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Characterization of Hamiltonian symmetries and their first integrals

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ABSTRACT

We provide explicit criteria when the Hamiltonian symmetries for a finite dimensional canonical Hamiltonian system correspond to their first integrals. There are two approaches used for the construction of the first integrals once the symmetry is known. In the standard classical approach the first integrals are obtained up to a distinguished function of time t. In the second, which is recent, the integrals are given by a formula which involves the determination of the divergence terms. In both methods utilized, the first integrals are not determined uniquely. Firstly we show what conditions need to be imposed on the Hamiltonian symmetry in order that it constructively and uniquely yields a first integral. Secondly we provide the extra condition on the first integral for the first approach and the integrability conditions on the divergence term for the second. As a consequence, we show that both methods are in fact equivalent. Furthermore, it is shown that when the Hamiltonian symmetries provide first integrals they form a Lie algebra. Moreover, we prove that the Hamilton first integral is invariant under the Hamilton action symmetry. Several examples taken from the literature are given to illustrate our results and conditions.

1. Introduction

Hamilton's canonical equations naturally arise in mechanics and much have been written on this topic (see, e.g. Abraham and Marsden [1], Goldstein [2], and Arnold [3]).

There is extensive use of dynamic optimization in economic models and these also invoke the Hamiltonian formulation. A Hamiltonian framework for several control, state and costate variables has recently been advocated in [4]. The method is applicable to an arbitrary system of first-order ODEs which has a nonstandard Hamiltonian formulation.

The classical correspondence between symmetries and first integrals for the canonical Hamilton equations is readily available. However, very less is known about the early insightful work of Levi-Civita of 1899 on this subject (see Levi-Civita [5] and the instructive translation of Saccomandi and Vitolo [6] which brings this to the fore) who first gave the link between Hamiltonian symmetries and first integrals. The idea of continuous symmetries emanates from the work of Lie [7]. The connection between symmetries and conservation laws for equations that admit a variational principle in a general sense dates back to Noether [8]. The Hamiltonian system and its reduction and first integrals since the initial work of Levi-Civita have been promulgated in many books and papers (see Whittaker [9], Smale [10], Marsden and Weinstein [11], Kozlov [12], Olver [13], and recently Dorodnitsyn and Kozlov [14]) with many significant findings.

In Levi-Civita [5,6] (see, e.g. Olver [13]), a Hamiltonian symmetry in evolutionary form determines a first integral of the canonical Hamilton equations up to a time-dependent function. Three *apparent* disadvantages of this approach have been clearly stated in Dorodnitsyn and Kozlov [14]. First is that some of the transformations lose their geometrical sense when considered in evolutionary or what is also referred to as in the canonical operator form by Ibragimov [15]. The second point mentioned (see [14]) is that there is a need to integrate to find the required first integrals once the canonical symmetries are known. Third, which is connected to the first point, it is stated that it is not clear why some point symmetries give rise to integrals whereas others not. It is important to highlight the fact that in the Hamiltonian framework one can regard a wide class of generalized symmetries of Lagrangian mechanics as point symmetries (e.g., the symmetries that yield the Laplace–Runge–Lenz vector [14]; see also herein Section 5 on applications) which geometrically is a nice enough advantage of the Hamiltonian viewpoint over the Lagrangian viewpoint.

In Dorodnitsyn and Kozlov [14], the authors remedy the apparent shortcomings stated above and study symmetries which are not

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restricted to phase space and therefore also transform the time t. In this approach, the Hamiltonian symmetries have a clear geometric sense in finite space. Moreover, in [14], the authors give a direct way to construct the first integrals which require the necessary gauge terms which require integration.

We re-visit the connection between symmetries and first integrals of the canonical Hamiltonian equations from the viewpoint of unifying the two seemingly disparate methods as well as clarifying in a simple way when the symmetries are characterized as Hamiltonian and the precise conditions which yield the first integrals. In the approaches up to date there is no simple characterization property of when a Hamiltonian symmetry will yield a first integral. This is done in an ad hoc manner (see [14]). Moreover, the first integral is not obtained uniquely (up to an ignorable constant) in either of the approaches. In the first method (see, e.g. Olver [13]), the first integral is determined up to a time dependent function whereas in the second method as in [14], the integral is constructed up to a suitable gauge or divergence term. Herein, we show how one can obtain the first integral uniquely by providing an extra integrability condition on the first integral for the first method and giving the integrability conditions on the gauge term for the second method. Furthermore, as a consequence of characterizing the symmetries that correspond to integrals and the added conditions on the integrals and divergence term, we show that both the approaches are equivalent. Moreover, we show that when the Hamiltonian symmetries give rise to first integrals they form a Lie algebra. We also prove the result that the Hamilton first integral is invariant under the symmetry that generates it. Applications of our results are presented as well.

In the next section we present the well-known operators as well as introduce some new ones. These are useful for the results that are unraveled in the sequel.

2. Operators in extended phase space

Let $\mathbb{R}\times T^{*}M$ be the extended phase space with the $2n+1$ coordinates as given by $(t,q,p)=(t,q^1,\dots,q^n,p_1,\dots,p_n)$ in which the time t is the independent variable and (q,p) the phase space coordinates of positions and conjugate momenta. There are some well-known operators in the first jet space of $\mathbb{R}\times T^{*}M$ (see, e.g. [14]) as well as others which we introduce below.

The derivatives of q^i and p_i with respect to t are

$$\dot{p}_i=D_t(p_i),\quad \dot{q}^i=D_t(q^i),\quad i=1,2,\dots,n, \tag{1}$$

where

$$D_t=\frac{\partial}{\partial t}+\dot{q}^i\frac{\partial}{\partial q^i}+\dot{p}_i\frac{\partial}{\partial p_i}+\cdots, \tag{2}$$

is the total derivative operator with respect to t. The summation convention applies for repeated indices unless otherwise specified. The differential variables q and p are independent and are connected only by the relations (1).

There are some well-known operators which are defined in the space of the variables (t,q,p) and its jet space. We now present them. Needless to say that these are required for the work at hand.

In addition to the well-known Euler operator

$$\frac{\delta}{\delta q^i}=\frac{\partial}{\partial q^i}-D_t\frac{\partial}{\partial \dot{q}^i},\quad i=1,2,\dots,n, \tag{3}$$

one also has the variational operator with respect to p

$$\frac{\delta}{\delta p_i}=\frac{\partial}{\partial p_i}-D_t\frac{\partial}{\partial \dot{p}_i},\quad i=1,2,\dots,n. \tag{4}$$

The action of the operators (3) and (4) on the Legendre transformation

$$p_i\dot{q}^i-H(t,q,p) \tag{5}$$

equated to zero yields the canonical Hamilton equations

$$\dot{q}^i=\frac{\partial H}{\partial p_i},$$
$$\dot{p}_i=-\frac{\partial H}{\partial q^i},\quad i=1,\dots,n. \tag{6}$$

A generator of symmetry of (6) in the space (t,q,p) is of the form

$$X=\xi(t,q,p)\frac{\partial}{\partial t}+\eta^i(t,q,p)\frac{\partial}{\partial q^i}+\zeta_i(t,q,p)\frac{\partial}{\partial p_i}. \tag{7}$$

This operator can also be written in characteristic form as

$$X=\xi D_t+\overline{\eta}^i\frac{\partial}{\partial q^i}+\overline{\zeta}_i\frac{\partial}{\partial p_i}, \tag{8}$$

where $\overline{\eta}^i$ and $\overline{\zeta}_i$ are the characteristic functions given by

$$\overline{\eta}^i=\eta^i-\xi\dot{q}^i,$$
$$\overline{\zeta}_i=\zeta_i-\xi\dot{p}_i,\quad i=1,\dots,n. \tag{9}$$

The operator (8) has the evolutionary representative

$$\overline{X}=\overline{\eta}^i\frac{\partial}{\partial q^i}+\overline{\zeta}_i\frac{\partial}{\partial p_i}, \tag{10}$$

or is called the canonical form of the operator X. The operators (8) and (10) are equivalent as $X-\overline{X}=\xi D_t$.

The first prolongation of the operator X is given by

$$X^{[1]}=\xi D_t+\overline{\eta}^i\frac{\partial}{\partial q^i}+\overline{\zeta}_i\frac{\partial}{\partial p_i}+D_t(\overline{\eta}^i)\frac{\partial}{\partial \dot{q}^i}+D_t(\overline{\zeta}_i)\frac{\partial}{\partial \dot{p}_i},$$
$$=\xi\frac{\partial}{\partial t}+\eta^i\frac{\partial}{\partial q^i}+\zeta_i\frac{\partial}{\partial p_i}+(D_t(\overline{\eta}^i)+\xi\ddot{q}^i)\frac{\partial}{\partial \dot{q}^i}+(D_t(\overline{\zeta}_i)+\xi\ddot{p}_i)\frac{\partial}{\partial \dot{p}_i} \tag{11}$$

We now introduce the Hamilton version of the Noether operator. We refer to it as the Hamilton operator. This is defined as

$$N_H=\xi+\overline{\eta}^i\frac{\delta}{\delta q^i}+\overline{\zeta}_i\frac{\delta}{\delta p_i}+D_t(\overline{\eta}^i)\frac{\delta}{\delta \dot{q}^i}+D_t(\overline{\zeta}_i)\frac{\delta}{\delta \dot{p}_i}. \tag{12}$$

By use of this operator one can prove the following identity:

$$X^{[1]}=N_H D_t. \tag{13}$$

The proof follows without labor by noting that $\delta D_t/\delta \dot{q}^i=\partial/\partial q^i$, $\delta D_t/\delta \ddot{q}^i=\partial/\partial \dot{q}^i$ and similarly for the variational derivatives with respect to p_i and $\dot{p}_i$, as well as by use of the characteristic form of $X^{[1]}$. We utilize this identity later in the proof of the invariance of the Hamilton first integral.

We also have the following operator commutator relation:

$$[X^{[1]},N_H]=N_H D_t(\xi), \tag{14}$$

where $[X^{[1]},N_H]=X^{1]}N_H-N_H X^{[1]}$. We note that $N_H D_t(\xi)$ acts on a differential function f as $N_H D_t(\xi)(f)=N_H(D_t(\xi)f)$. This relation follows directly by computation and utilization of the characteristic form of $X^{[1]}$ – one applies both sides of (14) to a differential function $f(t,q,p)$. We use this important relation (14) later in the proof of the invariance of the Hamiltonian first integral under a symmetry.

In the following section we briefly mention the main results of the two methods in use for the determination of the Hamiltonian first integrals once the symmetry generators are known.

3. Preliminaries on known results

We provide the salient features on known results which we extend in the next section.

The operator in (7) is a generator of symmetry of the canonical Hamilton system (6) if

$$\dot{\eta}^i-\dot{q}^i\dot{\xi}-X\left(\frac{\partial H}{\partial p_i}\right)=0,$$
$$\dot{\zeta}_i-\dot{p}_i\dot{\xi}+X\left(\frac{\partial H}{\partial q^i}\right)=0,\quad i=1,\dots,n \tag{15}$$

holds on the system (6). One merely applies the first prolongation (11) on the system (6) to deduce (15).

It is easy to see that X is a symmetry of the canonical Hamilton system if and only if its evolutionary representative $\overline{X}$ is. One merely invokes $X=\overline{X}+\xi D_t$. This is a useful property that we apply.

A conservation law of the system (6) is the relation

$$D_t I=0, \tag{16}$$

where $I=I(t,q,p)$, which is satisfied on all solutions to (6). We also have that

$$D_t I=Q^i(\dot{p}_i+H_{q^i})+R_i(\dot{q}^i-H_{p_i}) \tag{17}$$

which is called the characteristic form of the conservation law (16) with the functions Q^i and R_i, $i=1,\dots,n$, the associated characteristic functions. When the condition (16) is satisfied on the solutions to (6), I is referred to as a first integral of the system (6).

The usual approach given for example in Olver [13] to find first integrals of the canonical Hamilton equations (6) is to consider evolutionary symmetries

$$\tilde{X}=\eta^i(t,q,p)\frac{\partial}{\partial q^i}+\zeta_i(t,q,p)\frac{\partial}{\partial p_i} \tag{18}$$

with

$$\eta^i=\frac{\partial I}{\partial p_i},\quad \zeta_i=-\frac{\partial I}{\partial q^i},\ \ i=1,\dots,n \tag{19}$$

for some function $I(t,q,p)$ which determines a first integral up to a time dependent function.

Hamiltonian symmetries in evolutionary or canonical form have been considered (see e.g. [13]) as alluded to above. Furthermore, Hamiltonian symmetries in the space (t,q,p) have also been investigated by [14]. In the work cited, the authors consider the form of the symmetries (7) in (t,q,p) space and provide a Hamiltonian version of Noether's theorem.

The following important theorems (see [14] for a detailed discussion) were established.

Theorem 1 (*Hamilton action symmetries [14]*). *A Hamiltonian action*

$$p_i\,dq^i-H\,dt \tag{20}$$

is invariant up to divergence $B(t,q,p)$ with respect to a group generated by (7) if and only if the condition

$$\zeta_i\dot{q}^i+p_iD_t(\eta^i)-X(H)-HD_t(\xi)-D_t(B)=0, \tag{21}$$

is satisfied.

Theorem 2 (*Hamiltonian version of Noether's theorem [14]*). *The canonical Hamilton system (6) has first integral*

$$I=p_i\eta^i-\xi H-B \tag{22}$$

for some divergence function $B=B(t,q,p)$ if and only if the Hamiltonian action is invariant up to divergence with respect to the operator X given in (7) on the solutions to the canonical equations (6).

We now focus our attention on systems of canonical Hamilton equations in order to work out precise conditions of which symmetries yield conservation laws as well as to obtain further conditions on the first integrals relations in (19) which uniquely, up to an ignorable constant, specifies it and the conditions on the divergence term in (22) which enables one to uniquely construct it.

4. Characterization of Hamilton symmetry and related first integral conditions

In this section we provide a characterization of the Hamilton symmetry which directly corresponds to a first integral and as a consequence obtain the extra conditions on the first integral and divergence term that arise.

Theorem 3. *Necessary and sufficient conditions that the operator X of the form (8), which is a generator of symmetry for the canonical Hamilton system (6), yields a conservation law of the Hamilton equations (6) are that the characteristics $\overline{\eta}^i$ and $\overline{\zeta}_i$ of X are also the characteristics of the conservation law of (6) and additionally satisfy the relations*

$$\frac{\partial\overline{\eta}^i}{\partial q^j}+\frac{\partial\overline{\zeta}_j}{\partial p_i}=0, \tag{23}$$

$$\frac{\partial\overline{\zeta}_i}{\partial q^j}-\frac{\partial\overline{\zeta}_j}{\partial q^i}=0, \tag{24}$$

$$\frac{\partial\overline{\eta}^i}{\partial p_j}-\frac{\partial\overline{\eta}^j}{\partial p_i}=0,\quad i,j=1,\dots,n. \tag{25}$$

Proof. Any conservation law of the Hamilton canonical equations can be written in the equivalent characteristic form as

$$D_tI=Q^i(t,q,p)(\dot{p}_i+H_{q^i})+R_i(t,q,p)(\dot{q}^i-H_{p_i}) \tag{26}$$

for suitable multipliers Q^i and R_i yet to be determined. We take the variational derivative of (26) with respect to q and p in turn. In the first part of the proof we act on (26) with the Euler operator $\delta/\delta q^j, j=1,\dots,n$. That is, $\delta D_tI/\delta q^j=0$ must hold for all t,q and p. This immediately results in

$$\frac{\partial Q^i}{\partial q^j}(\dot{p}_i+H_{q^i})+Q^iH_{q^iq^j}$$
$$+\frac{\partial R_i}{\partial q^j}(\dot{q}^i-H_{p^i})-R_iH_{p_iq^j}-D_t(\delta^i_jR_i)=0,\quad j=1,\dots,n \tag{27}$$

where δ^i_j is the Kronecker delta. After expansion we anull the terms in $\dot{q}^i$ and $\dot{p}_i$ which directly give

$$\frac{\partial Q^i}{\partial q^j}-\frac{\partial R_j}{\partial p_i}=0, \tag{28}$$

$$\frac{\partial R_i}{\partial q^j}-\frac{\partial R_j}{\partial q^i}=0,\quad i,j=1,\dots,n \tag{29}$$

The remaining part of the expansion then results in

$$Q^iH_{q^iq^j}-R_iH_{p_iq^j}-\frac{\partial R^j}{\partial t}-\frac{\partial R_i}{\partial q^j}H_{p_i}+\frac{\partial Q^i}{\partial q^j}H_{q^i}=0,\quad j=1,\dots,n. \tag{30}$$

By use of (28) and (29), Eq. (30) becomes

$$Q^iH_{q^iq^j}-R_iH_{p_iq^j}-\frac{\partial R^j}{\partial t}-\frac{\partial R_j}{\partial q^i}H_{p_i}+\frac{\partial R_j}{\partial p_i}H_{q^i}=0,\quad j=1,\dots,n. \tag{31}$$

This is precisely one half of the symmetry conditions, viz.

$$D_t\overline{\zeta}_j+\overline{X}H_{q^j}=0,\quad j=1,\dots,n, \tag{32}$$

which hold on the solutions of (6), if and only if the following identifications are made, viz.

$$Q^i=\overline{\eta}^i,\quad R_i=-\overline{\zeta}_i,\ \ i=1,\dots,n \tag{33}$$

By means of (33), Eqs. (28) and (29) become precisely the required conditions (23) and (24).

To complete the proof of the second part, the action of the variational derivative operator with respect to p, $\delta/\delta p_j$, $j=1,\dots,n$ on (note that $\overline{\eta}^i$ and $\overline{\zeta}_i$ arise as a consequence of part one)

$$D_t I=\overline{\eta}^i(\dot{p}_i+H_{q^i})-\overline{\zeta}_i(\dot{q}^i-H_{p_i}) \quad (34)$$

is taken which at once gives rise to

$$\frac{\partial\overline{\eta}^i}{\partial p_j}(\dot{p}_i+H_{q^i})+\overline{\eta}^i H_{q^ip_j} -D_t(\delta^j_i\overline{\eta}^i)-\frac{\partial\overline{\zeta}_i}{\partial p_j}(\dot{q}^i-H_{p_i})+\overline{\zeta}_i H_{p_ip_j}=0,\quad j=1,\dots,n \quad (35)$$

We expand this via the D_t operator. The setting to zero of the coefficients of $\dot{p}_i$ and $\dot{q}^i$ yields

$$\frac{\partial\overline{\eta}^i}{\partial p_j}-\frac{\partial\overline{\eta}_j}{\partial p_i}=0, \quad (36)$$

$$\frac{\partial\eta^j}{\partial q^i}+\frac{\partial\overline{\zeta}_i}{\partial p_j}=0,\quad i,j=1,\dots,n \quad (37)$$

The rest of the terms in (35) implies

$$\overline{\eta}^i H_{q^ip_j}+\overline{\zeta}_i H_{p_ip_j}-\frac{\partial\overline{\eta}^j}{\partial t}+H_{p_i}\frac{\partial\overline{\zeta}_i}{\partial p_j}+H_{q^i}\frac{\partial\overline{\eta}^i}{\partial p_j}=0,\quad j=1,\dots,n \quad (38)$$

By replacing the respective terms of (36) and (37) in (38), the resultant equation is precisely the first half of the symmetry conditions of (6)

$$D_t\overline{\eta}^j-\overline{X}H_{p_j}=0,\quad j=1,\dots,n \quad (39)$$

which are satisfied on the solutions to (6). Here (37) is the same as what we deduced in the first part of the proof after interchanging i and j due to symmetry in the indices. This completes the proof. □

Remark 1. It is clear from Theorem 3 that for $n=1$ the conditions (23)–(25) reduce to just one relation, viz.

$$\frac{\partial\overline{\eta}}{\partial q}+\frac{\partial\overline{\zeta}}{\partial p}=0. \quad (40)$$

In expanded form (40) is

$$\frac{\partial\eta}{\partial q}+\frac{\partial\zeta}{\partial p}+\frac{\partial\xi}{\partial p}H_q-\frac{\partial\xi}{\partial q}H_p=0. \quad (41)$$

Also the expanded form for general n of the conditions (23)–(25) are easily

$$\frac{\partial\eta^i}{\partial q^j}+\frac{\partial\zeta_j}{\partial p_i}+\frac{\partial\xi}{\partial p_i}H_{q^j}-\frac{\partial\xi}{\partial q^j}H_{p_i}=0, \quad (42)$$

$$\frac{\partial\zeta_i}{\partial q^j}-\frac{\partial\zeta_j}{\partial q^i}+\frac{\partial\xi}{\partial q^j}H_{q^i}-\frac{\partial\xi}{\partial q^i}H_{q^j}=0, \quad (43)$$

$$\frac{\partial\eta^i}{\partial p_j}-\frac{\partial\eta^j}{\partial p_i}+\frac{\partial\xi}{\partial p_i}H_{p_j}-\frac{\partial\xi}{\partial p_j}H_{p_i}=0,\quad i,j=1,\dots,n. \quad (44)$$

There are n^2 conditions that arise from (23) and $n(n-1)/2$ conditions from each of the set of relations (24) and (25). This is due to the skew symmetry inherent in the conditions (24) and (25). These $2n^2-n$ conditions are with respect to a given Hamiltonian. Furthermore, it is clear from the relations (42)–(44) that the translation symmetries in t, q and p correspond to conservation laws. Also scaling operators $X=t\partial_t+\sum_i a_iq^i\partial_{q^i}+\sum_i b_ip_i\partial_{p_i}$ must have their constants constrained by $b_i-a_i=0$ for all i.

As a consequence of the preceding result, we have the following important theorem which provides the extra condition on the first integral.

Theorem 4. *For each Hamiltonian symmetry generator $X=\overline{X}+\xi D_t$ which satisfies the conditions (23)–(25), there corresponds a first integral I which is determined uniquely (up to an ignorable constant) from*

$$I_{q^i}=-\overline{\zeta}_i,\quad I_{p_i}=\overline{\eta}^i,\quad i=1,\dots,n \quad (45)$$

$$I_t=\overline{\eta}^j H_{q^j}+\overline{\zeta}_j H_{p_j}. \quad (46)$$

Proof. If the Hamilton symmetry generator X satisfies (23)–(25), then relation (34) applies. The expansion of the left-hand side of (34) and equating it to the right-hand side of (34) straightforwardly gives the results (45) and (46). This completes the proof. □

Remark 2. If the Hamilton vector field of system (6) of the form (7), viz.:

$$X=\xi(t,q,p)\frac{\partial}{\partial t}+\eta^i(t,q,p)\frac{\partial}{\partial q^i}+\zeta_i(t,q,p)\frac{\partial}{\partial p_i}$$

satisfy the $2n^2-n$ relations (42)–(44), then the first integral of (6) is uniquely (up to an ignorable constant) determined by integration from the $2n+1$ equations

$$I_{q^i}=-\zeta_i-\xi H_{q^i},\quad I_{p_i}=\eta^i-\xi H_{p_i},\quad i=1,\dots,n \quad (47)$$

$$I_t=\eta^j H_{q^j}+\zeta_j H_{p_j}. \quad (48)$$

The condition I_t in (48) is the extra condition which is missing in previous works. This provides the link to the other formulation as presented in [14]. We now focus on this.

There are 3 sets of relations that determines I as encapsulated in Theorem 4. We can utilize any one of them to obtain the form of I. It is prudent to use the relations involving I_{p_i} at first. This is dictated by the form of I given in Theorem 2 (due to [14]). Suppose that we can integrate $I_{p_i}=\overline{\eta}^i$ as given in Theorem 4. This implies

$$I=\overline{\eta}^i p_i-B \quad (49)$$

for a suitable function B. We now show that B is well-defined and satisfies the integrability conditions. The differentiation of I in (49) with respect to p_j and q^j (and invoking (45)) in turn results in

$$I_{p_j}=\overline{\eta}^j+p_i\frac{\partial\overline{\eta}^i}{\partial p_j}-B_{p_j}=\overline{\eta}^j,\quad j=1\dots,n$$

and

$$I_{q^i}=p_j\frac{\partial\overline{\eta}^j}{\partial q^i}-B_{q^i}=-\overline{\zeta}_i,\quad j=1\dots,n.$$

These straightforwardly give

$$B_{p_j}=p_i\frac{\partial\overline{\eta}^i}{\partial p_j},\quad j=1\dots,n \quad (50)$$

and

$$B_{q^i}=p_j\frac{\partial\overline{\eta}^j}{\partial q^i}+\overline{\zeta}_i,\quad i=1\dots,n. \quad (51)$$

Likewise by differentiating I with respect to t in (49) and invoking (46) we have

$$B_t=p_i\overline{\eta}^i_t-\overline{X}H. \quad (52)$$

It is a simple exercise to show that $B_{p_jq^i}=B_{q^ip_j}$ holds if the relation (23) is valid as it must be. Moreover $B_{p_it}=B_{tp_i}$ is satisfied provided

$$\overline{\eta}^i_t-(\overline{X}H)_{p_i}=0,\quad i=1\dots,n. \quad (53)$$

In a similar manner $B_{q^it}=B_{tq^i}$ holds if

$$\overline{\zeta}_{it}+(\overline{X}H)_{q^i}=0,\quad i=1\dots,n. \quad (54)$$

The conditions (53) and (54) are the symmetry conditions on (6) after use of (23). Therefore the B in (49) is well-defined and that the integrated form I is valid with B satisfying the differentiability conditions (50)–(52).

As a matter of fact one can essentially do the same by utilizing $I_{q^i} = -\overline{\zeta}_i$. Then we find

$$I = -\overline{\zeta}_j q^j - B \tag{55}$$

with B satisfying

$$B_{q^i} = -q^j \frac{\partial \overline{\zeta}_j}{\partial q^i},$$
$$B_{p_i} = q^j \frac{\partial \overline{\eta}^i}{\partial q^j} - \overline{\eta}^i, \quad i = 1, \ldots, n,$$
$$B_t = -q^j \frac{\partial \overline{\zeta}_j}{\partial t} - \overline{X}H. \tag{56}$$

Finally, if we invoke $I_t = \overline{\eta}^j H_{q^j} + \overline{\zeta}_j H_{p_j}$ we have that

$$I = t\overline{X}H - B \tag{57}$$

with B constrained by the relations

$$B_{q^i} = t(\overline{X}H)_{q^i} + \overline{\zeta}_i,$$
$$B_{p_i} = t(\overline{X}H)_{p_i} - \overline{\eta}^i, \quad i = 1, \ldots, n,$$
$$B_t = t(\overline{X}H)_t. \tag{58}$$

In both the above forms of (55) and (57), the compatibility conditions on the respective Bs are analogous to that for (49).

It transpires that one can have three integrated forms for I as directly dictated by Theorem 4. The question arises if one can have more. The answer is yes as long as the compatibility conditions on the new $\tilde{B}$ that comes from B hold. We see this when we link one of these to the divergence term contained in Theorem 2 below.

We state the following theorem which strengthens Theorem 2.

Theorem 5. *A Hamiltonian symmetry X of the form (7) corresponds to the first integral (of the form in (22))*

$$I = p_i \eta^i - \xi H - \tilde{B} \tag{59}$$

if and only if the coefficients ξ, η^i and ζ_i, $i = 1, \ldots, n$ of X satisfy (42)–(44) and the divergence term $\tilde{B}$ the relations

$$\tilde{B}_{q^i} = p_j \frac{\partial \eta^j}{\partial q^i} + \zeta_i - H\xi_{q^i},$$
$$\tilde{B}_{p_i} = p_j \frac{\partial \eta^j}{\partial p_i} - H\xi_{p_i}, \quad i = 1, \ldots, n,$$
$$\tilde{B}_t = p_j \frac{\partial \eta^j}{\partial t} - XH - \xi_t H. \tag{60}$$

Proof. We use (49). Alternatively one may invoke any one of the other two forms of I. Then $I = \eta^i p_i - \xi p_i H_{p_i} - B$. We equate this I to that of (59). As a result one has

$$B = -\xi p_i H_{p_i} + \xi H + \tilde{B} \tag{61}$$

We differentiate B in (61) with respect to t, q^i and p_i in turn and equate these respectively to the derived B in (52), (51) and (50). Then (60) follows as a consequence. Since the integrated form of (49) with the B constrained by the relations (50), (51) and (52) directly comes from Theorem 4, and I in (59) with $\tilde{B}$ in (60) follow from the aforementioned, this proves the result. □

Remark 3. If we utilize (55), then we need to define

$$B = -\zeta_j q^j - \xi q^j H_{q^j} - p_j \eta^j + \xi H + \tilde{B} \tag{62}$$

in order to arrive at the form (59). Also for I as in (57) we require that

$$B = t\eta^j H_{q^j} + t\zeta_j H_{p_j} - \eta^j p_j + \xi H + \tilde{B} \tag{63}$$

to deduce (59). Using any of these two forms of B gives the conditions (60) on $\tilde{B}$. Furthermore note that the I in (59) directly corresponds to the Hamilton version of the Noether integral as stated in [14]. This can alternatively be seen as follows. Indeed

$$I = N_H(p_i \dot{q}^i - H) - \tilde{B}$$

results in (59) as a consequence of the Hamilton operator's (12) action on the Legendre transformation (5) with the adjustment of the divergence $\tilde{B}$.

We have the following corollary.

Corollary. *If X, as in (7), is a Hamiltonian symmetry vector that yields a first integral, then so does the evolutionary representative $\overline{X} = X - \xi D_t$.*

Proof. The proof follows from Theorem 4, the ensuing discussion and Theorem 5. For if X is a Hamiltonian symmetry yielding an integral, then the condition (21) is satisfied. Let $X = \overline{X} + \xi D_t$, $\zeta_i = \overline{\zeta}_i + \xi \dot{p}_i$ and $\eta^i = \overline{\eta}^i + \xi \dot{q}^i$. Then (21) becomes

$$\overline{\zeta}_i \dot{q}^i + p_i D_t \overline{\eta}^i - \overline{X}H - D_t V = 0$$

with

$$V = \xi H + B - \xi p_i \dot{q}^i.$$

Note that this is the same form as (61). This proves the result. □

We now have the following important result on the Hamiltonian symmetries that correspond to a first integral forming a Lie algebra.

Theorem 6. *The Hamiltonian symmetry generators X which satisfy the conditions (23)–(25) form a Lie algebra.*

Proof. By the above Corollary we can prove the theorem for the canonical operators $\overline{X}$ and $\overline{Y}$. Suppose that $\overline{X} = \overline{\eta}^i_1 \partial/\partial q^i + \overline{\zeta}_{1i} \partial/\partial p_i + \cdots$ and $\overline{Y} = \overline{\eta}^i_2 \partial/\partial q^i + \overline{\zeta}_{2i} \partial/\partial p_i + \cdots$ (the dots refer to the prolongations) are Hamiltonian symmetry vectors in evolutionary form that yield first integrals. Then by acting $\overline{X}$ and $\overline{Y}$ on the Legendre transformation (5) we have

$$\overline{X}(p_i \dot{q}^i - H) = \overline{\zeta}_{1i} \dot{q}^i + p_i D_t(\overline{\eta}^i_1) - \overline{X}(H) - D_t(B)$$

and

$$\overline{Y}(p_i \dot{q}^i - H) = \overline{\zeta}_{2i} \dot{q}^i + p_i D_t(\overline{\eta}^i_2) - \overline{Y}(H) - D_t(C)$$

up to divergence terms B and C. From these we obtain

$$\overline{X}\overline{Y}(p_i \dot{q}^i - H) = \overline{X}(\overline{\zeta}_{2i}) \dot{q}^i + \overline{\zeta}_{2i} D_t \overline{\eta}^i_1 - \overline{X}\overline{Y}(H) - D_t(\overline{X}C) + p_i D_t \overline{X}\overline{\eta}^i_2 + \overline{\zeta}_{1i} D_t \overline{\eta}^2_i$$

and

$$\overline{Y}\overline{X}(p_i \dot{q}^i - H) = \overline{Y}(\overline{\zeta}_{1i}) \dot{q}^i + \overline{\zeta}_{1i} D_t \overline{\eta}^i_2 - \overline{Y}\overline{X}(H) - D_t(\overline{Y}B) + p_i D_t \overline{Y}\overline{\eta}^i_1 + \overline{\zeta}_{2i} D_t \overline{\eta}^i_1.$$

Subtraction of these imply

$$[\overline{X}, \overline{Y}](p_i \dot{q}^i - H) = (\overline{X}\overline{\zeta}_{2i} - \overline{Y}\overline{\zeta}_{1i}) \dot{q}^i - [\overline{X}, \overline{Y}]H - D_t(\overline{X}C - \overline{Y}B) + p_i D_t(\overline{X}\overline{\eta}^i_2 - \overline{Y}\overline{\eta}^i_1)$$

which states that $[\overline{X}, \overline{Y}]$ is also a canonical Hamiltonian vector field which gives a first integral. This proves the theorem. □

We refer to the algebra in Theorem 6 as the *Hamiltonian algebra*. The Hamiltonian algebra is a subalgebra of the Lie algebra of symmetries of the Hamilton canonical equations. This follows from the fact that the symmetries that correspond to a Hamiltonian first integral as in Theorem 4 or 5 are symmetries of Hamilton's equations.

We now prove an important invariant property of the first integral I of (59).

Theorem 7. *The first integral given in (59) is invariant under the symmetry generated by X.*

Proof. We make use of the operator commutation relation property (14). The application of this on the Legendre transformation (5) directly yields

$$[X^{[1]},N_H](p_i\dot{q}^i-H)=N_H(D_t(\xi)(p_i\dot{q}^i-H)) \tag{64}$$

Using the commutator relation (14) one has

$$X^{[1]}(N_H(p_i\dot{q}^i-H))=N_H((X^{[1]}+D_t(\xi))(p_i\dot{q}^i-H))$$
$$=N_H(\zeta_i\dot{q}^i+p_iD_t\eta^i-XH-HD_t\xi)$$
$$X^{[1]}(N_H(p_i\dot{q}^i-H))=N_HD_t(\tilde{B}) \tag{65}$$

$$X^{[1]}(N_H(p_i\dot{q}^i-H))=X^{[1]}(\tilde{B}). \tag{66}$$

Eq. (65) is due to (21) being satisfied and (66) is a consequence of the operator identity (13). Now the left-hand side of (66) is precisely $X^{[1]}(p_i\eta^i-\xi H)$ as a result of the action of the Hamilton operator (12) on the Legendre transformation (5). Thus we have

$$X^{[1]}(p_i\eta^i-\xi H-\tilde{B})=0 \tag{67}$$

which proves the result. We can replace $X^{[1]}$ by X as it operates only on the extended phase space coordinates t,q and p. □

5. Applications

We consider a number of examples taken from the literature.

It is instructive to first look at simple systems and then introduce more intricate systems.

1. A paradigm of the classical mechanical systems is the one-dimensional free particle equation which has Hamiltonian $H=p^2/2$ and canonical Hamilton equations

$$\dot{q}=p,\quad \dot{p}=0. \tag{68}$$

The point symmetries of (68) are

$$X_1=\partial_t,\quad X_2=\partial_q,\quad X_3=t\partial_t-p\partial_p,\quad X_4=q\partial_q+p\partial_p,\quad X_5=t\partial_q+\partial_p,$$
$$X_6=q\partial_t-p^2\partial_p,\quad X_7=t^2\partial_t+tq\partial_q+(q-tp)\partial_p,$$
$$X_8=tq\partial_t+q^2\partial_q+(qp-tp^2)\partial_p. \tag{69}$$

We take a linear combination of these symmetries (69) $Y=a^iX_i$. Then we test for which of the symmetries (69) satisfy the conditions of Theorem 3 and are symmetries that yield first integrals. Thus we need to check if condition (41) is satisfied for the linear combination. This straightforwardly gives

$$(-a^3+a^4-2a^6p-2a^8tp+a^8q)+(a^4+2a^8q)=0 \tag{70}$$

which implies that $a^3=2a^4$ and $a^6=a^8=0$. Hence there are five symmetries that correspond to first integrals and they are

$$Y_1=X_1,\quad Y_2=X_2;\quad Y_3=X_3+\tfrac{1}{2}X_4;\quad Y_4=X_5,\quad Y_5=X_7. \tag{71}$$

These operators span a five dimensional algebra by Theorem 6. This can easily be seen.

We now illustrate, using Y_3, how one constructs a first integral by utilizing Theorem 4.

The generator $Y_3=t\partial_t+\frac{1}{2}q\partial_q-\frac{1}{2}p\partial_p$ and $\overline{\eta}=\frac{1}{2}q-tp$, $\overline{\zeta}=-\frac{1}{2}p$. Indeed it is clear that I as in Theorem 4 now becomes

$$I_q=\tfrac{1}{2}p,\quad I_p=\tfrac{1}{2}q-tp,\quad I_t=-\tfrac{1}{2}p^2 \tag{72}$$

which integrates to

$$I=\tfrac{1}{2}(pq-tp^2). \tag{73}$$

Now we consider the forms of the first integral I as given by (49), (55) and (57) with the respective value of B as stated. This is to illustrate that the I obtained for a given symmetry, in this case Y_3, should be the same as that deduced from Theorem 4 as the forms of I in (49), (55) and (57) are direct consequences of Theorem 4 with suitable B.

Invoking (49), I is

$$I=\tfrac{1}{2}pq-tp^2-B, \tag{74}$$

where B satisfies (50)–(52)). That is

$$B_q=0,\quad B_p=-tp,\quad B_t=-\tfrac{1}{2}p^2.$$

From this $B=-\frac{1}{2}tp^2$ and thus I is exactly as in (73).

For the second form of I as given in (55), using Y_3 again, we have

$$I=\tfrac{1}{2}pq-B, \tag{75}$$

with B constrained by the relations (56). This results in

$$B_q=0,\quad B_p=tp,\quad B_t=\tfrac{1}{2}p^2$$

and hence $B=\frac{1}{2}tp^2$. Therefore, we have I as in (73).

Thirdly for I given in (57) we obtain

$$I=-\tfrac{1}{2}tp^2-B \tag{76}$$

with B satisfying

$$B_q=-\tfrac{1}{2}p,\quad B_p=-\tfrac{1}{2}q,\quad B_t=0,$$

thereby yielding $B=-\frac{1}{2}qp$ and I as in (73).

We see that for each of the three forms of I, B is different as it should be. However, the integral is the same as it must be. Moreover one can check that $Y_3I=0$ as it should be by Theorem 6.

Further, if we apply Theorem 5, we find I as in (73) with $B=0$.

The remaining integrals corresponding to the symmetries Y_1, Y_2, Y_4 and Y_5 are easily constructible, for example, by Theorem 5, and these together with (73) are

$$I_1=-H,\quad I_2=p,\quad I_3=\tfrac{1}{2}(pq-tp^2),\quad I_4=tp-q,$$
$$I_5=tqp-\tfrac{1}{2}(t^2p^2+q^2). \tag{77}$$

For each of the Is in (77), we have by Theorem 6 that they are invariant under the respective Y's.

One can consider more general symmetries which conform to the condition of Theorem 3. These can be suggested by the condition.

The purpose of the above elementary exercise was to illustrate the equivalence of the 5 approaches, two of which are encapsulated by which we take to be the main methods due to their prominence in the manner in which they arise in the literature, needless to say that we have a refinement in both methods.

2. We now consider the free particle system. First we dwell on the two-dimensional case and then state the result for the multi-dimensional case. The Hamiltonian for the n-dimensional free particle system is

$$H=\frac{1}{2}\sum_{j=1}^{n}p_j^2 \tag{78}$$

and corresponding Hamilton system is

$$\dot{q}^i=p_i,\quad \dot{p}_i=0,\quad i=1,\dots,n \tag{79}$$

For n, there are n^2+4n+3 point symmetries admitted by the system (79) which constitute the algebra $sl(n+2,\mathbb{R})$. We firstly look at the case $n=2$. To check which of the 15 symmetries, which are well-known, give first integrals we need that the symmetries satisfy the conditions of Theorem 3 which are stated in terms of ξ, η and ζ in (42)–(44) respectively. These become

$$\frac{\partial\eta^i}{\partial q^j}+\frac{\partial\zeta_j}{\partial p_i}-\frac{\partial\xi}{\partial q^j}p_i=0,\quad i,j=1,2,$$

$$\frac{\partial\zeta_1}{\partial q^2}-\frac{\partial\zeta_2}{\partial q^1}=0,$$
$$\frac{\partial\eta^1}{\partial p_2}-\frac{\partial\eta^2}{\partial p_1}=0. \quad (80)$$

The conditions (80) hold for 8 symmetries. We find the following 8 symmetries forming the Hamiltonian algebra which correspond to first integrals.

$$X_1=\partial_t,\quad X_2=t\partial_t+\tfrac{1}{2}q^i\partial_{q^i}-\tfrac{1}{2}p_i\partial_{p_i},\quad X_3=t^2\partial_t+tq^i\partial_{q^i}+(q^i-tp_i)\partial_{p_i},$$
$$X_{3+i}=\partial_{q^i},\ i=1.2,\quad X_{5+i}=t\partial_{q^i}+\partial_{p_i},$$
$$i=1,2, X_8=q^1\partial_{q^2}-q^2\partial_{q^1}+p_1\partial_{p_2}-p_2\partial_{p_1} \quad (81)$$

For example in the case of the scaling symmetry X_2 using Theorem 4 we have the conditions

$$I_{q^i}=\tfrac{1}{2}p_i,\quad I_{p_i}=\tfrac{1}{2}q^i-tp_i,\ i=1,2,\quad I_t=-H. \quad (82)$$

The solution of (82) easily yields the integral

$$I=\tfrac{1}{2}p_iq^i-tH. \quad (83)$$

The first integrals, including that for X_2, are

$$I_1=-H,\quad I_2=\tfrac{1}{2}p_iq^i-tH,\quad I_3=tq^ip_i-t^2H-\tfrac{1}{2}[(q^1)^2+(q^2)^2],$$
$$I_{3+i}=p_i,\quad i=1,2, I_{5+i}=tp_i-q^i,\ i=1,2,\quad I_8=q^1p_2-q^2p_1. \quad (84)$$

The integrals I_4 and I_5 are the components of linear momenta and I_8 is the nonzero component of the angular momentum.

It is now a simple step to extend the above discussion to that for the n-dimensional free particle system. We see that the symmetries X_1 to X_7 are extendible. For X_8 which is rotation, in the multi-dimensional case we have $n(n-1)/2$ symmetries. Therefore we have the following $(n^2+3n+6)/2$ symmetries:

$$X_1=\partial_t,\quad X_2=t\partial_t+\tfrac{1}{2}q^i\partial_{q^i}-\tfrac{1}{2}p_i\partial_{p_i},\quad X_3=t^2\partial_t+tq^i\partial_{q^i}+(q^i-tp_i)\partial_{p_i},$$
$$X_{3+i}=\partial_{q^i},\ i=1,\dots,n,\quad X_{n+3+i}=t\partial_{q^i}+\partial_{p_i},\ i=1,\dots,n,$$
$$X_{ij}=q^i\partial_{q^j}-q^j\partial_{q^i}+p_i\partial_{p_j}-p_j\partial_{p_i},\quad i<j,\ i,j=1,\dots,n(n-1)/2. \quad (85)$$

which are clearly composed of the $n+1$ translations, 1 scaling, 1 projective, n solution and $n(n-1)/2$ rotations. These constitute the Hamiltonian algebra. The first integrals are similar to (84) and are

$$I_1=-H,\quad I_2=\tfrac{1}{2}p_iq^i-tH,\quad I_3=tq^ip_i-t^2H-\tfrac{1}{2}[(q^1)^2+\cdots+(q^n)^2],$$
$$I_{3+i}=p_i,\ i=1,\dots,n,\quad I_{n+3+i}=tp_i-q^i,\ i=1,\dots,n,$$
$$I_{ij}=q^ip_j-q^jp_i,\quad i<j,\ i,j=1,\dots,n(n-1)/2. \quad (86)$$

3. The n-dimensional harmonic oscillator system is also a well-studied paradigm of mechanical systems and poses no more difficulty than the free particle system. The Hamiltonian is

$$H=\frac{1}{2}\left(\sum_{j=1}^{n}p_j^2+\sum_{j=1}^{n}(q^j)^2\right). \quad (87)$$

and canonical Hamilton system

$$\dot{q}^i=p_i,\quad \dot{p}_i+q^i=0,\ i=1,\dots,n. \quad (88)$$

The Hamiltonian symmetries which satisfy the conditions of Theorem 3 are again $(n^2+3n+6)/2$ symmetries which form the Hamiltonian algebra and are

$$X_1=\partial_t,\quad X_2=\sin 2t\partial_t+q^i\cos 2t\partial_{q^i}-(p_i\cos 2t+2q^i\sin 2t)\partial_{p_i},$$
$$X_3=\cos 2t\partial_t-q^i\sin 2t\partial_{q^i}+(p_i\sin 2t-2q^i\cos 2t)\partial_{p_i},$$
$$X_{3+i}=\sin t\partial_{q^i}+\cos t\partial_{p_i},\quad i=1,\dots,n,$$
$$X_{n+3+i}=\cos t\partial_{q^i}-\sin t\partial_{p_i},\quad i=1,\dots,n,$$
$$X_{ij}=q^i\partial_{q^j}-q^j\partial_{q^i}+p_i\partial_{p_j}-p_j\partial_{p_i},\quad i<j,\ i,j=1,\dots,n(n-1)/2. \quad (89)$$

which are clearly composed of the 1 translation, 2 symmetries that replace the scaling and projective symmetries of the free particle system, $2n$ solution and $n(n-1)/2$ rotations. The first integrals are similar to (84) and are

$$I_1=-H,\quad I_2=p_iq^i\cos 2t-\frac{1}{2}\sin 2t\left[\sum_{j=1}^{n}p_j^2-\sum_{j=1}^{n}(q^j)^2\right],$$
$$I_3=-p_iq^i\sin 2t-\frac{1}{2}\cos 2t\left[\sum_{j=1}^{n}p_j^2-\sum_{j=1}^{n}(q^j)^2\right],$$
$$I_{3+i}=p_i\sin t-q^i\cos t,\ i=1,\dots,n,$$
$$I_{3+n+i}=p_i\cos t+q^i\sin t,\ i=1,\dots,n,$$
$$I_{ij}=q^ip_j-q^jp_i,\quad i<j,\ i,j=1,\dots,n(n-1)/2. \quad (90)$$

4. The motion in a central force field in polar coordinates has Hamiltonian

$$H=\frac{1}{2}p_r^2+\frac{1}{2}\frac{p_\theta^2}{r^2}+f(r), \quad (91)$$

where $f(r)$ is the radially dependent central force. The Hamilton's equations are

$$\dot{r}=p_r,\quad \dot{\theta}=p_\theta/r^2,$$
$$\dot{p}_r=-f(r)+p_\theta^2/r^3,\quad \dot{p}_\theta=0. \quad (92)$$

Two well-known symmetries are

$$X_1=\partial_t, X_2=\partial_\theta \quad (93)$$

and both satisfy the conditions of Theorem 3. It is simple to observe that for X_1, Theorem 4 gives

$$I_{1r}=-f'(r)+p_\theta^2/r^3, I_{1\theta}=0, I_{1p_r}=-p_r, I_{1p_\theta}=-p_\theta/r^2, I_{1t}=0. \quad (94)$$

The integration of (94) results in

$$I_1=-H \quad (95)$$

which is negative of the energy. Also X_2 corresponds to the angular momentum

$$I_2=p_\theta=r^2\dot{\theta}, \quad (96)$$

which comes from the integration of

$$I_{2r}=I_{2\theta}=I_{2p_r}=I_{2t}=0,\quad I_{2p_\theta}=1. \quad (97)$$

Both these integrals are in involution.

5. We re-visit the examples as given in [14]. The Ermakov equation

$$\dot{q}=1/q^3$$

has Hamiltonian $H=p^2/2+1/(2q^2)$ and Hamiltons equations are

$$\dot{q}=p,\quad \dot{p}=1/q^3 \quad (98)$$

that admit the three symmetries

$$X_1=\partial_t,\quad X_2=2t\partial_t+q\partial_q-p\partial_p,\quad X_3=t^2\partial_t+tq\partial_q+(q-tp)\partial_p \quad (99)$$

which forms the algebra $sl(2,\mathbb{R})$. It is easy to check that the condition (40) of Theorem 3 holds for all symmetries (99). As an example we apply Theorem 4 on X_3. This gives

$$I_q=tp-q+t^2/q^3,\quad I_p=tq-t^2p,\quad I_t=-tq/q^3+p(q-tp)$$

which integrates to

$$I_3=-\tfrac{1}{2}(t^2/q^2+(q-tp)^2) \quad (100)$$

and is precisely the form obtained in [14].

For Kepler's problem in three dimensions, the Hamilton canonical equations are

$$\dot{q}^i=p_i,\quad \dot{p}_i=-K^2q^i/r^3,\ i=1,2,3, \quad (101)$$

where $r=\sqrt{(q^1)^2+((q^2)^2+(q^3)^2}$. The Hamiltonian is

$$H=\tfrac{1}{2}(p_1^2+p_2^2+p_3^2)-K^2/r. \quad (102)$$

The system admits the symmetries (we use the naming of the operators as given in [14])

$$X_0=\partial_t,\quad X_1=3t\partial_t+2q^i\partial_{q^i}-p_i\partial_{p_i},$$
$$X_{ij}=q^i\partial_{q^j}-q^j\partial_{q^i}+p_i\partial_{p_j}-p_j\partial_{p_i},\quad i<j,\ i,j=1,2,3,$$
$$Y_l=(2q^lp_i-q^ip_l-q^jp_j\delta_{li})\partial_{q^i}$$
$$+\left(p_lp_i-\sum_{j=1}^{3}(p_j)^2\delta_{li}-\frac{K^2}{r^3}\left(q^lq^i-\sum_{j=1}^{3}(q^j)^2\delta_{li}\right)\right)\partial_{p_i},\quad l=1,2,3. \tag{103}$$

As for the free particle and harmonic oscillator X_0 and X_{ij} satisfy the conditions of Theorem 3. They yield the integrals $I_1=-H$ and $I_{ij}=q^ip_j-q^jp_i$, $i<j$ as was obtained in [14].

The operator Y_l satisfy the conditions of Theorem 3. We now briefly mention this. There are 9 relations (42) and 3 each of (43) and (44) that need to be satisfied. For example

$$\frac{\partial\zeta_{12}}{\partial q^3}-\frac{\partial\zeta_{13}}{\partial q^2}=0$$

since $\zeta_{12}=p_1p_2-K^2q^1q^2/r^3$ and $\zeta_{13}=p_1p_3-K^2q^1q^3/r^3$ and $\partial r/\partial q^i=q^i/r$ for $i=1,2,3$. Similarly

$$\frac{\partial\eta_1^1}{\partial p_2}-\frac{\partial\eta_1^2}{\partial p_1}=0$$

holds. Likewise one finds by calculations that the remaining 15 conditions apply as well.

We apply Theorem 5 to deduce

$$I_l=\sum_{i=1}^{3}(2q^lp_i-q^ip_l-q^jp_j\delta_{li})p_i-B_l, \tag{104}$$

where B_l satisfies

$$B_{lq^i}=-K^2q^lq^i/r^3-p_lp_i+\delta_{li}\left(\sum_{j=1}^{3}p_j^2+K^2/r\right),$$
$$B_{lp_i}=q^lp_i-p_lq^i,$$
$$B_{lt}=0 \tag{105}$$

which in turn yields the same B_l in [14]

$$B_l=q^l\left(\sum_{j=1}^{3}p_j^2+K^2/r\right)-p_lq^jp_j \tag{106}$$

and hence the first integrals from (104) (see also [14])

$$I_l=q^l\left(\sum_{j=1}^{3}p_j^2-K^2/r\right)-p_lq^jp_j,\quad l=1,2,3 \tag{107}$$

These are components of the familiar Laplace–Runge–Lenz vector.

6. In [14] two symmetries are invoked to obtain reduction to quadrature for a system of two Hamilton equations. Here we show how one symmetry can be effective by applying Theorem 7. We take an example from [16] of the modified Emden equation, viz.

$$\ddot q+2\dot q/t-3q^5=0 \tag{108}$$

which has Hamiltonian $H=p^2/2t^2-(1/2)t^2q^6$ and the Hamilton equations are

$$\dot q=p/t^2,\dot p=3t^2q^5 \tag{109}$$

This system (109) has the sole symmetry

$$X=2t\partial_t-q\partial_q+p\partial_p \tag{110}$$

which has the right form to provide the integral

$$I=-p^2/t-pq+t^3q^6. \tag{111}$$

The symmetry X leaves the integral I invariant by Theorem 7. Thus $I=C$ is invariant, where C is an arbitrary constant. Using the approach of [17] we find an invariant function corresponding to X that is easily $v=q\sqrt{t}$, $t>0$ which in effect provides the reduction to quadrature of the system (108).

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