Creating and relating three-dimensional integrable maps

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Abstract

We show how some integrable third-order difference equations recently given in the literature are related to one another by the process of interchanging parameters and integrals. Using the same process, we then create a 21-parameter family of integrable third-order difference equations that contains the previous examples as special cases. Our methodology illustrates that the combination of finding 2-integrals (i.e. integrals of the second iterate of the map), exploiting linear parameter dependence and using the interchange process provides a powerful way to relate and create higher-dimensional discrete integrable systems.

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1. Introduction

Integrable systems in general are studied for various reasons: for their intrinsic physical and mathematical interest, as a starting point for a perturbative approach, as tests for various numerical methods. Discrete integrable systems in particular are studied because of their fundamental mathematical nature and their applications to various areas of physics (including statistical mechanics and quantum gravity) and because sometimes they are discrete analogues of integrable systems in classical mechanics or solid state physics [4, 14, 16–18, 23].

This letter is concerned with integrable mappings, in particular those that can be written as difference equations. Integrable mappings of the plane were first introduced by McMillan [13], with some precursors in the work of Lyness [11]. The McMillan and Lyness maps are generalized by the so-called QRT map [17, 18], which contains a large number of parameters, but is still a map of \( \mathbb{R}^2 \) (or a second-order difference equation). In recent years, some extensions
of integrable maps or difference equations to third (and higher) order have begun to appear
[1, 2, 6–9, 14, 16, 22], but a comprehensive approach has been elusive.

Building on recent advances, this letter represents a first attempt at a more comprehensive
approach to third-order integrable difference equations. In particular, it uses the technique
of [19] extended to the concept of 2-integrals [5] to create new integrable maps that, in their
most general form, are alternating [15]. Ultimately, we derive the fractional-linear third-order
difference equation \( \bar{L} \), of equation (52), which contains 21 parameters (6 of which alternate
from one iterate to another). Its integrability derives from the possession of two integrals (42)
and a measure-preservation property (62).

2. Interchange of 2-integrals and parameters to relate three existing integrable
third-order difference equations

The following third-order difference equation was derived in [6]:

\[
L : x_{n+3} = \frac{1}{x_n} p_3 x_{n+1} x_{n+2} + p_4 (x_{n+1} + x_{n+2}) + p_5,
\]

where \( p_1, p_2, \ldots, p_5 \) are arbitrary parameters. It has two integrals, as shown in [6], and is
also (anti) measure preserving since

\[
\det dL = \frac{\partial x_{n+3}}{\partial x_n} = -m(x_n, x_{n+1}, x_{n+2}) / m(x_{n+1}, x_{n+2}, x_{n+3})
\]
with density \( m(x, y, z) = (xyz)^{-1} \). Hence, it is integrable [5]. It turns out that underlying the
existence of the two integrals are two 2-integrals, \( I_1 \) and \( I_2 \), given by

\[
I_1(x_n, x_{n+1}, x_{n+2}) = \frac{p_1 x_n x_{n+1} x_{n+2} + (p_3 x_{n+1} + p_4)(x_{n} + x_{n+2}) + p_4 x_{n+1} + p_5}{x_n x_{n+2}}
\]

\[
I_2(x_n, x_{n+1}, x_{n+2}) = \frac{(p_2 x_{n+1} + p_1) x_n x_{n+2} + (p_1 x_{n+1} + p_3)(x_{n} + x_{n+2}) + p_4}{x_{n+1}}.
\]

It can be verified that

\[
I_1(n + 1) := I_1(x_{n+1}, x_{n+2}, x_{n+3}) = I_2(x_n, x_{n+1}, x_{n+2}) := I_2(n),
\]

\[
I_2(n + 1) := I_2(x_{n+1}, x_{n+2}, x_{n+3}) = I_1(x_n, x_{n+1}, x_{n+2}) := I_1(n),
\]

whence

\[
I_1(n + 2) = I_1(n) \quad \text{and} \quad I_2(n + 2) = I_2(n).
\]

The difference equation \( L \) has a time-reversal symmetry [20], being conjugate to its inverse
via the involution

\[
G : (x_n, x_{n+1}, x_{n+2}) \rightarrow (x_{n+2}, x_{n+1}, x_n).
\]

A systematic method for finding third-order difference equations that possess 2-integrals,
based upon assuming that the equations possess such time-reversal symmetry, will be detailed
elsewhere [21]. For the moment, we note that the existence of the 2-integrals \( I_1 \) and \( I_2 \) is a
more fundamental invariance property of \( L \) that implies the existence of the integrals. The
2-integrals are also substantially more compact building blocks than the integrals. Specifically,
we find that the integrals \( H_1 \) and \( H_2 \) of \( L \) given in [6] can be expressed as

\[4\] In [6], \( L \) is equation (Y1) of the appendix. We are grateful to Professor R Hirota and his collaborators for providing
us with \( H_1 \) and \( H_2 \). During the final preparation of our manuscript, [12] appeared in which the 2-integrals \( I_1 \) and \( I_2 \)
for \( (Y1) \) are also found, as are the 2-integrals for other integrable cases of [6]. This allows the authors of [12] to give
\[
H_1 = \frac{I_1 + I_2 - p_3 H_2}{p_4} \tag{8}
\]
\[
H_2 = \frac{I_1 I_2 - (4p_1 p_4 + 2p_1^2 + p_2 p_3) p_4}{p_1 p_4} \tag{9}
\]

In [19], a process was described whereby new maps with integrals could be obtained from an original map with integrals. This process works by interchanging parameters with (the values of) integrals in the original map. It can be generalized to cover the case of \(k\)-integrals depending on parameters, as we now illustrate (a complete description of this will be given in [21]).

In particular, we can use the two 2-integrals to solve for any two parameters present in them. This is facilitated by the fact that the dependence of \(I_1\) and \(I_2\) upon their five parameters is linear (in stark contrast to the case of the integrals \(H_1\) and \(H_2\)). Consider the equations

\[
I_1(n) = -\alpha(n), \quad I_2(n) = -\beta(n), \tag{10}
\]

where from (4)–(6)

\[
\alpha(n + 1) = \beta(n) \quad \text{and} \quad \beta(n + 1) = \alpha(n) \Rightarrow \alpha(n + 2) = \alpha(n) \quad \text{and} \quad \beta(n + 2) = \beta(n). \tag{11}
\]

Equations (10) define a linear system in the parameters \(p_1, p_2, \ldots, p_5\). Arbitrarily distinguishing two of these, calling them \(K_1\) and \(K_2\), we can write

\[
J(n) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = -\begin{pmatrix} \alpha(n) + \tilde{I}_1(n) \\ \beta(n) + \tilde{I}_2(n) \end{pmatrix}. \tag{12}
\]

Here,

\[
J(n) := \begin{pmatrix} \frac{\partial I_1(n)}{\partial K_1} & \frac{\partial I_1(n)}{\partial K_2} \\ \frac{\partial I_2(n)}{\partial K_1} & \frac{\partial I_2(n)}{\partial K_2} \end{pmatrix} \tag{13}
\]

and \(\tilde{I}_i(n), i = 1, 2\) are the parts of \(I_i(n)\) not involving \(K_1\) or \(K_2\). Solving (12) for \(K_1\) and \(K_2\), we obtain

\[
K_1 := k_1(x_n, x_{n+1}, x_{n+2}, \tilde{p}, \alpha(n), \beta(n)), \tag{14}
\]
\[
K_2 := k_2(x_n, x_{n+1}, x_{n+2}, \tilde{p}, \alpha(n), \beta(n)), \tag{15}
\]

where \(\tilde{p}\) represent the parameters not chosen as either \(K_1\) or \(K_2\). Similarly, the equations

\[
I_1(n + 1) = \alpha(n + 1) \quad \text{and} \quad I_2(n + 1) = \beta(n + 1)
\]

lead to the system

\[
J(n + 1) \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = -\begin{pmatrix} \beta(n) + \tilde{I}_1(n + 1) \\ \alpha(n) + \tilde{I}_2(n + 1) \end{pmatrix}, \tag{16}
\]

noting the use of (11) to update the parameters on the right-hand side. Since \(K_1\) and \(K_2\) must simultaneously satisfy both (12) and (16), we conclude that the difference equation \(\bar{L}\) formed from \(L\) by replacing, respectively, \(K_1\) and \(K_2\) with \(k_1\) and \(k_2\) has the integrals \(k_1\) and \(k_2\) (i.e. they equal their upshift). The map \(\bar{L}\) and \(k_1\) and \(k_2\) now contain the \(n\)-dependent parameters \(\alpha(n)\) and \(\beta(n)\) satisfying from (11)

\[
\alpha(n) = \alpha_0 + (-1)^n \alpha_1, \quad \beta(n) = \alpha_0 - (-1)^n \alpha_1 \tag{17}
\]

where \(\alpha_0\) and \(\alpha_1\) are constants.

5 The negative sign is introduced simply for aesthetic reasons.
In summary, the map \( L \) containing constant parameters and possessing two 2-integrals becomes the map \( \tilde{L} \) containing a 2-cycle of parameters and possessing two integrals. This interchange can be encoded as

\[
\{ (I_1, I_2); K_1, K_2 \} \rightarrow \{ k_1, k_2; (\alpha, \beta) \},
\]

where the semicolon in each bracket separates integrals to its left from parameters on its right, and the round brackets represent a 2-cycle in integrals or in parameters (in two dimensions, alternating versions of the QRT map with one integral have been recently studied in \[15\]).

To exemplify the above process, first consider solving (3) and (12) for \( K_1 = p_3 \) and \( K_2 = p_2 \). This yields

\[
k_1 = -\alpha(n)x_nx_{n+2} - p_1x_nx_{n+1}x_{n+2} - p_3x_{n+1}(x_n + x_{n+2}) - p_4(x_n + x_{n+1} + x_{n+2}),
\]

\[
k_2 = -\frac{p_1(x_nx_{n+1} + x_nx_{n+2} + x_{n+1}x_{n+2}) + p_3(x_n + x_{n+2}) + p_4 + \beta(n)x_{n+1}}{x_nx_{n+1}x_{n+2}}.
\]

where \( \alpha(n) \) and \( \beta(n) \) are given by (17). With these replacements, \( \tilde{L} \) becomes

\[
L_1 : x_{n+3} = -\frac{p_4 + p_3x_{n+1} + \alpha(n)x_{n+2} + p_1x_{n+1}x_{n+2}}{p_4 + p_3x_{n+2} + \beta(n)x_{n+1} + p_1x_{n+1}x_{n+2}}.
\]

The third-order difference equation \( L_1 \) is a generalization of the one studied in \[22\]. It is measure preserving with density \( m_1(x_n, x_{n+1}, x_{n+2}) = (x_nx_{n+1}x_{n+2})^{-\frac{1}{2}} \).

Another possibility is constructing (12) from (3) with \( K_1 = p_3 \) and \( K_2 = p_4 \). This yields

\[
k_1 = -\alpha(n)x_nx_{n+2} - p_1x_nx_{n+1}x_{n+2} - p_3x_{n+1}(x_n + x_{n+2}) - (x_n + x_{n+1} + x_{n+2})k_2,
\]

\[
k_2 = -\beta(n)x_{n+1} - p_2x_nx_{n+1}x_{n+2} - p_3(x_nx_{n+2} + x_nx_{n+1} + x_{n+1}x_{n+2}) - 3(x_n + x_{n+2}).
\]

With these replacements, \( \tilde{L} \) becomes

\[
L_2 : x_{n+3} = x_n - \frac{p_3(x_{n+1} - x_{n+2}) - \beta(n)x_{n+1} + \alpha(n)x_{n+2}}{p_4 + p_3x_{n+1} + x_{n+2} + p_2x_{n+1}x_{n+2}}.
\]

The equation \( L_2 \) is volume preserving and represents a generalization, with alternating parameters, of equation (12) of \[8\] (in particular, the latter is obtained by taking \( p_1 = 0 \) and \( \alpha(n) = \beta(n) = 0 \) in \( L_2 \)). Note that both integrals of \( L_2 \) are now polynomials. It is also the case that \( L_2 \) can be obtained directly from \( L_1 \), an application of the interchange of parameters and integrals as originally advanced in \[19\]. This is achieved by solving the second equation of (19) for \( p_4 \) and renaming the value of \( k_2 \) there by \( p_2 \). Replacing for \( p_4 \) in \( L_1 \) then yields \( L_2 \). It is also seen that the expression obtained by solving for \( p_4 \) is precisely the integral \( k_2 \) in (21), whereas \( k_1 \) of (21) follows from \( k_1 \) of (19) with \( p_4 \) also replaced. Summarizing, the three integrable difference equations \( L, L_1 \) and \( L_2 \) can all be obtained from one another via the interchange process. In other words, any two can be generated from the third. This is highlighted schematically in figure 1.
3. Reparametrization and interchange of 2-integrals and parameters to create a new class of integrable third-order difference equations

In fuller generality, analogous to the case of integrals described in [19], we can reparametrize all parameters in $L$ and its two 2-integrals, in terms of two parameters $K_1$ and $K_2$. That is, we let

$$p_i \rightarrow p_i^0 + p_i^1 K_1 + p_i^2 K_2, \quad i = 1, \ldots, 5,$$

i.e. each parameter $p_i$ is ‘replaced’ by three parameters $p_i^0$, $p_i^1$ and $p_i^2$ (note that the superscripts here do not denote powers). Similarly, we can reparametrize

$$I_1 + \alpha \rightarrow (I_1^0 + I_1^1 K_1 + I_1^2 K_2) + (\alpha^0 + \alpha^1 K_1 + \alpha^2 K_2),$$

$$I_2 + \beta \rightarrow (I_2^0 + I_2^1 K_1 + I_2^2 K_2) + (\beta^0 + \beta^1 K_1 + \beta^2 K_2).$$

In (24), $I_1^0$ is the 2-integral $I_1$ of $L$ with $p_i \rightarrow p_i^0$ and

$$\alpha^j (n + 1) = \beta^j (n) \quad \text{and} \quad \beta^j (n + 1) = \alpha^j (n), \quad j = 0, 1, 2.$$  

It is seen that (24) and (25) incorporate the $p_i$ reparametrization in terms of $K_1$ and $K_2$ as well as the ‘expansion’ of $\alpha$ and $\beta$, respectively, satisfying (26). This extra reparametrization is simply a reflection of the fact that the addition of an alternating parameter, or a linear combination of alternating parameters, to a 2-integral still gives a 2-integral of the map. At this point, the expanded $p_i$, $\alpha$ and $\beta$, together with $K_1$ and $K_2$, should all be considered parameters on an equal footing. Substituting these reparametrizations into (1) and (3), the following trivial reformulation of the properties of $L$: the third-order difference equation

$$L : x_{n+3} = \frac{(N_0, N_1, N_2) \cdot (1, K_1, K_2)}{(D_0, D_1, D_2) \cdot (1, K_1, K_2)} = \frac{N \cdot K}{D \cdot K}$$

possesses the 2-integrals

$$\hat{I}_1 = (I_1^0 + \alpha^0, I_1^1 + \alpha^1, I_1^2 + \alpha^2) \cdot (1, K_1, K_2) = I_1 \cdot K$$

$$\hat{I}_2 = (I_2^0 + \beta^0, I_2^1 + \beta^1, I_2^2 + \beta^2) \cdot (1, K_1, K_2) = I_2 \cdot K.$$  

In (27), the vector $N := (N_0, N_1, N_2)$, where $N_0$ is the numerator on the rhs of $L$ with $p_i \rightarrow p_i^0$ etc, and similarly for $D$, the corresponding vector built from copies of the denominator. The vector $K := (1, K_1, K_2)$. For brevity, we define $I_1 = (I_{10}, I_{11}, I_{12})$ and similarly for $I_2$, where

$$I_{1j} := I_1^j + \alpha^j, \quad I_{2j} := I_2^j + \beta^j, \quad j = 0, 1, 2.$$  

Now, as in the similar philosophy for the case of integrals as discussed in [19], we now take $K_1$ and $K_2$ as distinguished parameters chosen to satisfy $I_1(n) = 0$ and $I_2(n) = 0$ respectively. These two equations, the generalizations of (10), define a linear system (12) for $K_1$ and $K_2$ where now $J = (I_{1i}, I_{2i})$ and the right-hand side of (12) is now $-L_{ij}^{-1}$. The ensuing discussion concerning generating a new difference equation by replacing $K_1$ and $K_2$ applies, this new difference equation preserving integrals $k_1$ and $k_2$ given by solving, respectively, for $K_1$ and $K_2$. The reparametrizations we used above will mean that this new difference equation is much more general than the examples discussed in section 2 as we have tripled the number of parameters. Ultimately, we will derive $L$ of equation (52), which contains 21 arbitrary

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6 One sees that e.g. $\alpha^0$ and $I_1^0$ now play the role of $\alpha(n)$ and $I_1(n)$, respectively, in (12).
parameters (6 of which alternate from one iterate to another). Nevertheless, the examples in section 2 will be special cases of our final result \( \hat{L} \), for example, \( L_1 \) of (20) corresponds to taking \( \{ p_1^i = p_2^i = 0, i = 1, 3, 4; p_2^5 = p_3^5 = p_4^5 = p_1^5 = 0; \alpha = \beta = 0, i = 1, 2 \} \). Moreover, the original equation \( \hat{L} \) of (1) corresponds to taking \( \{ p_1^i = p_2^i = 0, i = 1, \ldots, 5 \} \) so that no replacement of parameters in \( L \) actually occurs (in this case, \( \alpha_i = \beta_i = 0, i = 1, 2 \)).

We proceed to the derivation of \( \hat{L} \) of equation (52). From (28)–(29), we see that the solutions of \( \hat{I}_1(n) = 0 \) and \( \hat{I}_2(n) = 0 \) can be expressed vectorially as

\[
\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{bmatrix}^{-1} \left( I_1 \times I_2 \right),
\]

and equivalently, by Cramer’s rule, as

\[
k_1 = -\begin{vmatrix} I_{10} & I_{12} \\ I_{11} & I_{12} \end{vmatrix} / \begin{vmatrix} I_{11} & I_{12} \\ I_{11} & I_{12} \end{vmatrix}, \quad k_2 = -\begin{vmatrix} I_{10} & I_{12} \\ I_{11} & I_{12} \end{vmatrix} / \begin{vmatrix} I_{11} & I_{12} \\ I_{11} & I_{12} \end{vmatrix}.
\]

The third-order difference equation \( \hat{L} \) preserving these integrals follows from (27) with (31), i.e.

\[
\hat{L} : x_{n+3} = \frac{N \cdot I_1 \times I_2}{D \cdot I_1 \times I_2},
\]

equivalently,

\[
L : x_{n+3} = \begin{vmatrix} N_0 & N_1 & N_2 \\ I_{20} & I_{21} & I_{22} \\ I_{30} & I_{31} & I_{32} \end{vmatrix}.
\]

Note that equations (32) and (34) actually hold more generally than those derived from just (1). That is, suppose we are given an original third-order rational difference equation \( L : x_{n+3} = \frac{N}{D} \), where \( N \) and \( D \) are general polynomial expressions in \( \{ x_n, x_{n+1}, x_{n+2} \} \) which have linear dependence on parameters. If, additionally, there are two 2-integrals \( I_1 \) and \( I_2 \) with linear dependence on parameters, (34) describes a new third-order rational difference equation preserving the integrals \( k_1 \) and \( k_2 \) given by (32) that can be obtained from the original by the processes of reparametrization and replacement.

Turning back to the specific starting points (1) and (3), we now give the explicit details of (31) and (34) in this case. The calculation of \( k_1 \), \( k_2 \) and \( \hat{L} \) both involve the calculation of determinants whose entries are bilinear or trilinear forms, so it is useful to have a formalism for the products of such forms.

For \( i \geq 1 \), define the \((i + 1)\)-dimensional vector

\[
x^i := \begin{pmatrix} x^i \\ x^{i-1} \\ \vdots \\ 1 \end{pmatrix}.
\]
With $A_i$ as an $(i+1) \times (i+1)$ matrix, $x_i' \cdot A_i' z_i'$ defines a form of degree $i$ in $x$ and $z$.

The multiplication of two forms can be achieved using the Kronecker product [3] (or tensor product $\otimes$):

\[
(x_i' \cdot A_i' z_i')(x_j' \cdot A_j' z_j') = (x_i' \otimes x_j') \cdot (A_i \otimes A_j)(z_i' \otimes z_j').
\] (36)

Note that $(x_i' \otimes x_j')$ is $(i+1)(j+1)$-dimensional and the matrix $A_i \otimes A_j$ is $(i+1)(j+1) \times (i+1)(j+1)$. Here, we wish to use a certain contraction of the Kronecker product of $A_i$ and $A_j$ which we denote by $\otimes$. We define $A_i \otimes A_j$ in the following way:

\[
(x_i' \cdot A_i' z_i')(x_j' \cdot A_j' z_j') = (x_i' \otimes x_j') \cdot (A_i \otimes A_j)(z_i' \otimes z_j') =: x_{i+j}^+ \cdot (A_i \otimes A_j)z_{i+j}^+.
\] (37)

This uniquely defines $A_i \otimes A_j$ as a $(i+j+1) \times (i+j+1)$ matrix. Its entries come from certain sums of the entries in appropriate rows and columns of $A_i \otimes A_j$. Note that $A_i \otimes A_j$ is a commutative operation, unlike $A_i \otimes A_j$. Nevertheless, many properties of the product $\otimes$ we use below can be inferred from those of $\otimes$.

The 2-integral $I_1 + \alpha$ is a trilinear form in $x_n, x_{n+1}$ and $x_{n+2}$. Two ways in which it can be written are

\[
I_1 + \alpha = \frac{x_{n+1}^1 \cdot A_1 x_{n+2}^1}{x_n x_{n+2}} = \frac{x_{n+1}^1 \cdot B_1 x_{n+2}^1}{x_n x_{n+2}},
\] (38)

where $x_i^1 = (x_i)$ etc, following the notation of (35), and

\[
A_1 := \begin{pmatrix} p_1 x_{n+1} + \alpha & p_3 x_{n+1} + p_4 \\ p_3 x_{n+1} + p_4 & p_4 x_{n+1} + p_5 \end{pmatrix}, \quad B_1 := \begin{pmatrix} p_1 x_n + p_3 & p_3 x_n + p_4 \\ \alpha x_n + p_4 & p_4 x_n + p_5 \end{pmatrix}.
\] (39)

Similarly,

\[
I_2 + \beta = \frac{x_{n+1}^1 \cdot A_2 x_{n+2}^1}{x_n x_{n+1}} = \frac{x_{n+1}^1 \cdot B_2 x_{n+2}^1}{x_n x_{n+1}},
\] (40)

with

\[
A_2 := \begin{pmatrix} p_2 x_{n+1} + p_1 & p_1 x_{n+1} + p_3 \\ p_1 x_{n+1} + p_3 & p_3 x_{n+1} + p_4 \end{pmatrix}, \quad B_2 := \begin{pmatrix} p_2 x_n + p_1 & p_1 x_n + \beta \\ p_1 x_n + \beta & p_3 x_n + p_4 \end{pmatrix}.
\] (41)

The denominators in (38) and (40) are independent of parameters, and hence will cancel in the rational expressions for $k_1$ and $k_2$ in (32). Consequently, the entries in the determinants in the latter can be taken to be the bilinear forms in the numerators of the first expressions in (38) and (40), with the matrix $A_i$, $i = 1, 2$, replaced by e.g. $A_i^0$ in $I_{i0}$ so as to depend on $p_i^0$ etc.

We obtain that the integrals $k_1$ and $k_2$ are ratios of symmetric trivariate forms in particular, biquadratic in $x_n$ and $x_{n+2}$, which, using the product $\otimes$, we can write as

\[
k_1 = -\frac{x_n^2 \cdot A_1^* A_2^*}{A_1 \otimes A_2} \frac{x_{n+2}^2}{x_n x_{n+2}} = -\frac{x_n^2}{x_n} \cdot \left( M_2^{0,2} x_{n+1}^2 + M_1^{0,2} x_{n+1} + M_0^{0,2} \right) x_{n+2}^2
\] (42)

\[
k_2 = -\frac{x_n^2 \cdot A_1^* A_2^*}{A_1 \otimes A_2} \frac{x_{n+2}^2}{x_n x_{n+2}} = -\frac{x_n^2}{x_n} \cdot \left( M_2^{1,2} x_{n+1}^2 + M_1^{1,2} x_{n+1} + M_0^{1,2} \right) x_{n+2}^2
\] (42)

\footnote{For example, when $i = 2, j = 1$, the $4 \times 4$ matrix $A_i \otimes A_j$ is derived from the $6 \times 6$ matrix $A_i \otimes A_j$ in the following way: (i) simultaneously merge, under addition, the second and third rows of the latter into one row and the fourth and fifth rows into another row, yielding an intermediary $4 \times 6$ matrix; (ii) convert the $4 \times 6$ matrix into $A_2 \otimes A_1$ by repeating (i) on the corresponding columns.}
Here, $\cdots$ refers to a determinant with respect to the matrix product $\otimes$. This matrix determinant, and the linear dependence of (39) and (41) on the parameters, induces a $(2 \times 2)$-determinantal structure in the parameters of each entry of the resulting symmetric $3 \times 3$ matrices defining each biquadratic form. For ease of notation, we define e.g.

$$31^{1,2} := \begin{vmatrix} p_1^1 & p_1^3 \\ p_1^4 & p_1^2 \end{vmatrix}, \quad 1\beta^{0,2} := \begin{vmatrix} p_0^1 & p_1^2 \\ \beta^0 & \beta^2 \end{vmatrix},$$

and the symmetric matrices $M_{ij}^{k,l}$ take the form

$$M_{2}^{k,l} = \begin{pmatrix} 12 & 32 & 31 \\ * & * & 4\beta \\ 21 & 31 & 41 \end{pmatrix},$$

$$M_{1}^{k,l} = \begin{pmatrix} \alpha 2 & \alpha 1 + 42 & 41 \\ * & * & 5\beta \\ 12 & 51 & 53 \end{pmatrix},$$

$$M_{0}^{k,l} = \begin{pmatrix} \alpha 1 & \alpha 3 + 41 & 43 \\ * & * & 54 \\ 12 & 43 & 54 \end{pmatrix},$$

where the superscripts for the matrices indicate that they should be applied to each entry so as to create determinants like (43) and $\ast$ entries follow from the symmetry of the matrices (note also $2 \cdot 31^{k,l} = 31^{k,l} + 31^{k,l}$ etc). For notational convenience, the $n$-dependence of $\alpha_i$ and $\beta_j$ in the matrices $M_{ij}^{k,l}$ is suppressed, but (26) should be used in the upshifts of $k_1$ and $k_2$.

Now we give the details of $\bar{L}$ of (34). We write the numerator $N$ and denominator $D$ of $L$ of (1), both bilinear forms in $x_{n+1}$ and $x_{n+2}$, as

$$N = x_{n+1}^i \cdot B_N x_{n+2}^i, \quad D = x_{n+1}^i \cdot (x_n B_D) x_{n+2}^i$$

with

$$B_N := \begin{pmatrix} p_3 & p_4 \\ p_4 & p_5 \end{pmatrix}, \quad B_D := \begin{pmatrix} p_2 & p_1 \\ p_1 & p_3 \end{pmatrix}.$$  

Noting the second expressions for $I_1 + \alpha$ and $I_2 + \beta$ in (38) and (40) respectively, it follows from (34) that

$$\bar{L} : x_{n+3} = \begin{vmatrix} x_{n+1}^3 & B_{n+1} \\ x_{n+2}^3 & B_{n+2} \end{vmatrix} \begin{vmatrix} B_N & x_{n+1}^3 \\ B_D & x_{n+2}^3 \end{vmatrix}$$

$$= \begin{vmatrix} x_{n+1}^3 & B_{n+1} \\ x_{n+2}^3 & B_{n+2} \end{vmatrix} \begin{vmatrix} B_N & x_{n+1}^3 \\ B_D & x_{n+2}^3 \end{vmatrix}$$

where $B_N$ represents the row vector in the matrix determinant. The numerator and denominator of $\bar{L}$ are bicubic expressions in $x_{n+1}$ and $x_{n+2}$, respectively. We now show that they can be taken as linear in $x_n$ via a cancellation. Note the decompositions

$$B_1 := x_n B_{11} + B_N,$$

$$B_2 := x_n B_D + B_{12},$$

where

$$B_{11} := \begin{pmatrix} p_1 & p_3 \\ \alpha & p_4 \end{pmatrix}.$$
the superscript $t$ denotes transpose and the superscript $^\leftrightarrow$ means the exchange $\alpha \leftrightarrow \beta$. These allow simplifications in the matrix determinants of (46):

$$\begin{vmatrix} B_N \\ B_1 \\ B_2 \end{vmatrix} = \begin{vmatrix} B_N \\ x_nB_1 + B_N \\ x_nB_D + B'_1 \end{vmatrix} = \begin{vmatrix} B_N \\ x_nB_1 + B_N \\ x_nB_D + B'_1 \end{vmatrix} = x_n^3 \begin{vmatrix} B_N \\ B_D \end{vmatrix} + x_n \begin{vmatrix} B_N \\ B_D \end{vmatrix} \begin{vmatrix} B_1 \end{vmatrix} ,$$

(50)

which can be viewed as the result of row operations on the matrix determinant. This gives the fractional-linear form

$$\tilde{L} : x_{n+3} = \frac{f_1(x_{n+1}, x_{n+2})x_n + f_2(x_{n+1}, x_{n+2})}{f_3(x_{n+1}, x_{n+2})x_n + \tilde{f}_1(x_{n+2}, x_{n+1})},$$

(52)

with the functions $f_i$ bicubic in their arguments:

$$f_1(x_{n+1}, x_{n+2}) = x_{n+1}^3 \begin{vmatrix} B_N \\ B_1 \\ B_2 \end{vmatrix} ,$$

$$f_2(x_{n+1}, x_{n+2}) = x_{n+1}^3 \begin{vmatrix} B_N \\ B_1 \\ B_2 \end{vmatrix} ,$$

$$f_3(x_{n+1}, x_{n+2}) = x_{n+1}^3 \begin{vmatrix} B_N \\ B_1 \\ B_2 \end{vmatrix} ,$$

(53)

(54)

(55)

The function $\tilde{f}_1$ is $f_1$ with the exchange $\alpha^j \leftrightarrow \beta^j$. This follows since the term independent of $x_n$ on the denominator of $L$ is, from (51),

$$x_{n+1}^3 \begin{vmatrix} B_D \\ B_N \end{vmatrix} x_{n+2}^3 = x_{n+2}^3 \begin{vmatrix} B_D \\ B_N \end{vmatrix} x_{n+1}^3 = x_{n+2}^3 \begin{vmatrix} B_D \\ B_N \end{vmatrix} = \tilde{f}_1(x_{n+2}, x_{n+1}),$$

(56)

noting that $B_D$ and $B_N$ are symmetric and independent of $\alpha, \beta$, and that the transpose distributes over the product. Similar manipulations show that the $4 \times 4$ matrices $H_2$ and $H_3$ satisfy a type of skew symmetry

$$H_j^j = -H_j, \quad j = 2, 3.$$

(57)

The entries of $H_1, H_2$ and $H_3$ can be expressed in terms of $(3 \times 3)$ determinants in the parameters. Define e.g.

$$123 := \begin{vmatrix} p_1^0 & p_1^1 & p_1^2 \\ p_2^0 & p_2^1 & p_2^2 \\ p_3^0 & p_3^1 & p_3^2 \end{vmatrix} , \quad 34\beta := \begin{vmatrix} p_3^0 & p_4^1 & p_5^2 \\ p_4^0 & p_4^1 & p_5^2 \\ \beta^0 & \beta^1 & \beta^2 \end{vmatrix} .$$

(58)

The rigorous justification for the validity of such operations can be inferred from properties of the Kronecker product $\otimes$. 

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showing that the triple indices are equal to their cyclic permutations. Then,

\( H_1 = \begin{pmatrix}
123 & 124 & 243 & 143 \\
0 & 123 & 243 & 143 \\
243 & 123 & 134 & 254 \\
134 & 243 & 254 & 154
\end{pmatrix} \). \quad (59)

\( H_2 = \begin{pmatrix}
0 & 31\beta & 143 + 41\beta & 43\beta \\
* & 14\alpha + 41\beta + 3\alpha\beta & 153 + 51\beta + 4\alpha\beta & 53\beta \\
* & * & 5\alpha\beta & 345 + 54\beta \\
* & * & * & 0
\end{pmatrix} \). \quad (60)

and

\( H_3 = \begin{pmatrix}
0 & 123 + 21\beta & 23\beta & 13\beta \\
* & 2\alpha\beta & 234 + 24\beta + 1\alpha\beta & 134 + 14\beta \\
* & * & 41\alpha + 14\beta + 3\alpha\beta & 34\beta \\
* & * & * & 0
\end{pmatrix} \). \quad (61)

where property (57) can be used to complete the below-diagonal entries of \( H_2 \) and \( H_3 \), e.g.

\( (H_2)_{32} = -(153 + 51\alpha + 4\beta\alpha) = 135 + 15\alpha + 4\alpha\beta \) (note that the diagonal entries of \( H_2 \) and \( H_3 \) change sign under interchange of \( \alpha \) and \( \beta \)).

Finally, we point out that \( \bar{L} \) of (52) satisfies a form of (alternating) measure preservation, namely

\[
\det \, d\bar{L} = \frac{\partial x_{n+2}}{\partial x_n} = \frac{\ell(x_n, x_{n+1}, x_{n+2})}{\ell(x_{n+1}, x_{n+2}, x_{n+3})},
\]

where the density \( \ell \) is

\[
\ell(x_n, x_{n+1}, x_{n+2}) = (x_n x_{n+1} x_{n+2})^{-1} |I_{12}| |I_{23}|^{-1} \left( (x_n x_{n+1} x_{n+2})^{-1} \right)^{-1} \left( \frac{\alpha^2}{a_n^2} : (M_2^{1.2} x_{n+1}^{2} + M_1^{1.2} x_{n+1} + M_0^{1.2}) a_{n+2}^2 \right).
\]

In (62), \( \ell \) again refers to the exchange \( \alpha^j \leftrightarrow \beta^j \) and the result represents a generalization of theorem 1 of [19], noting that \( (x_n x_{n+1} x_{n+2})^{-1} \) is the density of \( L \) of (1) and \( \bar{L} \) arises from \( L \) by the interchange of parameters and 2-integrals. The composition of \( \bar{L} \) with its upshift, i.e. \( \bar{L} \circ \bar{L} \), is no longer alternating and is measure preserving in the usual sense, with density \( \ell \). When we take \( \alpha^j = \beta^j \), \( \bar{L} \) itself ceases to be alternating and is also measure preserving with the corresponding density \( \ell \).

4. Concluding remarks

We conclude with the following remarks.

- The maps \( L \) of (1), \( L_1 \) of (20) and \( L_2 \) of (22) are all contained as special cases of the general map \( \bar{L} \) of (52). In general, \( \bar{L} \) is an alternating map but when we choose \( \alpha^j = \beta^j \), it becomes non-alternating.
- The determinantal structure of \( \bar{L} \) and \( k_1 \) and \( k_2 \) of (42) and the fractional-linear form of \( \bar{L} \) have analogues in the symmetric QRT map in two dimensions [21] when it is obtained from the McMillan map via reparametrization and interchange [10].

This letter highlights the importance and usefulness of several recent developments in discrete integrable systems: 2-integrals [5], interchange of parameters and integrals [19] and alternating maps [15].
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References