INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 3965–3980 doi:10.1088/0305-4470/38/18/007

Duality for discrete integrable systems

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Received 20 December 2004, in final form 21 March 2005
Published 18 April 2005
Online at stacks.iop.org/JPhysA/38/3965

Abstract

A new class of discrete dynamical systems is introduced via a duality relation for discrete dynamical systems with a number of explicitly known integrals. The dual equation can be defined via the difference of an arbitrary linear combination of integrals and its upshifted version. We give an example of an integrable mapping with two parameters and four integrals leading to a (four-dimensional) dual mapping with four parameters and two integrals. We also consider a more general class of higher-dimensional mappings arising via a travelling-wave reduction from the (integrable) MKdV partial-difference equation. By differencing the trace of the monodromy matrix we obtain a class of novel dual mappings which is shown to be integrable as level-set-dependent versions of the original ones.

PACS numbers: 02.30.Ik, 05.45.−a

1. Introduction

Discrete integrable systems have received a lot of attention in the last two decades. Areas of physics in which discrete integrable systems prominently feature include statistical mechanics and discrete analogues of integrable systems in classical mechanics or solid state physics [5–8, 22–25].

Some of the early papers dealt with the problem of discretizing integrable partial differential equations, such as the (modified) Korteweg–de Vries (MKdV) equation and the sine-Gordon equation, while retaining their integrability. This led to integrable partial difference equations (PΔEs) [12–15]. Later papers studied integrable ordinary difference equations (OΔEs), both autonomous (i.e. integrable maps) [5–8, 10, 21] and non-autonomous (e.g. discrete Painlevé equations, for a review see [11]).

In this paper we start from a (primary) OΔE with one or more first integrals, and construct a dual OΔE which also has one or more integrals. While neither the primary OΔE nor the dual need be integrable in order for this construction to work, here we will be mainly interested in the case where the primary OΔE is integrable. The question we subsequently seek to answer is whether or not the ensuing dual equation is also integrable.

Accordingly, we focus on a new class of discrete dynamical systems which can be obtained by means of a *duality* relation from a known discrete dynamical system which possesses a number of integrals. The general idea is as follows. We consider a discrete dynamical system given by the dth-order OΔE:

$$u_{n+d} = f(u_n, u_{n+1}, \ldots, u_{n+d-1}, \{p_i\}), \tag{1}$$

where $\{p_i\}$ is a set of l parameters occurring in the system and $f : \mathbb{R}^{d+l} \to \mathbb{R}$. By the standard method, we can alternatively view (1) as defining a map

$$\mathbf{V}_{n+1} = \mathbf{F}[\mathbf{V}_n, \{p_i\}], \tag{2}$$

in which

$$\mathbf{V}_n := (u_n, u_{n+1}, \ldots, u_{n+d-1}) \tag{3}$$

is a d-dimensional vector and $\mathbf{F} : \mathbb{R}^{d+l} \to \mathbb{R}^d$.

Suppose the dynamical system is known to have the integrals

$$I_j := I_j(\mathbf{V}_n, \{p_i\}) = I_j(\mathbf{V}_{n+1}, \{p_i\}), \tag{4}$$

with $j = 1, \ldots, m$. Then we can form a linear combination

$$\sum_{i=1}^{m} \alpha_j I_j =: I(\mathbf{V}_n, \{p_i\}, \{\alpha_j\}), \tag{5}$$

which, for arbitrary α_j, is an integral of system (1).

Taking the difference between the integral $\mathbf{I}$ and its upshifted version, we derive a relation of the type

$$I(\mathbf{V}_{n+1}, \{p_i\}, \{\alpha_j\}) - I(\mathbf{V}_n, \{p_i\}, \{\alpha_j\}) = L(\mathbf{V}_n, \mathbf{V}_{n+1}, \{p_i\})\, L^*(\mathbf{V}_n, \mathbf{V}_{n+1}, \{p_i\}, \{\alpha_j\}), \tag{6}$$

in which

$$L(\mathbf{V}_n, \mathbf{V}_{n+1}, \{p_i\}) = u_{n+d} - f(u_n, u_{n+1}, \ldots, u_{n+d-1}, \{p_i\}) = 0 \tag{7}$$

is equivalent to the original dynamical equation (1).

In fact, for system (1), the left-hand side of (6) must vanish and one would expect the right-hand side to contain a factor such that the vanishing of that factor is equivalent to the dynamical equation (1). Apart from this, the right-hand side may contain another factor such that the vanishing of this second factor also ensures that the left-hand side of (6) is zero.

Starting from a specific example of a dynamical mapping (1) possessing integrals, it is not clear to which extent the second factor would contain an interesting dependence on the field $\mathbf{V}_n$. However, if this dependence is interesting, the vanishing of the second factor could be equivalent to another dynamical equation which by construction of (6) may be called the *dual equation* of the original (1). The dual equation automatically has one integral which is given by $I(\mathbf{V}_n, \{p_i\}, \{\alpha_j\})$ but, depending on the presence or absence of the original parameters $\{p_i\}$ in L^*, there may be more integrals.

At this stage the description of how to obtain the dual equation and the nature of the resulting dual equation is rather general[5]. However, in this paper, we will show on the basis of

[5] For example, the dual equation may be almost trivial.

some more sophisticated specific examples that indeed new dynamical systems with interesting properties can be derived from the differencing of integrals described by (6).

In section 2, we will first consider what we call a motivating example. Our starting point is a low-dimensional mapping arising from the integrable partial difference MKdV equation of [6] by a travelling-wave reduction. It has four integrals and the dual mapping has four parameters and two integrals but may not be integrable.

The following sections 3–6 are devoted to a more general class of higher-dimensional mappings arising from the MKdV partial-difference equations treated in [6]. We consider the integrals arising from different powers of the spectral parameter occurring in the trace of the monodromy matrix T. Defining the dual equation by (6) with $I = \mathrm{Trace}\, T$, we find that these provide some new dynamical systems which can be considered as generalizations of known integrable mappings and which have a number of interesting integrals. In fact, we can establish the integrability of the ensuing dual equations. We do this by deriving a Lax representation for it which can be obtained from the Lax representation of the original system by some simple substitutions. Furthermore, it is interesting to note that in this case the dual equation is a so-called *level-set-dependent* (LSD) version of the original equation. This terminology refers to a higher-dimensional generalization of the work of [3, 4] on QRT [7, 8] and other mappings, where it was shown that a large class of such mappings amounts to a LSD version of the McMillan mapping. The work of [3, 4] and of [9] represents another way of associating two dynamical systems with integrals to one another.

The treatment given in this paper can also be applied to a large variety of other dynamical systems and various possible extensions and specific comments are given in a final discussion.

2. A motivating example: creating the dual of a 4D map

Consider the following fourth-order difference equation:

$$V_4 = V_0 \frac{qV_1 - pV_3}{qV_3 - pV_1}. \tag{8}$$

In (8) and throughout this section, V_j, $j = 0, 1, 2, 3, 4$ is shorthand for V_{n+j}, $n \in \mathbb{Z}$, and p and q are parameters. This equation can be obtained as a reduction of the so-called MKdV PΔE [6]. It is equivalent to the following map $(V_0, V_1, V_2, V_3) \mapsto (V_0', V_1', V_2', V_3')$ in four dimensions:

$$L:\quad \begin{aligned} V_0' &= V_1 \\ V_1' &= V_2 \\ V_2' &= V_3 \\ V_3' &= V_0 \frac{qV_1 - pV_3}{qV_3 - pV_1}. \end{aligned} \tag{9}$$

One checks that the map L has the following four integrals of motion (i.e. $I_\alpha(V_1, V_2, V_3, V_4) = I_\alpha(V_0, V_1, V_2, V_3)$ etc). They each depend linearly on the parameters p and q, which we highlight by writing them:

$$I_\alpha = I_{\alpha,q} q - I_{\alpha,p} p \tag{10}$$

$$I_\beta = I_{\beta,q} q - I_{\beta,p} p \tag{11}$$

$$I_\gamma = I_{\gamma,q} q - I_{\gamma,p} p \tag{12}$$

$$I_\delta = I_{\delta,q} q - I_{\delta,p} p, \tag{13}$$

with

$$I_{\alpha,p} = \frac{V_1}{V_0} + \frac{V_2}{V_1} + \frac{V_3}{V_2} + \frac{V_0}{V_1} + \frac{V_1}{V_2} + \frac{V_2}{V_3} \tag{14}$$

$$I_{\beta,p} = \frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_0}{V_2} + \frac{V_1}{V_3} + \frac{V_3V_0}{V_2V_1} + \frac{V_2V_1}{V_3V_0} \tag{15}$$

$$I_{\gamma,p} = V_3V_0 \tag{16}$$

$$I_{\delta,p} = V_3^{-1}V_0^{-1} \tag{17}$$

$$I_{\alpha,q} = \frac{V_3}{V_0} + \frac{V_0}{V_3} \tag{18}$$

$$I_{\beta,q} = \frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_0}{V_2} + \frac{V_1}{V_3} + \frac{V_2V_3}{V_0V_1} + \frac{V_0V_1}{V_2V_3} \tag{19}$$

$$I_{\gamma,q} = V_0V_1 + V_1V_2 + V_2V_3 \tag{20}$$

$$I_{\delta,q} = V_0^{-1}V_1^{-1} + V_1^{-1}V_2^{-1} + V_2^{-1}V_3^{-1}. \tag{21}$$

These integrals may be inferred from the work of Hydon [2], but they can also be checked directly using two obvious symmetries of (8): $S_1 : V_i \mapsto \lambda V_i, \lambda \in \mathbb{R}$ and $S_2 : V_i \mapsto V_i^{-1}$. This is equivalent to saying that L of (9) commutes with S_1 and S_2, whence if L has an integral I it also has an integral $I \circ S_i, i = 1, 2$. Using this, the integrals follow by constructing homogeneous expressions in the V_i of degree 0 (i.e. I_α, I_β), 2 (i.e. I_γ) and -2 (i.e. I_δ). We now take the linear combination of these integrals[6]:

$$I(V_0, V_1, V_2, V_3; p, q; \alpha, \beta, \gamma, \delta) = \alpha I_\alpha + \beta I_\beta + \gamma I_\gamma + \delta I_\delta, \tag{22}$$

and *difference* it, meaning we consider the difference between I and its upshifted version I' with $V_j \mapsto V_{j+1}$. Since the separate integrals satisfy

$$I_\alpha - I'_\alpha = \left[\left(q\frac{V_3}{V_1} - p\right) - \frac{V_0}{V_4}\left(q - p\frac{V_3}{V_1}\right)\right]\left(\frac{V_1}{V_0} - \frac{V_4}{V_3}\right), \tag{23}$$

$$I_\beta - I'_\beta = \left[\left(q\frac{V_3}{V_1} - p\right) - \frac{V_0}{V_4}\left(q - p\frac{V_3}{V_1}\right)\right]\left(\frac{V_2}{V_0}\left(1 + \frac{V_1}{V_3}\right) - \frac{V_4}{V_2}\right)\left(1 + \frac{V_1}{V_3}\right), \tag{24}$$

$$I_\gamma - I'_\gamma = \left[\left(q\frac{V_3}{V_1} - p\right) - \frac{V_0}{V_4}\left(q - p\frac{V_3}{V_1}\right)\right](-V_1V_4), \tag{25}$$

$$I_\delta - I'_\delta = \left[\left(q\frac{V_3}{V_1} - p\right) - \frac{V_0}{V_4}\left(q - p\frac{V_3}{V_1}\right)\right]\left(V_0^{-1}V_3^{-1}\right), \tag{26}$$

and so vanish if (8) is satisfied, we find

$$\begin{aligned} &I(V_1, V_2, V_3, V_4; p, q; \alpha, \beta, \gamma, \delta) - I(V_0, V_1, V_2, V_3; p, q; \alpha, \beta, \gamma, \delta) \\ &\quad = L(V_0, V_1, V_2, V_3, V_4; p, q)\, L^*(V_0, V_1, V_2, V_3, V_4; \alpha, \beta, \gamma, \delta), \end{aligned} \tag{27}$$

where

$$L(V_0, V_1, V_2, V_3, V_4; p, q) = \left[\left(q\frac{V_3}{V_1} - p\right) - \frac{V_0}{V_4}\left(q - p\frac{V_3}{V_1}\right)\right] \tag{28}$$

[6] Note that the four integrals are not independent, i.e. $I_\gamma I_\delta = 3q^2 + p^2 + qI_\beta$. However, since I_β is linear in p and q we still use this integral in the construction of a dual mapping.

$$L^*(V_0, V_1, V_2, V_3, V_4; \alpha, \beta, \gamma, \delta) = \left[\frac{1}{V_0}(\alpha V_1 V_3 + \beta V_2 V_3 + \beta V_1 V_2 + \delta) \right.$$
$$\left. - \frac{V_4}{V_2}(\alpha V_2 + \beta(V_3 + V_1) + \gamma V_1 V_2 V_3) \right] \frac{1}{V_3}. \tag{29}$$

The equation

$$L(V_0, V_1, V_2, V_3, V_4; p, q) = 0, \tag{30}$$

solved for V_4 gives precisely (8). With (27), this reminds us that I of (22) is an integral of this map. On the other hand, the equation

$$L^*(V_0, V_1, V_2, V_3, V_4; \alpha, \beta, \gamma, \delta) = 0, \tag{31}$$

defines a *different* fourth-order difference equation:

$$V_4 = \frac{V_2}{V_0} \frac{\alpha V_1 V_3 + \beta(V_1 V_2 + V_2 V_3) + \delta}{\alpha V_2 + \beta(V_1 + V_3) + \gamma V_1 V_2 V_3}. \tag{32}$$

We call (32) the *dual map* corresponding to (8). It follows from (27) that I is also an integral of the dual. But significantly in equation (27), the parameter sets $\{p, q\}$ and $\{\alpha, \beta, \gamma, \delta\}$ *dissociate* from one another on the right-hand side. Since p and q do not appear in the dual map (32), we can conclude that their coefficients in I are separately integrals of (32). More precisely, we can use (10)–(13) to rewrite the expression for I of (22) as

$$I(V_0, V_1, V_2, V_3; p, q; \alpha, \beta, \gamma, \delta) = q I_q - p I_p, \tag{33}$$

where I_q and I_p given by

$$\begin{aligned} I_q &= (\alpha I_{\alpha,q} + \beta I_{\beta,q} + \gamma I_{\gamma,q} + \delta I_{\delta,q}) \\ &= \alpha\left(\frac{V_3}{V_0} + \frac{V_0}{V_3}\right) + \beta\left(\frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_0}{V_2} + \frac{V_1}{V_3} + \frac{V_2 V_3}{V_0 V_1} + \frac{V_0 V_1}{V_2 V_3}\right) \\ &\quad + \gamma(V_0 V_1 + V_1 V_2 + V_2 V_3) + \delta\left(V_0^{-1} V_1^{-1} + V_1^{-1} V_2^{-1} + V_2^{-1} V_3^{-1}\right) \end{aligned} \tag{34}$$

$$\begin{aligned} I_p &= (\alpha I_{\alpha,p} + \beta I_{\beta,p} + \gamma I_{\gamma,p} + \delta I_{\delta,p}) \\ &= \alpha\left(\frac{V_1}{V_0} + \frac{V_2}{V_1} + \frac{V_3}{V_2} + \frac{V_0}{V_1} + \frac{V_1}{V_2} + \frac{V_2}{V_3}\right) \\ &\quad + \beta\left(\frac{V_2}{V_0} + \frac{V_3}{V_1} + \frac{V_0}{V_2} + \frac{V_1}{V_3} + \frac{V_0 V_3}{V_1 V_2} + \frac{V_1 V_2}{V_0 V_3}\right) + \gamma V_0 V_3 + \delta V_0^{-1} V_3^{-1} \end{aligned} \tag{35}$$

are integrals of the dual.

Let us make some remarks about this process.

Remark.

(1) It has been convenient that the parameters p and q entered the integrals of (10)–(13) in a linear way. One sees that the number of integrals of the dual is equal to the number of parameters appearing in the original map and vice versa.

(2) The original map is actually degenerate, and can be reduced to a second-order difference equation. This is achieved by introducing the reduced variables: $W_0 = \frac{V_2}{V_0}$, $W_1 = \frac{V_3}{V_1}$. Then (8) reduces to

$$W_2 = \frac{1}{W_0} \frac{q - p W_1}{q W_1 - p}. \tag{36}$$

Of the four integrals I_α, I_β, I_γ, I_δ of the original map, only I_β can be expressed in terms of the reduced variables. However, differencing just I_β in (22)–(27) would lead to a completely trivial dual equation determined by $L^*(V_0, V_1, V_2, V_3, V_4; 0, \beta, 0, 0) = 0$.

(3) For general parameters α, β, γ and δ, the dual map (32) has two integrals. It can also be checked to be measure preserving. But this is not enough to ensure integrability. In fact, no symplectic structure and no additional integrals have been found so far for the general case. Numerically, some special cases of (32) have been shown to have zero algebraic entropy [20]. It may also be interesting to investigate the possible integrability of the dual by extending to four dimensions the arithmetic integrability tests of [16, 17].

We mention some special cases of the dual map that we have proved to be integrable.

- For $\beta = 0$, (32) reduces to the four-dimensional sine-Gordon (SG) mapping [1]. It can be derived from the SG difference equation using a periodicity constraint [6, 1].
- For $\gamma = \delta = 0$, (32) reduces to a three-dimensional mapping in terms of the variables $W_0 = \frac{V_1}{V_0}$, $W_1 = \frac{V_2}{V_1}$, $W_2 = \frac{V_3}{V_2}$:
$$W_3 = \frac{W_1 W_0}{W_2} \frac{\alpha W_2 + \beta(1 + W_1 W_2)}{\alpha W_1 + \beta(1 + W_1 W_2)}. \tag{37}$$
For this reduced mapping, there are two integrals I_q and I_p with $\gamma = \delta = 0$, both of which can be expressed in terms of W_0, W_1 and W_2. In combination with measure preservation this implies integrability.
- For $\alpha = 0$, (32) reduces to a three-dimensional mapping in terms of the variables $W_0 = V_1 V_0$, $W_1 = V_2 V_1$, $W_2 = V_3 V_2$:
$$W_3 = \frac{W_1 W_2}{W_0} \frac{\beta(W_1 + W_2) + \delta}{\beta(W_1 + W_2) + \gamma W_1 W_2}. \tag{38}$$
Again, for $\alpha = 0$, both of the integrals I_q and I_p of (32) can be expressed in terms of W_0, W_1 and W_2.

(4) Finally, it is worthwhile to note that we can achieve four normal forms for (32) with $\gamma = \delta = 0$; $\gamma = \delta = 1$; $\gamma = 1, \delta = 0$; resp $\gamma = 1, \delta = -1$. This can be done using the rescaling $V_i \mapsto \lambda V_i$ and the symmetry $V_i \mapsto V_i^{-1}$, $\gamma \leftrightarrow \delta$.

The exercise of constructing the dual of (8) is not without some mystery! From (36) it follows that (8) is in a sense a trivial four-dimensional mapping since it can be reduced to a two-dimensional one. And yet, ignoring this fact and taking its four non-independent integrals I_α, I_β, I_γ and I_δ produces, in special cases, duals (37) and (38) which appear to be genuinely new and nontrivial integrable three-dimensional mappings (see [18] for some other examples of three-dimensional integrable mappings).

3. Integrable O∆Es derived from soliton equations have a dual

In this section and in section 4, we present a method for obtaining the dual equations of a general class of higher-dimensional mappings arising from the MKdV partial difference equation [6]. However, we first use a general formulation on the basis of a given Lax representation for general integrable partial difference equations. This is done to make the treatment directly applicable to other integrable dynamical systems as well. The remaining sections 5 and 6 will specialize to the MKdV case.

A (scalar) P∆E on a two-dimensional lattice $f_{\ell,m} = 0$, $\ell, m \in \mathbb{Z}$ has a *Lax representation* if there are matrices $L_{\ell,m}(k)$, $M_{\ell,m}^{-1}(k)$, $N_{\ell,m}(k)$ depending on a spectral parameter k such that

$$L_{\ell,m}(k) M_{\ell,m}^{-1}(k) - M_{\ell+1,m}^{-1}(k)\, L_{\ell,m+1}(k) = f_{\ell,m} N_{\ell,m}(k) \tag{39}$$

in which $f_{\ell,m}$ does not depend on k, and $N_{\ell,m}$ is nonsingular on $f_{\ell,m} = 0$.

We restrict ourselves to the PΔE

$$f_{\ell,m} = 0$$

with $f_{\ell,m}$ of the form

$$f_{\ell,m} = f(u_{\ell,m}, u_{\ell+1,m}, u_{\ell,m+1}, u_{\ell+1,m+1}, \underline{p}_{\ell,m}) \qquad (40)$$

where the $\underline{p}_{\ell,m}$ denote any additional parameters arising from the matrices L and M, and the fields $u_{\ell,m}$, for simplicity, are taken to be scalars. The subscripts ℓ, m allow for the possibility that the parameters $\underline{p}$ depend on the lattice sites (ℓ, m). This general setting is investigated in order to obtain dual equations and their integrals with a sufficient amount of generality.

A PΔE can be reduced to an ordinary difference equation (OΔE) through *travelling-wave reductions* [6]. This can be done considering two integers z_1 and $z_2 > z_1$ which are relatively prime. In the (z_1, z_2) travelling-wave reduction the parameters $\underline{p}_{\ell,m}$ in the matrices $L_{\ell,m}(k)$ and $M^{-1}_{\ell,m}(k)$ depend on the sites (ℓ, m) via the similarity variable $n = z_1\ell + z_2 m$ and we consider periodic solutions of $f_{\ell,m} = 0$ satisfying $u_{\ell,m} = u_{\ell-z_2,m+z_1}$, i.e.

$$\underline{p}_{\ell,m} = \underline{p}_n, \qquad u_{\ell,m} = u_n, \qquad n = z_1\ell + z_2 m \qquad (41)$$

and these solutions can be obtained from the OΔE

$$f_n = f\left(u_n, u_{n+z_1}, u_{n+z_2}, u_{n+z_1+z_2}, \underline{p}_n\right) = 0. \qquad (42)$$

The OΔE can be solved specifying *initial values* on a *standard staircase* [6] consisting of points $(\ell_i, m_i), i = 0, 1, \ldots, z_1 + z_2 - 1$ with $n(\ell_i, m_i) = n + n_i$ such that every value $n_i = 0, 1, \ldots, z_1 + z_2 - 1$ occurs exactly once among the n_i values on the staircase. In fact, the u_n with $n \geqslant z_1 + z_2$ can be obtained from the OΔE $f_{n+\nu} = 0$ $(\nu \geqslant 0)$, expressing $u_{n+\nu+z_1+z_2}$ in terms of $u_{n+\nu}, u_{n+\nu+z_1}, u_{n+\nu+z_2}$.

The *monodromy matrix* $\mathcal{L}_n$ is defined to be the ordered product of Lax matrices along a standard staircase (more detail will follow in the next section). From (39) it can be shown that

$$\text{Trace}\,\mathcal{L}_n^\nu = \text{Trace}\,\mathcal{L}_{n+1}^\nu, \qquad \nu = 1, 2, \ldots. \qquad (43)$$

In the special case of 2×2 matrices L_n, M_n^{-1}, such that $\det \mathcal{L}_n$ is trivial, we can restrict ourselves to $\nu = 1$.

Equation (43) is satisfied independently of the value of the spectral parameter k, and the coefficients of the various powers of k appearing in Trace $\mathcal{L}_n$ give integrals of the OΔE $f_n = 0$. (Note that it is not generally true that all integrals of the OΔE can be obtained that way. In fact of the four integrals $I_\alpha, I_\beta, I_\gamma, I_\delta$ of the mapping (8) in the example of section 2, only the integral I_β follows from the Lax representation of the MKdV reduction associated with (8)). On the other hand, taking the difference between Trace $\mathcal{L}_n$ and its shifted version we obtain a relation

$$\text{Trace}\,\mathcal{L}_n - \text{Trace}\,\mathcal{L}_{n+1} = f_n f_n^* \qquad (44)$$

containing a factor f_n^*, ensuring that $f_n^* = 0$ implies the vanishing of the lhs as well. By analogy with (6), the equation $f_n^* = 0$ will be called the dual OΔE.

4. Standard staircase and dual OΔE

We now give a prescription for the dual OΔE in the case that $z_1 = 1, z_2 = z, n = l + zm$. In this case, the standard staircase as introduced in [6] can be constructed in the following steps, see, e.g., figure 1.

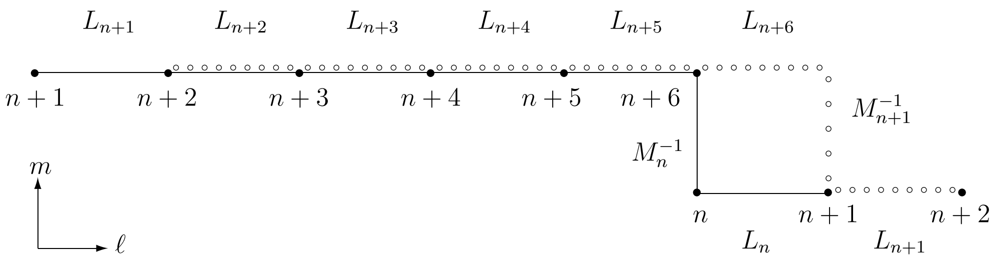

Figure 1. Standard staircase (———) and shifted staircase (○ ○ ○ ○ ○) under $n \to n+1$, in the case $z_1 = 1, z_2 = 6$.

To construct a standard staircase for $z_1 = 1, z_2 = z$:

(a) we start with the point (ℓ_0, m_0) with $n(\ell_0, m_0) = n + n_0, n_0 = 1$;

(b) we do a step to the left, to the point $(\ell_1, m_1) = (\ell_0 - 1, m_0)$ with $n(\ell_1, m_1) = n + n_1$, $n_1 = 0$;

(c) then we do a step upward to the point $(\ell_2, m_2) = (\ell_1, m_1 + 1)$ with $n(\ell_2, m_2) = n + n_2$, $n_2 = z$;

(d) next we do $z - 2$ steps to the left via points $(\ell_i, m_i) = (\ell_0 - i + 1, m_0 + 1)$ with $n(\ell_i, m_i) = n + n_i, n_i = z - i + 2, i = 3, 4, \ldots, z - 1$ to reach the point $(\ell_z, m_z) = (\ell_0 - z + 1, m_0 + 1)$ with $n(\ell_z, m_z) = n + n_z, n_z = 2$;

(e) a final step to the left brings us to $(\ell_{z+1}, m_{z+1}) = (\ell_0 - z, m_0 + 1)$ with $n(\ell_{z+1}, m_{z+1}) = 1$. The staircase is thus completed. See figure 1 for an example.

To obtain the monodromy matrix $\mathcal{L}_n$ we associate Lax matrices with the steps of the staircase in the following way:

(i) we associate with the first step from (ℓ_0, m_0) to (ℓ_1, m_1) the Lax matrix $S(n_0, n_1) = L_n$;

(ii) with the second step from (ℓ_1, m_1) to (ℓ_2, m_2) we associated the Lax matrix $S(n_1, n_2) = M_n^{-1}$;

(iii) with the $z - 2$ steps to the left from (ℓ_i, m_i) to (ℓ_{i+1}, m_{i+1}) with $i = 2, \ldots, z - 1$ we associate the Lax matrices $S(n_i, n_{i+1}) = L_{n+z-i+1}$;

(iv) with the final step from (ℓ_z, m_z) to (ℓ_{z+1}, m_{z+1}) we associate the Lax matrix $S(n_z, n_{z+1}) = L_{n+1}$.

The monodromy matrix $\mathcal{L}_n$ is the ordered product of Lax matrices along the standard staircase

$$\mathcal{L}_n = \prod_{i=0}^{z} S(n_i, n_{i+1}). \tag{45}$$

From the explicit $S(n_i, n_{i+1})$ we have the factorization property

$$\mathcal{L}_n = L_n M_n^{-1} A_n L_{n+1} \tag{46a}$$

with

$$A_n = L_{n+z-1} L_{n+z-2} \ldots L_{n+2}. \tag{46b}$$

The shifted monodromy matrix

$$\mathcal{L}_{n+1} = \prod_{i=0}^{z} S(n_i + 1, n_{i+1} + 1), \tag{47a}$$

cf figure 1, can be expressed as

$$\mathcal{L}_{n+1} = L_{n+1} M_{n+1}^{-1} L_{n+z} A_n \tag{47b}$$

and therefore, cf (39)

$$\mathcal{L}_n L_{n+1}^{-1} - L_{n+1}^{-1} \mathcal{L}_{n+1} = \left(L_n M_n^{-1} - M_{n+1}^{-1} L_{n+z}\right) A_n = f_n N_n A_n. \tag{48}$$

Hence

$$\text{Trace}\,\mathcal{L}_n - \text{Trace}\,\mathcal{L}_{n+1} = f_n\,\text{Trace}\,N_nT_n \tag{49a}$$

with

$$T_n = A_nL_{n+1} = L_{n+z-1}L_{n+z-2}\ldots L_{n+1}. \tag{49b}$$

The dual equation is given by

$$f_n^* = \text{Trace}\,N_nT_n = 0. \tag{50}$$

It is completely determined by the matrix $N_n(k)$ in equation (39) and the matrices L in equation (49b). Since Trace $\mathcal{L}_n$ is also an integral of the dual equation, we can use the difference to define a dual–dual equation $f_n^{**} = 0$ via Trace $\mathcal{L}_n$ − Trace $\mathcal{L}_{n+1} = f_n^*f_n^{**}$. In comparing this to equation (44) it is clear that the dual–dual equation $f_n^{**} = 0$ is just the original one $f_n = 0$, provided that the same integrals are used in the construction of the dual and the dual–dual equation.

5. Dual MKdV OΔE in the case $z_1 = 1, z_2 = z$

We now specialize the discussion of the previous two sections to consider the dual OΔE associated with the MKdV PΔE $f_{\ell,m} = 0$, in the case that $z_1 = 1, z_2 = z, n = \ell + zm$. The dual equation associated with the more general reduction of the MKdV PΔE with $z_2 > z_1 > 1$ can also be investigated. However, this is more complicated and will not be explicitly pursued in this paper.

In the investigation of the dual equation of the $z_1 = 1, z_2 = z$ reduction of the MKdV PΔE, one may anticipate that the number of parameters appearing in the dual equation will depend on the number of parameters in the original MKdV PΔE, or in the Lax matrices occurring in the associated Lax representation. We therefore choose a rather general inhomogeneous setting in which the Lax matrices $L_{\ell,m}(k), M^{-1}_{\ell,m}(k)$ contain parameters $p_{\ell,m}, r_{\ell,m}, a_{\ell,m}, b_{\ell,m}, q_{\ell,m}, s_{\ell,m}, c_{\ell,m}, d_{\ell,m}$. The subscripts indicate a possible dependence of the parameters on the sites (ℓ, m) of the two-dimensional lattice; albeit with such a dependence satisfying compatibility conditions to ensure that a consistent difference equation $f_{\ell,m} = 0$ follows from the Lax equation

$$L_{\ell,m}(k)M^{-1}_{\ell,m}(k) - M^{-1}_{\ell+1,m}(k)L_{\ell,m+1}(k) = 0, \tag{51}$$

i.e. equation (39) with the right-hand side replaced by 0.

To investigate the $z_1 = 1, z_2 = z$ reduction of the MKdV PΔE with $u_{\ell,m}$ satisfying the periodicity constraint $u_{l,m} = u_{l-z,m+1} = u_n, n = l + zm$, we impose the conditions $\underline{p}_{l,m} = \underline{p}_{l-z,m+1} = \underline{p}_n$ for $\underline{p} = (p, r, a, b, q, s, c, d)$. That is, all parameters occurring in the Lax matrices satisfy the periodicity constraint also.

Hence we consider the Lax matrices

$$L_{\ell,m}(k) = \begin{pmatrix} p_n & -a_nu_{\ell+1,m} \\ \frac{-k^2b_n}{u_{\ell,m}} & r_n\frac{u_{\ell+1,m}}{u_{\ell,m}} \end{pmatrix}, \qquad M^{-1}_{\ell,m}(k) = \begin{pmatrix} s_n & d_nu_{\ell,m} \\ \frac{k^2c_n}{u_{\ell,m+1}} & q_n\frac{u_{\ell,m}}{u_{\ell,m+1}} \end{pmatrix}. \tag{52}$$

The special case of (52): $p_{\ell,m} = p, r_{\ell,m} = r, q_{\ell,m} = q, s_{\ell,m} = s, a_{\ell,m} = b_{\ell,m} = c_{\ell,m} = d_{\ell,m} = 1$, was investigated in [1] in an investigation of four-dimensional mappings satisfying a specific symplectic structure (as a generalization of [6]). Working out the (1, 1) and (2, 2) elements of condition (51) and insisting that the (1, 2) and (2, 1) elements of this condition lead to the same difference equation, we are led to the compatibility conditions

$$s_{n+1} = \frac{p_n}{p_{n+z}}s_n, \qquad q_{n+1} = \frac{r_n}{r_{n+z}}q_n, \qquad d_{n+1} = \frac{a_n}{b_{n+z}}c_n, \qquad c_{n+1} = \frac{b_n}{a_{n+z}}d_n \tag{53}$$

and

$$k_n := k^2 \frac{a_n b_n}{p_n r_n} = k_{n+z}, \tag{54a}$$

or equivalently with (53)

$$\mathtt{l}_n := \frac{c_n d_n}{q_n s_n} = \mathtt{l}_{n+1} = \mathtt{l}. \tag{54b}$$

Under these conditions, we have equation (39) with

$$f_{\ell,m} = p_n d_n - a_n q_n \frac{u_{\ell+1,m}}{u_{\ell,m+1}} + \frac{u_{\ell+1,m+1}}{u_{\ell,m}} \left(a_{n+z} s_{n+1} - d_{n+1} r_{n+z} \frac{u_{\ell+1,m}}{u_{\ell,m+1}} \right). \tag{55a}$$

The equation $f_{\ell,m} = 0$ with $f_{\ell,m}$ given by (55*a*) is a general inhomogeneous version of the MKdV partial-difference equation of [1, 6] with parameters $\underline{p}$ depending on the sites (ℓ, m) via the similarity variable n. Under conditions (53), (54) for the parameters, it has the Lax representation (39) with the matrices $L_{\ell,m}(k)$ and $M^{-1}_{\ell,m}(k)$ given by (52) and the matrix $N_{\ell,m}(k)$ given by

$$N_{\ell,m}(k) = \begin{pmatrix} 0 & u_{\ell,m} \\ \frac{-k^2}{u_{\ell+1,m+1}} \frac{b_n}{a_{n+z}} \frac{p_{n+z}}{p_n} & 0 \end{pmatrix}. \tag{55b}$$

In the travelling-wave reduction $z_1 = 1$, $z_2 = z$, (55) reduces to the primary MKdV OΔE

$$f_n = 0 \tag{56a}$$

with

$$f_n = p_n d_n - a_n q_n \frac{u_{n+1}}{u_{n+z}} + \frac{u_{n+1+z}}{u_n} \left(a_{n+z} s_{n+1} - d_{n+1} r_{n+z} \frac{u_{n+1}}{u_{n+z}} \right). \tag{56b}$$

The dual equation follows from (50) and (55*b*) and is given by

$$f^*_n = 0, \tag{57a}$$

where

$$f^*_n = t_{21} u_n - \frac{b_n}{a_{n+z}} \frac{p_{n+z}}{p_n} \frac{k^2}{u_{n+z+1}} t_{12} \tag{57b}$$

and t_{12} and t_{21} are the off-diagonal elements of the matrix T_n, defined by (49*b*).

Evaluating the off-diagonal elements of the matrix T_n it can be shown that the dual MKdV OΔE (57) can be expressed as a $(z-1)$-dimensional mapping in terms of the reduced variables

$$W_{n+\nu} = \frac{r_{n+\nu}}{a_{n+\nu}} \frac{a_{n+\nu+1}}{p_{n+\nu+1}} \frac{u_{n+\nu+2}}{u_{n+\nu}}. \tag{58}$$

We have

$$\frac{W_{n+z-1}}{W_n} = k_n \frac{Z_{12}(W_{n+1}, W_{n+2}, \ldots, W_{n+z-2})}{Z_{21}(W_{n+1}, W_{n+2}, \ldots, W_{n+z-2})} \tag{59}$$

in which Z_{12} and Z_{21} are the off-diagonal elements of the matrix

$$Z(W_{n+1}, W_{n+2}, \ldots, W_{n+z-2}) = \begin{pmatrix} 1 & 1 \\ k_{n+z-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ k_{n+z-2} W_{n+z-2} & W_{n+z-2} \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 1 \\ k_{n+z-3} W_{n+z-3} & W_{n+z-3} \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ k_{n+1} W_{n+1} & W_{n+1} \end{pmatrix}, \tag{60}$$

cf appendix A for some details of the derivation.

In equation (59) the numerator Z_{12} and denominator Z_{21} are multilinear functions containing all 2^{z-2} terms $W^{\mu_1}_{n+1} W^{\mu_2}_{n+2} \ldots W^{\mu_{z-2}}_{n+z-2}$, $\mu_1, \mu_2, \ldots, \mu_{z-2} = 0, 1$, with coefficients depending on $k_{n+1}, k_{n+2}, \ldots, k_{n+z-1}$.

For example for $z = 3, 4, 5$, we have respectively

$$
\begin{aligned}
z = 3:\quad & Z_{12}(W_{n+1}) = 1 + W_{n+1} \\
& Z_{21}(W_{n+1}) = k_{n+2} + k_{n+1} W_{n+1} \\
z = 4:\quad & Z_{12}(W_{n+1}, W_{n+2}) = 1 + W_{n+1} + W_{n+2}(k_{n+2} + W_{n+1}) \\
& Z_{21}(W_{n+1}, W_{n+2}) = k_{n+3}(1 + k_{n+1} W_{n+1}) + W_{n+2}(k_{n+2} + k_{n+1} W_{n+1}) \\
z = 5:\quad & Z_{12}(W_{n+1}, W_{n+2}, W_{n+3}) = (1 + k_{n+3} W_{n+3})(1 + W_{n+1}) \\
& \qquad + W_{n+2}(1 + W_{n+3})(k_{n+2} + W_{n+1}) \\
& Z_{21}(W_{n+1}, W_{n+2}, W_{n+3}) = (k_{n+4} + k_{n+3} W_{n+3})(1 + k_{n+1} W_{n+1}) \\
& \qquad + W_{n+2}(k_{n+4} + W_{n+3})(k_{n+2} + k_{n+1} W_{n+1}).
\end{aligned}
\tag{61}
$$

6. Integrability of the dual MKdV

To prove that the dual MKdV OΔE is integrable we first construct two integrals $\mathcal{J}_s, \mathcal{J}_q$ and two 2-integrals[7] $\mathcal{J}_{c_n}, \mathcal{J}_{d_n}$ satisfying $\mathcal{J}_{c_{n+1}} = \mathcal{J}_{d_n}, \mathcal{J}_{d_{n+1}} = \mathcal{J}_{c_n}$.

From equations (46*a*), (49*b*), (50), it is clear that the parameters $\{s_n, c_n, d_n, q_n\}$ in the matrix M_n^{-1} in Trace $\mathcal{L}_n$ dissociate from the parameters p_n, a_n, b_n, r_n occurring in the matrices L and the dual equation. To investigate the integrals of the dual equation it is therefore worthwhile to consider the decomposition

$$
\operatorname{Trace} \mathcal{L}_n = \mathcal{J}_{s_n} + \mathcal{J}_{c_n} + \mathcal{J}_{d_n} + \mathcal{J}_{q_n} \tag{62}
$$

in which $\mathcal{J}_{s_n}, \mathcal{J}_{c_n}, \mathcal{J}_{d_n}, \mathcal{J}_{q_n}$ are terms arising from elements s_n, c_n, d_n, q_n of the matrix M_n^{-1} in Trace $\mathcal{L}_n$ = Trace $L_n M_n^{-1} T_n$.

Here using (52), and involving the matrix elements t_{ij} of T_n, we have

$$
\begin{pmatrix} \frac{\mathcal{J}_{s_n}}{s_n} & \frac{\mathcal{J}_{c_n}}{k^2 c_n} \\ \frac{\mathcal{J}_{d_n}}{d_n} & \frac{\mathcal{J}_{q_n}}{q_n} \end{pmatrix} = \begin{pmatrix} t_{11} & \frac{t_{12}}{u_n} \\ t_{21} u_n & t_{22} \end{pmatrix} \begin{pmatrix} p_n & -a_n \frac{u_{n+1}}{u_{n+z}} \\ -k^2 b_n & r_n \frac{u_{n+1}}{u_{n+z}} \end{pmatrix}. \tag{63}
$$

Introducing

$$
J_{s_n} = \mathcal{J}_{s_n}/s_n, \qquad J_{c_n} = \mathcal{J}_{c_n}/c_n, \qquad J_{d_n} = \mathcal{J}_{d_n}/d_n, \qquad J_{q_n} = \mathcal{J}_{q_n}/q_n \tag{64}
$$

and using equation (53) to express $s_{n+1}, q_{n+1}, d_{n+1}, c_{n+1}$ in terms of s_n, q_n, c_n, d_n, we find from (49*a*) and (50) that if the dual equation is satisfied then

$$
\begin{aligned}
0 = \operatorname{Trace} \mathcal{L}_n - \operatorname{Trace} \mathcal{L}_{n+1} &= s_n \left\{ J_{s_n} - \frac{p_n}{p_{n+z}} J_{s_{n+1}} \right\} + q_n \left\{ J_{q_n} - \frac{r_n}{r_{n+z}} J_{q_{n+1}} \right\} \\
&\quad + c_n \left\{ J_{c_n} - \frac{a_n}{b_{n+z}} J_{d_{n+1}} \right\} + d_n \left\{ J_{d_n} - \frac{b_n}{a_{n+z}} J_{c_{n+1}} \right\}.
\end{aligned}
\tag{65}
$$

Since the matrices $L_{n+1}, L_{n+2}, \ldots, L_{n+z-1}$ and N_n occurring in the dual equation do not contain any of the parameters s_n, q_n, c_n, d_n, equation (65) must be valid independently of the choice of s_n, q_n, c_n, d_n. This means that

$$
J_{s_n} = \frac{p_n}{p_{n+z}} J_{s_{n+1}}, \qquad J_{q_n} = \frac{r_n}{r_{n+z}} J_{q_{n+1}}, \qquad J_{c_n} = \frac{a_n}{b_{n+z}} J_{d_{n+1}}, \qquad J_{d_n} = \frac{b_n}{a_{n+z}} J_{c_{n+1}} \tag{66}
$$

and hence, using again (53) and (64),

$$
\mathcal{J}_{s_n} = \mathcal{J}_{s_{n+1}} := \mathcal{J}_s, \qquad \mathcal{J}_{q_n} = \mathcal{J}_{q_{n+1}} := \mathcal{J}_q, \qquad \mathcal{J}_{c_n} = \mathcal{J}_{d_{n+1}}, \qquad \mathcal{J}_{d_n} = \mathcal{J}_{c_{n+1}}. \tag{67}
$$

[7] More generally, k-integrals of a mapping are defined as integrals of the kth iterate of the mapping [19], here corresponding to the shift $n \to n + k$.

From equation (63) we can solve the off-diagonal elements t_{12}, t_{21} of the matrix T_n to obtain

$$\frac{1}{u_n^2}\frac{t_{12}}{t_{21}} = \frac{a_n \frac{\mathcal{J}_{s_n}}{s_n}\frac{u_{n+1}}{u_{n+z}} + \frac{p_n}{k^2}\frac{\mathcal{J}_{c_n}}{c_n}}{k^2 b_n \frac{\mathcal{J}_{q_n}}{q_n} + r_n \frac{\mathcal{J}_{d_n}}{d_n}\frac{u_{n+1}}{u_{n+z}}}. \tag{68}$$

Inserting this in the dual MKdV (57), including the factor a_{n+z}/b_n in the denominator and the factor p_{n+z}/p_n in the numerator and using (53) yields

$$\frac{u_{n+z+1}}{u_n} = \frac{k^2 a_n \frac{\mathcal{J}_{s_n}}{s_{n+1}}\frac{u_{n+1}}{u_{n+z}} + p_{n+z}\frac{\mathcal{J}_{c_n}}{c_n}}{k^2 a_{n+z}\frac{\mathcal{J}_{q_n}}{q_n} + r_n \frac{\mathcal{J}_{d_n}}{c_{n+1}}\frac{u_{n+1}}{u_{n+z}}}. \tag{69}$$

Equation (69) is very similar to the MKdV OΔE (56). To prove its integrability we rewrite it as follows, recalling (54*b*):

$$\frac{u_{n+z+1}}{u_n} = \frac{k^2 a_n q_n \mathcal{J}_{s_n}\frac{u_{n+1}}{u_{n+z}} + p_n d_n \mathtt{1}^{-1}\mathcal{J}_{c_n}}{k^2 a_{n+z} s_{n+1}\mathcal{J}_{q_n} + r_{n+z} d_{n+1}\mathtt{1}^{-1}\mathcal{J}_{d_n}\frac{u_{n+1}}{u_{n+z}}}. \tag{70}$$

Equation (70) is the travelling-wave reduction $u_{\ell,m} \to u_{\ell+zm} = u_n$ of the PΔE

$$\frac{u_{\ell+1,m+1}}{u_{\ell,m}} = \frac{a_n q_n^* \frac{u_{\ell+1,m}}{u_{\ell,m+1}} - p_n d_n^*}{a_{n+z}s_{n+1}^* - r_{n+z}d_{n+1}^*\frac{u_{\ell+1,m}}{u_{\ell,m+1}}} \tag{71a}$$

with

$$\begin{aligned} q_n^* &= k^2 q_n \mathcal{J}_s, & d_n^* &= -d_n \mathtt{1}^{-1}\mathcal{J}_{c_n}, \\ s_{n+1}^* &= k^2 s_{n+1}\mathcal{J}_q, & d_{n+1}^* &= -d_{n+1}\mathtt{1}^{-1}\mathcal{J}_{d_n}. \end{aligned} \tag{71b}$$

Since (71*a*) is identical to (55*a*), it follows that in order for (71*a*) to be integrable via a Lax representation, we must impose the compatibility conditions

$$\frac{{s_{n+1}}^*}{{s_n}^*} = \frac{p_n}{p_{n+z}}, \qquad \frac{{q_{n+1}}^*}{{q_n}^*} = \frac{r_n}{r_{n+z}}, \qquad \frac{{d_{n+1}}^*}{{c_n}^*} = \frac{a_n}{b_{n+z}}, \qquad \frac{{c_{n+1}}^*}{{d_n}^*} = \frac{b_n}{a_{n+z}}. \tag{72}$$

Defining

$$c_n^* = -c_n \mathcal{J}_{d_n}\mathtt{1}^{-1} \tag{73}$$

and using (53) and (67) it follows with (71*b*) that this condition is indeed satisfied. Comparing (71) and (72) with (55*a*) and (53) it follows that the matrices

$$L_{\ell,m}(k^*) = \begin{pmatrix} p_n & -a_n u_{\ell+1,m} \\ \frac{-k^{*2}b_n}{u_{\ell,m}} & r_n \frac{u_{\ell+1,m}}{u_{\ell,m}} \end{pmatrix} \tag{74a}$$

and

$$M^{*-1}_{\ell,m}(k^*) = \begin{pmatrix} s_n^* & d_n^* u_{\ell,m} \\ \frac{k^{*2}c_n^*}{u_{\ell,m+1}} & q_n^* \frac{u_{\ell,m}}{u_{\ell,m+1}} \end{pmatrix}, \tag{74b}$$

in which k^* is a new spectral parameter, satisfy the relation

$$\begin{aligned} L_{\ell,m}(k^*)M^{*-1}_{\ell,m}(k^*) &- M^{*-1}_{\ell+1,m}(k^*)L_{\ell,m+1}(k^*) \\ &= \left(p_n d_n^* - a_n q_n^* \frac{u_{\ell+1,m}}{u_{\ell,m+1}} + \frac{u_{\ell+1,m+1}}{u_{\ell,m}}\left(a_{n+z}s_{n+1}^* - d_{n+1}^* r_{n+z}\frac{u_{\ell+1,m}}{u_{\ell,m+1}} \right) \right) N_{\ell,m}(k^*). \end{aligned} \tag{75}$$

This means that equation (71) has a Lax representation in terms of the matrices $L_{\ell,m}(k^*)$ and $M^{*-1}_{\ell,m}(k^*)$ and this can be used to evaluate the monodromy matrix $\mathcal{L}_n^*$ for the travelling-wave reduction (70) built up from these matrices. From Trace $\mathcal{L}_n^*$, one can obtain additional integrals of the dual MKdV OΔE for larger values of z.

The parameters $\mathcal{J}_s$, $\mathcal{J}_{c_n}$, $\mathcal{J}_{d_n}$ and $\mathcal{J}_q$ can be determined specifying the initial conditions for $u_n, u_{n+1}, \ldots, u_{n+z}$ on a standard staircase. Because of this, equation (29) will be called the *level-set-dependent* (LSD) MKdV OΔE.

The term *level-set-dependent* relates to a concept introduced in [3, 4]. Given two OΔEs, with OΔE (2) possessing integrals, we will say

$$\mathrm{O\Delta E}(2) = \mathrm{LSD\ O\Delta E}(1), \tag{76}$$

if OΔE (2) acts on the level sets of its integrals as OΔE (1). The idea is that an initial condition for OΔE (2) determines the values of the integrals of OΔE (2). Then the ensuing motion can be described by a simpler (and possibly integrable) OΔE (1). This is completely analogous to the two-dimensional setting of [3, 4] in which it was shown that each QRT map actually acts as a generalized McMillan map on the level sets of the QRT integral. Here we have shown that the *dual* MKdV map acts as a MKdV map on the level sets of the integrals $\mathcal{J}_s$, $\mathcal{J}_q$ and the 2-integrals $\mathcal{J}_{c_n} = \mathcal{J}_{d_{n+1}}$ and $\mathcal{J}_{d_n} = \mathcal{J}_{c_{n+1}}$.

The explicit expressions for $\mathcal{J}_s$, $\mathcal{J}_q$, $\mathcal{J}_{c_n}$ and $\mathcal{J}_{d_n}$ are given by[8]

$$\begin{aligned} &\mathcal{J}_s = S(Z_{11} + k_n W_n Z_{12}), && \mathcal{J}_q = \frac{Q(Z_{21} + W_n Z_{22})}{W_n W_{n+1} \ldots W_{n+z-2}} \\ &\mathcal{J}_{c_n} = -k^2 C_n S(Z_{11} + W_n Z_{12}) \frac{u_{n+1}}{u_{n+z}}, && \mathcal{J}_{d_n} = -C_n^{-1} 1 Q \frac{(Z_{21} + k_n W_n Z_{22})}{W_n W_{n+1} \ldots W_{n+z-2}} \frac{u_{n+z}}{u_{n+1}}, \end{aligned} \tag{77}$$

in which the $W_{n+\nu}$ have been defined by (58) and $Z_{\alpha\beta} = Z_{\alpha\beta}(W_{n+1}, W_{n+2}, \ldots, W_{n+z-2})$ are matrix elements of the matrix Z defined by (60).

The parameter C_n is defined by

$$C_n = \frac{c_n a_n}{s_n p_n} \tag{78}$$

and the constants Q and S by

$$\begin{aligned} S_n &= p_n p_{n+1} \ldots p_{n+z-1} s_n = S \qquad (S_{n+1}/S_n = 1) \\ Q_n &= r_n r_{n+1} \ldots r_{n+z-1} \cdot q_n = Q \qquad (Q_{n+1}/Q_n = 1). \end{aligned} \tag{79}$$

Some details of the derivation of (77) are presented in appendix B.

In summary, we have considered the dual MKdV OΔE associated with the $z_1 = 1$, $z_2 = z$ similarity reduction of the MKdV PΔE. These considerations can be extended to more general similarity reductions $z_1 > 1$. Also there, the dual MKdV OΔE can be expressed as a generalized MKdV with parameters $\mathcal{J}$ depending on the initial conditions but in the general case we do not anticipate relations like (67) ensuring the existence of a Lax representation and consequently the integrability of the dual MKdV OΔE. The present treatment can also be adapted to the SG case with only minor variations.

7. Concluding remarks

We have studied the dual MKdV OΔE associated with the $z_1 = 1$, $z_2 = z$ similarity reduction of the MKdV PΔE. This dual equation is an integrable system, since it is the LSD version of the MKdV OΔE. In this LSD version, there appear the integrals $\mathcal{J}_s$, $\mathcal{J}_q$ and the 2-integrals $\mathcal{J}_{c_n} = \mathcal{J}_{d_{n+1}}$ and $\mathcal{J}_{d_n} = \mathcal{J}_{c_{n+1}}$ of the dual MKdV which follow from the trace of the monodromy matrix. The dual MKdV has a Lax representation that is obtained from the Lax representation (52) of the MKdV PΔE using the substitutions (71*b*) which involve the integrals $\mathcal{J}_s$, $\mathcal{J}_q$ and

[8] Note that $\mathcal{J}_s$ and $\mathcal{J}_q$ depend only on the variables W_n.

the 2-integrals $\mathcal{J}_{c_n}, \mathcal{J}_{d_n}$, and introduces a new spectral parameter. The (reduced variable) dual MKdV (59), (60) contains the parameters $k_n, \ldots, k_{n+z-1}$, defined by (54a). In the homogeneous case, this set of parameters reduces only to the single parameter k. However, it is worthwhile to note that the dual equation has been obtained using the trace of the monodromy matrix

$$\text{Trace } T_n = \sum_m k^{2m} I_m \tag{80}$$

in which the integrals I_m of the original MKdV O∆E are the coefficients of different powers of the spectral parameter. Rather than considering the special case of (80), one can also start from an arbitrary linear combination of integrals

$$I = \sum_m \alpha_m I_m \tag{81}$$

and the dual MKdV in the homogeneous case can be inferred from (59)–(60) by replacing k^{2m} by the parameter α_m. In this way, the family of dual MKdV mappings for increasing z will contain an increasing number of parameters α_m, but the existence of a Lax representation is not obvious in this more general case.

Finally, it may be noted that the dual equations (59), (60) have been obtained taking into account only the integrals of the MKdV O∆E that appear in the trace of the monodromy matrix. However, in special cases, the MKdV O∆E $f_n = 0$ of (56) may have additional integrals which can be inferred from the work of Hydon [2] or by considering homogeneous expressions in the variables u of various degrees. This would then lead to a dual equation in the variables u with more parameters, similar to the case of the motivating example of section 2. But in that case the Lax representation (74) and LSD description (70) are no longer valid and the integrability of the dual equation is open.

We conclude with some remarks:

- The considerations in this paper can be applied to the $z_1 = 1, z_2 = z$ reductions of other P∆E as well. For instance, for the reduction of the integrable SG P∆E, the dual equation can be expressed as a LSD SG O∆E and a Lax representation is obtained by simple substitutions from the Lax representation of the SG O∆E. Also the dual of the $z_1 = 1, z_2 = z$ reduction of the KdV P∆E can be investigated in a similar way.
- The treatment in this paper can also be extended to more general reductions with $z_1 > 1$ of the MKdV P∆E. The dual equation can be obtained from the trace of the monodromy matrix and one can show that the dual equation can be regarded as an extended version of the MKdV O∆E as given by (70) with the coefficients $\mathcal{J}_{s_n}, \mathcal{J}_{q_n}, \mathcal{J}_{c_n}$ and $\mathcal{J}_{d_n}$ explicitly given in terms of the variables u_n. In this general case, however, the reasoning leading to a condition of the type (66) is no longer applicable and there is no direct relation between the coefficients $\mathcal{J}_{s_n}, \mathcal{J}_{q_n}, \mathcal{J}_{c_n}$ and $\mathcal{J}_{d_n}$ and the integrals of the dual MKdV. So in this case the extended MKdV equation (70) cannot be interpreted as an LSD MKdV equation and also a Lax representation like (75) does not apply. Hence, at this stage, the integrability of the dual MKdV equation for $z_1 > 1$ has not been proved. It is interesting to note, however, that the $z_1 = 2, z_2 = 3$ reduction of the MKdV P∆E leads to a dual equation which is a four-dimensional mapping that is superintegrable by the existence of two integrals $I_n = I_{n+1}, J_n = J_{n+1}$ and two coupled 2-integrals $G_n = H_{n+1}, H_n = G_{n+1}$. We hope to investigate the integrability of the more general cases in the near future.
- From (76), it follows that each orbit of O∆E (2) corresponds to an orbit of O∆E (1) restricted to a particular intersection of level sets of the integrals of O∆E (2). This shows that considered 'orbit-by-orbit' O∆E (2) is solvable, and in that sense integrable, if O∆E

(1) is. For example, if OΔE (2) is a rational map, each of its orbits will have vanishing algebraic entropy, thus passing this test. However, on the face of it, LSD integrability does not imply symplecticity or Liouville integrability of OΔE (2), as these refer to global structures of OΔE (2) by which orbits are assembled together.

Appendix A

To prove equation (59) we first express the Lax matrix $L_{n+\nu}(k)$, cf (52), as

$$L_{n+\nu}(k) = p_{n+\nu}\begin{pmatrix}1 & 0\\ 0 & -\frac{p_{n+\nu+1}}{a_{n+\nu+1}}\frac{1}{u_{n+\nu+2}}.\end{pmatrix}\begin{pmatrix}1 & 1\\ k_{n+\nu}W_{n+\nu} & W_{n+\nu}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -\frac{a_{n+\nu}}{p_{n+\nu}}u_{n+\nu+1}\end{pmatrix}. \tag{A1}$$

Then from (49b) we have

$$\begin{aligned}T_n = p_{n+1}p_{n+2}\dots p_{n+z-1}&\begin{pmatrix}1 & 0\\ 0 & -\frac{p_{n+z}}{a_{n+z}}\frac{1}{u_{n+z+1}}\end{pmatrix}\begin{pmatrix}1 & 1\\ k_{n+z-1}W_{n+z-1} & W_{n+z-1}\end{pmatrix}\\ &\times\begin{pmatrix}1 & 1\\ k_{n+z-2}W_{n+z-2} & W_{n+z-2}\end{pmatrix}\cdots\begin{pmatrix}1 & 1\\ k_{n+1}W_{n+1} & W_{n+1}\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -\frac{a_{n+1}}{p_{n+1}}u_{n+2}\end{pmatrix}\end{aligned} \tag{A2}$$

which can be expressed as

$$T_n = p_{n+1}\dots p_{n+z-1}\begin{pmatrix}1 & 0\\ 0 & -\frac{r_{n+z-1}}{a_{n+z-1}}\frac{1}{u_{n+z-1}}\end{pmatrix}Z\begin{pmatrix}1 & 0\\ 0 & -\frac{a_{n+1}}{p_{n+1}}u_{n+2}\end{pmatrix} \tag{A3}$$

in which the matrix Z has been defined by (60).

From (A3) and (57) we then obtain

$$u_{n+z+1} = \frac{k^2 b_n}{a_{n+z}}\frac{p_{n+z}}{p_n}\frac{Z_{12}}{Z_{21}}\frac{a_{n+1}}{p_{n+1}}\frac{a_{n+z-1}}{r_{n+z-1}}\frac{u_{n+2}u_{n+z-1}}{u_n} \tag{A4}$$

which with (54a) and (58) yields (59).

Appendix B

From equations (63) and (A3) we obtain the relations

$$\begin{aligned}\mathcal{J}_{s_n} &= S\left(Z_{11} + k_n\frac{r_n}{a_n}\frac{a_{n+1}}{p_{n+1}}\frac{u_{n+2}}{u_n}Z_{12}\right)\\ \mathcal{J}_{q_n} &= S\frac{q_n}{s_n}\frac{a_n}{p_n}\frac{r_{n+z-1}}{a_{n+z-1}}\frac{u_n u_{n+1}}{u_{n+z-1}u_{n+z}}\left(Z_{21} + \frac{r_n}{a_n}\frac{a_{n+1}}{p_{n+1}}\frac{u_{n+2}}{u_n}Z_{22}\right)\\ \mathcal{J}_{c_n} &= -k^2 S\frac{c_n a_n}{s_n p_n}\left(Z_{11} + \frac{r_n}{a_n}\frac{a_{n+1}}{p_{n+1}}\frac{u_{n+2}}{u_n}Z_{12}\right)\frac{u_{n+1}}{u_{n+z}}\\ \mathcal{J}_{d_n} &= -S\frac{d_n}{s_n}\frac{r_{n+z-1}}{a_{n+z-1}}\frac{u_n}{u_{n+z-1}}\left(Z_{21} + k_n\frac{r_n}{a_n}\frac{a_{n+1}}{p_{n+1}}\frac{u_{n+2}}{u_n}Z_{22}\right)\end{aligned} \tag{B1}$$

and equations (77) can be derived using (58) and the relation

$$\frac{u_n u_{n+1}}{u_{n+z-1}u_{n+z}} = \frac{a_{n+z-1}}{a_n}\frac{r_n r_{n+1}\dots r_{n+z-2}}{p_{n+1}p_{n+2}\dots p_{n+z-1}}\frac{1}{W_n W_{n+1}\dots W_{n+z-2}}. \tag{B2}$$

Acknowledgments

GRWQ acknowledges helpful discussions with P Hydon and C Viallet and support by the Australian Research Council. HWC gratefully acknowledges the hospitality of the

Mathematics Department, La Trobe University. JAGR gratefully acknowledges the hospitality of the Instituut voor Theoretische Fysica, Universiteit van Amsterdam.

References

[1] Capel H W and Sahadevan R 2001 A new family of four dimensional symplectic and integrable mappings *Physica* A **289** 86–106

[2] Hydon P E 2001 Private communication

[3] Iatrou A and Roberts J A G 2001 Integrable mappings of the plane preserving biquadratic invariant curves *J. Phys. A: Math. Gen.* **34** 6617–36

[4] Iatrou A and Roberts J A G 2002 Integrable mappings of the plane preserving biquadratic invariant curves II *Nonlinearity* **15** 459–89

[5] Papageorgiou V G, Nijhoff F W and Capel H W 1990 Integrable mappings and nonlinear integrable lattice equations *Phys. Lett.* A **147** 106–14

[6] Quispel G R W, Capel H W, Papageorgiou V G and Nijhoff F W 1991 Integrable mappings derived from soliton equations *Physica* A **173** 243–66

[7] Quispel G R W, Roberts J A G and Thompson C J 1988 Integrable mappings and soliton equations *Phys. Lett.* A **126** 419–21

[8] Quispel G R W, Roberts J A G and Thompson C J 1989 Integrable mappings and soliton equations II *Physica* D **34** 183–92

[9] Roberts J A G, Iatrou A and Quispel G R W 2002 Interchanging parameters and integrals in dynamical systems: the mapping case *J. Phys. A: Math. Gen.* **35** 2309–25

[10] Bruschi M, Ragnisco O, Santini P M and Gui-Zhang Tu 1991 Integrable symplectic maps *Physica* D **49** 273–94

[11] Grammaticos B, Nijhoff F W and Ramani A 1999 Discrete Painleve equations *The Painleve Property: One Century Later* ed R Conte (New York: Springer) pp 413–516

[12] Quispel G R W, Nijhoff F W, Capel N W and Van der Linden J 1984 Linear integral equations and nonlinear difference-difference equations *Physica* A **125** 344–80

[13] Ablowitz M J and Ladik F J 1976 A nonlinear difference scheme and inverse scattering *Stud. Appl. Math.* **55** 213–29

Ablowitz M J and Ladik F J 1977 On the solution of a class of nonlinear partial difference equations *Stud. Appl. Math.* **57** 1–12

[14] Hirota R 1977 Nonlinear partial difference equations I-III *J. Phys. Soc. Japan* **43** 1424–33, 2074–86

[15] Date E, Jimbo M and Miwa T 1982 Method for generating discrete soliton equations I-II *J. Phys. Soc. Japan* **51** 4116–31

Date E, Jimbo M and Miwa T 1983 Method for generating discrete soliton equations III *J. Phys. Soc. Japan* **52** 388–93

Date E, Jimbo M and Miwa T 1983 Method for generating discrete soliton equations IV-V *J. Phys. Soc. Japan* **52** 761–71

[16] Roberts J A G and Vivaldi F 2003 Arithmetical method to detect integrability in maps *Phys. Rev. Lett.* **90** 034102

[17] Roberts J A G, Jogia D and Vivaldi F 2003 The hasse-weil bound and integrability detection in rational maps *J. Nonl. Math. Phys.* **10** 166–80

[18] Iatrou A 2003 Higher-dimensional integrable mappings *Physica* D **179** 229–53

[19] Haggar F A, Byrnes G B, Quispel G R W and Capel H W 1996 k-integrals and k-symmetries in discrete dynamical systems *Physica* A **233** 379–94

[20] Viallet C 2001 Private communication

[21] Veselov A P 1991 Integrable maps *Russ. Math. Surv.* **46** 1–51

[22] Abarenkova N, Anglès d'Auriac J-Ch, Boukraa S, Hassani S and Maillard J-M 1999 From Yang–Baxter equations to dynamical zeta functions for birational transformations *Statistical Physics on the Eve of the 21st Century Ser. Adv. Statist. Mech.* vol 14 (River Edge, NJ: World Scientific) pp 436–90

[23] Kutz N 1996 The doubly discrete sine–gordon equation in geometry and physics *PhD Thesis* Mathematics Department, TU Berlin

[24] Roberts J A G and Thompson C J 1988 Dynamics of the classical heisenberg spin chain *J. Phys. A: Math. Gen.* **21** 1769–80

[25] Veselov A P 1991 What is an integrable mapping? *What is Integrability? (Springer Series in Nonlinear Dynamics)* ed V E Zakharov (New York: Springer) pp 251–70