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Interchanging parameters and integrals in dynamical systems: the mapping case

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Abstract
We consider dynamical systems with discrete time (maps) that possess one or more integrals depending upon parameters. We show that integrals can be used to *replace* parameters in the original map so as to construct a different map with different integrals. We also highlight a process of *reparametrization* that can be used to increase the number of parameters in the original map prior to using integrals to replace them. Properties of the original map and the new map are compared. The theory is motivated by, and illustrated with, examples of a three-dimensional trace map and some four-dimensional maps previously shown to be integrable.

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1. Introduction

Dynamical systems come in two kinds: continuous or discrete. We will restrict our discussion in this paper to finite-dimensional systems. Continuous dynamical systems (ordinary differential equations) have been studied since the time of Newton, and they come in a number of different classes, each with their own characteristic properties [11].

In recent years much effort has been devoted to finding discrete analogues of the various classes of ODEs. This has resulted in the discovery and study of integrable mappings [2, 13, 14, 16, 17], discrete Painleve equations [4], continuous symmetries of difference equations [6, 18, 22], etc. (Note that there has also been much work on preserving properties such as being symplectic, or divergence-free, or possession of integrals or symmetries in numerical integration algorithms [12].)

Many dynamical systems that are of interest, either for theoretical or for practical reasons, possess integrals (i.e. conserved quantities). Possible examples of such integrals are energy, momentum, angular momentum, but there are many other possibilities. Many of these systems with integrals also contain one or more parameters, i.e. they occur in l-parameter families, where some or all of the integrals may depend on the parameters.

In this paper we will focus on discrete-time dynamical systems (also called maps or mappings). We hope to treat the case of continuous-time dynamical systems (i.e. ordinary differential equations) in a future paper.

This then is the setting we are interested in: l-parameter families of discrete dynamical systems with j first integrals. What we will show in this paper is that parameters and integrals are, in a sense, interchangeable, i.e. we will show how from a map $L : \boldsymbol{x} \mapsto \boldsymbol{x}'$, $\boldsymbol{x} \in \mathbb{R}^n$, with parameter K and integral $I(\boldsymbol{x})$ we can construct another map $\tilde{L}$ with parameter I and integral $K(\boldsymbol{x})$ (under some mild technical conditions).

We will generalize this basic idea to interchanging p parameters and integrals, and also show how in the process the number of parameters in the map may be increased, such that the new map $\tilde{L}$ contains the old one as a special case. We will also investigate to what extent other properties of L, such as volume-preservation, symmetries and time-reversal symmetries, carry over to the new map $\tilde{L}$. This paper is an extension to higher dimensions of [7,8] which studied two-dimensional maps with one biquadratic integral. In the latter papers, it was shown that application of these ideas can, for example, show how to obtain the 18-parameter QRT family of integrable maps [16, 17] from a nine-parameter family of McMillan-like maps.

The outline of this paper is as follows: in section 2 we present three motivating examples. They all illustrate the interchange of a single parameter and an integral, a process we call *replacement*. The second and third examples also illustrate an increase in the number of parameters, a process we call *reparametrization*. In section 3 we present a general theorem, of which the examples of section 2 are a special case. In section 4 we then give two additional examples, with replacement of two parameters, as a further illustration of the theorem, before concluding in section 5.

2. Two motivating examples of 3D and 4D maps illustrating replacement and reparametrization

Example 1. Consider the following three-dimensional map (this example is a rescaling of the so-called Fibonacci trace map [19, 20]):

$$L_1 : \quad \begin{aligned} x' &= y \\ y' &= z \\ z' &= -x - \frac{yz}{K} \end{aligned} \tag{1}$$

where K is an arbitrary parameter. The map L_1 has the following properties:

- L_1 has an integral, i.e.

$$I(x', y', z', K) = I(x, y, z, K) \tag{2}$$

where

$$I(x, y, z, K) = K(x^2 + y^2 + z^2) + xyz \tag{3}$$

- L_1 is volume preserving but orientation-reversing, i.e. $\det dL_1 = -1$,

- L_1 is reversible[3], i.e. there exists a reversing symmetry G such that $G \circ L_1 \circ G^{-1} = L_1^{-1}$, where

$$G: \quad \begin{aligned} x' &= z \\ y' &= y \\ z' &= x. \end{aligned} \tag{4}$$

Notice that I is linear in K and from (2)

$$I(x, y, z, K) = 0 \qquad \Rightarrow \qquad I(x', y', z', K) = 0. \tag{5}$$

The left-hand side of (5) together with (3) can be used to solve for K as a function of x, y and z, i.e. we get $K = k(x, y, z)$, where

$$k(x, y, z) = -\frac{xyz}{x^2 + y^2 + z^2} \tag{6}$$

and it follows that L_1 with the replacement $K = k$ satisfies $k(x', y', z') = k(x, y, z)$. Explicitly, L_1 with the replacement $K = k$ yields the map

$$\tilde{L}_1: \quad \begin{aligned} x' &= y \\ y' &= z \\ z' &= -x - \frac{yz}{k(x, y, z)} = \frac{y^2 + z^2}{x}. \end{aligned} \tag{7}$$

The map $\tilde{L}_1$ has the following properties:

- $\tilde{L}_1$ has an integral,

$$k(x, y, z) = -\frac{xyz}{x^2 + y^2 + z^2} \tag{8}$$

which, as indicated above, follows from (5).

- $\tilde{L}_1$ is measure preserving and orientation-reversing (or anti-measure-preserving), which means [21]

$$\det d\tilde{L}_1 = -\frac{\tilde{\rho}(x, y, z)}{\tilde{\rho}(x', y', z')} \tag{9}$$

where the so-called density $\tilde{\rho}$ is given by

$$\tilde{\rho}(x, y, z) = \frac{1}{x^2 + y^2 + z^2}. \tag{10}$$

Note for future reference that $\tilde{\rho}(x, y, z) = \left[\frac{\partial I}{\partial K}\right]^{-1}$.

- $\tilde{L}_1$ has the symmetry $\tilde{S}$, i.e. $\tilde{S} \circ \tilde{L}_1 \circ \tilde{S}^{-1} = \tilde{L}_1$, where

$$\tilde{S}: \quad \begin{aligned} x' &= -x \\ y' &= -y \\ z' &= -z. \end{aligned} \tag{11}$$

Note that L_1 does not have this symmetry.

- $\tilde{L}_1$ is reversible, where a reversing symmetry is

$$\tilde{G}: \quad \begin{aligned} x' &= z \\ y' &= y \\ z' &= x. \end{aligned} \tag{12}$$

Note that $\tilde{G} = G$.

[3] L_1 also has some k-symmetries [19, 20], which we disregard in this paper.

Example 2. The map,

$$L_2: \quad \begin{aligned} x' &= y \\ y' &= z \\ z' &= -x - \frac{yz}{a_0 + a_1 K} \end{aligned} \tag{13}$$

has the following integral

$$I(x, y, z, K) = (a_0 + a_1 K)(x^2 + y^2 + z^2) + xyz + b_0 + b_1 K. \tag{14}$$

The map L_2 and integral (14) can clearly be obtained from L_1 of example 1 and its integral (3) by reparametrizing $K \to a_0 + a_1 K$ and $I \to I + b_0 + b_1 K$. Again, for I of (14) it follows that

$$I(x, y, z, K) = 0 \qquad \Rightarrow \qquad I(x', y', z', K) = 0, \tag{15}$$

where primes denote images under L_2. Since I of (14) is linear in K, the left-hand side of (15) can be used to solve for $K = k(x, y, z)$, i.e.

$$k(x, y, z) = -\frac{xyz + a_0(x^2 + y^2 + z^2) + b_0}{a_1(x^2 + y^2 + z^2) + b_1} \tag{16}$$

and it follows that L_2 with the replacement $K = k$ satisfies $k(x', y', z') = k(x, y, z)$. This new map is given by

$$\tilde{L}_2: \quad \begin{aligned} x' &= y \\ y' &= z \\ z' &= -x + \frac{yz[a_1(x^2 + y^2 + z^2) + b_1]}{a_1(xyz + b_0) - a_0 b_1}. \end{aligned} \tag{17}$$

The map $\tilde{L}_2$ has the following properties:

- $\tilde{L}_2$ has, by construction, the integral $k(x, y, z)$ given by (16).
- $\tilde{L}_2$ is (anti-) measure preserving with density

$$\tilde{\rho}(x, y, z) = \frac{1}{a_1(x^2 + y^2 + z^2) + b_1} \tag{18}$$

  noting that $\tilde{\rho}(x, y, z) = \left[\frac{\partial I}{\partial K}\right]^{-1}$ for I of (14).
- $\tilde{L}_2$ is reversible, where the reversing symmetry is

$$\tilde{G}: \quad \begin{aligned} x' &= z \\ y' &= y \\ z' &= x. \end{aligned} \tag{19}$$

Notice that in example 1 we began with a map L_1, depending on one parameter, and replacement yielded $\tilde{L}_1$, which contains no parameter. In example 2 we first used a reparametrization of L_1 to introduce an additional four parameters into L_2 and its integral (14). This ensures that $\tilde{L}_2$ of (17) represents a four-parameter family of mappings, which includes $\tilde{L}_1$ of example 1 as a special case ($a_0 = b_0 = b_1 = 0, a_1 = 1$). Significantly, $\tilde{L}_2$ also includes the original L_1 as a special case if we take $a_0 = K$, $a_1 = 0$ in (17) which corresponds in (14) to trivially shifting the value of the original integral (3). Hence we have been able to embed L_1 in a larger family of measure-preserving and reversible 3D maps with an integral.

Example 3. The following four-dimensional discrete KdV map is given in [4]

$$L_3: \quad \begin{aligned} w' &= y \\ x' &= z \\ y' &= x + K\left[\frac{1}{(1+y)} - \frac{1}{(1+z)}\right] \\ z' &= -w - x + K\left[\frac{1}{(1+z)} - \frac{1}{(1-y-z)}\right] \end{aligned} \tag{20}$$

where K is an arbitrary parameter. The map L_3 has the following properties:

- L_3 has two integrals, i.e.

$$\begin{aligned} I_1(w,x,y,z,K) &= 2w^2 + 2wx + 2x^2 - (w+1)(w+2x)y - (w+1)(w-2)y^2 \\ &\quad + (w+x-1)(x-w)z - (w+x+2)(w+x-1)z^2 \\ &\quad - 2(w+1)(w+x-1)zy + 3(wy+wz+xz)K \end{aligned} \tag{21}$$

$$\begin{aligned} I_2(w,x,y,z,K) &= wx - (w+1)(x+1)(w+x-1)[(1+z)y^2 + (y+1)z^2 + zy] \\ &\quad + (x+1)w^2 + (w+1)x^2 + [(2w+3wx+w^2+2x+x^2-1)yz \\ &\quad - wx - w^2 - x^2 + (x+1)(2w+x)y + (w+1)(w+x-1)y^2 \\ &\quad + (w+1)(w+2x)z + (x+1)(w+x-1)z^2]K \\ &\quad - (2wy+wz+xy+2xz)K^2 \end{aligned} \tag{22}$$

- L_3 is volume preserving,
- L_3 is reversible, with reversing symmetry given by

$$G: \quad \begin{aligned} w' &= z \\ x' &= y \\ y' &= x \\ z' &= w. \end{aligned} \tag{23}$$

Reparametrizing the parameter in (20) via $K \to a_0 + a_1 K$ and reparametrizing (21) via $I_1 \to I_1 + (b_0 + b_1 K)$, it follows immediately that the map

$$\hat{L}_3: \quad \begin{aligned} w' &= y \\ x' &= z \\ y' &= x + (a_0 + a_1 K)\left[\frac{1}{(1+y)} - \frac{1}{(1+z)}\right] \\ z' &= -w - x + (a_0 + a_1 K)\left[\frac{1}{(1+z)} - \frac{1}{(1-y-z)}\right] \end{aligned} \tag{24}$$

possesses the integrals

$$\begin{aligned} \hat{I}_1(w,x,y,z,K) &= 2w^2 + 2wx + 2x^2 - (w+1)(w+2x)y - (w+1)(w-2)y^2 \\ &\quad + (w+x-1)(x-w)z - (w+x+2)(w+x-1)z^2 \\ &\quad - 2(w+1)(w+x-1)zy + 3(wy+wz+xz)(a_0 + a_1 K) \\ &\quad + (b_0 + b_1 K) \end{aligned} \tag{25}$$

$$\begin{aligned} \hat{I}_2(w,x,y,z,K) &= wx - (w+1)(x+1)(w+x-1)[(1+z)y^2 + (y+1)z^2 + zy] \\ &\quad + (x+1)w^2 + (w+1)x^2 + [(2w+3wx+w^2+2x+x^2-1)yz \\ &\quad - wx - w^2 - x^2 + (x+1)(2w+x)y + (w+1)(w+x-1)y^2 \\ &\quad + (w+1)(w+2x)z + (x+1)(w+x-1)z^2](a_0 + a_1 K) \\ &\quad - (2wy+wz+xy+2xz)(a_0 + a_1 K)^2. \end{aligned} \tag{26}$$

Since $\hat{I}_1$ is linear in K, we can solve $\hat{I}_1(w, x, y, z, K) = 0$ for $K = k(w, x, y, z)$, i.e.

$$\begin{aligned} k(w, x, y, z) = -[&2w^2 + 2wx + 2x^2 - (w+1)(w+2x)y - (w+1)(w-2)y^2 \\ &+ (w+x-1)(x-w)z - (w+x+2)(w+x-1)z^2 \\ &- 2(w+1)(w+x-1)zy + \\ &3a_0(wy + wz + xz) + b_0] \,/\, [3a_1(wy + wz + xz) + b_1]. \end{aligned} \tag{27}$$

Define the map $\tilde{L}_3$ by substituting (27) into (24)

$$\begin{aligned} \tilde{L}_3: \quad w' &= y \\ x' &= z \\ y' &= x + \{a_0 - a_1[2w^2 + 2wx + 2x^2 - (w+1)(w+2x)y - (w+1)(w-2)y^2 \\ &\quad + (w+x-1)(x-w)z - (w+x+2)(w+x-1)z^2 \\ &\quad - 2(w+1)(w+x-1)zy + 3a_0(wy + wz + xz) \\ &\quad + b_0]/[3a_1(wy + wz + xz) + b_1]\}\,(z-y)\,/\,[(1+y)(1+z)] \\ z' &= -w - x - \{a_0 - a_1[2w^2 + 2wx + 2x^2 - (w+1)(w+2x)y - (w+1)(w-2)y^2 \\ &\quad + (w+x-1)(x-w)z - (w+x+2)(w+x-1)z^2 \\ &\quad - 2(w+1)(w+x-1)zy + 3a_0(wy + wz + xz) \\ &\quad + b_0]/[3a_1(wy + wz + xz) + b_1]\}\,(y+2z)\,/\,[(1+z)(1-y-z)]. \end{aligned} \tag{28}$$

The map $\tilde{L}_3$ has the following properties:

- $\tilde{L}_3$ has two integrals,

$$\begin{aligned} \tilde{I}_1(w, x, y, z) = -[&2w^2 + 2wx + 2x^2 - (w+1)(w+2x)y - (w+1)(w-2)y^2 \\ &+ (w+x-1)(x-w)z - (w+x+2)(w+x-1)z^2 \\ &- 2(w+1)(w+x-1)zy + 3a_0(wy + wz + xz) \\ &+ b_0] \,/\, [3a_1(wy + wz + xz) + b_1] \end{aligned} \tag{29}$$

$$\begin{aligned} \tilde{I}_2(w, x, y, z) = {}&wx - (w+1)(x+1)(w+x-1)[(1+z)y^2 + (y+1)z^2 + zy] \\ &+ (x+1)w^2 + (w+1)x^2 + [(2w + 3wx + w^2 + 2x + x^2 - 1)yz \\ &- wx - w^2 - x^2 + (x+1)(2w+x)y + (w+1)(w+x-1)y^2 \\ &+ (w+1)(w+2x)z + (x+1)(w+x-1)z^2](a_0 + a_1 k) \\ &- (2wy + wz + xy + 2xz)(a_0 + a_1 k)^2 \end{aligned} \tag{30}$$

  noting that $\tilde{I}_1 = k(w, x, y, z)$ and $\tilde{I}_2(w, x, y, z) = \hat{I}_2(w, x, y, z, k(w, x, y, z))$,
- $\tilde{L}_3$ is measure preserving with

$$\tilde{\rho}(w, x, y, z) = \left[\frac{\partial \hat{I}_1}{\partial K}\right]^{-1} = \frac{1}{3a_1(wy + wz + xz) + b_1}. \tag{31}$$

- $\tilde{L}_3$ is reversible, with reversing symmetry given by

$$\begin{aligned} \tilde{G}: \quad w' &= z \\ x' &= y \\ y' &= x \\ z' &= w. \end{aligned} \tag{32}$$

The dynamics of a particular example of the map $\tilde{L}_3$ of (28) is given in figure 1. Embedded in the four-parameter map $\tilde{L}_3$ is the original one-parameter map L_3 of (20) which arises from (28) with the choice $a_1 = 0$ and identification of a_0 as the parameter.

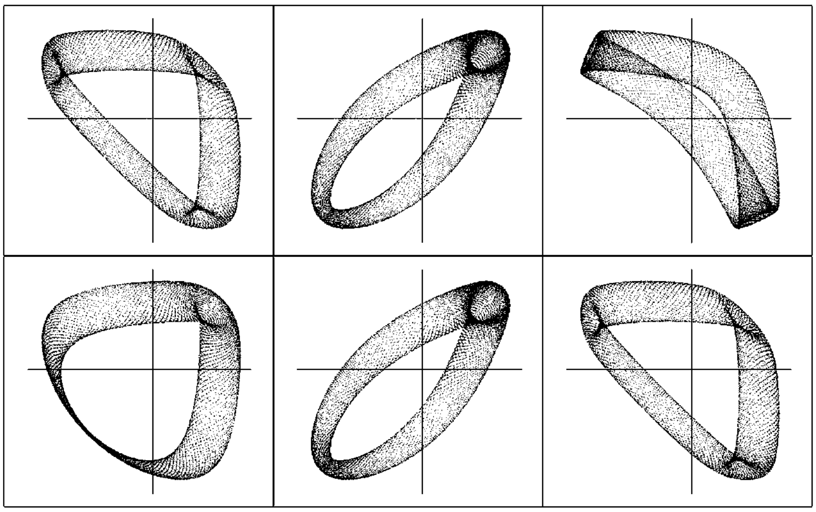

Figure 1. Two-dimensional projections of the phase portrait of $\tilde{L}_3$ of (28) on the square $[-0.5, 0.4] \times [-0.5, 0.4]$ are shown, with $(a_0, a_1, b_0, b_1) = (1, 2.1, 3, 4.2)$ and initial conditions $(w_0, x_0, y_0, z_0) = (0.21, 0.2, 0.2, 0.2)$. The projections are $[wx, wy, wz]$ across from the top left-hand corner and $[xy, xz, yz]$ across from the bottom left-hand corner.

3. Theoretical basis for replacement and reparametrization

In the preceding examples, we have seen that new 3D and 4D maps have been created from existing ones by using one of the original map's integrals to eliminate or *replace* one parameter from it. In examples 2 and 3 we have prepared an original map using *reparametrization* prior to the replacement procedure. In this section, we generalize these procedures and give a theoretical justification for the observed properties of the new maps.

3.1. Replacement

Consider a map $L : \boldsymbol{x} \mapsto \boldsymbol{x}'$ where $\boldsymbol{x} := (x_1, \ldots, x_n) \in \mathbb{R}^n$ which depends on l parameters $\boldsymbol{K} := (K_1, \ldots, K_l)$ so:

$$L : \boldsymbol{x} \mapsto \boldsymbol{x}' := (f_1(\boldsymbol{x}, \boldsymbol{K}), \ldots, f_n(\boldsymbol{x}, \boldsymbol{K})). \tag{33}$$

Suppose L possesses j functionally-independent integrals $I_1, \ldots, I_j$ dependent on the parameters so that

$$I_i(\boldsymbol{x}', \boldsymbol{K}) = I_i(\boldsymbol{x}, \boldsymbol{K}) \qquad i = 1, \ldots, j. \tag{34}$$

We assume that there exists $1 \leqslant p \leqslant j$ such that the p equations

$$I_i(\boldsymbol{x}, \boldsymbol{K}) = 0 \qquad i = 1, \ldots, p \tag{35}$$

uniquely determine $K_1, \ldots, K_p$ via

$$K_i = k_i(\boldsymbol{x}, K_{p+1}, \ldots, K_l) \qquad i = 1, \ldots, p \tag{36}$$

where k_i are smooth functions. The general condition for (35) to be solvable (implicitly or explicitly) for K_i, $i = 1, \ldots, p$, will not be discussed here. In the examples of the present paper, equations (35) will always be affine in the parameters $K_1, \ldots, K_p$ so that the functions

k_i can be found explicitly when a certain determinant is non-vanishing (which is generically true).

We create a new map $\tilde{L}$ of $\mathbb{R}^n$ from L by *replacement* of the *parameters* $K_1, \ldots, K_p$ in (33) by the corresponding *functions* $k_1, \ldots, k_p$ of (36). That is, using $\hat{\boldsymbol{K}} := (K_{p+1}, \ldots, K_l)$ for the parameters not replaced and $\boldsymbol{k} := (k_1, \ldots, k_p)$ for the vector of functions determined by solving (35), we have

$$\tilde{L} : \boldsymbol{x} \mapsto \boldsymbol{x}' := (\tilde{f}_1(\boldsymbol{x}, \hat{\boldsymbol{K}}), \ldots, \tilde{f}_n(\boldsymbol{x}, \hat{\boldsymbol{K}})) = (f_1(\boldsymbol{x}, \boldsymbol{k}, \hat{\boldsymbol{K}}), \ldots, f_n(\boldsymbol{x}, \boldsymbol{k}, \hat{\boldsymbol{K}})) \tag{37}$$

It is clear that replacement in L of the 'distinguished' parameters $K_1, \ldots, K_p$ solved from (35) means that $\tilde{L}$ depends on $l - p$ parameters (versus the l parameters in L).

The theorem below relates properties of L to those of $\tilde{L}$. It turns out to be instructive to compare L and $\tilde{L}$ by introducing an intermediate map L^{ext} of the extended space $\mathbb{R}^{p+n} = (\bar{\boldsymbol{K}}, \boldsymbol{x}) := (K_1, \ldots, K_p, x_1, \ldots, x_n)$ obtained from adding the distinguished parameters to the existing phase space so that

$$L^{ext} : \boldsymbol{y} := (\bar{\boldsymbol{K}}, \boldsymbol{x}) \mapsto \boldsymbol{y}' := (\bar{\boldsymbol{K}}', \boldsymbol{x}') := (\bar{\boldsymbol{K}}, f_1(\boldsymbol{x}, \boldsymbol{K}), \ldots, f_n(\boldsymbol{x}, \boldsymbol{K})). \tag{38}$$

Note that L^{ext} acts as the identity map in its first p components and as L in the remaining n components with the remaining (undistinguished) parameters $K_{p+1}, \ldots, K_l$ remaining in L^{ext} as standard parameters. It follows from (34) that L^{ext} has the j integrals

$$I_i(\boldsymbol{x}, \bar{\boldsymbol{K}}, \hat{\boldsymbol{K}}) \qquad i = 1, \ldots, j. \tag{39}$$

Consequently, L^{ext} induces a map on the intersections of the level sets $I_i = c_i$ $(i = 1, \ldots j)$, where c_i are determined from initial conditions. In particular, for arbitrary $\boldsymbol{x} = (x_1, \ldots, x_n)$, suppose we choose initial conditions of $\bar{\boldsymbol{K}} = (K_1, \ldots, K_p)$ so as to lie in the following set in the extended $(\bar{\boldsymbol{K}}, \boldsymbol{x})$ space

$$\mathcal{I}_0 := \bigcap_{i=1}^{p} \{ I_i(\boldsymbol{x}, \bar{\boldsymbol{K}}, \hat{\boldsymbol{K}}) = 0 \}. \tag{40}$$

Then we remain in this set under iteration of L^{ext} with the heights $I_i = c_i$ $(i = p + 1, \ldots, j)$ also fixed at their initial values. Note by assumption (cf (35) and (36)), $\mathcal{I}_0$ is well-defined and equal to the intersection of the graphs of the p functions $k_i(x_1, \ldots, x_n, K_{p+1}, \ldots, K_l)$ defined by (35).

By construction, the map $\tilde{L}$ of (37) is, geometrically, simply the projection onto the $\mathbb{R}^n$ space $(x_1, \ldots, x_n)$ of the restriction of L^{ext} to its invariant set $\mathcal{I}_0$, utilizing (36) to replace for K_i $(i = 1, \ldots, p)$ in the expressions for x_i' in L^{ext} of (38). We denote this relationship by

$$\tilde{L} = \pi_{\boldsymbol{x}} \, (L^{ext} \,|_{\mathcal{I}_0}). \tag{41}$$

Using the projection $\pi_{\boldsymbol{x}}$ means that to obtain $\tilde{L}$ we discard the first p mapping equations of (38) when L^{ext} is restricted to $\mathcal{I}_0$.

Theorem 1. *If the maps L and $\tilde{L}$ are defined as above, then:*

(i) $\tilde{L}$ has j integrals:

$$\tilde{I}_i(\boldsymbol{x}, \hat{\boldsymbol{K}}) = k_i(\boldsymbol{x}, \hat{\boldsymbol{K}}) \qquad i = 1, \ldots, p$$
$$\tilde{I}_i(\boldsymbol{x}, \hat{\boldsymbol{K}}) = I_i(\boldsymbol{x}, \boldsymbol{k}, \hat{\boldsymbol{K}}) \qquad i = p + 1, \ldots, j.$$

(ii) If L is (anti-) measure preserving with density $\rho(\boldsymbol{x}, \boldsymbol{K})$, then $\tilde{L}$ is also (anti-) measure preserving with density

$$\tilde{\rho}(\boldsymbol{x}, \hat{\boldsymbol{K}}) = \frac{\rho(\boldsymbol{x}, \boldsymbol{k}, \hat{\boldsymbol{K}})}{J} \tag{42}$$

where $J := J(\boldsymbol{x}, \hat{\boldsymbol{K}})$ *is the Jacobian determinant of the map* $(K_1, \ldots, K_p) \mapsto (I_1, \ldots, I_p)$ *evaluated at* (36) *i.e.*

$$J = \left. \frac{\partial(I_1, \ldots, I_p)}{\partial(K_1, \ldots, K_p)} \right|_{(\boldsymbol{x}, \boldsymbol{k}, \hat{\boldsymbol{K}})}. \tag{43}$$

(iii) $\tilde{L}$ *has a symmetry* $\tilde{S}$ *(i.e.* $\tilde{S} \circ \tilde{L} = \tilde{L} \circ \tilde{S}$*), with* $\tilde{S}$ *derived from a map* S^{ext} *of the extended phase space if and only if:*

(a) S^{ext} *preserves* $\mathcal{I}_0$ *of* (40)*, whence* $\tilde{S} = \pi_{\boldsymbol{x}}\ (S^{ext}\,|_{\mathcal{I}_0})$*; and*

(b) $S^{ext} \circ L^{ext} - L^{ext} \circ S^{ext}$ *vanishes on* $\mathcal{I}_0$*.*

In particular, if L has a symmetry S

$$S : \boldsymbol{x} \mapsto \boldsymbol{x}' := (s_1(\boldsymbol{x}, \boldsymbol{K}), \ldots, s_n(\boldsymbol{x}, \boldsymbol{K})) \tag{44}$$

and S also satisfies

$$I_i(\boldsymbol{x}, \boldsymbol{K}) = 0 \qquad \Rightarrow \qquad (I_i \circ S)\ (\boldsymbol{x}, \boldsymbol{K}) = 0 \tag{45}$$

for $i = 1, \ldots, p$*, then* $\tilde{L}$ *has the symmetry* $\tilde{S}$ *where*

$$\tilde{S} : \boldsymbol{x} \mapsto \boldsymbol{x}' := (s_1(\boldsymbol{x}, \boldsymbol{k}, \hat{\boldsymbol{K}}), \ldots, s_n(\boldsymbol{x}, \boldsymbol{k}, \hat{\boldsymbol{K}})). \tag{46}$$

(iv) $\tilde{L}$ *has a reversing symmetry* $\tilde{G}$ *(i.e.* $\tilde{G} \circ \tilde{L} \circ \tilde{G}^{-1} = \tilde{L}^{-1}$*), with* $\tilde{G}$ *derived from a map* G^{ext} *of the extended phase space if and only if:*

(a) G^{ext} *preserves* $\mathcal{I}_0$ *of* (40)*, whence* $\tilde{G} = \pi_{\boldsymbol{x}}\ (G^{ext}\,|_{\mathcal{I}_0})$*; and*

(b) $G^{ext} \circ L^{ext} - L^{ext\,-1} \circ G^{ext}$ *vanishes on* $\mathcal{I}_0$*.*

In particular, if L has a reversing symmetry G

$$G : \boldsymbol{x} \mapsto \boldsymbol{x}' := (g_1(\boldsymbol{x}, \boldsymbol{K}), \ldots, g_n(\boldsymbol{x}, \boldsymbol{K})) \tag{47}$$

and G also satisfies

$$I_i(\boldsymbol{x}, \boldsymbol{K}) = 0 \qquad \Rightarrow \qquad (I_i \circ G)\ (\boldsymbol{x}, \boldsymbol{K}) = 0 \tag{48}$$

for $i = 1, \ldots, p$*, then* $\tilde{L}$ *is reversible with reversing symmetry* $\tilde{G}$ *where*

$$\tilde{G} : \boldsymbol{x} \mapsto \boldsymbol{x}' := (g_1(\boldsymbol{x}, \boldsymbol{k}, \hat{\boldsymbol{K}}), \ldots, g_n(\boldsymbol{x}, \boldsymbol{k}, \hat{\boldsymbol{K}})). \tag{49}$$

Proof. (i) By construction, L^{ext} of (38) also has the trivial integrals $K_1, \ldots, K_p$. When we choose $K_i = k_i\ (i = 1, \ldots, p)$ to lie on $\mathcal{I}_0$, it follows that the values of k_i are preserved by L^{ext} and hence $\tilde{L}$. This accounts for $\tilde{I}_i = k_i\ (i = 1, \ldots, p)$. The remaining integrals $\tilde{I}_i\ \ (i = p + 1, \ldots, j)$ of L are just the remaining integrals (39) of L^{ext} with $K_i = k_i\ (i = 1, \ldots, p)$.

(ii) Since L is (anti-) measure-preserving, we have for $V \subset \mathbb{R}^n$

$$\int_V \rho(\boldsymbol{x}, \boldsymbol{K})\ \mathrm{d}x_1 \ldots \mathrm{d}x_n = (-) \int_{L(V)} \rho(\boldsymbol{x}', \boldsymbol{K})\ \mathrm{d}x'_1 \ldots \mathrm{d}x'_n. \tag{50}$$

Clearly L^{ext} is measure-preserving with density ρ via the obvious extension of (50) (with $U \subset \mathbb{R}^{n+p}$):

$$\int_U \rho(\boldsymbol{x}, \boldsymbol{K})\ \mathrm{d}K_1 \ldots \mathrm{d}K_p\, \mathrm{d}x_1 \ldots \mathrm{d}x_n = (-) \int_{L^{ext}(U)} \rho(\boldsymbol{x}', \boldsymbol{K})\ \mathrm{d}K'_1 \ldots \mathrm{d}K'_p\, \mathrm{d}x'_1 \ldots \mathrm{d}x'_n. \tag{51}$$

Now consider a volume element $W \subset \mathbb{R}^{n+p}$ constructed in the following way. It has an arbitrary projection $W^* = \mathrm{d}x_1 \ldots \mathrm{d}x_n$ onto the $(x_1, \ldots, x_n)$ coordinates but is bounded by $\mathcal{I}_0$ of (40) and

$$\mathcal{I}_\epsilon := \bigcap_{i=1}^{p} \{\, I_i(\boldsymbol{x}, \bar{\boldsymbol{K}}, \hat{\boldsymbol{K}}) = \epsilon_i \,\} \tag{52}$$

i.e. the intersection of the level sets $(I_1, \dots, I_p) = (\epsilon_1, \dots, \epsilon_p)$, where ϵ_i are arbitrarily small (the existence of $\mathcal{I}_\epsilon$ for any $(x_1, \dots, x_n)$ is guaranteed by the implicit function theorem). From (39), the image $L^{ext}(W)$ of this volume element will continue to have a face on $\mathcal{I}_0$ and one on $\mathcal{I}_\epsilon$. For each point of $\mathcal{I}_0$ in W, the relationship between the variation of I_i $(i = 1, \dots, p)$ to the adjacent level sets $\mathcal{I}_\epsilon$ and the variation of K_i $(i = 1, \dots, p)$ is given by

$$\epsilon_i \; = \; \mathrm{d}I_i = \sum_{k=1}^{p} \left.\frac{\partial I_i}{\partial K_k}\right|_{\mathcal{I}_0} \mathrm{d}K_k \qquad i = 1, \dots, p. \tag{53}$$

In particular,

$$\prod_{i=1}^{p} \epsilon_i \; = \; \mathrm{d}I_1 \dots \mathrm{d}I_p = \left.\frac{\partial(I_1, \dots, I_p)}{\partial(K_1, \dots, K_p)}\right|_{\mathcal{I}_0} \mathrm{d}K_1 \dots \mathrm{d}K_p. \tag{54}$$

The Jacobian determinant evaluated on $\mathcal{I}_0$ in (54) can be denoted $J(\boldsymbol{x}, \hat{\boldsymbol{K}})$ since (36) are used to eliminate $K_1, \dots, K_p$. Using the volume W in (51) together with (54) to replace $\mathrm{d}K_1 \dots \mathrm{d}K_n$ and its primed version, and taking the limit as $\epsilon_i \to 0$ leads to

$$\int_{W^*} \frac{\rho(\boldsymbol{x}, \boldsymbol{k}, \hat{\boldsymbol{K}})}{J(\boldsymbol{x}, \hat{\boldsymbol{K}})} \, \mathrm{d}x_1 \dots \mathrm{d}x_n = (-) \int_{\tilde{L}(W^*)} \frac{\rho(\boldsymbol{x}', \boldsymbol{k}', \hat{\boldsymbol{K}})}{J(\boldsymbol{x}', \hat{\boldsymbol{K}})} \, \mathrm{d}x_1' \dots \mathrm{d}x_n'. \tag{55}$$

In (55), $k_1' = k_1(\boldsymbol{x}', \hat{\boldsymbol{K}})$ etc is evaluated at the images of $\tilde{L}$. Since W^* is arbitrary, (55) establishes the result in (ii) above.

(iii) From (41), $\tilde{L}$ is obtained from projection of the map $L^{ext}|_{\mathcal{I}_0}$ which is a map from $\mathcal{I}_0$ to itself. If a symmetry $\tilde{S}$ of $\tilde{L}$ is derived from a map S^{ext} of the extended phase space $(K_1, \dots, K_p, x_1, \dots, x_n)$, then S^{ext} must preserve the set $\mathcal{I}_0$ and, by definition, the following is satisfied:

$$\pi_{(x_1,\dots,x_n)} \, (S^{ext}|_{\mathcal{I}_0} \circ L^{ext}|_{\mathcal{I}_0}) = \pi_{(x_1,\dots,x_n)} \, (L^{ext}|_{\mathcal{I}_0} \circ S^{ext}|_{\mathcal{I}_0}). \tag{56}$$

By assumptions (35), (36), we can cancel the projection operators from both sides of (56) since two points on $\mathcal{I}_0$ *cannot* share the same values of $(x_1, \dots, x_n)$ but have different values for $(K_1, \dots, K_p)$. Therefore, we have

$$\begin{aligned}(S^{ext} \circ L^{ext})|_{\mathcal{I}_0} = (L^{ext} \circ S^{ext})|_{\mathcal{I}_0} \; \Rightarrow \; & S^{ext} \circ L^{ext} - L^{ext} \circ S^{ext} \\ & = F(K_1, \dots, K_p, x_1, \dots, x_n)\end{aligned} \tag{57}$$

where $F : \mathbb{R}^{n+p} \mapsto \mathbb{R}^{n+p}$ vanishes on $\mathcal{I}_0$. If the original map L has a symmetry S that satisfies (45) then clearly its extension S^{ext} preserves $\mathcal{I}_0$. Furthermore, (57) is satisfied in this case with $F \equiv 0$ since L^{ext} *and* S^{ext} both act as the identity on $K_1, \dots, K_p$.

(iv) The details of the proof for reversing symmetries are exactly analogous to those for symmetries and we omit the details. □

Example 1 of the previous section corresponds to application of the above result with L given by L_1 of (1). The number of parameters in L_1 is $l = 1$ with $K_1 = K$. The number of integrals of L_1 is $j = 1$ and since I of (3) is linear in K, we can solve $I(x, y, z, K) = 0$ for $K = k(x, y, z)$ (so $p = 1$). The fact that $\tilde{L}_1$ of (7) has the symmetry $\tilde{S}$ of (11), whereas L_1 does not, illustrates part (iii) of the theorem. For $\tilde{S}$ of (11), we have $\tilde{S} = \pi_{\boldsymbol{x}} \, (S^{ext}|_{\mathcal{I}_0})$ with $\mathcal{I}_0 = \{I(x, y, z, K) = 0\}$ and I of (3). The map S^{ext} is not unique and one can take

$$S^{ext} : \begin{array}{l} K' = -K + f(I) \\ x' = -x \\ y' = -y \\ z' = -z \end{array} \tag{58}$$

where f is an arbitrary function that satisfies $f(0) = 0$. The map L_1^{ext} corresponding to L_1 of (1) is

$$L_1^{ext}: \begin{array}{l} K' = K \\ x' = y \\ y' = z \\ z' = -x - \frac{yz}{K} \end{array} \tag{59}$$

and one finds that

$$S^{ext} \circ L_1^{ext} - L_1^{ext} \circ S^{ext} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ yz(\frac{1}{K} + \frac{1}{f(I)-K}) \end{pmatrix}. \tag{60}$$

The latter vanishes when $I = 0$, and since S^{ext} also preserves this set, $\tilde{S}$ of (11) becomes a symmetry of $\tilde{L}_1$ of (7).

Example 1 illustrates the general result that the original map L of (33) and the converted map $\tilde{L}$ of (37) need not share the same symmetries or reversing symmetries.

3.2. Reparametrization

Examples 2 and 3 of the previous section illustrate a *reparametrization* process that can be applied *prior* to application of the replacement theorem above. This process *increases* the number of parameters present in the original map L. In general it works as follows. Given L of the form (33) satisfying (34) and assuming (35), (36), we reparametrize the parameters $K_1, \ldots, K_r, r \geqslant p$ using the affine transformation

$$R_K : \begin{pmatrix} K_1 \\ \vdots \\ K_r \end{pmatrix} \mapsto \begin{pmatrix} K_1' \\ \vdots \\ K_r' \end{pmatrix} := \boldsymbol{A_K} \begin{pmatrix} K_1 \\ \vdots \\ K_p \end{pmatrix} + \boldsymbol{b} \tag{61}$$

where A_K is a $r \times p$ matrix and $\boldsymbol{b}$ is a $r \times 1$ column vector. Note that we may be able to reparametrize more parameters than just the distinguished ones if the integrals $I_i, i = 1, \ldots, p$ are linear in more than p parameters (so there is some choice available as to which parameters to make distinguished). We can also reparametrize integrals (34) of L according to:

$$R_I : \begin{pmatrix} I_1 \\ \vdots \\ I_p \end{pmatrix} \mapsto \begin{pmatrix} I_1' \\ \vdots \\ I_p' \end{pmatrix} := \boldsymbol{A_I} \begin{pmatrix} I_1 \\ \vdots \\ I_p \end{pmatrix} + \boldsymbol{C} \begin{pmatrix} K_1 \\ \vdots \\ K_p \end{pmatrix} + \boldsymbol{d} \tag{62}$$

where $\boldsymbol{A_I}$ and $\boldsymbol{C}$ are $p \times p$ matrices and $\boldsymbol{d}$ is a $p \times 1$ column vector. The replacement theorem of the previous section can now be applied with

$$L \to L \circ R_k \qquad I \to R_I(I \circ R_K). \tag{63}$$

In the examples seen in section 2, $p = 1$ but in the next section we will see an example where we can take $p = 1$ or 2 in (61) and (62). In all the examples of this paper, $\boldsymbol{A_I}$ will be the $p \times p$ identity matrix.

It is clear that the process of reparametrization applied before replacement compensates for the parameters removed from the map by the latter process. Furthermore, taking all the values of the introduced parameters in $\boldsymbol{A_K}$ to be zero in the new map $\tilde{L}$ obtained from replacement will lead to a map equivalent to the original map L (since this is equivalent to there having been no replacement in the first place).

We conclude this section with the following:

Remarks.

(1) The curve-dependent McMillan maps of [7, 8] are a special case of the above theory when $n = 2$ and $j = p = 1$. In these papers the assumptions (35), (36) are called condition F and the case where the integral I is *nonlinear* in the distinguished parameter K is treated.
(2) The idea of the measure-preservation result (ii) of the theorem is a discrete version of the concept of integral invariants on integral submanifolds in ordinary differential equations [9].

4. Examples of 4D maps with replacement of two parameters

In this section we provide two further illustrations of the results of section 3.

Example 4. The following map is a rescaling of the discrete sine–Gordon map [3, 14]:

$$L_4: \quad \begin{aligned} w' &= x \\ x' &= y \\ y' &= z \\ z' &= \frac{pxz - b}{w(p - cxz)} \end{aligned} \tag{64}$$

where b, c, and p are arbitrary parameters. It has the following properties:

- L_4 has two integrals,

$$\begin{aligned} I_1 &= p\left(\frac{w}{z} + \frac{z}{w}\right) - b\left(\frac{1}{wx} + \frac{1}{xy} + \frac{1}{yz}\right) - c(wx + xy + yz) \\ I_2 &= p\left(\frac{w}{x} + \frac{x}{y} + \frac{y}{z} + \frac{z}{y} + \frac{x}{w} + \frac{y}{x}\right) - \frac{b}{wz} - cwz \end{aligned} \tag{65}$$

- L_4 is measure preserving, with

$$\rho(w, x, y, z) = \frac{1}{wxyz} \tag{66}$$

- L_4 has symmetry S, where

$$S: \quad \begin{aligned} w' &= -w \\ x' &= -x \\ y' &= -y \\ z' &= -z \end{aligned} \tag{67}$$

- L_4 is reversible, with reversing symmetry given by

$$G: \quad \begin{aligned} w' &= z \\ x' &= y \\ y' &= x \\ z' &= w. \end{aligned} \tag{68}$$

Reparametrizing the parameters, i.e. $p \to p_0 + p_1 K$, $b \to b_0 + b_1 K$, $c \to c_0 + c_1 K$ and the first integral $I_1 \to I_1 + f_0 + f_1 K$, we obtain the integrals

$$
\begin{aligned}
\hat{I}_1(w, x, y, z, K) &= (p_0 + p_1 K)\left(\frac{w}{z} + \frac{z}{w}\right) - (b_0 + b_1 K)\left(\frac{1}{wx} + \frac{1}{xy} + \frac{1}{yz}\right) \\
&\quad - (c_0 + c_1 K)(wx + xy + yz) + (f_0 + f_1 K) \\
\hat{I}_2(w, x, y, z, K) &= (p_0 + p_1 K)\left(\frac{w}{x} + \frac{x}{y} + \frac{y}{z} + \frac{z}{y} + \frac{x}{w} + \frac{y}{x}\right) - \frac{(b_0 + b_1 K)}{wz} \\
&\quad - (c_0 + c_1 K) wz.
\end{aligned}
\tag{69}
$$

The map preserving $\hat{I}_1$ and $\hat{I}_2$ is

$$
\begin{aligned}
\hat{L}_4: \quad w' &= x \\
x' &= y \\
y' &= z \\
z' &= \frac{(p_0 + p_1 K) xz - (b_0 + b_1 K)}{w(p_0 + p_1 K - (c_0 + c_1 K) xz)}.
\end{aligned}
\tag{70}
$$

Using $\hat{I}_1(w, x, y, z, K) = 0$ we can define a new integral $K = k(w, x, y, z)$ which is preserved by the map (70). Define the map $\tilde{L}_4$ by

$$
\begin{aligned}
\tilde{L}_4: \quad w' &= x \\
x' &= y \\
y' &= z \\
z' &= \frac{(p_0 + p_1 k(w, x, y, z)) xz - (b_0 + b_1 k(w, x, y, z))}{w(p_0 + p_1 k(w, x, y, z) - (c_0 + c_1 k(w, x, y, z)) xz)}.
\end{aligned}
\tag{71}
$$

The map $\tilde{L}_4$ has the following properties:

- $\tilde{L}_4$ has two integrals,

$$
\begin{aligned}
\tilde{I}_1 &= k(w, x, y, z) \\
\tilde{I}_2 &= \hat{I}_2(w, x, y, z, k(w, x, y, z))
\end{aligned}
\tag{72}
$$

- $\tilde{L}_4$ is measure preserving, with

$$
\tilde{\rho}(w, x, y, z) = \frac{1}{wxyz}\left[\frac{\partial \hat{I}_1}{\partial K}\right]^{-1}
\tag{73}
$$

  where

$$
\frac{\partial \hat{I}_1}{\partial K} = p_1\left(\frac{w}{z} + \frac{z}{w}\right) - b_1\left(\frac{1}{wx} + \frac{1}{xy} + \frac{1}{yz}\right) - c_1(wx + xy + yz) + f_1.
\tag{74}
$$

- $\tilde{L}_4$ has symmetry

$$
\begin{aligned}
\tilde{S}: \quad w' &= -w \\
x' &= -x \\
y' &= -y \\
z' &= -z
\end{aligned}
\tag{75}
$$

- $\tilde{L}_4$ is reversible, with reversing symmetry given by

$$\tilde{G}:\quad \begin{aligned} w' &= z\\ x' &= y\\ y' &= x\\ z' &= w. \end{aligned} \tag{76}$$

All of these properties of $\tilde{L}_4$ are in agreement with theorem 1.

Example 5. Consider the map L_4 given in example 4. We observe that both the integrals (65) are linear in p, b and c so there is the possibility to solve for any two of these parameters (i.e. we have $j = p = 2$ in (34) and (36)).

We first reparametrize to introduce K_1 *and* K_2, i.e. the existing parameters now become $p \to p_0 + p_1K_1 + p_2K_2$, $b \to b_0 + b_1K_1 + b_2K_2$, $c \to c_0 + c_1K_1 + c_2K_2$ and the integrals become $I_1 \to \hat{I}_1 = I_1 + f_0 + f_1K_1 + f_2K_2$ and $I_2 \to \hat{I}_2 = I_2 + g_0 + g_1K_1 + g_2K_2$. The two integrals $\hat{I}_1(w, x, y, z, K_1, K_2)$ and $\hat{I}_2(w, x, y, z, K_1, K_2)$ are preserved by

$$\hat{L}_5:\quad \begin{aligned} w' &= x\\ x' &= y\\ y' &= z\\ z' &= \frac{(p_0 + p_1K_1 + p_2K_2)xz - (b_0 + b_1K_1 + b_2K_2)}{w(p_0 + p_1K_1 + p_2K_2 - (c_0 + c_1K_1 + c_2K_2)xz)}. \end{aligned} \tag{77}$$

Setting $\hat{I}_1(w, x, y, z, K_1, K_2) = 0$ and $\hat{I}_2(w, x, y, z, K_1, K_2) = 0$ we can solve for $K_1 = k_1(w, x, y, z)$ and $K_2 = k_2(w, x, y, z)$ since $\hat{I}_1$ and $\hat{I}_2$ are linear in K_1 and K_2. Define the map $\tilde{L}_5$ to be the map (77) with the replacements $K_1 = k_1(w, x, y, z)$ and $K_2 = k_2(w, x, y, z)$. The map $\tilde{L}_5$ has the following properties:

- $\tilde{L}_5$ has two integrals

$$\begin{aligned} \tilde{I}_1 &= k_1(w, x, y, z)\\ \tilde{I}_2 &= k_2(w, x, y, z) \end{aligned} \tag{78}$$

- $\tilde{L}_5$ is measure preserving, with

$$\tilde{\rho}(w, x, y, z) = \frac{1}{wxyz}\, J^{-1} \tag{79}$$

where

$$J := \begin{vmatrix} \frac{\partial \hat{I}_1}{\partial K_1} & \frac{\partial \hat{I}_1}{\partial K_2}\\ \frac{\partial \hat{I}_2}{\partial K_1} & \frac{\partial \hat{I}_2}{\partial K_2} \end{vmatrix} \tag{80}$$

and

$$\begin{aligned} \frac{\partial \hat{I}_1}{\partial K_1} &= p_1\left(\frac{w}{z} + \frac{z}{w}\right) - b_1\left(\frac{1}{wx} + \frac{1}{xy} + \frac{1}{yz}\right) - c_1(wx + xy + yz) + f_1\\ \frac{\partial \hat{I}_1}{\partial K_2} &= p_2\left(\frac{w}{z} + \frac{z}{w}\right) - b_2\left(\frac{1}{wx} + \frac{1}{xy} + \frac{1}{yz}\right) - c_2(wx + xy + yz) + f_2\\ \frac{\partial \hat{I}_2}{\partial K_1} &= p_1\left(\frac{w}{x} + \frac{x}{y} + \frac{y}{z} + \frac{z}{y} + \frac{x}{w} + \frac{y}{x}\right) - \frac{b_1}{wz} - c_1wz + g_1\\ \frac{\partial \hat{I}_2}{\partial K_2} &= p_2\left(\frac{w}{x} + \frac{x}{y} + \frac{y}{z} + \frac{z}{y} + \frac{x}{w} + \frac{y}{x}\right) - \frac{b_2}{wz} - c_2wz + g_2. \end{aligned} \tag{81}$$

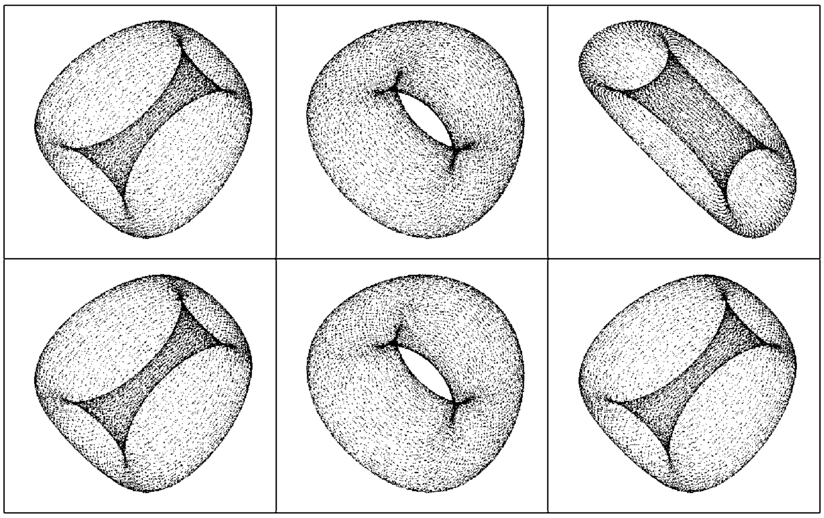

Figure 2. Two-dimensional projections of the phase portrait of $\tilde{L}_5$ on the square $[-1.5, -1] \times [-1.5, -1]$ are shown, with $(b_0, b_1, b_2, c_0, c_1, c_2, f_0, f_1, f_2, g_0, g_1, g_2, p_0, p_1, p_2) = (10, 8, 99, 1, 1, 8, 1, 4, 1, 9, 5, 3, 3, 1, 7)$ and initial conditions $(w_0, x_0, y_0, z_0) = (-1.33, -1.37, -1.3, -1.3)$. The projections are $[wx, wy, wz]$ from the top left-hand corner and $[xy, xz, yz]$ from the bottom left-hand corner.

- $\tilde{L}_5$ has symmetry $\tilde{S}$, where

$$\tilde{S}: \quad \begin{aligned} w' &= -w \\ x' &= -x \\ y' &= -y \\ z' &= -z \end{aligned} \tag{82}$$

- $\tilde{L}_5$ is reversible, with $\tilde{G}$ given by

$$\tilde{G}: \quad \begin{aligned} w' &= z \\ x' &= y \\ y' &= x \\ z' &= w. \end{aligned} \tag{83}$$

The dynamics of a particular example of the map $\tilde{L}_5$ is given in figure 2.

5. Concluding remarks

In this paper, we have illustrated how mappings that possess integrals dependent on parameters can be used to construct other mappings with integrals. Via the processes of reparametrization and replacement, the original mapping can be embedded in a larger family of maps with integrals. This is interesting in the sense that *a priori* it seems non-trivial to embed a map with integrals into a larger parameter family where the integrals are also embedded. The process described here works in any number of dimensions.

We close with the following remarks:

(1) A geometric understanding of the results presented here comes from considering L^{ext} of section 3 as a way of studying maps with integrals dependent on parameters. By considering the projection π_x of the dynamics of L^{ext} on the invariant surfaces $K_i = c_i$, $i = 1, \ldots, p$, one obtains the dynamics of the original map L. The dynamics of the map $\tilde{L}$ is simply the projection π_x of the dynamics of L^{ext} on the alternative invariant surfaces $I_i = c_i, i = 1, \ldots, p$.

(2) The theorem of section 3 compares the properties of measure-preservation, possession of symmetries and reversing symmetries between L and $\tilde{L}$. The maps L_3 and L_4 of examples 3–5 have the additional properties of being symplectic and integrable. An open question is whether, in general, $\tilde{L}$ will also inherit such properties as symplecticity and integrability from L.

(3) Our observation is that integrals of maps are often linear or affine in their parameters. This means that the replacement process is relatively easy to manage. However, with additional conditions, it should be possible to solve for parameters that occur in a nonlinear way (cf [7, 8]). One can also imagine using nonlinear reparametrizations instead of the affine ones presented in (62) and (61).

(4) A similar procedure of interchanging parameters and integrals can be applied to dynamical systems with continuous time (i.e. ordinary differential equations). We hope to report on this in the near future. Significantly, in the case of integrable Hamiltonian systems, this has already been investigated in [5]. There it is shown that the interchange procedure does indeed lead to a *duality* between the original integrable system and the constructed one since the latter can also be shown to be integrable (e.g. the Henon-Heiles and Holt Hamiltonians are shown to be related by just such a process [5]).

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