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Fast Track Communication

Birational maps that send biquadratic curves to biquadratic curves

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Abstract
Recently, many papers have begun to consider so-called non-Quispel–Roberts–Thompson (QRT) birational maps of the plane. Compared to the QRT family of maps which preserve each biquadratic curve in a fibration of the plane, non-QRT maps send a biquadratic curve to another biquadratic curve belonging to the same fibration or to a biquadratic curve from a different fibration of the plane. In this communication, we give the general form of a birational map derived from a difference equation that sends a biquadratic curve to another. The necessary and sufficient condition for such a map to exist is that the discriminants of the two biquadratic curves are the same (and hence so are the j-invariants). The result allows existing examples in the literature to be better understood and allows some statements to be made concerning their generality.

Keywords: integrable map, QRT map, biquadratic curve, elliptic fibration

1. Introduction

This communication concerns generalizations of discrete integrable systems in the form of integrable maps of the plane that have received a lot of attention in the recent literature. We consider birational maps $M: (x, y) \mapsto (x', y')$ of the form

$$M: x' = y, \quad y' = R(x, y). \tag{1}$$

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Such maps arise naturally from second order difference equations

$$z_{n+1} = R(z_{n-1}, z_n) \tag{2}$$

with the identification $(x, y) := (z_{n-1}, z_n)$ and $(x', y') := (z_n, z_{n+1})$. Conversely, the identification can be used to rewrite any map of the form M as the difference equation (2). We are particularly interested in maps of the form (1) that send a one-parameter family of biquadratic curves $B(x, y; t) = 0$ (the *source* family) to another such family $B'(x, y; \tau) = 0$ (the *target* family), where

$$\begin{aligned} B(x, y; t) \ := \ & (\alpha_0 + \alpha_1 t)x^2y^2 + \left(\beta_0 + \beta_1 t\right)x^2y + \left(\delta_0 + \delta_1 t\right)xy^2 \\ & + \left(\gamma_0 + \gamma_1 t\right)x^2 + (\kappa_0 + \kappa_1 t)y^2 + (\epsilon_0 + \epsilon_1 t)xy \\ & + \left(\zeta_0 + \zeta_1 t\right)x + \left(\lambda_0 + \lambda_1 t\right)y + \left(\mu_0 + \mu_1 t\right) = 0. \end{aligned} \tag{3}$$

So the coefficients in $B(x, y; t)$ are affine functions of the parameter t and $B'(x, y; \tau) = 0$ has primed versions of the coefficients and is affine in the parameter τ. We are interested in maps

$$M: B(x, y; t) = 0 \Rightarrow B'(x', y'; \tau) = 0. \tag{4}$$

The affine dependence on t and τ means that each family of curves $B = 0$ and $B' = 0$ is a *fibration* of the plane (if there are no base points of the fibration, which all fibres pass through, then the family is a *foliation* of the plane). Birational maps acting on fibrations is a well-studied topic in algebraic geometry [2]. For generic values of t, the biquadratic curve (3) is an elliptic curve [3].

One possibility in (4) is that $B' = B$ and $\tau = t$ and M preserves the family, mapping each curve of the fibration $B = 0$ to itself. A paradigm example of this is the so-called symmetric Quispel–Roberts–Thompson (QRT) map M_{QRT} [15] that preserves each curve of the family (3) when it is symmetric under interchange of x and y, i.e. M_{QRT}: $B_{\mathrm{sym}}(x, y; t) = 0 \Rightarrow B_{\mathrm{sym}}(x', y'; t) = 0$. Since B is affine in t, $B = 0$ can be solved for t to give $t(x, y)$, a ratio of biquadratics, which is an *integral* of the dynamics as $t(x, y) = t(x', y')$ [15].

In recent years, there has been a great interest in the case when the family $B' = 0$ is not the same as the family $B = 0$ and many examples of the form M have been presented (some authors have called these 'non-QRT' maps [11, 12]). These examples fall into two categories, which we label the *intrafibration* case and the *interfibration* case[4].

In the intrafibration case, M sends a biquadratic curve labelled t in (3) to a different one in the same family, labelled $\tau = f(t) \neq t$, so

$$M: B(x, y; t) = 0 \Rightarrow B(x', y'; \tau) = 0. \tag{5}$$

Repeated application of M induces an action on the parameter space of the family, necessarily a finite order action when the curve is elliptic [3, 13].

Example 1. Kassotakis and Joshi [12] give an example when (3) is symmetric in x and y, so $\delta_i = \beta_i$, $\kappa_i = \gamma_i$, $\lambda_i = \zeta_i$ $(i = 0, 1)$. With the remaining parameters satisfying:

$$\alpha_1 = \alpha_0\epsilon_0 - \beta_0^2,\ \beta_1 = 2\left(\alpha_0\zeta_0 - \beta_0\gamma_0\right),\ \gamma_1 = \beta_0\zeta_0 - \gamma_0\epsilon_0, \tag{6}$$

[4] We point out that this division, whilst convenient to reference the examples in the literature, is artificial because an intrafamily case becomes an instance of the interfamily case under the process of parameter interchange outlined in [9].

$$\epsilon_1 = 4\left(\alpha_0\mu_0 - \gamma_0^2\right),\ \zeta_1 = 2\left(\beta_0\mu_0 - \gamma_0\zeta_0\right),\ \mu_1 = \epsilon_0\mu_0 - \zeta_0^2. \tag{7}$$

then (5) is achieved with $\tau = -t$ by the map

$$M_{KJ}\colon x' = y,\ \ y' = -\frac{2\left(\gamma_0 y^2 + \zeta_0 y + \mu_0\right) + x\left(\beta_0 y^2 + \epsilon_0 y + \zeta_0\right)}{\left(\beta_0 y^2 + \epsilon_0 y + \zeta_0\right) + 2x\left(\alpha_0 y^2 + \beta_0 y + \gamma_0\right)}. \tag{8}$$

Hence M_{KJ}^2 preserves each curve of the family. Equivalently, it is said that $B(x, y, t) = 0$ solved for $t = t(x, y)$ provides a two-integral for M_{KJ} since $t(x'', y'') = -t(x', y') = t(x, y)$. The study of so-called alternating maps and k-integrals in the literature pertains to the intrafibration case [6, 7, 11, 12, 19].

In the interfibration case, a map of the form (1) sends a biquadratic curve of the form (3) with parameter t_i to a biquadratic curve with parameter t_{i+1} in a different biquadratic family:

$$M_i\colon B_i(x, y; t_i) = 0 \Rightarrow B_{i+1}\left(x', y'; t_{i+1}\right) = 0. \tag{9}$$

Examples in the literature have included cases where there is a finite sequence of p biquadratic families $(B_i(x, y; t_i) = 0,\ \ i = 1, \ldots, p)$ and a finite sequence of p maps $(M_i,\ \ i = 1, \ldots, p)$, such that $M_p \circ \ldots M_2 \circ M_1$ returns the original biquadratic family B_1 to itself. The alternating integrable maps of [14] provide an illustration of this with $p = 2$; for $p = 3$ see [8]. One avenue to the periodic families of biquadratics and associated maps of [17] and [5] was the prior studies of discrete Painlevé equations (see also [16]).

Example 2. An example of (9) with $p = 15$ is given in [4]:

$$M_i\colon x' = y,\ y' = \frac{1}{x}\frac{a_i\, y + b_i}{e\, y + d_i}, \tag{10}$$

with

$$\begin{aligned} B_i(x, y; t) = {} & e d_{i+1} b_{i+2}\, x^2 y^2 + \left(d_{i+1} d_{i+2} b_{i+3} + e a_i b_{i+2}\right) x y^2 + \left(a_i d_{i+2} b_{i+3}\right) y^2 \\ & + \left(d_i d_{i+1} b_{i+2} + e a_{i-1} b_{i+1}\right) x^2 y + \left(a_{i-1} d_i b_{i+1}\right) x^2 \\ & + \left(d_{i+2} b_i b_{i+3} + a_i a_{i+1} b_{i-1}\right) y + \left(d_i b_{i-1} b_{i+1} + a_{i-1} a_{i-2} b_i\right) x \\ & + a_{i-2} b_i b_{i+2} - t\, xy. \end{aligned} \tag{11}$$

Here b_i is a sequence of period 5 $(b_{i+5} = b_i)$, d_i of period 3 $(d_{i+3} = d_i)$, e is an arbitrary constant with $e a_i = d_{i+1} d_{i+2}$ and $t_i = t_{i+1} = t$. A special case of (10) with period 5 was studied in Ramani *et al* [17]:

$$M_i\colon x' = y, \quad y' = \frac{1}{x}\frac{y + b_i}{y - 1}, \quad 1 \leqslant i \leqslant 5. \tag{12}$$

Our aim here is to ask when can a map of the form (1) sends a biquadratic curve to another, under what conditions on the two curves can this occur and what form does the resulting map M take (including what is the simplest form it can take)? By considering biquadratics over a function field, we cover the same questions addressed to fibrations like (4). Theorem 3 below answers these questions and explains the form of the many examples in the recent literature. It is well-known that the so-called j-invariant of two elliptic curves mapped birationally to each other is preserved and a discriminantal condition is given in theorem 3 that implies preservation of the j-invariant. In effect, theorem 3 provides the generalisaton of the symmetric QRT map to the case that the source and target biquadratics are not the same.

Theorem 3 has some consequences that we explore in section 3. It allows us to see that sometimes the examples given in the literature are not the simplest way to map between the biquadratic curves. It also allows us to systematically approach questions like what is the most general intrafibration map for various scenarios (see proposition 4).

2. Characterizing theorem

The image of an algebraic curve under a birational map will also be an algebraic curve. The problem becomes more restrictive if we require the source and target curves to be of the same type—here we impose that they are both biquadratic curves—and fix the type of birational map that transforms one curve to the other—here we impose the form M of (1). Already, one sees some ambiguity in the answer (as we will see below) because if there exists an M that sends biquadratic $B = 0$ to biquadratic $B' = 0$, then

$$\text{Auto}' \circ M \circ \text{Auto} \tag{13}$$

is another such mapping if Auto (Auto′) is a birational automorphism of B (B') and the composition (13) is also of the form (1).

Let $B(x, y) = 0$ and $B'(x, y) = 0$ be the equations for two biquadratic curves which will be denoted

$$B(x, y) = \alpha x^2y^2 + \beta x^2y + \delta xy^2 + \gamma x^2 + \epsilon xy + \kappa y^2 + \zeta x + \lambda y + \mu \tag{14}$$

and

$$B'(x, y) = \alpha' x^2y^2 + \beta' x^2y + \delta' xy^2 + \gamma' x^2 + \epsilon' xy + \kappa' y^2 + \zeta' x + \lambda' y + \mu'. \tag{15}$$

We consider x, y to belong to some field K. The entries of the vector of curve parameters $(\alpha, \beta, \ldots, \mu)$ and their primed counterparts can be in K, or more generally in the *rational function field* $K(t)$. The latter case means curve parameters are a rational function of some parameter t, i.e. $\alpha = \alpha(t)$ etc so the biquadratic curve over the function field represents a family of curves in K^2 as we let t vary across some field K (in our previous work [9], we used the term *curve-dependent maps* to refer to maps sending biquadratics over rational function fields to themselves, see also [10]). Here we take $K = \mathbb{R}$ or $K = \mathbb{C}$. Recall the notation for the multiplicative group $K^* = K \backslash \{0\}$.

By definition, a biquadratic expression can be written as a quadratic in x or in y, so we can write

$$\begin{aligned} B(x, y) &= X_2(x)y^2 + X_1(x)y + X_0(x) \\ &= Y_2(y)x^2 + Y_1(y)x + Y_0(y) = 0, \end{aligned} \tag{16}$$

where

$$\begin{aligned} &X_2(x) = \alpha x^2 + \delta x + \kappa, \quad X_1(x) = \beta x^2 + \epsilon x + \lambda, \quad X_0(x) = \gamma x^2 + \zeta x + \mu, \\ &Y_2(y) = \alpha y^2 + \beta y + \gamma, \quad Y_1(y) = \delta y^2 + \epsilon y + \zeta, \quad Y_0(y) = \kappa y^2 + \lambda y + \mu. \end{aligned} \tag{17}$$

We shall denote the discriminant of the quadratic B with respect to y, which is a polynomial in x, by $\Delta_y(B)(x)$:

$$\Delta_y(B)(x) = X_1(x)^2 - 4\,X_0(x)X_2(x)$$
$$= \left(\beta^2 - 4\alpha\gamma\right)x^4 + (2\beta\epsilon - 4\alpha\zeta - 4\delta\gamma)x^3 + \left[2\beta\lambda + \epsilon^2 - 4(\delta\zeta + \gamma\kappa + \alpha\mu)\right]x^2$$
$$+ (2\epsilon\lambda - 4\kappa\zeta - 4\delta\mu)x + \left(\lambda^2 - 4\kappa\mu\right). \quad (18)$$

Similarly, we use $\Delta_x(B)(y)$ to denote the polynomial in y that is the discriminant of the quadratic B with respect to x:

$$\Delta_x(B)(y) = Y_1(y)^2 - 4\,Y_0(y)Y_2(y)$$
$$= \left(\delta^2 - 4\alpha\kappa\right)y^4 + (2\delta\epsilon - 4\alpha\lambda - 4\beta\kappa)y^3 + \left[2\delta\zeta + \epsilon^2 - 4(\beta\lambda + \gamma\kappa + \alpha\mu)\right]y^2$$
$$+ (2\epsilon\zeta - 4\gamma\lambda - 4\beta\mu)y + \left(\zeta^2 - 4\gamma\mu\right). \quad (19)$$

To guarantee that both discriminants are genuine quartics, we make the following:

Assumption. The leading coefficients $\beta^2 - 4\alpha\gamma$ and $\delta^2 - 4\alpha\kappa$ of $\Delta_y(B)(x)$, respectively $\Delta_x(B)(y)$, are non-zero.

As quartic polynomials, $\Delta_y(B)(x)$ and $\Delta_x(B)(y)$ each have their own discriminants, expressible in terms of their coefficients, which vanish if and only if the quartics have a repeated root over their *splitting field*. It turns out that these discriminants are one and the same expression for each quartic, which we denote $\Delta_{xy}(B)$. We call the biquadratic *non-singular* when $\Delta_{xy}(B) \neq 0$. This means that $\Delta_y(B)$ and $\Delta_x(B)$ can then only have simple roots. It also means that the biquadratic curve $B = 0$ is a non-singular elliptic curve, transferrable to Weierstrass cubic form $W\colon Y^2 = X^3 + W_1X + W_0$ with discriminant $\Delta(W) = \Delta_{xy}(B)/256 \neq 0$ [3, 13].

Analogous expressions to (17)–(19) can be written for the biquadratic $B'(x, y) = 0$ given by

$$B'(x, y) = X_2'(x)y^2 + X_1'(x)y + X_0'(x)$$
$$= Y_2'(y)x^2 + Y_1'(y)x + Y_0'(y) = 0, \quad (20)$$

and its discriminants. One simple way to send the biquadratic $B(x, y)$ to the particular case of $B'(x, y) = B(y, x) = 0$ is to use the switch

$$S\colon x' = y, \;\; y' = x, \quad (21)$$

since $B' = B \circ S$. We remark that the effect of the switch S of (21) on the biquadratic is to switch the coefficients of the respective discriminants, i.e.

$$\Delta_y(B \circ S)(y) = \Delta_x(B)(y), \quad \Delta_x(B \circ S)(x) = \Delta_y(B)(x). \quad (22)$$

Furthermore, any map of the form M of (1) can be written

$$M = P_x \circ S, \quad (23)$$

where P_x fixes x:

$$P_x\colon x' = x, \;\; y' = R(y, x). \quad (24)$$

Lemma 1. *The map M sends $B(x, y) = 0$ to $B'(x, y) = 0$ if and only if P_x sends $B(y, x) = 0$ to $B'(x, y) = 0$.*

Proof. Suppose M sends $B(x, y) = 0$ to $B'(x, y) = 0$. Since S sends $B(x, y) = 0$ to the biquadratic $B(y, x) = 0$ and M can be decomposed as (23), it follows that P_x sends $B(y, x) = 0$ to $B'(x, y) = 0$. Conversely, if $P_x = M \circ S$ sends $B(y, x) = 0$ to $B'(x, y) = 0$, then since S sends $B(x, y) = 0$ to $B(y, x) = 0$, $M = P_x \circ S$ sends $B(x, y) = 0$ to $B'(x, y) = 0$. □

Provided $X_2 \neq 0$ in (17), which excludes possibly two points on the curve

$$B(x, y) = 0 \quad \Leftrightarrow \quad \left(y + \frac{X_1(x)}{2\, X_2(x)}\right)^2 - \frac{\Delta_y(B)(x)}{4\, X_2(x)^2} = 0. \tag{25}$$

Similarly, provided $X_2' \neq 0$ in (20), which excludes possibly two points on the curve

$$B'(x, y) = 0 \quad \Leftrightarrow \quad \left(y + \frac{X_1'(x)}{2\, X_2'(x)}\right)^2 - \frac{\Delta_y(B')(x)}{4\, X_2'(x)^2} = 0. \tag{26}$$

Observe $B(x, y) = 0 \Leftrightarrow B'(x, y') = 0$ when y′ and y are related by

$$P_x^{\pm}\colon \; y' + \frac{X_1'(x)}{2\, X_2'(x)} \; = \; \pm \sqrt{\frac{\Delta_y(B')(x)}{\Delta_y(B)(x)}}\, \frac{X_2(x)}{X_2'(x)} \left(y + \frac{X_1(x)}{2\, X_2(x)}\right). \tag{27}$$

Note the subscript ‘x’ in $P_x^{\pm}$ is used to show each choice of sign is a transformation $(x, y) \mapsto (x', y')$ that fixes x. Recall that there are the well-known *involutions* I_x and I_x' [15] which fix x and switch the roots in y for, respectively, $B(x, y) = 0$ and $B'(x, y) = 0$:

$$I_x\colon \; y' = -y - \frac{X_1(x)}{X_2(x)} \quad \Leftrightarrow \quad y' + \frac{X_1(x)}{2\, X_2(x)} = -\left(y + \frac{X_1(x)}{2\, X_2(x)}\right) \tag{28}$$

and

$$I_x'\colon \; y' = -y - \frac{X_1'(x)}{X_2'(x)} \quad \Leftrightarrow \quad y' + \frac{X_1'(x)}{2X_2'(x)} = -\left(y + \frac{X_1'(x)}{2\, X_2'(x)}\right). \tag{29}$$

Note that I_x (I_x') can be taken to be the representative non-identity (birational) curve automorphism fixing x and preserving $B = 0$ ($B' = 0$), and we see that $P_x^{\pm}$ becomes I_x or the identity when $B' \equiv B$. Of course, I_x (I_x') is the asymmetric QRT involution on $B = 0$ ($B' = 0$) and the QRT map follows from the composition of I_x and $I_y = S \circ I_x \circ S$, the corresponding automorphism on $B = 0$ with y fixed.

In general, $P_x^{\pm}$ are maps between the biquadratic curves defined over $\mathbb{C}$ due to the presence of the square root of a polynomial being required. The following result gives conditions for the birationality of $P_x^{\pm}$ and establishes their generality.

Theorem 2. *Let $B = 0$ be a nonsingular biquadratic curve over the field $\mathbb{C}$. There exists a birational map P over $\mathbb{C}^2$ fixing x and sending $B = 0$ to the biquadratic $B' = 0$ if and only if*

$$\Delta_y(B')(x) = c^2\, \Delta_y(B)(x), \qquad c \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}. \tag{30}$$

The biquadratic $B' = 0$ is also nonsingular and, modulo equivalence on $B = 0$, the map P can be chosen to be $P_x^{\pm}$ above, i.e. either of the maps

$$P_x^{\pm}: \quad x' = x, \quad y' = \pm\left(\frac{c\ X_2(x)}{X_2'(x)}\right)y + \left(\frac{\pm c\ X_1(x) - X_1'(x)}{2\ X_2'(x)}\right).$$

The two maps are not independent since:

$$P_x^{\pm} \circ I_x = P_x^{\mp} \ \text{ and } \ I_x' \circ P_x^{\pm} = P_x^{\mp} \Rightarrow I_x' \circ P_x^{\pm} \circ I_x = P_x^{\pm} \tag{31}$$

with I_x and I_x' of , respectively, (28) and (29).

Proof. Denote the roots of the quadratic (25) in y by

$$y = r_{\pm}(x) = \frac{-X_1 \pm \sqrt{\Delta_y(B)}}{2X_2}, \tag{32}$$

and similarly for the quadratic (26)

$$y = R_{\pm}(x) = \frac{-X_1' \pm \sqrt{\Delta_y(B')}}{2X_2'}. \tag{33}$$

For each fixed x, we want to invertibly map the set of roots $y = \{r_+, r_-\}$ to $y' = \{R_+, R_-\}$, giving one possibility where the parity of the sign labels is preserved and a second possibility where they are switched. We require y' to be a rational function of y with coefficients rational in x, and likewise for the inverse map taking y' to y. Since quadratic (and higher) terms in y and y' in these maps can always be removed using the equations of the biquadratics (25) and (26), the map P from y to y' can be taken necessarily to be a Möbius or fractional linear transformation

$$P: \ y' = \frac{\Lambda\, y + \Sigma}{\Omega\, y + \Gamma}, \quad \Lambda = \Lambda(x),\ \Sigma = \Sigma(x),\ \Omega = \Omega(x),\ \Gamma = \Gamma(x). \tag{34}$$

Möbius transformations constitute the group of birational maps of the projective line. Without loss of generality, we can study the case

$$P(r_+) = R_+, \quad P(r_-) = R_-, \tag{35}$$

since the case with the right-hand sides interchanged is given by $I_x' \circ P$. Any solution for the map P will satisfy

$$P \circ I_x = I_x' \circ P, \tag{36}$$

with I_x of (28) and I_x' of (29). Imposing (35) on P of (34) gives two linear equations which can be solved for

$$\Lambda = \frac{R_+\left(\Omega\, r_+ + \Gamma\right) - R_-\left(\Omega\, r_- + \Gamma\right)}{r_+ - r_-},$$
$$\Sigma = \frac{r_+ R_-\left(\Omega\, r_- + \Gamma\right) - r_- R_+\left(\Omega\, r_+ + \Gamma\right)}{r_+ - r_-}, \tag{37}$$

with Ω and Γ free. If $\Omega = 0$, then Γ cancels in P and we have the affine map:

$$P_0\!: y' = \left(\frac{R_+ - R_-}{r_+ - r_-}\right) y + \left(\frac{r_+ R_- - r_- R_+}{r_+ - r_-}\right). \tag{38}$$

If $\Omega \neq 0$, we can take $\Omega = 1$ without loss of generality in (37) and (34) together with Γ free. This gives a genuine fractional linear transformation which we denote as P_1. However, some manipulation shows that P_1 is equivalent to P_0 modulo the biquadratic $B(x, y) = 0$. In other words, one finds

$$P_0 \cdot \text{ denominator}\left(P_1\right) - \text{ numerator}\left(P_1\right) \equiv 0 \quad \left(\text{mod } B = 0\right), \tag{39}$$

by replacing y^2, r_+^2 and r_-^2 on the left-hand side using $B(x, y) = B(x, r_+) = B(x, r_-) = 0$.

We now simplify the form of P_0 of (38) by using $r_- = -r_+ - X_1/X_2$ and $R_- = -R_+ - X_1'/X_2'$ and then substituting the definition of r_+ via (32) and R_+ via (33). We readily find that P_0 is P_x^+ of (27), whereas $P_0 \circ I_x = I_x' \circ P_0 = P_x^-$ of (27), as expected. For $P_x^\pm$ of (27) to be rational in x, and thus to be a birational map of $\mathbb{C}^2$, we require the ratio of the quartic discriminants to be a perfect square in $\mathbb{C}(x)$, i.e.

$$\frac{\Delta_y(B')(x)}{\Delta_y(B)(x)} = \frac{N^2(x)}{D^2(x)}, \tag{40}$$

where $N(x)$ and $D(x)$ are relatively prime polynomials. However, this implies that any root of $N(x)$, if $N(x)$ is non-constant, is at least a double root of $\Delta_y(B')$ and any root of $D(x)$, if $D(x)$ is non-constant, is at least a double root of $\Delta_y(B)$. By assumption $B(x, y) = 0$ and $B'(x, y) = 0$ are non-singular curves with $\Delta_{xy}(B) \neq 0$ and $\Delta_{xy}(B') \neq 0$, respectively, so their discriminants do not have multiple roots. Consequently, $N(x)$ and $D(x)$ must be constants so that $\Delta_y(B')(x)$ and $\Delta_y(B)(x)$ differ by a constant, which must be a positive constant in the real case. Substituting this into (27) gives the forms advertised in the statement of the theorem. In particular, in the real case, the quartics $\Delta_y(B')(x)$ and $\Delta_y(B)(x)$ are positive in the same x-intervals, so that $B(x, y) = 0$ and $B'(x, y) = 0$ occupy the same real domains in x.

This shows that a birational map from $B = 0$ to $B' = 0$ exists only if the curve discriminants obey (30) and it then takes the form $P_x^\pm$. Conversely, if the curve discriminants satisfy (30), $P_x^\pm$ constitute birational maps between them. □

Remarks.

(1) Note that $c\,B(x, y) = 0$ for $c \in K^*$ defines the same curve as $B(x, y) = 0$, so two biquadratics are *equivalent* if the ratio of their corresponding coefficients is the same constant, or their vectors of parameters are parallel. Since $\Delta_x(cB) = c^2\Delta_x(B)$ and $\Delta_y(cB) = c^2\Delta_y(B)$, the discriminants of a biquadratic curve are defined up to their multiplication by c^2. This means that if the curve parameters are real constants, one of the quartic discriminants can always be taken to have leading coefficient ±1; if the curve is over $\mathbb{C}$, one of the discriminants can be taken to be monic. If (30) is satisfied, we can always replace $B = 0$ by the equivalent biquadratic $cB = 0$, in which case c can be taken to be 1. So the theorem could alternatively be stated in terms of birational maps between equivalence classes of biquadratics, in which case $c = 1$ without loss of generality. But practically speaking, when dealing with biquadratics, as presented, it is useful to retain the 'up to a multiplicative constant' freedom of (30).

(2) The conditions of the theorem arising from (30) are that the ratios of corresponding coefficients of the two quartic discriminants are the same. We make this more explicit in theorem 3 below.

(3) Equation (31) illustrates (13), noting also that the identity on each curve is an obvious automorphism. It shows that the two x-preserving transformations $P_x^\pm$ from the biquadratic $B = 0$ to the biquadratic $B' = 0$ are related to each other via the automorphisms I_x and I_x' but are a closed set of transformations between the curves up to

these root-switching freedoms. The requirement that x is fixed, restricts the possibilities. The reason we list both P_x^+ and P_x^- is that one of them may have a simpler form than the other in a particular example, and so is the more fundamental way to move between the curves.

Example 3. Here are some simple maps that fix x and obviously map the biquadratic (16) to the biquadratic (20), leading to (30) being satisfied (cases (a) and (b) are included to illustrate remark (1) above).

(a) $y' = Ey\ (E \in K^*) \quad \Rightarrow \quad X_2'(x) = X_2(x)/E^2,\ X_1'(x) = X_1(x)/E,\ X_0'(x) = X_0(x)$ so $c = 1/E$.
(b) $y' = Ey\ (E \in K^*) \quad \Rightarrow \quad X_2'(x) = X_2(x)/E,\ X_1'(x) = X_1(x),\ X_0'(x) = E\, X_0(x)$ so $c = 1$.
(c) $y' = 1/y \quad \Rightarrow \quad X_2'(x) = X_0(x),\ X_1'(x) = X_1(x),\ X_0'(x) = X_2(x)$ so $c = 1$.

We observe that the maps in (a)–(b) correspond to the form P_x^+. The map in (c) is not affine in y but on $B = 0$ we have

$$1/y \equiv -\left(\frac{X_2(x)}{X_0(x)}\right) y - \left(\frac{X_1(x)}{X_0(x)}\right),$$

which is P_x^- as here $X_2'(x) = X_0(x)$. In this example, P_x^+ is $y' = \left(\frac{X_2(x)}{X_0(x)}\right) y$ and one confirms that this also sends (16) to the biquadratic (20) in this case.

To build more exotic examples of birational maps that send biquadratics to biquadratics but do not fix x, we can involve the switch S. We know that the switch S exchanges x and y in $B = 0$. In the process, this leads to $X_i(x)$ for $B = 0$ being the same as $Y_i(x)$ for $B \circ S = 0$ and vice versa. The discriminant changes according to (22). Via lemma 1 and theorem 2, we then have:

Theorem 3. *There exists a birational map M of the form (1) over $\mathbb{C}^2$ sending the nonsingular biquadratic $B = 0$ to the nonsingular biquadratic $B' = 0$ if and only if*

$$\Delta_y(B')(x) = c^2 \Delta_y(B \circ S)(x) = c^2 \Delta_x(B)(x), \tag{41}$$

with $c \in \mathbb{C}^*$. Hence from (19)

$$\frac{\text{Coeff}\left[\Delta_x(B)(y),\, i\right]}{\text{Coeff}\left[\Delta_y(B')(x),\, i\right]} = \frac{\text{Coeff}\left[\Delta_x(B)(y),\, i+1\right]}{\text{Coeff}\left[\Delta_y(B')(x),\, i+1\right]},\ i = 0, 1, 2, 3, \tag{42}$$

where $\text{Coeff}[\Delta_x(B)(y), i]$ is the coefficient of y^i in $\Delta_x(B)(y)$ of (19) and $\text{Coeff}[\Delta_y(B')(x), i]$ is the coefficient of x^i in the primed version of $\Delta_y(B)(x)$ of (18). Modulo equivalence on $B = 0$, and noting (16) and (20), the map M is either of

$$M^\pm\!: x' = y, \quad y' = \pm\left(\frac{c\ Y_2(y)}{X_2'(y)}\right) x + \left(\frac{\pm c\ Y_1(y) - X_1'(y)}{2\, X_2'(y)}\right).$$

The two maps are related by

$$M^\pm \circ I_y = M^\mp \quad \text{and} \quad I_x' \circ M^\pm = M^\mp \Rightarrow I_x' \circ M^\pm \circ I_y = M^\pm \tag{43}$$

with I_x and I_x' of, respectively, (28) and (29) and $I_y = S \circ I_x \circ S$.

We finish this section with the following

Remarks.

(1) Our main interest is the case of fibrations of biquadratic curves as exemplified by the examples in the Introduction. Each fibration is an elliptic curve over the *function field*. theorem 3 holds in this case as well, as the arguments of the proof of theorem 2 hold. The parameters in the biquadratics (25) and (26) are now taken to be affine in their respective fibration parameters t and τ (the intrafibration case of (5)) or t_i and t_{i+1} (the interfibration case of (9)). This follows through to the associated discriminants being quadratic in their respective fibration parameters so that (42) represents four conditions linking the fibration parameters of source and target biquadratics. The maps $M^{\pm}$ notionally depend on the respective fibration parameters, but the parameters can be eliminated using the equation of the biquadratic in each case. In the fibration case, we can always reparametrize the fibration parameter in a source or target biquadratic by a Möbius transformation $T\colon t' \mapsto t$

$$T\colon t = \frac{At' + E}{Ct' + D}, \tag{44}$$

which transforms (3) to another fibration, affine in t'. Such reparametrization is actually encompassed by our theorems as we can take $B'(x, y, t') = (Ct' + D)B(x, y, T(t')) = 0$, in which case (30) is satisfied with $c = (Ct' + D)$ and P_x^+ of theorem 2 becomes the identity and P_x^- the QRT involution I_x of (28). Conversely, to test if two biquadratics $B'(x, y, t') = 0$ and $B(x, y, t) = 0$ are related by a reparametrization, one must have that equality of their coefficient ratios $\alpha'(t')/\alpha(t) = \beta'(t')/\beta(t) = \ldots = \mu'(t')/\mu(t)$ leads to one and the same Möbius function for t' in terms of t. It is clear this then gives $P_x^+ = id$ and satisfaction of (30).

(2) The form of $M^{\pm}$ is invariant under multiplication of B and B' by (possibly different) constants, relating to remark (1) after theorem 2. When we take $B' = B$ together with $B(x, y) = B(y, x)$ in the theorem, we find that M^+ becomes the switch S of (21) and M^- becomes $I_x \circ S$, the symmetric McMillan map on the curve: $x' = y$, $y' = -x - \frac{X_1(y)}{X_2(y)}$ (or the curve-dependent-McMillan version of the symmetric QRT map if the curve is a fibration [9]).

(3) The j-invariant of the biquadratic B is given by

$$j(B) = j(W) = -\frac{1728\, I^3}{I^3 - 27J^2}. \tag{45}$$

The quantities I and J are the (Eisenstein) invariants [3, 13] expressed in the coefficients of the quartics $\Delta_y(B)$ or $\Delta_x(B)$—they are the same in each case. The condition (41) ensures that the corresponding invariants for B' satisfy $I' = c^4 I$ and $J' = c^6 J$ so that $j(B') = j(B)$. As it must be, theorem 3 is consistent with the well-known fact from algebraic geometry that if elliptic curves share the same j-invariant, then there is a birational map between them and vice versa.

3. Examples and applications

Note that the condition (41) says

$$X_1'(x)^2 - 4\,X_0'(x)X_2'(x) = c^2\left[\,Y_1(x)^2 - 4\,Y_0(x)Y_2(x)\right]. \tag{46}$$

Example 4. One simple way to satisfy (46) is for $X_1'(x) = \epsilon\, c\, Y_1(x)$ with $\epsilon = \pm 1$ and also

$$X_0'(x)X_2'(x) = c^2\,Y_0(x)Y_2(x). \tag{47}$$

The latter condition implies, for instance if $X_0'(x) \neq 0$, that

$$X_2'(x) = \frac{c^2\,Y_0(x)Y_2(x)}{X_0'(x)}. \tag{48}$$

Since all the functions involved are polynomials of maximal degree 2, we see that there must be the divisibililty $X_0'(x)|\,Y_0(x)Y_2(x)$. From theorem 3, we have:

$$M^\epsilon\colon x' = y, \quad y' = \epsilon\left(\frac{c\,Y_2(y)}{X_2'(y)}\right)x. \tag{49}$$

It turns out that many cases we have checked in the literature, e.g. [1, 4, 19], follow the form of the previous example. However, sometimes the birational maps advertised in the literature are not of the form (49), appearing more complicated because they take the alternative form offered by theorem 3, the two forms related via the relations (43) (the alternative forms might be said to borrow some of their non-QRT features from the associated QRT map).

For instance, using unprimed coefficient functions for B_i and primed coefficient functions for B_{i+1} of (11), we have: $X_1'(x) = Y_1(x) = [d_{n+1}d_{n+2}b_{n+3} + e a_n b_{n+2}]\,x^2 - tx + d_n b_{n-1}b_{n+1} + b_n a_{n-1}a_{n-2}$; $Y_2(x) = (e\,x + d_n)(d_{n+1}b_{n+2}\,x + a_{n-1}b_{n+1})$; $Y_0(x) = (a_n x + b_n)(d_{n+2}b_{n+3}\,x + a_{n+1}b_{n-1})$; $X_2'(x) = \frac{d_{n+2}}{e}(e\,x + d_n)(e\,b_{n+3}\,x + d_n b_{n+4})$;$X_0'(x) = (a_n x + b_n)(d_{n+1}b_{n+2}\,x + a_{n-1}b_{n+1})$. Note how the factorizations (47) and (48) are achieved with $c^2 = 1$. The discriminants of B_i and B_{i+1} are equal. We then find that the map presented by [4] is M^- of theorem 3. On the other hand, M^+ follows from (49) with $\epsilon = c = 1$ as:

$$M_i^+\colon x' = y, \quad y' = \frac{d_{n+1}}{d_{n-1}}\left(\frac{e\,b_{n+2}\,y + d_n b_{n+1}}{e\,b_{n-2}\,y + d_n b_{n-1}}\right)x, \tag{50}$$

recalling the periodicities of d_n (i.e. 3) and b_n (i.e. 5) and noting $a_{n-1} = d_n d_{n+1}/e$.

Finally, we point out that the use of the discriminant enables us to make general statements about the possibility to have intrafibration mappings, their orders and their forms. We call the fibration (3) *symmetric* when $B(x, y; t) = B(y, x; t)$, so $\{\delta_i = \beta_i, \kappa_i = \gamma_i, \lambda_i = \zeta_i;\ i = 0, 1\}$. We say the fibration (3) is generalized McMillan when the only non-zero coefficient of t is μ_1; otherwise, we call it a QRT fibration.

Proposition 4. *Consider the fibration $B_t := B(x, y; t) = 0$ of (3) and maps of the form P of (24) or M of (1) that preserve the fibration (e.g. as in (5)). Using theorems 2 and 3, we have:*

(1) When the fibration is generalized McMillan and symmetric, maps of the form P or M necessarily preserve each fibre.

(2) When the fibration is QRT, the possible intrafibration period for a map P is necessarily 2 and we can take $\tau = -t$, possibly after a Möbius reparametrization. The map P^+ of theorem 2 is then $S \circ h_1 \circ S$ where h_1 is from proposition 4.1 of [12].

(3) When the fibration is QRT and symmetric, the possible intrafibration period for a map M is necessarily 2 and we can take $\tau = -t$, possibly after a Möbius reparametrization. The map M^+ of theorem 3 is then M_{KJ} of (8).

Proof. If the fibration B_t is symmetric in x and y and preserved by M, as in cases (1) and (3), this is equivalent to the fibration being preserved by P using (23). The condition follows from (30), giving a modification of (42) with $\Delta_y(B_t)(x)$ now in the numerators of (42) and $\Delta_y(B')(x) = \Delta_y(B_\tau)(x)$ in the denominators, i.e. the corresponding numerators and denominators differ by the replacement $t \mapsto \tau$ giving

$$\frac{\mathrm{Coeff}\left[\Delta_y\left(B_t\right)(x),\, i\right]}{\mathrm{Coeff}\left[\Delta_y\left(B_\tau\right)(x),\, i\right]} = \frac{\mathrm{Coeff}\left[\Delta_y\left(B_t\right)(x),\, i+1\right]}{\mathrm{Coeff}\left[\Delta_y\left(B_\tau\right)(x),\, i+1\right]},\ i = 0, 1, 2, 3. \tag{51}$$

This is also the condition for an asymmetric B_t to be preserved by a map of form P, as in case (2) of the proposition. From (19), the numerators (denominators) in (51) are generally quadratic in t (τ) because they appear linearly in (3) so that cross-multiplying in each of the four conditions produces polynomial conditions

$$Q_i(t, \tau) = 0, \quad i = 0, 1, 2, 3,$$

that are most generally quadratic in t and τ and antisymmetric in them. One solution in each case is clearly $\tau = t$, corresponding to the existence of I_x that preserves each fibre. Factoring out $(t - \tau)$ from the left-hand side of $Q_i(t, \tau) = 0$ results in four equations, most generally affine in t and τ and symmetric in them:

$$A_i\,(T + t) + B_i\,Tt + C_i = 0, \quad i = 0, 1, 2, 3. \tag{52}$$

For preservation of the fibration, these four conditions need to be consistent, which defines relations between the parameters in B_t. When this occurs, one sees the symmetry means that the map $t \mapsto \tau$ is necessarily an involution (order 2) map in the fibration parameter space. But Möbius involutions are conjugate within the group of (real or complex, as the case may be) Möbius transformations to $\tau = -t$. This provides the intrafibration period of 2 in (2) and (3). The form of the P in (2) and hence the M in (3) follows from [12] which solves when maps of this type change the sign of the fibration parameter (or integral). For case (1), it is easy to check that there are only three non-trivial conditions following from (51) and the coefficients are maximally of degree one in τ and t. The assumption of section 2 that $\delta^2 - 4\alpha\kappa \neq 0$ forces then $\tau = t$ so each fibre is preserved.

In closing, we remark that there are birational maps not of the form (1) that can, for instance, achieve (5), e.g. see [18]. We are presently investigating how these maps might be understood in a similar vein to the class given by (1).

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