{pmatrix}$.[6] The ensuing discussion concerning generating a new difference equation by *replacing* K_1 and K_2 applies, this new difference equation preserving integrals k_1 and k_2 given by solving, respectively, for K_1 and K_2. The reparametrizations we used above will mean that this new difference equation is much more general than the examples discussed in section 2 as we have tripled the number of parameters. Ultimately, we will derive $\bar{L}$ of equation (52), which contains 21 arbitrary

[6] One sees that e.g. α^0 and I_1^0 now play the role of $\alpha(n)$ and $\tilde{I}_1(n)$, respectively, in (12).

parameters (6 of which alternate from one iterate to another). Nevertheless, the examples in section 2 will be special cases of our final result $\bar{L}$, for example, L_1 of (20) corresponds to taking $\{p_i^1 = p_i^2 = 0, i = 1, 3, 4; p_5^1 = p_2^2 = 1; p_5^0 = p_5^2 = p_2^0 = p_2^1 = 0; \alpha^i = \beta^i = 0, i = 1, 2\}$. Moreover, the original equation L of (1) corresponds to taking $\{p_i^1 = p_i^2 = 0, i = 1, \ldots, 5\}$ so that no replacement of parameters in L actually occurs (in this case, $J = \begin{pmatrix} \alpha^1 & \alpha^2 \\ \beta^1 & \beta^2 \end{pmatrix}$ so that the integrals k_1 and k_2 are, in general, certain L-invariant linear combinations of the 2-integrals I_1 and I_2). This further highlights the benefits of 'expanding' α and β in (24)–(26) to include α^i and β^i for $i = 1, 2$, as it allows the final result to include the starting point as a special case.

We proceed to the derivation of $\bar{L}$ of equation (52). From (28)–(29), we see that the solutions of $\hat{I}_1(n) = 0$ and $\hat{I}_2(n) = 0$ can be expressed vectorially as

$$\boldsymbol{k} := (1, k_1, k_2) = \begin{vmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{vmatrix}^{-1} (\boldsymbol{I}_1 \times \boldsymbol{I}_2), \tag{31}$$

and equivalently, by Cramer's rule, as

$$k_1 = -\frac{\begin{vmatrix} I_{10} & I_{12} \\ I_{20} & I_{22} \end{vmatrix}}{\begin{vmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{vmatrix}}, \qquad k_2 = -\frac{\begin{vmatrix} I_{11} & I_{10} \\ I_{21} & I_{20} \end{vmatrix}}{\begin{vmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{vmatrix}}. \tag{32}$$

The third-order difference equation $\bar{L}$ preserving these integrals follows from (27) with (31), i.e.

$$\bar{L} : x_{n+3} = \frac{\boldsymbol{N} \cdot \boldsymbol{I}_1 \times \boldsymbol{I}_2}{\boldsymbol{D} \cdot \boldsymbol{I}_1 \times \boldsymbol{I}_2}, \tag{33}$$

equivalently,

$$\bar{L} : x_{n+3} = \frac{\begin{vmatrix} N_0 & N_1 & N_2 \\ I_{10} & I_{11} & I_{12} \\ I_{20} & I_{21} & I_{22} \end{vmatrix}}{\begin{vmatrix} D_0 & D_1 & D_2 \\ I_{10} & I_{11} & I_{12} \\ I_{20} & I_{21} & I_{22} \end{vmatrix}}. \tag{34}$$

Note that equations (32) and (34) actually hold more generally than those derived from just (1). That is, suppose we are given an original third-order rational difference equation $L : x_{n+3} = \frac{N}{D}$, where N and D are general polynomial expressions in $\{x_n, x_{n+1}, x_{n+2}\}$ which have linear dependence on parameters. If, additionally, there are two 2-integrals I_1 and I_2 with linear dependence on parameters, (34) describes a new third-order rational difference equation preserving the integrals k_1 and k_2 given by (32) that can be obtained from the original by the processes of reparametrization and replacement.

Turning back to the specific starting points (1) and (3), we now give the explicit details of (31) and (34) in this case. The calculation of k_1, k_2 and $\bar{L}$ both involve the calculation of determinants whose entries are bilinear or trilinear forms, so it is useful to have a formalism for the products of such forms.

For $i \geqslant 1$, define the $(i + 1)$-dimensional vector

$$\boldsymbol{x}^i := \begin{pmatrix} x^i \\ x^{i-1} \\ \vdots \\ 1 \end{pmatrix}. \tag{35}$$

With A_i as an $(i+1)\times(i+1)$ matrix, $\boldsymbol{x}^i\cdot A_i\boldsymbol{z}^i$ defines a form of degree i in x and z. The multiplication of two forms can be achieved using the Kronecker product [3] (or tensor product) $\otimes$:

$$(\boldsymbol{x}^i\cdot A_i\boldsymbol{z}^i)(\boldsymbol{x}^j\cdot A_j\boldsymbol{z}^j)=(\boldsymbol{x}^i\otimes\boldsymbol{x}^j)\cdot(A_i\otimes A_j)(\boldsymbol{z}^i\otimes\boldsymbol{z}^j). \tag{36}$$

Note that $(\boldsymbol{x}^i\otimes\boldsymbol{x}^j)$ is $(i+1)(j+1)$-dimensional and the matrix $A_i\otimes A_j$ is $(i+1)(j+1)\times(i+1)(j+1)$. Here, we wish to use a certain contraction of the Kronecker product of A_i and A_j which we denote by $\boxtimes$. We define $A_i\boxtimes A_j$ in the following way:

$$(\boldsymbol{x}^i\cdot A_i\boldsymbol{z}^i)(\boldsymbol{x}^j\cdot A_j\boldsymbol{z}^j)=(\boldsymbol{x}^i\otimes\boldsymbol{x}^j)\cdot(A_i\otimes A_j)(\boldsymbol{z}^i\otimes\boldsymbol{z}^j)=:\boldsymbol{x}^{i+j}\cdot(A_i\boxtimes A_j)\boldsymbol{z}^{i+j}. \tag{37}$$

This uniquely defines $A_i\boxtimes A_j$ as a $(i+j+1)\times(i+j+1)$ matrix. Its entries come from certain sums of the entries in appropriate rows and columns of $A_i\otimes A_j$.[7] Note that $A_i\boxtimes A_j$ is a commutative operation, unlike $A_i\otimes A_j$. Nevertheless, many properties of the product $\boxtimes$ we use below can be inferred from those of $\otimes$.

The 2-integral $I_1+\alpha$ is a trilinear form in x_n, x_{n+1} and x_{n+2}. Two ways in which it can be written are

$$I_1+\alpha=\frac{\boldsymbol{x}_n^1\cdot A_1\boldsymbol{x}_{n+2}^1}{x_nx_{n+2}}=\frac{\boldsymbol{x}_{n+1}^1\cdot B_1\boldsymbol{x}_{n+2}^1}{x_nx_{n+2}}, \tag{38}$$

where $\boldsymbol{x}_n^1=\binom{x_n}{1}$ etc, following the notation of (35), and

$$A_1:=\begin{pmatrix}p_1x_{n+1}+\alpha & p_3x_{n+1}+p_4\\ p_3x_{n+1}+p_4 & p_4x_{n+1}+p_5\end{pmatrix},\qquad B_1:=\begin{pmatrix}p_1x_n+p_3 & p_3x_n+p_4\\ \alpha x_n+p_4 & p_4x_n+p_5\end{pmatrix}. \tag{39}$$

Similarly,

$$I_2+\beta=\frac{\boldsymbol{x}_n^1\cdot A_2\boldsymbol{x}_{n+2}^1}{x_{n+1}}=\frac{\boldsymbol{x}_{n+1}^1\cdot B_2\boldsymbol{x}_{n+2}^1}{x_{n+1}}, \tag{40}$$

with

$$A_2:=\begin{pmatrix}p_2x_{n+1}+p_1 & p_1x_{n+1}+p_3\\ p_1x_{n+1}+p_3 & \beta x_{n+1}+p_4\end{pmatrix},\qquad B_2:=\begin{pmatrix}p_2x_n+p_1 & p_1x_n+\beta\\ p_1x_n+p_3 & p_3x_n+p_4\end{pmatrix}. \tag{41}$$

The denominators in (38) and (40) are independent of parameters, and hence will cancel in the rational expressions for k_1 and k_2 in (32). Consequently, the entries in the determinants in the latter can be taken to be the bilinear forms in the numerators of the first expressions in (38) and (40), with the matrix A_i, $i=1,2$, replaced by e.g. A_i^0 in I_{i0} so as to depend on p_i^0 etc.

We obtain that the integrals k_1 and k_2 are ratios of symmetric triquadratic forms (in particular, biquadratic in x_n and x_{n+2}), which, using the product $\boxtimes$, we can write as

$$k_1=-\frac{\boldsymbol{x}_n^2\cdot\begin{vmatrix}A_1^0 & A_1^2\\ A_2^0 & A_2^2\end{vmatrix}_{\boxtimes}\boldsymbol{x}_{n+2}^2}{\boldsymbol{x}_n^2\cdot\begin{vmatrix}A_1^1 & A_1^2\\ A_2^1 & A_2^2\end{vmatrix}_{\boxtimes}\boldsymbol{x}_{n+2}^2}=-\frac{\boldsymbol{x}_n^2\cdot\left(M_2^{0,2}x_{n+1}^2+M_1^{0,2}x_{n+1}+M_0^{0,2}\right)\boldsymbol{x}_{n+2}^2}{\boldsymbol{x}_n^2\cdot\left(M_2^{1,2}x_{n+1}^2+M_1^{1,2}x_{n+1}+M_0^{1,2}\right)\boldsymbol{x}_{n+2}^2} \tag{42}$$

$$k_2=-\frac{\boldsymbol{x}_n^2\cdot\begin{vmatrix}A_1^1 & A_1^0\\ A_2^1 & A_2^0\end{vmatrix}_{\boxtimes}\boldsymbol{x}_{n+2}^2}{\boldsymbol{x}_n^2\cdot\begin{vmatrix}A_1^1 & A_1^2\\ A_2^1 & A_2^2\end{vmatrix}_{\boxtimes}\boldsymbol{x}_{n+2}^2}=-\frac{\boldsymbol{x}_n^2\cdot\left(M_2^{1,0}x_{n+1}^2+M_1^{1,0}x_{n+1}+M_0^{1,0}\right)\boldsymbol{x}_{n+2}^2}{\boldsymbol{x}_n^2\cdot\left(M_2^{1,2}x_{n+1}^2+M_1^{1,2}x_{n+1}+M_0^{1,2}\right)\boldsymbol{x}_{n+2}^2}. \tag{42}$$

[7] For example, when $i=2$, $j=1$, the 4×4 matrix $A_i\boxtimes A_j$ is derived from the 6×6 matrix $A_i\otimes A_j$ in the following way: (i) simultaneously merge, under addition, the second and third rows of the latter into one row and the fourth and fifth rows into another row, yielding an intermediary 4×6 matrix; (ii) convert the 4×6 matrix into $A_2\boxtimes A_1$ by repeating (i) on the corresponding columns.

Here, $|\cdots|_{\boxtimes}$ refers to a determinant with respect to the matrix product $\boxtimes$. This matrix determinant, and the linear dependence of (39) and (41) on the parameters, induces a (2×2)-determinantal structure in the parameters of each entry of the resulting symmetric 3×3 matrices defining each biquadratic form. For ease of notation, we define e.g.

$$31^{1,2} := \begin{vmatrix} p_3^1 & p_3^2 \\ p_1^1 & p_1^2 \end{vmatrix}, \qquad 1\beta^{0,2} := \begin{vmatrix} p_1^0 & p_1^2 \\ \beta^0 & \beta^2 \end{vmatrix}, \tag{43}$$

and the symmetric matrices $M_i^{k,l}$ take the form

$$M_2^{k,l} = \begin{pmatrix} 12 & 32 & 31 \\ \star & 42 + 1\beta + 2 \cdot 31 & 3\beta + 41 \\ \star & \star & 4\beta \end{pmatrix}^{k,l}$$

$$M_1^{k,l} = \begin{pmatrix} \alpha 2 & \alpha 1 + 42 & 41 \\ \star & \alpha\beta + 2 \cdot 41 + 52 & 4\beta + 51 \\ \star & \star & 5\beta \end{pmatrix}^{k,l}$$

$$M_0^{k,l} = \begin{pmatrix} \alpha 1 & \alpha 3 + 41 & 43 \\ \star & 51 + 2 \cdot 43 + \alpha 4 & 53 \\ \star & \star & 54 \end{pmatrix}^{k,l}$$

where the superscripts for the matrices indicate that they should be applied to each entry so as to create determinants like (43) and $\star$ entries follow from the symmetry of the matrices (note also $2 \cdot 31^{k,l} = 31^{k,l} + 31^{k,l}$ etc). For notational convenience, the n-dependence of α^j and β^j in the matrices $M_i^{k,l}$ is suppressed, but (26) should be used in the upshifts of k_1 and k_2.

Now we give the details of $\bar{L}$ of (34). We write the numerator N and denominator D of L of (1), both bilinear forms in x_{n+1} and x_{n+2}, as

$$N = \boldsymbol{x}_{n+1}^1 \cdot B_N \boldsymbol{x}_{n+2}^1, \qquad D = \boldsymbol{x}_{n+1}^1 \cdot (x_n B_D) \boldsymbol{x}_{n+2}^1 \tag{44}$$

with

$$B_N := \begin{pmatrix} p_3 & p_4 \\ p_4 & p_5 \end{pmatrix}, \qquad B_D := \begin{pmatrix} p_2 & p_1 \\ p_1 & p_3 \end{pmatrix}. \tag{45}$$

Noting the second expressions for $I_1 + \alpha$ and $I_2 + \beta$ in (38) and (40) respectively, it follows from (34) that

$$\bar{L} : x_{n+3} = \frac{\boldsymbol{x}_{n+1}^3 \cdot \begin{vmatrix} B_N^0 & B_N^1 & B_N^2 \\ B_1^0 & B_1^1 & B_1^2 \\ B_2^0 & B_2^1 & B_2^2 \end{vmatrix}_{\boxtimes} \boldsymbol{x}_{n+2}^3}{\boldsymbol{x}_{n+1}^3 \cdot \begin{vmatrix} x_n B_D^0 & x_n B_D^1 & x_n B_D^2 \\ B_1^0 & B_1^1 & B_1^2 \\ B_2^0 & B_2^1 & B_2^2 \end{vmatrix}_{\boxtimes} \boldsymbol{x}_{n+2}^3} =: \frac{\boldsymbol{x}_{n+1}^3 \cdot \begin{vmatrix} \boldsymbol{B}_N \\ \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{vmatrix}_{\boxtimes} \boldsymbol{x}_{n+2}^3}{\boldsymbol{x}_{n+1}^3 \cdot \begin{vmatrix} x_n \boldsymbol{B}_D \\ \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{vmatrix}_{\boxtimes} \boldsymbol{x}_{n+2}^3} \tag{46}$$

where $\boldsymbol{B}_N$ represents the row vector in the matrix determinant. The numerator and denominator of $\bar{L}$ are bicubic expressions in x_{n+1} and x_{n+2}, respectively. We now show that they can be taken as linear in x_n via a cancellation. Note the decompositions

$$B_1 := x_n B_{11} + B_N, \tag{47}$$

$$B_2 := x_n B_D + B_{11}^{\tilde{t}}, \tag{48}$$

where

$$B_{11} := \begin{pmatrix} p_1 & p_3 \\ \alpha & p_4 \end{pmatrix}; \tag{49}$$

the superscript t denotes transpose and the superscript $\tilde{}$ means the exchange $\alpha \leftrightarrow \beta$. These allow simplifications in the matrix determinants of (46):

$$\begin{vmatrix} \boldsymbol{B}_N \\ \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{vmatrix}_{\boxtimes} = \begin{vmatrix} \boldsymbol{B}_N \\ x_n\boldsymbol{B}_{11} + \boldsymbol{B}_N \\ x_n\boldsymbol{B}_D + \boldsymbol{B}_{11}^{\tilde{t}} \end{vmatrix}_{\boxtimes} = \begin{vmatrix} \boldsymbol{B}_N \\ x_n\boldsymbol{B}_{11} \\ x_n\boldsymbol{B}_D + \boldsymbol{B}_{11}^{\tilde{t}} \end{vmatrix}_{\boxtimes} = x_n^2 \begin{vmatrix} \boldsymbol{B}_N \\ \boldsymbol{B}_{11} \\ \boldsymbol{B}_D \end{vmatrix}_{\boxtimes} + x_n \begin{vmatrix} \boldsymbol{B}_N \\ \boldsymbol{B}_{11} \\ \boldsymbol{B}_{11}^{\tilde{t}} \end{vmatrix}_{\boxtimes}, \tag{50}$$

$$\begin{vmatrix} x_n\boldsymbol{B}_D \\ \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{vmatrix}_{\boxtimes} = \begin{vmatrix} x_n\boldsymbol{B}_D \\ x_n\boldsymbol{B}_{11} + \boldsymbol{B}_N \\ x_n\boldsymbol{B}_D + \boldsymbol{B}_{11}^{\tilde{t}} \end{vmatrix}_{\boxtimes} = \begin{vmatrix} x_n\boldsymbol{B}_D \\ x_n\boldsymbol{B}_{11} + \boldsymbol{B}_N \\ \boldsymbol{B}_{11}^{\tilde{t}} \end{vmatrix}_{\boxtimes} = x_n^2 \begin{vmatrix} \boldsymbol{B}_D \\ \boldsymbol{B}_{11} \\ \boldsymbol{B}_{11}^{\tilde{t}} \end{vmatrix}_{\boxtimes} + x_n \begin{vmatrix} \boldsymbol{B}_D \\ \boldsymbol{B}_N \\ \boldsymbol{B}_{11}^{\tilde{t}} \end{vmatrix}_{\boxtimes}, \tag{51}$$

which can be viewed as the result of row operations on the matrix determinant[8]. This gives the fractional-linear form

$$\bar{L} : x_{n+3} = \frac{f_1(x_{n+1}, x_{n+2})x_n + f_2(x_{n+1}, x_{n+2})}{f_3(x_{n+1}, x_{n+2})x_n + \tilde{f}_1(x_{n+2}, x_{n+1})}, \tag{52}$$

with the functions f_i bicubic in their arguments:

$$f_1(x_{n+1}, x_{n+2}) = \boldsymbol{x}_{n+1}^3 \cdot \begin{vmatrix} \boldsymbol{B}_N \\ \boldsymbol{B}_{11} \\ \boldsymbol{B}_D \end{vmatrix}_{\boxtimes} \boldsymbol{x}_{n+2}^3 = \boldsymbol{x}_{n+1}^3 \cdot H_1 \boldsymbol{x}_{n+2}^3, \tag{53}$$

$$f_2(x_{n+1}, x_{n+2}) = \boldsymbol{x}_{n+1}^3 \cdot \begin{vmatrix} \boldsymbol{B}_N \\ \boldsymbol{B}_{11} \\ \boldsymbol{B}_{11}^{\tilde{t}} \end{vmatrix}_{\boxtimes} \boldsymbol{x}_{n+2}^3 = \boldsymbol{x}_{n+1}^3 \cdot H_2 \boldsymbol{x}_{n+2}^3, \tag{54}$$

$$f_3(x_{n+1}, x_{n+2}) = \boldsymbol{x}_{n+1}^3 \cdot \begin{vmatrix} \boldsymbol{B}_D \\ \boldsymbol{B}_{11} \\ \boldsymbol{B}_{11}^{\tilde{t}} \end{vmatrix}_{\boxtimes} \boldsymbol{x}_{n+2}^3 = \boldsymbol{x}_{n+1}^3 \cdot H_3 \boldsymbol{x}_{n+2}^3. \tag{55}$$

The function $\tilde{f}_1$ is f_1 with the exchange $\alpha^j \leftrightarrow \beta^j$. This follows since the term independent of x_n on the denominator of $\bar{L}$ is, from (51),

$$\boldsymbol{x}_{n+1}^3 \cdot \begin{vmatrix} \boldsymbol{B}_D \\ \boldsymbol{B}_N \\ \boldsymbol{B}_{11}^{\tilde{t}} \end{vmatrix}_{\boxtimes} \boldsymbol{x}_{n+2}^3 = \boldsymbol{x}_{n+2}^3 \cdot \begin{vmatrix} \boldsymbol{B}_D \\ \boldsymbol{B}_N \\ \boldsymbol{B}_{11}^{\tilde{t}} \end{vmatrix}_{\boxtimes}^{t} \boldsymbol{x}_{n+1}^3 = \boldsymbol{x}_{n+2}^3 \cdot \begin{vmatrix} \boldsymbol{B}_D \\ \boldsymbol{B}_N \\ \boldsymbol{B}_{11}^{\tilde{}} \end{vmatrix}_{\boxtimes} \boldsymbol{x}_{n+1}^3 = \tilde{f}_1(x_{n+2}, x_{n+1}), \tag{56}$$

noting that B_D and B_N are symmetric and independent of α, β, and that the transpose distributes over the product $\boxtimes$. Similar manipulations show that the 4×4 matrices H_2 and H_3 satisfy a type of skew symmetry

$$H_j^{\tilde{t}} = -H_j, \qquad j = 2, 3. \tag{57}$$

The entries of H_1, H_2 and H_3 can be expressed in terms of (3×3) determinants in the parameters. Define e.g.

$$123 := \begin{vmatrix} p_1^0 & p_1^1 & p_1^2 \\ p_2^0 & p_2^1 & p_2^2 \\ p_3^0 & p_3^1 & p_3^2 \end{vmatrix}, \qquad 34\beta := \begin{vmatrix} p_3^0 & p_3^1 & p_3^2 \\ p_4^0 & p_4^1 & p_4^2 \\ \beta^0 & \beta^1 & \beta^2 \end{vmatrix}, \tag{58}$$

[8] The rigorous justification for the validity of such operations can be inferred from properties of the Kronecker product $\otimes$.

showing that the triple indices are equal to their cyclic permutations. Then,

$$H_1 = \begin{pmatrix} 123 & 124 & 243 & 143 \\ 124 + 23\alpha & 24\alpha + 13\alpha + 125 & 253 + 14\alpha & 153 \\ 13\alpha + 24\alpha & 134 + 25\alpha + 2 \cdot 14\alpha & 254 + 15\alpha + 34\alpha & 154 \\ 14\alpha & 15\alpha + 34\alpha & 154 + 35\alpha & 354 \end{pmatrix}, \tag{59}$$

$$H_2 = \begin{pmatrix} 0 & 31\beta & 143 + 41\beta & 43\beta \\ \star & 14\alpha + 41\beta + 3\alpha\beta & 153 + 51\beta + 4\alpha\beta & 53\beta \\ \star & \star & 5\alpha\beta & 345 + 54\beta \\ \star & \star & \star & 0 \end{pmatrix} \tag{60}$$

and

$$H_3 = \begin{pmatrix} 0 & 123 + 21\beta & 23\beta & 13\beta \\ \star & 2\alpha\beta & 234 + 24\beta + 1\alpha\beta & 134 + 14\beta \\ \star & \star & 41\alpha + 14\beta + 3\alpha\beta & 34\beta \\ \star & \star & \star & 0 \end{pmatrix}, \tag{61}$$

where property (57) can be used to complete the below-diagonal entries of H_2 and H_3, e.g. $(H_2)_{32} = -(153 + 51\alpha + 4\beta\alpha) = 135 + 15\alpha + 4\alpha\beta$ (note that the diagonal entries of H_2 and H_3 change sign under interchange of α and β).

Finally, we point out that $\bar{L}$ of (52) satisfies a form of (alternating) measure preservation, namely

$$\det \mathrm{d}\bar{L} = \frac{\partial x_{n+3}}{\partial x_n} = \frac{\ell(x_n, x_{n+1}, x_{n+2})}{\tilde{\ell}(x_{n+1}, x_{n+2}, x_{n+3})}, \tag{62}$$

where the density ℓ is

$$\ell(x_n, x_{n+1}, x_{n+2}) = (x_n x_{n+1} x_{n+2})^{-1} \begin{vmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{vmatrix}^{-1} = \frac{(x_n x_{n+1} x_{n+2})^{-1}}{\boldsymbol{x}_n^2 \cdot \left(M_2^{1,2} x_{n+1}^2 + M_1^{1,2} x_{n+1} + M_0^{1,2}\right) \boldsymbol{x}_{n+2}^2}.$$

In (62), $\tilde{\ell}$ again refers to the exchange $\alpha^j \leftrightarrow \beta^j$ and the result represents a generalization of theorem 1 of [19], noting that $(x_n x_{n+1} x_{n+2})^{-1}$ is the density of L of (1) and $\bar{L}$ arises from L by the interchange of parameters and 2-integrals. The composition of $\bar{L}$ with its upshift, i.e. $\tilde{\bar{L}} \circ \bar{L}$, is no longer alternating and is measure preserving in the usual sense, with density ℓ. When we take $\alpha^j = \beta^j$, $\bar{L}$ itself ceases to be alternating and is also measure preserving with the corresponding density ℓ.

4. Concluding remarks

We conclude with the following remarks.

- The maps L of (1), L_1 of (20) and L_2 of (22) are all contained as special cases of the general map $\bar{L}$ of (52). In general, $\bar{L}$ is an alternating map but when we choose $\alpha^j = \beta^j$, it becomes non-alternating.
- The determinantal structure of $\bar{L}$ and k_1 and k_2 of (42) and the fractional-linear form of $\bar{L}$ have analogues in the symmetric QRT map in two dimensions [21] when it is obtained from the McMillan map via reparametrization and interchange [10].

This letter highlights the importance and usefulness of several recent developments in discrete integrable systems: 2-integrals [5], interchange of parameters and integrals [19] and alternating maps [15].

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