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# Integrable mappings of the plane preserving biquadratic invariant curves II

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**Abstract**
We review recently introduced curve-dependent McMillan maps which are mappings of the plane that preserve biquadratic foliations. We show that they are measure preserving and thus integrable. We discuss the geometry of these maps including their fixed points and their stability. We consider the normal forms of symmetric and asymmetric biquadratic curves and the normal forms for their associated McMillan maps. We further discuss the dynamics on biquadratic curves by considering the possibility of parametrizing them by Jacobian elliptic or rational functions.

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## 1. Introduction

In recent years there has been a growing interest in the study of time-discrete integrable systems (integrable maps); see [22], the collection [4] and the excellent review [7] and references therein. Some of the topics that have been investigated for such systems include necessary and sufficient conditions for integrability, discrete Painleve equations, Lax pairs, discrete analogues of Hamiltonian structures and quantization.

In all these areas of investigation, it has proved useful to have example classes of integrable maps with which to work. This paper aims to present a large class of planar (i.e. two-dimensional) integrable maps (cf also [10]). We call a planar mapping integrable if it is measure preserving and leaves invariant each curve of a one-parameter family of non-intersecting curves $C(x, y, K) = 0$ that foliate the plane. Each curve of the family is parametrized by $K$ and is left invariant, so that $K = k(x, y)$ defined explicitly or implicitly by solving $C(x, y, K) = 0$ for $K$ is an integral for the map. That is, $k(x', y') = k(x, y)$, where primes denote application of the map.

The first family of integrable mappings of the plane was introduced by McMillan [14]:

$$x' = y, \qquad y' = -x - \frac{\beta y^2 + \epsilon y + \xi}{\alpha y^2 + \beta y + \gamma}. \tag{1}$$

This family is an area-preserving rational family of mappings preserving the biquadratic foliation

$$\alpha x^2 y^2 + \beta(x^2 y + x y^2) + \gamma(x^2 + y^2) + \epsilon x y + \xi(x + y) + K = 0, \tag{2}$$

where $K$ is the parameter which parametrizes each invariant curve in the plane. This family of mappings was generalized by Quispel, Roberts and Thompson (QRT) in [15, 16]. The QRT family of measure-preserving integrable rational mappings of the plane takes the form

$$x' = \frac{f_1(y) - x f_2(y)}{f_2(y) - x f_3(y)}, \qquad y' = \frac{g_1(x') - y g_2(x')}{g_2(x') - y g_3(x')}, \tag{3}$$

where the functions $f_i$ and $g_i$ $(i = 1, 2, 3)$ are certain quartic polynomials whose coefficients are functions of 18 parameters (given explicitly in (27) below). Each QRT map preserves a biquadratic foliation

$$(\alpha_0 + \alpha_1 K)x^2 y^2 + (\beta_0 + \beta_1 K)x^2 y + (\delta_0 + \delta_1 K)x y^2 + (\gamma_0 + \gamma_1 K)x^2 + (\kappa_0 + \kappa_1 K)y^2$$
$$+(\epsilon_0 + \epsilon_1 K)xy + (\xi_0 + \xi_1 K)x + (\lambda_0 + \lambda_1 K)y + (\mu_0 + \mu_1 K) = 0. \tag{4}$$

When $\delta_i = \beta_i, \kappa_i = \gamma_i, \lambda_i = \xi_i$ in (4), each curve of the foliation becomes symmetric in $x$ and $y$. In this case, $g_i = f_i$ in (3) and the mapping (3) corresponds to two applications of the integrable symmetric QRT mapping,

$$x' = y, \qquad y' = \frac{f_1(y) - x f_2(y)}{f_2(y) - x f_3(y)}, \tag{5}$$

of which (1) is a special case.

In [10] we provided a general framework to construct integrable mappings of the plane that preserve a one-parameter family $B(x, y, K) = 0$ of biquadratic invariant curves where the parametrization of the coefficient functions by $K$ is very general (and not necessarily affine in $K$ as in (4)). We called these maps curve-dependent McMillan maps (CDM maps). They are reversible by construction (i.e. they are the composition of two involutions) and will be shown below to be measure preserving. They generalize the above-mentioned integrable maps previously given by McMillan and Quispel, Roberts and Thompson.

The purpose of this paper is to further elucidate the properties of these new integrable planar maps. The plan of the paper is as follows. In section 2 we review some details of [10] and show a new relationship between CDM maps and QRT and McMillan maps. In section 3 we prove that the CDM maps are measure preserving and show, more generally, how new integrable maps can be generated from one-parameter families of known integrable maps. In section 4 we discuss how to find the fixed points of CDM maps and how to obtain the stability of these points. In section 5 we discuss the normal forms for a symmetric or asymmetric biquadratic curve. In section 6 we discuss the dynamics of biquadratic curves by considering the possibility of parametrizing them by Jacobian elliptic functions.

## 2. Overview of CDM maps

In this section, we first review the definition of CDM maps and highlight their relationship to the previously studied integrable McMillan maps and QRT maps; cf also [10]. We then introduce a new relationship between the CDM maps and QRT and McMillan maps which shows that the dynamics of a CDM map in a neighbourhood of an invariant curve (neighbourhood of a point) can be approximated by a QRT map (McMillan map).

Consider the function

$$B(x, y, K) = \alpha(K)x^2y^2 + \beta(K)x^2y + \delta(K)xy^2 + \gamma(K)x^2 + \kappa(K)y^2 + \epsilon(K)xy + \xi(K)x + \lambda(K)y + \mu(K), \tag{6}$$

where the nine coefficients $\alpha, \dots, \mu$ are, in general, functions of a parameter $K$. For each $K$,

$$B(x, y, K) = 0 \tag{7}$$

defines a biquadratic curve in the $(x, y)$ plane. We can alternatively write (7) using a dot product

$$B(x, y, K) = X \cdot A(K)Y = 0, \tag{8}$$

where

$$X := \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}, \qquad Y := \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix} \tag{9}$$

and

$$A(K) := \begin{pmatrix} \alpha(K) & \beta(K) & \gamma(K) \\ \delta(K) & \epsilon(K) & \xi(K) \\ \kappa(K) & \lambda(K) & \mu(K) \end{pmatrix}. \tag{10}$$

In [10], we identified two automorphisms of the curve (7).

**Proposition 1.** *Let $(x', y') = G_i(K)\,(x, y)$ be the image of $(x, y)$ under either of the following involutions, parametrized by $K$:*

$$G_1(K): \qquad x' = x, \quad y' = -y - \frac{\beta(K)x^2 + \epsilon(K)x + \lambda(K)}{\alpha(K)x^2 + \delta(K)x + \kappa(K)} \tag{11}$$

*and*

$$G_2(K): \qquad x' = -x - \frac{\delta(K)y^2 + \epsilon(K)y + \xi(K)}{\alpha(K)y^2 + \beta(K)y + \gamma(K)}, \quad y' = y. \tag{12}$$

*Both involutions satisfy*

$$B(x', y', K) = B(x, y, K), \tag{13}$$

*with $B(x, y, K)$ given by (6). In particular, if $(x, y)$ satisfy $B(x, y, K) = 0$ for some fixed $K$, then $B(x', y', K) = 0$. $G_1$ and $G_2$ are the most general non-identity automorphisms fixing one coordinate that satisfy (13).*

**Proof.** See [10]. □

Since $G_1(K)$ and $G_2(K)$ preserve the biquadratic curve (8) for each $K$, then so does their composition $M_{\mathrm{a}}(K) := G_1(K) \circ G_2(K)$ given by

$$M_{\mathrm{a}}(K): \qquad x' = -x - \frac{\delta(K)y^2 + \epsilon(K)y + \xi(K)}{\alpha(K)y^2 + \beta(K)y + \gamma(K)}, \quad y' = -y - \frac{\beta(K)x'^2 + \epsilon(K)x' + \lambda(K)}{\alpha(K)x'^2 + \delta(K)x' + \kappa(K)}. \tag{14}$$

We use the subscript 'a' in $M_a(K)$ to refer to the fact that the curve (7) or (8) is *asymmetric* in $x$ and $y$ (equivalently, the matrix $A$ in (10) is asymmetric). When the biquadratic (7) is symmetric in $x$ and $y$, we can write it with six coefficients,

$$B_s(x, y, K) = \alpha(K)x^2y^2 + \beta(K)(x^2y + xy^2) + \gamma(K)(x^2 + y^2) + \epsilon(K)xy + \xi(K)(x + y) + \mu(K) = 0; \tag{15}$$

equivalently, the matrix of (10) is now symmetric: $A(K) = A(K)^{\mathrm{T}}$ (superscript T denotes matrix transpose) with $\delta(K) = \beta(K)$, $\kappa(K) = \gamma(K)$ and $\lambda(K) = \xi(K)$. In this case, another involutory automorphism of the curve $B_s(x, y, K) = 0$ is clearly

$$G_s: \qquad x' = y, \quad y' = x. \tag{16}$$

It follows that the mapping $M_s(K) := G_1(K) \circ G_s$ given by

$$M_s(K): \qquad x' = y, \quad y' = -x - \frac{\beta(K)y^2 + \epsilon(K)y + \xi(K)}{\alpha(K)y^2 + \beta(K)y + \gamma(K)} \tag{17}$$

is also an automorphism of the curve (and in fact in this case (14) corresponds to $M_s(K)^2$).

Consider varying $K$ in (8)–(10) to give a one-parameter family of curves in the plane. Then $M_a(K)$ or $M_s(K)$ provide a one-parameter family of curve automorphisms, each preserving the corresponding biquadratic curve in the family. We will be interested in the case where the family of curves forms a foliation of the plane so that most points $(x, y)$ can be associated with a single value of $K$ satisfying (7). To be more precise, we *assume* the following condition is satisfied.

**Condition F.** *The equation* $B(x, y, K) = 0$ *defines globally* $K = k(x, y)$*, a smooth real-valued function for all but a finite number of exceptional points* $(x, y)$.

With condition F satisfied, $M_a(K)$ and $M_s(K)$ can be used to create smooth global maps of the plane that preserve the foliation by biquadratic curves (7), respectively (15). Specifically, these new maps of the plane denoted $L_a$, respectively $L_s$, are formed by substituting $K = k(x, y)$ into $M_a(K)$, respectively $M_s(K)$, so that

$$\begin{aligned} L_a: \qquad x' &= -x - \left.\frac{\delta(K)y^2 + \epsilon(K)y + \xi(K)}{\alpha(K)y^2 + \beta(K)y + \gamma(K)}\right|_{K=k(x,y)}, \\ y' &= -y - \left.\frac{\beta(K)x'^2 + \epsilon(K)x' + \lambda(K)}{\alpha(K)x'^2 + \delta(K)x' + \kappa(K)}\right|_{K=k(x,y)} \end{aligned} \tag{18}$$

and

$$\begin{aligned} L_s: \qquad x' &= y, \\ y' &= -x - \left.\frac{\beta(K)y^2 + \epsilon(K)y + \xi(K)}{\alpha(K)y^2 + \beta(K)y + \gamma(K)}\right|_{K=k(x,y)}. \end{aligned} \tag{19}$$

For brevity, we will use the notation $|_{K=k(x,y)}$ to indicate that $K$ is to be replaced by $k(x, y)$ defined explicitly or implicitly by solving (7). It is not necessary in condition F or (18) or (19) to know $k(x, y)$ explicitly. It is seen from the definition of $L_a$ and $L_s$ that the orbit of the point $(x_0, y_0)$ is found by solving $B(x_0, y_0, K) = 0$ to find $K = K_0 = k(x_0, y_0)$ and then iterating $M_a(K_0)$ or $M_s(K_0)$ for $(x, y) = (x_0, y_0)$. This $K_0$ value labels the curve on which the orbit lies; alternatively, it is the height of the level set of the integral $k(x, y)$ defined explicitly or implicitly by solving (7). When (7) can be solved explicitly for $k(x, y)$, this can be substituted directly into (18) ((19)) so as to write $L_a$ ($L_s$) in closed form $(x, y) \mapsto (x', y')$. Since $L_a$ ($L_s$) are measure preserving (see section 3), we find that they are integrable mappings of the plane which we call asymmetric (symmetric) CDM maps (remark 2 explains this terminology).

Note that the maps $L_a$ and $L_s$ are *reversible* [20, 13] since they are the composition of two involutions,

$$L_a = G_1|_{K=k(x,y)} \circ G_2|_{K=k(x,y)} \tag{20}$$

and

$$L_s = G_1|_{K=k(x,y)} \circ G_s. \tag{21}$$

Some remarks about CDM maps, summarizing and further illustrating points of [10], are as follows.

**Remark 1.** In [10], we showed that imposing

$$\frac{\partial B}{\partial K}(x, y, K) = X \cdot \frac{\mathrm{d}A(K)}{\mathrm{d}K} Y > 0 \tag{22}$$

for all $(x, y, K)$ is a sufficient condition that gives condition F when $\alpha(K), \ldots, \mu(K)$ in (7) are polynomial or rational functions of $K$. We also showed that (22) can be used to help construct examples when the coefficient functions are smooth but not rational by additionally verifying that $B(x, y, K)$ has, for any $(x, y)$, a zero as a function of $K$. The condition (22) is admittedly quite strong as it forces $B(x, y, K)$ to be a monotone increasing function of $K$ for all fixed $x$ and fixed $y$ (as noted in [10], choosing $(\partial B/\partial K)(x, y, K) < 0$ for all $(x, y, K)$ is an equally acceptable condition).

Another possible condition on $B(x, y, K)$ which is less restrictive is investigated in [11].

**Remark 2.** For each $K$, the maps $M_a(K)$ of (14) and $M_s(K)$ of (17) are constructed to preserve a biquadratic curve (7), respectively (15). However, clearly $M_a(K)$ and $M_s(K)$ are rational maps that can be extended to the whole plane. As maps of the plane, they are also seen to be area preserving and *integrable*, which follows from (13) of proposition 1. So, for example, $M_a(K_0)$ preserves the one-parameter family of biquadratics

$$\alpha_0 x^2 y^2 + \beta_0 x^2 y + \delta_0 x y^2 + \gamma_0 x^2 + \kappa_0 y^2 + \epsilon_0 x y + \xi_0 x + \lambda_0 y + \mu_0 = C, \tag{23}$$

where $\alpha_0 = \alpha(K_0)$, etc and $C$ is an *arbitrary* parameter that can be varied so that (23) foliates the plane. For fixed $K$, $M_s$ is recognized as the integrable mapping previously found by McMillan [14] which we call the *symmetric McMillan map*. The corresponding case of $M_a$ for a fixed $K$ value was given in appendix A of [20] and we will call it the *asymmetric McMillan map*.

Consequently, assuming condition F is satisfied, the phase portrait of e.g. the asymmetric CDM map $L_a$ of (18) is built up from a one-parameter family $M_a(K)$ of asymmetric McMillan maps, choosing one curve (that with $C = 0$ in (23)) from the phase portrait of each McMillan map in the family. Condition F ensures that this set of chosen curves is itself a foliation. On each curve the CDM map acts as a particular McMillan map. The range of $K$ taken in this one-parameter family of McMillan maps to create the CDM map is determined from the range of $K$ generated by solving $B(x, y, K) = 0$ throughout the $(x, y)$ plane.

This point is illustrated pictorially in figure 1 for the following example.

**Example 1.**

$$B_1(x, y, K) = 2x^2y^2 + (3K + 6)(x^2 + y^2) - 2(1 + K + \cos K)xy - 8x - 0.2y + 1 + K = 0. \tag{24}$$

This asymmetric biquadratic foliation defines a CDM map $L_a$ after substituting into (18) the appropriate coefficient functions $\alpha(K) = 2$, $\beta(K) = \delta(K) = 0$, $\gamma(K) = \kappa(K) = 3K + 6$,

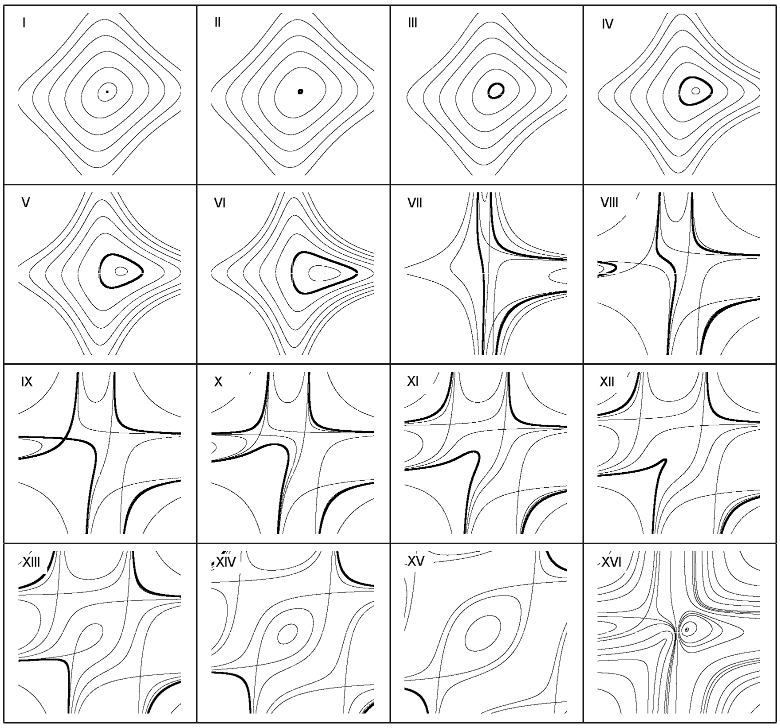



**Figure 1.** The phase portrait of the asymmetric CDM map $L_a$ of example 1 on the square $[-5, 5] \times [-5, 5]$ is shown in picture XVI. Pictures I–XV show the phase portraits of the 15 McMillan maps $M_a(K)$ obtained from respectively fixing $K = K_0$ with $K_0 \in \{0.9696, 0.9169, 0.25, -0.6923, -1, -1.327, -2.079, -2.586, -2.757, -2.933, -3.723, -4.38, -5.087, -7.211, -10.9\}$. The bold curve in each of pictures I–XV is $B_1(x, y, K_0) = 0$, and the sequence shows how the phase portrait of the CDM map is built up from one curve taken from the phase portrait of each McMillan map.

$\epsilon(K) = -2(1 + K + \cos K)$, $\xi(K) = -8$, $\lambda(K) = -0.2$ and $\mu(K) = 1 + K$. The value of $K = k(x, y)$ for each initial condition is determined by solving (24).

The previously discovered symmetric and asymmetric McMillan maps are a special case of CDM maps. For example, if the eight coefficient functions $\alpha(K), \ldots, \lambda(K)$ in (7) are $K$-independent, then the asymmetric CDM map $L_a(K) = M_a(K)$ is independent of $K$ and hence the same on each curve of the foliation $B(x, y, K) = 0$. This foliation depends on $K$ via $\mu(K)$ which appears in (7) but not in (14) or (18).

The creation of a new integrable map from a one-parameter family of integrable maps by taking one curve from each phase portrait in the family is investigated more generally in theorem 3 of section 3.

**Remark 3.** The previously studied QRT maps preserving (4) are a special case of CDM maps where the $K$-dependence in $L_a$ or $L_s$ can be explicitly eliminated by substituting for $k(x, y)$ to give an alternative closed form for the map[3]. This can be achieved since all the coefficient functions in (6) and (7) are linear in $K$. In this case, equation (7) can be written as

$$B_{\mathrm{QRT}}(x, y, K) = X \cdot (A_0 + K A_1)Y = X \cdot A_0 Y + K(X \cdot A_1 Y) = 0, \tag{25}$$

[3] We have recently become aware that this way of constructing QRT maps is also given on page 35 of [22].

where $A_0$ and $A_1$ are constant coefficient matrices of the form (10). Provided $X \cdot A_1 Y \neq 0$, we can solve (25) explicitly for $K$ to find

$$k(x, y) = -\frac{X \cdot A_0 Y}{X \cdot A_1 Y}, \tag{26}$$

so that condition F is satisfied. If we substitute this expression for $k(x, y)$ into (18) with $\alpha(K) = \alpha_0 + K\alpha_1, \ldots, \lambda(K) = \lambda_0 + K\lambda_1$, we obtain after manipulation the asymmetric QRT mapping form (3) where $f_i$ and $g_i$ can be neatly expressed as components of cross products:

$$(f_1, f_2, f_3)(y) = (A_0 Y) \times (A_1 Y), \qquad (g_1, g_2, g_3)(x') = (A_0^{\mathrm{T}} X') \times (A_1^{\mathrm{T}} X'). \tag{27}$$

A similar substitution in the symmetric case recovers (5) (see also example 1 of [10]).

It is clear that the CDM maps $L_{\mathrm{a}}$ of (18) and $L_{\mathrm{s}}$ of (19) are completely determined once the coefficient functions in (6) and (7) are specified in such a way that condition F is satisfied. Equation (7) then defines a surface in $(x, y, K)$ space, which by condition F is the graph of $k(x, y)$. As well as being special cases of CDM maps, the QRT and McMillan maps can also be related to the former in terms of approximating their dynamics (equivalently, in terms of approximating $k(x, y)$ that the CDM map preserves).

Suppose $B(x_0, y_0, K_0) = 0$ so that $k(x_0, y_0) = K_0 \in Range(K)$. In a *neighbourhood of the curve* $B(x, y, K_0) = 0$ containing the point $(x_0, y_0)$, a first-order expansion in $K$ about $K_0$ yields

$$B(x, y, K_0) + \frac{\partial B}{\partial K}(x, y, K_0)\,(K - K_0) = 0. \tag{28}$$

It is seen that (28) is a biquadratic foliation of the plane of QRT type; cf (25). Alternatively, in a neighbourhood of $K_0$, (28) solved for $K$ gives

$$K = K_0 - \frac{B(x, y, K_0)}{(\partial B/\partial K)(x, y, K_0)} = K_0 - \frac{X \cdot A(K_0) Y}{X \cdot A'(K_0) Y} \tag{29}$$

as an approximation to the (implicitly defined) *true* solution $k(x, y)$ of (7). Figure 2 shows a comparison in a neighbourhood of a particular curve of a CDM map and the approximating QRT map corresponding to (28).

On the other hand, we can approximate the surface $B(x, y, K) = 0$ in a *neighbourhood of the point* $(x_0, y_0)$ by evaluating the derivative in (28) at this point, to give

$$B(x, y, K_0) + \frac{\partial B}{\partial K}(x_0, y_0, K_0)\,(K - K_0) = 0. \tag{30}$$

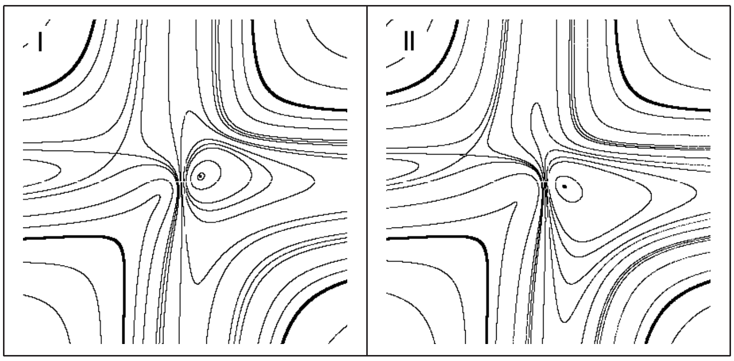



**Figure 2.** Comparison of phase portraits in the square $[-5, 5] \times [-5, 5]$ of the asymmetric CDM map $L_{\mathrm{a}}$ of example 1, picture I, and the approximating QRT map, picture II, in the neighbourhood of the bold curve. The bold curve has $K_0 = -5.087$ (and is the bold curve in picture XIII of figure 1).

Now (30) is a biquadratic foliation of the plane of McMillan type (i.e. (23) with $C = -(\partial B(x_0, y_0, K_0)/\partial K)\, K$). Solving for $K$, (30) now gives

$$K = K_0 - \frac{B(x, y, K_0)}{(\partial B/\partial K)(x_0, y_0, K_0)} = K_0 - \frac{X \cdot A(K_0)Y}{X_0 \cdot A'(K_0)Y_0} \tag{31}$$

as the approximation to the true solution in a neighbourhood of a point of a CDM map (in fact, (31) is the first in an iterative sequence of approximations that can be found to converge uniformly to the true solution in a neighbourhood of $(x_0, y_0)$ [6, theorem 1.5]). The approximation (30)–(31) can be used for the stability analysis of fixed points of CDM maps (see section 4).

## 3. Measure preservation and integral invariants on integral submanifolds

Recall that $L : \boldsymbol{x} \mapsto \boldsymbol{x}'$ with $\boldsymbol{x} \in \mathbb{R}^n$ is (anti-)measure preserving with density $m(\boldsymbol{x})$ if the Jacobian determinant $J(\boldsymbol{x}) := \det \mathrm{d}L(\boldsymbol{x})$ can be written as

$$J(\boldsymbol{x}) = (-)\frac{m(\boldsymbol{x})}{m(\boldsymbol{x}')} \Rightarrow \int_V m(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} = (-)\int_{L(V)} m(\boldsymbol{x}')\, \mathrm{d}\boldsymbol{x}' \tag{32}$$

for any region $V$ in $\mathbb{R}^n$ (anti-measure-preservation corresponds to $L$ being measure preserving and orientation reversing). A special case of measure preservation is when $L$ is volume preserving, which corresponds to taking $m(\boldsymbol{x}) \equiv 1$. For the planar case $n = 2$, measure preservation implies conjugacy to area preservation [20, section 2.2] and hence essentially a symplectic structure (in higher (even) dimensions, measure preservation of $L$ does not guarantee that it is a symplectic map).

In this section, we first establish that the CDM maps are measure preserving in a direct analytic way that exploits their reversibility (a similar approach was first used to establish measure preservation of the QRT maps [18]). Subsequently, we give an alternative proof which is geometric and shows more generally how *new* integrable maps can be generated under certain conditions by considering one-parameter families of *known* integrable maps.

**Theorem 2.** *The maps $L_\mathrm{a}$ of (18) and $L_\mathrm{s}$ of (19) are measure preserving with density*

$$m(x, y) = \left[\frac{\partial B}{\partial K}(x, y, k(x, y))\right]^{-1} = \left[X \cdot \frac{\mathrm{d}A}{\mathrm{d}K}(k(x, y))Y\right]^{-1}. \tag{33}$$

**Proof.** We will prove the result by exploiting the reversibility (20) of $L_\mathrm{a}$ and (21) of $L_\mathrm{s}$ and showing that their component involutions are measure preserving with the same density.

Consider $G_1|_{K=k(x,y)}$. From proposition 1 and condition F of the previous section, we have

$$B(x, y, k(x, y)) = X \cdot A(k(x, y))Y = 0 \tag{34}$$

and

$$B(x', y', k(x, y)) = X' \cdot A(k(x, y))\, Y' = X \cdot A(k(x, y))\, Y' = 0, \tag{35}$$

with $x'$ and $y'$ given by $G_1$ of (11) with $K = k(x, y)$, and noting $k(x', y') = k(x, y)$. From (11) it follows that

$$\det \mathrm{d}G_1\Big|_{K=k(x,y)} = \frac{\partial y'}{\partial y}\Big|_{K=k(x,y)} = -1 - \frac{\partial}{\partial K}\left(\frac{\beta(K)x^2 + \epsilon(K)x + \lambda(K)}{\alpha(K)x^2 + \delta(K)x + \kappa(K)}\right)\Big|_{K=k(x,y)} \frac{\partial k}{\partial y}. \tag{36}$$

If we differentiate (34) with respect to $y$, we get

$$X \cdot A(k(x,y)) \begin{pmatrix} 2y \\ 1 \\ 0 \end{pmatrix} + \frac{\partial B}{\partial K}(x, y, k(x,y)) \frac{\partial k}{\partial y} = 0, \tag{37}$$

and similarly for (35)

$$X \cdot A(k(x,y)) \begin{pmatrix} 2y' \\ 1 \\ 0 \end{pmatrix} \frac{\partial y'}{\partial y}\bigg|_{K=k(x,y)} + \frac{\partial B}{\partial K}(x, y', k(x,y)) \frac{\partial k}{\partial y} = 0. \tag{38}$$

We can now use (38) to solve for $(\partial y'/\partial y)|_{K=k(x,y)}$ and then use (37) to eliminate $\partial k/\partial y$ in this expression, obtaining

$$\frac{\partial y'}{\partial y}\bigg|_{K=k(x,y)} = \left( \frac{X \cdot A(k(x,y))(2y \quad 1 \quad 0)^{\mathrm{T}}}{X \cdot A(k(x,y))(2y' \quad 1 \quad 0)^{\mathrm{T}}} \right) \frac{[(\partial B/\partial K)(x, y, k(x,y))]^{-1}}{[(\partial B/\partial K)(x, y', k(x,y))]^{-1}}. \tag{39}$$

The leading bracketed factor in (39) equals $-1$ since

$$X \cdot A(K) \begin{pmatrix} 2y \\ 1 \\ 0 \end{pmatrix} + X \cdot A(K) \begin{pmatrix} 2y' \\ 1 \\ 0 \end{pmatrix} = 2\, X \cdot A(K) \begin{pmatrix} y + y' \\ 1 \\ 0 \end{pmatrix} \tag{40}$$

and the latter expression can be rewritten as (suppressing the $K$-dependence for convenience)

$$2 \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha[y+y'] + \beta \\ \delta[y+y'] + \epsilon \\ \kappa[y+y'] + \lambda \end{pmatrix} = 2\,[(y+y')(\alpha x^2 + \delta x + \kappa) + (\beta x^2 + \epsilon x + \lambda)] = 0 \tag{41}$$

from the definition of $G_1$. Therefore, from (36) with (39) we have established that $G_1$ is anti-measure-preserving, i.e.

$$\det \mathrm{d}G_1|_{K=k(x,y)} = -\frac{m(x,y)}{m(x',y')} \tag{42}$$

with $m(x,y)$ as stated in theorem 2.

By entirely similar means, one can consider $G_2|_{K=k(x,y)}$ and, by differentiating (34) and (35) with respect to $x$, can establish that it is anti-measure-preserving with the *same* density. It follows that the composition $L_{\mathrm{a}} = G_1|_{K=k(x,y)} \circ G_2|_{K=k(x,y)}$ is measure preserving.

If $A(K)$ is a symmetric matrix, then $B(x, y, K)$ and $(\partial B(x, y, K)/\partial K)$ are symmetric in $x$ and $y$. We see from (16) that

$$\det \mathrm{d}G_{\mathrm{s}} = -1 = -\frac{m(x,y)}{m(x',y')} = -\frac{m(x,y)}{m(y,x)} \tag{43}$$

and again the composition $L_{\mathrm{s}} = G_1|_{K=k(x,y)} \circ G_{\mathrm{s}}$ is measure preserving with the stated density. □

The measure preservation of the CDM maps can also be understood in a geometric way as a special case of a more general result. In fact, this more general perspective also gives a nice geometric interpretation of how to create new integrable maps of the plane from known ones, generalizing the creation of the CDM maps $L_{\mathrm{a}}$ and $L_{\mathrm{s}}$ of (18) and (19) from one-parameter families of McMillan maps $M_{\mathrm{a}}(K)$ and $M_{\mathrm{s}}(K)$ of (14) and (17).

The idea is to consider an integrable two-dimensional map $P$ depending on a parameter $K$ in an extended three-dimensional $(x, y, K)$ space. The obvious extended map $M$ in this three-dimensional space also has an integral which leads to the three-dimensional space being foliated by invariant surfaces. If one of these invariant surfaces $I(x, y, K) = 0$ is the graph of a function $K = k(x, y)$, then the action of $M$ on $I(x, y, K) = 0$ can be expressed as a new two-dimensional map $L$ by substituting $K = k(x, y)$ back into $P$. The precise result is as follows (cf also figure 3).

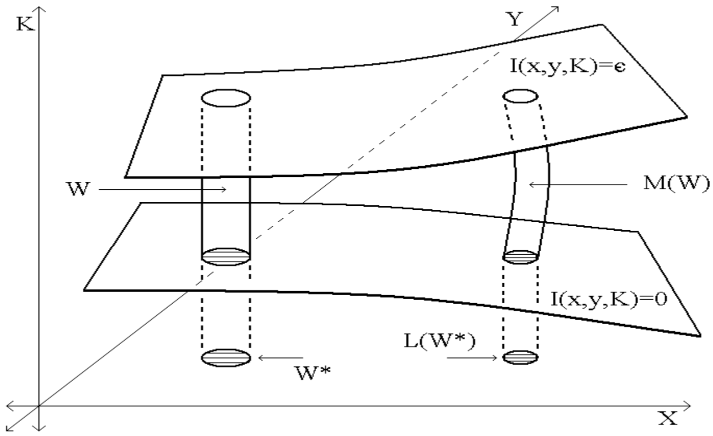



**Figure 3.** Illustration of the measure preservation proof of theorem 3.

**Theorem 3.** *Let $P : (x, y) \mapsto (x', y') := (f(x, y, K), g(x, y, K))$ be a map from $\mathbb{R}^2$ to itself depending on a parameter $K$. Suppose $P$ is (anti-)measure preserving with density $\rho(x, y, K)$ and that it possesses the smooth integral $I$ so that*

$$I(x', y', K) = I(x, y, K). \tag{44}$$

*Suppose the equation $I(x, y, K) = 0$ uniquely determines $K$ via $K = k(x, y)$, where $k$ is a smooth function, and that $(\partial I(x, y, k(x, y))/\partial K) \neq 0$. Then the two-dimensional map $L$ defined by $L : (x, y) \mapsto (x', y') := (f(x, y, k(x, y)), g(x, y, k(x, y)))$ has the integral $k(x, y)$ and is also (anti-)measure preserving with density*

$$m(x, y) = \rho(x, y, k(x, y)) \left[ \frac{\partial I}{\partial K}(x, y, k(x, y)) \right]^{-1}. \tag{45}$$

**Proof.** Since $P$ is (anti-)measure preserving, we have for arbitrary $V \subset \mathbb{R}^2$

$$\int_V \rho(x, y, K)\, \mathrm{d}x\, \mathrm{d}y = (-) \int_{P(V)} \rho(x', y', K)\, \mathrm{d}x'\, \mathrm{d}y'. \tag{46}$$

Extend $P$ into a map $M$ of $\mathbb{R}^3 := (K, x, y)$ by treating the parameter $K$ as an additional coordinate so that $M : (K, x, y) \mapsto (K', x', y') := (K, f(x, y, K), g(x, y, K))$. Clearly, $M$ is (anti-)measure preserving with density $\rho(x, y, K)$ via the obvious extension of (46) (with $U \subset \mathbb{R}^3$):

$$\int_U \rho(x, y, K)\, \mathrm{d}K\, \mathrm{d}x\, \mathrm{d}y = (-) \int_{M(U)} \rho(x', y', K')\, \mathrm{d}K'\, \mathrm{d}x'\, \mathrm{d}y'. \tag{47}$$

From (44), $M$ has, in addition to the trivial integral $K$, the integral $I$ and so induces a map on the level sets $I = c$, the constant $c$ determined from initial conditions. By construction, the map $L$ is simply the projection onto the $\mathbb{R}^2$ space $(x, y)$ of the action of $M$ on $\boldsymbol{I}_0 := \{(x, y, K)$

$\in \mathbb{R}^3 \mid I(x, y, K) = 0\}$, which by assumption equals $\{(x, y, k(x, y)) \in \mathbb{R}^3 \mid (x, y) \in \mathbb{R}^2\}$. The map $L$ is obtained by using $k(x, y)$ to replace for $K$ in the expressions for $f(x, y, K)$ and $g(x, y, K)$ in $P$, and it has the integral $k(x, y)$.

Consider a volume element $W \subset \mathbb{R}^3$ constructed in the following way (cf figure 3). It is topologically a cylinder with arbitrary projection $W^*$ onto the $(x, y)$ plane and is bounded vertically by $\mathbf{I}_0$ and $\mathbf{I}_\epsilon := \{(x, y, K) \in \mathbb{R}^3 \mid I(x, y, K) = \epsilon\}$, where $\epsilon$ is chosen small enough for $I_\epsilon$ to also be the graph of a function (that this can be done is guaranteed by the implicit function theorem). From (44), the image $M(W)$ of this volume element will continue to have a face on $\mathbf{I}_0$ and one on $\mathbf{I}_\epsilon$. For each point of $\mathbf{I}_0$ in $W$, the relationship between the variation of $I$ to the adjacent level set $\mathbf{I}_\epsilon$ and the variation of $K$ is given to first order by

$$\epsilon = \mathrm{d}I = \left.\frac{\partial I}{\partial K}\right|_{\mathbf{I}_0} \mathrm{d}K = \frac{\partial I}{\partial K}(x, y, k(x, y))\,\mathrm{d}K. \tag{48}$$

Using the constructed volume $W$ for $U$ in (47), together with (48) to replace $\mathrm{d}K$ and its primed version, and taking the limit as $\epsilon \to 0$ leads to

$$\int_{W^*} \rho(x, y, k(x, y)) \left[\frac{\partial I}{\partial K}(x, y, k(x, y))\right]^{-1} \mathrm{d}x\,\mathrm{d}y$$
$$= (-) \int_{L(W^*)} \rho(x', y', k(x', y')) \left[\frac{\partial I}{\partial K}(x', y', k(x', y'))\right]^{-1} \mathrm{d}x'\,\mathrm{d}y'.$$

Since $W^*$ is arbitrary, this establishes the result. □

Some remarks concerning this result are as follows.

**Remark 1.** The idea of the proof is a discrete version of the concept of integral invariants on integral submanifolds in ordinary differential equations [12]. The same map $M$ of the three-dimensional space induces different integrable maps on the invariant (planar) foliation $K =$ constant (where it induces a map of the form $P$ when projected onto the $x$–$y$ plane) as compared to the invariant foliation $I =$ constant (where it induces a map of the form $L$ when projected onto the $x$–$y$ plane). From a purely geometric viewpoint, it is irrelevant whether $k(x, y)$ can be explicitly found since $L$ is well defined in any case.

**Remark 2.** The CDM maps of the previous section and theorem 2 represent a special case of theorem 3 when $P$ is the one-parameter family of McMillan maps $M_{\mathrm{a}}(K)$ of (14) or $M_{\mathrm{s}}(K)$ of (17), which are area preserving so $\rho(x, y, K) = 1$, and have the biquadratic integral $I(x, y, K) = B(x, y, K)$ of (7) or (15). The assumption that the equation $B = 0$ can be solved uniquely for $K$ for (most) fixed $(x, y)$ is condition F of section 2, and (22) of remark 1 of section 2 and [11] indicate that the sufficient conditions to guarantee condition F keep the density non-singular. The QRT maps are a case where $k(x, y)$ can be calculated explicitly.

**Remark 3.** Theorem 3 can be generalized to higher-dimensional maps depending on parameters and possessing integrals dependent on (some of) the parameters [19].

## 4. Fixed points and stability analysis

In this section we will look at the fixed points of CDM maps and their stability.

We denote the fixed points of our mappings by $(x^*, y^*, K^*)$. These points must simultaneously satisy

$$B(x^*, y^*, K^*) = \alpha(K^*)x^{*2}y^{*2} + \beta(K^*)x^{*2}y^* + \delta(K^*)x^*y^{*2} + \gamma(K^*)x^{*2} + \kappa(K^*)y^{*2}$$
$$+ \epsilon(K^*)x^*y^* + \xi(K^*)x^* + \lambda(K^*)y^* + \mu(K^*) = 0, \tag{49}$$

together with the two equations that follow from the mapping equations. For the maps $L_{\rm a}$, these two additional equations are

$$\begin{aligned} x^* &= -\frac{1}{2}\frac{\delta(K^*)y^{*2}+\epsilon(K^*)y^*+\xi(K^*)}{\alpha(K^*)y^{*2}+\beta(K^*)y^*+\gamma(K^*)}, \\ y^* &= -\frac{1}{2}\frac{\beta(K^*)x^{*2}+\epsilon(K^*)x^*+\lambda(K^*)}{\alpha(K^*)x^{*2}+\delta(K^*)x^*+\kappa(K^*)}, \end{aligned} \tag{50}$$

and presuming the denominators are non-zero, then these two equations can be written in a more compact form,

$$\begin{pmatrix} 2x^* \\ 1 \\ 0 \end{pmatrix} \cdot A(K^*) \begin{pmatrix} y^{*2} \\ y^* \\ 1 \end{pmatrix} = 0, \qquad \begin{pmatrix} x^{*2} \\ x^* \\ 1 \end{pmatrix} \cdot A(K^*) \begin{pmatrix} 2y^* \\ 1 \\ 0 \end{pmatrix} = 0. \tag{51}$$

For the maps $L_{\rm s}$ (noting that $\beta(K^*) = \delta(K^*)$, $\gamma(K^*) = \kappa(K^*)$ and $\xi(K^*) = \lambda(K^*)$), it is seen that these two additional equations are

$$x^* = y^*, \qquad y^* = -x^* - \frac{\beta(K^*)y^{*2}+\epsilon(K^*)y^*+\xi(K^*)}{\alpha(K^*)y^{*2}+\beta(K^*)y^*+\gamma(K^*)}, \tag{52}$$

which is the special case of (51) with $x^* = y^*$, i.e.

$$\begin{pmatrix} 2y^* \\ 1 \\ 0 \end{pmatrix} \cdot A(K^*) \begin{pmatrix} y^{*2} \\ y^* \\ 1 \end{pmatrix} = 0. \tag{53}$$

The form of the equations (51) is not surprising. Differentiating both sides of (8) with respect to $x$ gives

$$\begin{pmatrix} 2x \\ 1 \\ 0 \end{pmatrix} \cdot A(K) \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix} + \frac{\partial B}{\partial K}(x, y, k(x, y)) \ \frac{\partial k}{\partial x} = 0, \tag{54}$$

and with respect to $y$ gives

$$\begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \cdot A(K) \begin{pmatrix} 2y \\ 1 \\ 0 \end{pmatrix} + \frac{\partial B}{\partial K}(x, y, k(x, y)) \ \frac{\partial k}{\partial y} = 0. \tag{55}$$

Since $(\partial B(x, y, k(x, y))/\partial K) \neq 0$ by assumption (cf remark 1 of section 2), we see that equations (51) are satisfied if and only if the fixed point $(x^*, y^*)$ satisfies

$$\frac{\partial k}{\partial x}(x^*, y^*) = \frac{\partial k}{\partial y}(x^*, y^*) = 0, \tag{56}$$

i.e. all fixed points of $L_{\rm a}$ are critical points of the integral $k(x, y)$ and vice versa. For $L_{\rm s}$, all fixed points are also critical points, but critical points of $k(x, y)$ off the line $y = x$ come in symmetric pairs with respect to reflection in this line and form 2-cycles of $L_{\rm s}$ (e.g. examples 2 and 6 of [10]). In appendix A, the relationship between isolated critical points and isolated $n$-cycles for a general mapping with an integral is discussed.

The Jacobian matrix of the mappings $L_{\rm a}$ or $L_{\rm s}$ is given by

$$\mathrm{d}L(x, y) = \begin{pmatrix} \dfrac{\partial x'}{\partial x} & \dfrac{\partial x'}{\partial y} \\ \dfrac{\partial y'}{\partial x} & \dfrac{\partial y'}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \dagger}{\partial x} + \dfrac{\partial \dagger}{\partial K}\dfrac{\partial k}{\partial x} & \dfrac{\partial \dagger}{\partial y} + \dfrac{\partial \dagger}{\partial K}\dfrac{\partial k}{\partial y} \\ \dfrac{\partial \ddagger}{\partial x} + \dfrac{\partial \ddagger}{\partial x'}\dfrac{\partial x'}{\partial x} + \dfrac{\partial \ddagger}{\partial K}\dfrac{\partial k}{\partial x} & \dfrac{\partial \ddagger}{\partial y} + \dfrac{\partial \ddagger}{\partial x'}\dfrac{\partial x'}{\partial y} + \dfrac{\partial \ddagger}{\partial K}\dfrac{\partial k}{\partial y} \end{pmatrix}, \tag{57}$$

where † and ‡ denote the right-hand sides of $x'$ and $y'$ in (14), respectively (17). The stability of fixed points is determined by looking at the eigenvalues of (57) evaluated at $(x^*, y^*, K^*)$. It is seen that the property (56) brings considerable simplification to (57) evaluated at fixed points. In fact, in this case (57) becomes the Jacobian matrix of the McMillan map $M_a(K)$ of (14), respectively of the McMillan map $M_s(K)$ of (17), with parameter $K$ fixed at $K^*$ evaluated at $(x^*, y^*)$. This means that the fixed points of a CDM map occur as fixed points of a certain McMillan map belonging to the one-parameter family and share the stability that the fixed point has for that McMillan map, a relationship which can also be seen from first principles. This relationship is clearly seen in figure 1: the fixed points in the phase portrait XVI and their stability are inherited from fixed points in the McMillan phase portraits I and IX.

Since $L_a$ and $L_s$ are measure preserving from the previous section, $\det \mathrm{d}L(x^*, y^*) = 1$ and the stability is totally determined by Trace $\mathrm{d}L(x^*, y^*)$. For example, if $|\mathrm{Trace}\, \mathrm{d}L(x^*, y^*)| > 2$, the fixed point is a saddle, whereas if $|\mathrm{Trace}\, \mathrm{d}L(x^*, y^*)| < 2$, the fixed point is a centre. For $L_a$ we have

$$\begin{aligned}&\mathrm{Trace}\, \mathrm{d}L_a(x^*, y^*, K^*)\\ &\quad = -2 + \frac{(4\alpha(K^*)x^*y^* + 2(\beta(K^*)x^* + \delta(K^*)y^*) + \epsilon(K^*))^2}{(\alpha(K^*)x^{*2} + \delta(K^*)x^* + \kappa(K^*))(\alpha(K^*)y^{*2} + \beta(K^*)y^* + \gamma(K^*))},\end{aligned} \tag{58}$$

while for $L_s$ we have

$$\mathrm{Trace}\, \mathrm{d}L_s(x^*, y^*, K^*) = -4 - \frac{\epsilon(K^*) - 4\gamma(K^*)}{\alpha(K^*)y^{*2} + \beta(K^*)y^* + \gamma(K^*)}. \tag{59}$$

Again, from (56) and the relationship between fixed points of the maps and critical points of the integral, it follows that the stability of the fixed points can also be deduced from studying the stability of the critical points of $k(x, y)$ to which they correspond. Thus, equivalent expressions for (58) and (59) are

$$\mathrm{Trace}\, \mathrm{d}L_a(x^*, y^*, K^*) = -2 + \left.\frac{4B_{xy}^2}{B_{xx}B_{yy}}\right|_{(x^*,y^*,K^*)} \tag{60}$$

and

$$\mathrm{Trace}\, \mathrm{d}L_s(x^*, y^*, K^*) = -2\left.\frac{B_{xy}}{B_{xx}}\right|_{(x^*,x^*,K^*)}, \tag{61}$$

where $B_{xy} := (\partial^2 B(x, y, K)/\partial y\, \partial x)$, etc. These formulae show that e.g. the fixed point is a saddle (centre) when $\Delta := (B_{xx}B_{yy} - B_{xy}^2)|_{(x^*,y^*,K^*)} < 0$ (>0). Note the appearance of $B(x, y, K)$ or, equivalently, $B(x, y, K^*)$ in (60) and (61) instead of $k(x, y)$. The latter can be replaced with the former to calculate stability of critical points since (31) shows the local approximation to $k(x, y)$ near $(x_0, y_0) = (x^*, y^*)$ with $K_0 = K^*$ is a rescaled/translated version of the graph of $B(x, y, K^*)$. Again, $B(x, y, K^*)$ is just the integral of the particular McMillan map that $L_a$ or $L_s$ becomes for $K = K^*$, once more emphasizing that the stability of fixed points follows from that of an underlying McMillan map.

When the dependence of the coefficient matrix $A(K)$ on $K$ is polynomial, the simultaneous solution of (49) and (51) or (49) and (53) can be aided by using resultant theory, cf [5, chapter 3.6]. For example, in the symmetric case, the left-hand side of (49) with $x^* = y^*$ is a quartic polynomial $f(y^*)$ in $y^*$ and the left-hand side of (53) is a cubic polynomial $g(y^*)$ in $y^*$, with both having coefficients polynomial in $K^*$. When their resultant $\mathrm{Res}(f, g, y^*)$, a polynomial in $K^*$, vanishes it implies existence of a simultaneous solution of $(y^*, K^*)$ to (49) and (53) (provided $\alpha(K^*)$ does not also vanish). This theory is used in the following example.

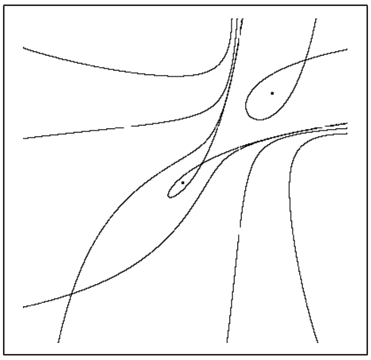


**Figure 4.** Phase portrait of the symmetric CDM map $L_s$ of example 2 on the square $[-2, 1] \times [-2, 1]$.

**Example 2.**

$$\begin{aligned} B_2(x, y, K) = {} & 5(K^3 - 24)x^2y^2 - 20(K^3 - 3K^2 + 12)(x^2y + xy^2) \\ & + 20(K^3 - 6K^2 + 12K + 16)(x^2 + y^2) \\ & - 10(3K^4 - 4K^3 + 24K^2 - 48K + 32)xy \\ & + 20(3K^4 - 8K^3 - 6)(x + y) + 8K(6K^4 + 2K^2 + 3) = 0. \end{aligned} \tag{62}$$

The above symmetric biquadratic satisfies (22) (cf [10, section 3] for a method to construct such examples). The symmetric CDM map corresponding to this biquadratic is

$$\begin{aligned} L_s : \quad & x' = y, \\ & y' = -x - \{[-20(K^3 - 3K^2 + 12)y^2 - 10(3K^4 - 4K^3 + 24K^2 - 48K + 32)y \\ & \quad + 20(3K^4 - 8K^3 - 6)]/[5(K^3 - 24)y^2 - 20(K^3 - 3K^2 + 12)y \\ & \quad + 20(K^3 - 6K^2 + 12K + 16)]\}\big|_{K=k_2(x,y)}. \end{aligned} \tag{63}$$

Using the resultant of the left-hand sides of (49) and (53) for this example, as described above, we find six fixed points $(y^*, y^*)$ of $L_s$ with $y^* \in \{-1.549\,43, -0.521\,11, -0.343\,39, 0.303\,83, 0.614\,94, 6.586\,17\}$ and corresponding $K^* \in \{-0.606\,14, -0.585\,48, -0.586\,60, 0.836\,56, 0.608\,61, 8.055\,42\}$. From (59) or (61), we find, for the stability of the respective fixed points, $\mathrm{Trace}\, \mathrm{d}L_s(x^*, y^*, K^*) = \{3.324, 1.765, 2.362, 1.004, 3.429, -1.992\}$. Five of the fixed points (three saddles and two centres) are shown in figure 4.

## 5. Normal form for symmetric and asymmetric biquadratics

In this section we consider the question of a normal form for a *particular* biquadratic curve belonging to the real symmetric or asymmetric biquadratic foliations (15), (6) and (7). Since the parameter $K$ is then fixed, we will *suppress* the $K$-dependence in $B(x, y, K) = 0$ and write e.g. $B(x, y) = 0$. Similarly, we fix $K$ in the McMillan maps (17) and (14) and write $M_s$ and $M_a$. These McMillan maps are automorphisms of the particular biquadratic curve, and normalizing the curve leads to associated normal forms for these McMillan mappings which we also describe below. Note that condition F of section 2 is irrelevant to the discussion below because we are considering a normal form for a single biquadratic curve and the dynamics upon it without caring whether the curve belongs to a foliation[4]. Nevertheless, to illustrate our results, the examples below are chosen from CDM maps $L_a$ or $L_s$.

Consider the equation

$$R(x) = 0, \tag{64}$$

where the left-hand side is a real quartic polynomial

$$R(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = a_4(x - z_1)(x - z_2)(x - z_3)(x - z_4), \tag{65}$$

with $a_i$ constants, $a_4 \neq 0$, and $z_i$ are the (real and/or complex) zeros of $R(x)$. In the theory of elliptic integrals, there is a method due to Legendre that can reduce (64) and (65) to a form in which the quartic polynomial contains only even terms [8, 9]. That is, in the generic situation there exist real $p$ and $q$ such that if we set

$$x = v_1(\bar{x}) = \frac{p + q\bar{x}}{1 + \bar{x}}, \tag{66}$$

then if $x$ satisfies (64), $\bar{x}$ satisfies

$$\bar{R}(\bar{x}) = 0, \tag{67}$$

where

$$\bar{R}(\bar{x}) := (1 + \bar{x})^4\ R(v_1(\bar{x})) = \bar{a}_4\bar{x}^4 + \bar{a}_2\bar{x}^2 + \bar{a}_0. \tag{68}$$

In (66), the desired $p$ and $q$ are (real) functions of the zeros of $R(x)$. They can be obtained as the solutions of the real quadratic equation

$$t^2 - St + P = 0, \tag{69}$$

where

$$S = p + q = 2\frac{z_1z_2 - z_3z_4}{z_1 + z_2 - z_3 - z_4} \tag{70}$$

and

$$P = pq = \frac{z_1z_2(z_3 + z_4) - z_3z_4(z_1 + z_2)}{z_1 + z_2 - z_3 - z_4}. \tag{71}$$

If the sum of any two pairs of zeros of $R(x)$ is equal ($z_1 + z_2 = z_3 + z_4$), (66) cannot be used; instead a translation

$$x = v_2(\bar{x}) = \bar{x} - \frac{z_1 + z_2}{2}, \tag{72}$$

with $(z_1 + z_2)/2$ real, can be used to give (67) with $\bar{R}(\bar{x}) = R(v_2(\bar{x}))$ (see [9, article 2] for details). Note that (66) is a bilinear transformation with inverse

$$\bar{x} = f_1(x) = v_1^{-1}(x) = \frac{p - x}{x - q}. \tag{73}$$

The inverse of (72) is

$$\bar{x} = f_2(x) = v_2^{-1}(x) = x + \frac{z_1 + z_2}{2}. \tag{74}$$

Significantly, (66) and (72)–(74) are modular (or homographic) transformations and, as is well known, the set of modular transformations

$$\left\{\bar{t} = f(t) = \frac{at + b}{ct + d} : a, b, c, d \in \mathbb{R}, ad - bc \neq 0\right\} \tag{75}$$

form a group under composition.

[4] As illustrated in [10, section 4], it is not possible, in general, to find a global transformation of the plane that will simultaneously achieve the normal forms below for each curve of the biquadratic foliation.

In this section, we will outline two methods that exploit Legendre's method and can be used to reduce a particular asymmetric or symmetric biquadratic curve (7) or (15) to canonical form. The methods use the fact that applying to $B(x, y) = 0$ a transformation $T : (x, y) \mapsto (\bar{x}, \bar{y}) = (f(x), g(y))$ where $f$ and $g$ are arbitrary modular transformations (75) will preserve the biquadratic form, possibly after clearing a denominator, leading to a transformed biquadratic $\bar{B}(\bar{x}, \bar{y}) = 0$. If $B(x, y) = B_s(x, y)$ is symmetric and $g = f$, then applying $T$ will lead to a transformed *symmetric* biquadratic $\bar{B}_s(\bar{x}, \bar{y}) = 0$. Furthermore, we use the following lemma to show that the McMillan maps that preserve the biquadratic curve are transformed to the obvious McMillan maps preserving the transformed biquadratic curve.

**Lemma 4.** *Let $M_a : (x, y) \mapsto (x', y')$ be the asymmetric McMillan map (14) that preserves the curve $B_a(x, y) = 0$. Suppose $T : (x, y) \mapsto (\bar{x}, \bar{y}) = (f(x), g(y))$, with $f$ and $g$ modular, transforms the biquadratic into $\bar{B}_a(\bar{x}, \bar{y}) = 0$. Then $\bar{M}_a = T \circ M_a \circ T^{-1} : (\bar{x}, \bar{y}) \mapsto (\bar{x}', \bar{y}')$ which preserves the transformed biquadratic is the asymmetric McMillan map of the form (14) with barred coordinates and barred coefficients.*

**Proof.** To establish that $\bar{M}_a = T \circ M_a \circ T^{-1}$ which preserves the curve $\bar{B}_a(\bar{x}, \bar{y}) = 0$ has the required form, we use the fact that $M_a$ is reversible. Since $M_a = G_1 \circ G_2$ with the involutions $G_1$ of (11) and $G_s$ of (12) also preserving $B_a(x, y) = 0$, we have $\bar{M}_a$ is also reversible being the composition of the involutions $\bar{G}_1 := T \circ G_1 \circ T^{-1}$ and $\bar{G}_2 := T \circ G_2 \circ T^{-1}$ which must preserve $\bar{B}_a(\bar{x}, \bar{y}) = 0$. With $T(x, y) : (x, y) \mapsto (\bar{x}, \bar{y}) = (f(x), g(y))$, it is easy to check that the first component of $\bar{G}_1$ is $\bar{x}' = \bar{x}$ whereas the second component of $\bar{G}_2$ is $\bar{y}' = \bar{y}$. From proposition 1 of section 2, it follows that an automorphism that preserves a biquadratic and fixes one coordinate is uniquely determined, so $\bar{G}_1$ and $\bar{G}_2$ must be, respectively, the obvious involutions of the form (11) and (12) (with barred coordinates and coefficients) that preserve $\bar{B}_a(\bar{x}, \bar{y}) = 0$. Their composition produces the corresponding barred version of the asymmetric map (14). □

Lemma 4 has an entirely analogous statement in the symmetric case provided $g = f$ so that the $x$–$y$ symmetry of the original biquadratic is retained in the transformed one. The proof uses the reversibility of $M_s$ of (17) and the fact that $\bar{G}_s := T \circ G_s \circ T^{-1} = G_s$ since $T(x, y) : (x, y) \mapsto (\bar{x}, \bar{y}) = (f(x), f(y))$ and $G_s$ commute.

The first method that we have developed is as follows.

**Method I.** *Take either the symmetric or the asymmetric biquadratic, (15) or (6) respectively, set $y = x$ to obtain a quartic, and then apply Legendre's method to this quartic to determine the appropriate transformation $x = v(\bar{x})$ of the form (66) or (72). Use the transformations $x = v(\bar{x})$, $y = v(\bar{y})$ in (15) or (6), respectively, to produce a transformed symmetric or asymmetric biquadratic.*

Method I is based upon the following reasoning, which also shows the simplification that can be achieved in the transformed biquadratic. Consider the transformation $t_1 : (x, y) \mapsto (x, x)$ and let $t_2 : (x, y) \mapsto (u(x), u(y))$, where $u$ is any smooth real bijection. Clearly, $t_1$ and $t_2$ commute, i.e. $t_1 \circ t_2 = t_2 \circ t_1$. Considering $B_a(x, y)$ (the symmetric biquadratic (15) being a special case), we have

$$(B_a \circ t_1) \circ t_2 = B_a \circ (t_1 \circ t_2) = B_a \circ (t_2 \circ t_1) = (B_a \circ t_2) \circ t_1. \quad (76)$$

Both the left- and right-hand sides of (76) equal $B_a(u(x), u(x))$.

If $\alpha \neq 0$ in $B_a(x, y) = 0$, then $B_a \circ t_1 = B_a(x, x) = 0$ is of the form (64) and (65) with $a_4 = \alpha$, $a_3 = \beta + \delta$, $a_2 = \gamma + \kappa + \epsilon$, $a_1 = \xi + \lambda$ and $a_0 = \mu$. It follows from the Legendre method that, via calculating the zeros of $B_a(x, x)$, there exists a transformation $x = v(\bar{x})$ with

$v$ of the form $v_1$ of (66) or $v_2$ of (72) such that $(B_a \circ t_1) \circ t_2(\bar{x}, \bar{y}) = B_a(v(\bar{x}), v(\bar{x})) = 0$ implies the form (67) and (68) with no odd terms. However, the right-hand side of (76) shows that taking $B_a \circ t_2(\bar{x}, \bar{y}) = B_a(v(\bar{x}), v(\bar{y})) = 0$ and then setting $\bar{y} = \bar{x}$ must yield the same result. In other words, if $\alpha \neq 0$, we can always achieve from $B_a(v(\bar{x}), v(\bar{y})) = 0$ a biquadratic in $\bar{x}$ and $\bar{y}$ where the transformed odd terms $\bar{\beta}\,\bar{x}^2\bar{y}$ and $\bar{\delta}\,\bar{x}\bar{y}^2$ cancel each other when $\bar{y} = \bar{x}$, and similarly for the transformed odd terms $\bar{\xi}\,\bar{x}$ and $\bar{\lambda}\,\bar{y}$. For the symmetric biquadratic, since $x = v(\bar{x})$, $y = v(\bar{y})$ respects the $x$–$y$ symmetry, this means that we can achieve $\bar{\beta} = \bar{\xi} = 0$. For the asymmetric biquadratic, this means that we can make the coefficients $\bar{\beta}$ and $\bar{\delta}$ ($\bar{\xi}$ and $\bar{\lambda}$) equal and opposite.

In summary, also noting lemma 4, we obtain from the application of method I the following two results (the symmetric result of proposition 5 was previously stated in [10]).

**Proposition 5.** *Suppose $(x, y)$ satisfies $B_s(x, y) = 0$ with $B_s(x, y)$ the symmetric biquadratic (15). If $\alpha \neq 0$ there exists an invertible transformation $T(x, y) : (x, y) \mapsto (\bar{x}, \bar{y}) = (f(x), f(y))$ with $f$ of the form (73) or (74) such that*

$$\bar{B}_s^{\text{can}}(\bar{x}, \bar{y}) = \bar{x}^2\bar{y}^2 + \bar{\gamma}(\bar{x}^2 + \bar{y}^2) + \bar{\epsilon}\bar{x}\bar{y} + \bar{\mu} = 0. \tag{77}$$

*The transformed McMillan map $\bar{M}_s^{\text{can}} = T \circ M_s \circ T^{-1}$ preserving the transformed biquadratic is*

$$\bar{M}_s^{\text{can}}: \quad \bar{x}' = \bar{y}, \quad \bar{y}' = -\bar{x} - \frac{\bar{\epsilon}\bar{y}}{\bar{y}^2 + \bar{\gamma}}. \tag{78}$$

**Proposition 6.** *Suppose $(x, y)$ satisfies $B_a(x, y) = 0$ with $B_a(x, y)$ the asymmetric biquadratic (6). If $\alpha \neq 0$, there exists an invertible transformation $T(x, y) : (x, y) \mapsto (\bar{x}, \bar{y}) = (-f(x), f(y))$ with $f$ of the form (73) or (74) such that*

$$\bar{B}_a^{\text{can}}(\bar{x}, \bar{y}) = \bar{x}^2\bar{y}^2 + \bar{\beta}(\bar{x}^2\bar{y} + \bar{x}\bar{y}^2) + \bar{\gamma}\bar{x}^2 + \bar{\epsilon}\bar{x}\bar{y} + \bar{\kappa}\bar{y}^2 + \bar{\xi}(\bar{x} + \bar{y}) + \bar{\mu} = 0. \tag{79}$$

*The transformed McMillan map $\bar{M}_a^{\text{can}} = T \circ M_a \circ T^{-1}$ preserving the transformed biquadratic is*

$$\bar{M}_a^{\text{can}}: \qquad \bar{x}' = -\bar{x} - \frac{\bar{\beta}\bar{y}^2 + \bar{\epsilon}\bar{y} + \bar{\xi}}{\bar{y}^2 + \bar{\beta}\bar{y} + \bar{\gamma}}, \quad \bar{y}' = -\bar{y} - \frac{\bar{\beta}\bar{x}'^2 + \bar{\epsilon}\bar{x}' + \bar{\xi}}{\bar{x}'^2 + \bar{\beta}\bar{x}' + \bar{\kappa}}. \tag{80}$$

With reference to these propositions, we note the following:

(i) The transformation $f$ is of the form (73) or (74) as determined by studying the zeros of $B_a(x, x)$ or $B_s(x, x)$. For the asymmetric case, this guarantees that the transformed coefficients $\bar{\beta}$ and $\bar{\delta}$ are equal and opposite, as are $\bar{\xi}$ and $\bar{\lambda}$. Further use of the transformation $(\bar{x}, \bar{y}) \mapsto (-\bar{x}, \bar{y})$ makes the coefficients of the cubic terms equal and the coefficients of the linear terms equal, recovering (79). In both the symmetric and asymmetric cases, we divide through in the converted biquadratic if necessary to ensure the coefficient of $\bar{x}^2\bar{y}^2$ is 1.

(ii) If $\bar{\mu} \neq 0$, a rescaling

$$(\bar{x}, \bar{y}) = |\bar{\mu}|^{1/4}(\hat{x}, \hat{y}) \tag{81}$$

can be used in (79) or (77) to obtain versions with $\hat{}$ variables and $\hat{\mu} = \pm 1$ according as $\bar{\mu} \gtrless 0$.

(iii) If $\bar{\mu} = 0$ and $\bar{\gamma} \neq 0$ in (77), a transformation

$$(\bar{x}, \bar{y}) = a(1/\hat{x}, 1/\hat{y}) \tag{82}$$

with $a = \sqrt{|\bar{\gamma}|}$ can be used to obtain a biquadratic $\hat{B}_s^{\text{can}}(\hat{x}, \hat{y}) = 0$ of the form

$$\hat{B}_s^{\text{can}}(\hat{x}, \hat{y}) = \hat{x}^2 + \hat{y}^2 + \hat{\epsilon}\hat{x}\hat{y} \pm 1 = 0 \tag{83}$$

according as $\bar{\gamma} \gtrless 0$, where $\hat{\epsilon} = \bar{\epsilon}/\bar{\gamma}$. Similarly, if $\bar{\mu} = 0$ and $\bar{\xi} \neq 0$ in (79), application of (82) with $a = \bar{\xi}^{1/3}$ gives the biquadratic

$$\hat{B}_a^{\text{can}}(\hat{x}, \hat{y}) = (\hat{x}^2\hat{y} + \hat{x}\hat{y}^2) + \hat{\gamma}\hat{x}^2 + \hat{\epsilon}\hat{x}\hat{y} + \hat{\kappa}\hat{y}^2 + \hat{\xi}(\hat{x} + \hat{y}) + 1 = 0. \tag{84}$$

(iv) If the original asymmetric biquadratic (6) or symmetric biquadratic (15) has $\alpha = 0$ but possesses at least one term containing $x^2$ and another term containing $y^2$, it can be transformed to a biquadratic with $\bar{\alpha} \neq 0$ through application of a transformation $(x, y) \mapsto (\bar{x}, \bar{y}) = (f(x), f(y))$, where $f$ is now an arbitrary modular transformation (75).

Note the transformed curve $\bar{B}_s^{\rm can}(\bar{x}, \bar{y}) = 0$ in (77) has a double symmetry, being invariant under interchange of $\bar{x}$ and $\bar{y}$ together with rotation by $\pi$ (so that $\bar{M}_s^{\rm can}$ commutes with $-Id$). An example of the application of proposition 5 to a symmetric biquadratic is the following (all numerical values are given to five significant figures).

**Example 3.** Consider the symmetric foliation

$$B_3(x, y, K) = x^2y^2 + (x^2y + xy^2) + 5K(x^2 + y^2) - xy + 7(x + y) + K = 0. \tag{85}$$

Using method I on the biquadratic curve with $K = 0.685\,60$, we obtain the transformation $T_1 : (x, y) \mapsto ((p - x)/(x - q), (p - y)/(y - q))$, where $(p, q) = (-0.956\,39, 5.9451)$, which when used in (85) produces the curve

$$\bar{B}_3(\bar{x}, \bar{y}, 0.685\,60) = 1960.3\bar{x}^2\bar{y}^2 + 169.55(\bar{x}^2 + \bar{y}^2) - 22.535\bar{x}\bar{y} - 8.2604 = 0. \tag{86}$$

Finally, using the rescaling transformation $T_2 : (\bar{x}, \bar{y}) \mapsto (a\bar{x}, a\bar{y})$, where $a = 3.9249$, produces the curve

$$\hat{B}_3(\hat{x}, \hat{y}, 0.685\,60) = \hat{x}^2\hat{y}^2 + 1.3324(\hat{x}^2 + \hat{y}^2) - 0.177\,09\hat{x}\hat{y} - 1 = 0. \tag{87}$$

In summary, the biquadratic (85) with $K = 0.685\,60$, becomes (87) with the transformation $T = T_2 \circ T_1 : (x, y) \mapsto (a(p - x)/(x - q), a(p - y)/(y - q))$. The transformed McMillan map preserving (87) is given by

$$\hat{x}' = \hat{y}, \qquad \hat{y}' = -\hat{x} + 0.177\,09\,\frac{\hat{y}}{\hat{y}^2 + 1.3324}. \tag{88}$$

See figure 5.

We have developed a second method exploiting Legendre's method that is particularly useful for studying asymmetric biquadratics (although it could also be applied to symmetric

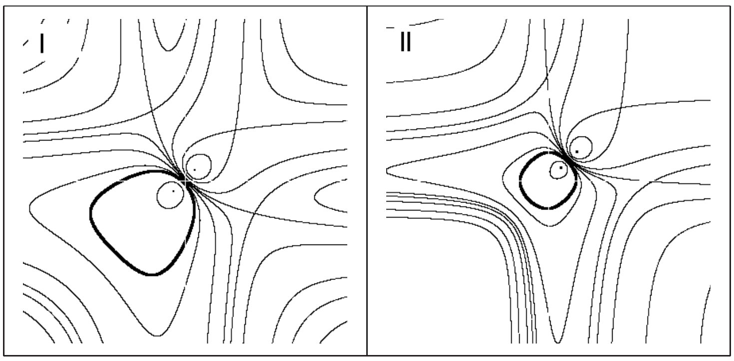



**Figure 5.** Picture I shows the phase portrait of the original foliation (85) with the curve of example 3 to be reduced to canonical form in bold. Picture II shows the phase portrait of the transformed foliation with the transformation $T$ needed to transform the bold curve given in picture I to canonical form applied globally. The transformed curve, in bold, now has a double symmetry. Both phase portraits are on the square $[-5, 5] \times [-5, 5]$.

ones). The motivation for this method comes from taking the asymmetric biquadratic (6) and (7) and writing $x$ as a function of $y$ or $y$ as a function of $x$, i.e.

$$x = \frac{-Q_2 \pm \sqrt{Q_2^2 - 4Q_1Q_3}}{2Q_1}, \qquad y = \frac{-\tilde{Q}_2 \pm \sqrt{\tilde{Q}_2^2 - 4\tilde{Q}_1\tilde{Q}_3}}{2\tilde{Q}_1}, \tag{89}$$

where

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \end{pmatrix} = A(K) \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} \tilde{Q}_1 \\ \tilde{Q}_2 \\ \tilde{Q}_3 \end{pmatrix} = A^{\mathrm{T}}(K) \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix}. \tag{90}$$

In general, the polynomials under the square root signs in (89) are a quartic in $y$ in the first square root and a quartic in $x$ in the second square root. Two possibilities can occur. We classify the biquadratic as *rational* if at least one of the square root signs disappears, meaning $y$ is a rational function of $x$ or $x$ is a rational function of $y$ (a special case is when the biquadratic is factorizable: $B_{\mathrm{a}}(x, y) = (a_0xy+b_0x+c_0y+d_0)(a_1xy+b_1x+c_1y+d_1) = 0$). On the other hand, if neither square root sign disappears, the generic case for fixed $K$, we call the biquadratic *elliptic*. This is because, anticipating the next section, elliptic function parametrizations of $x$ and $y$ are, in general, then needed to remove the square root sign. The main purpose of method II is to use Legendre's method under the square root signs of (89) to simplify the quartic polynomials and provide a normal form for the elliptic biquadratic.

**Method II.** *Take the asymmetric biquadratic (6) and solve for x as a function of y, and y as a function of x as in (89). Apply Legendre's method to the quartic in y (x) under the first (second) square root sign in (89) to determine the appropriate transformation* $y = v_y(\bar{y})$ $(x = v_x(\bar{x}))$ *of the form (66) or (72) that removes the cubic and linear terms. Use these transformations in (6) to produce a transformed asymmetric biquadratic* $\bar{B}_{\mathrm{a}}(\bar{x}, \bar{y}) = 0$.

Note that the coefficients of the quartic terms under the square roots in (89) are $\delta^2 - 4\alpha\kappa$ and $\beta^2 - 4\alpha\gamma$, respectively, and the use of Legendre's method in method II presumes that neither of these coefficients vanishes. If $\alpha = \beta = \gamma = 0$ ($\alpha = \delta = \kappa = 0$), the biquadratic in consideration is linear in $x$ ($y$), so $x$ ($y$) can be solved as a rational function of $y$ ($x$), e.g.

$$x = -\frac{\kappa y^2 + \lambda y + \mu}{\delta y^2 + \epsilon y + \xi}. \tag{91}$$

Method II is inapplicable/irrelevant in this case. Otherwise if $\beta^2 - 4\alpha\gamma = 0$ ($\delta^2 - 4\alpha\kappa = 0$) without the three coefficients in each expression vanishing, a transformation of $x$ ($y$) by an arbitrary modular tranformation achieves in general $\bar{\beta}^2 - 4\bar{\alpha}\bar{\gamma} \neq 0$ ($\bar{\delta}^2 - 4\bar{\alpha}\bar{\kappa} \neq 0$), whence method II can be applied to this new biquadratic.

Method II is based upon the following reasoning. Consider the first equation of (89). Under an *arbitrary* modular transformation $y = v_y(\bar{y})$ on the right-hand side of the equals sign, together with an *arbitrary* modular transformation $x = v_x(\bar{x})$ on the left-hand side, the form of the original equation can be retained in a barred version (after possible manipulation). The corresponding statement is true for the second equation of (89). In particular, the modular transformations can be chosen to be the desired ones of method II that remove odd terms under the respective square roots of the equations of (89).

The removal of the cubic and linear terms under both square root signs in (89) is equivalent to satisfying four simultaneous nonlinear equations in the transformed coefficients:

$$\begin{aligned} &\bar{\beta}\bar{\epsilon} - 2(\bar{\alpha}\bar{\xi} + \bar{\gamma}\bar{\delta}) = 0, \qquad && \bar{\epsilon}\bar{\lambda} - 2(\bar{\delta}\bar{\mu} + \bar{\xi}\bar{\kappa}) = 0, \\ &\bar{\delta}\bar{\epsilon} - 2(\bar{\alpha}\bar{\lambda} + \bar{\beta}\bar{\kappa}) = 0, \qquad && \bar{\epsilon}\bar{\xi} - 2(\bar{\beta}\bar{\mu} + \bar{\gamma}\bar{\lambda}) = 0. \end{aligned} \tag{92}$$

The left-hand sides of these equations are just the expressions for coefficients of cubic and linear terms. Written in terms of the original coefficients and the parameters of the modular transformations, they can also be thought of as the defining equations for these parameters. In general, they are nonlinear in the parameters and not solvable (except numerically for given numerical values of the original coefficients).

Our approach is to use equations (92) to infer the resultant transformed biquadratic. An exhaustive partition of the transformed asymmetric biquadratics $\bar{B}_a(\bar{x}, \bar{y}) = 0$ that satisfy equations (92) is given in appendix B. The transformed biquadratic being rational implies that the original one was (since the modular transformations are birational). In appendix B, it can be checked that cases (I), (IIc), (IIIa), (IIIc), (IV), (V) and (VII) are always rational since they are linear in $\bar{x}$ or $\bar{y}$ or else the square-root sign disappears when solved for $\bar{x}$ or $\bar{y}$. The remaining cases (IIa), (IIb), (IIIb) and (VI) of appendix B can contain elliptic biquadratics. These represent the reduced forms that can be obtained from method II for originally elliptic biquadratics and we collect them together in the result that follows.

**Proposition 7.** *Suppose* $(x, y)$ *satisfies* $B_a(x, y) = 0$ *with* $B_a(x, y)$ *an elliptic asymmetric biquadratic (6). There exists an invertible transformation* $T(x, y) : (x, y) \mapsto (\bar{x}, \bar{y}) = (f(x), g(y))$*, with* $f$ *and* $g$ *of the form (73) or (74), such that the transformed elliptic biquadratic,* $\bar{B}_a(\bar{x}, \bar{y})$*, is one of the following:*

(1) $$\bar{B}_a(\bar{x}, \bar{y}) = \bar{\alpha}\bar{x}^2\bar{y}^2 + \bar{\gamma}\bar{x}^2 + \bar{\kappa}\bar{y}^2 + \bar{\epsilon}\bar{x}\bar{y} + \bar{\mu} = 0. \qquad (93)$$

(2) $$\bar{B}_a(\bar{x}, \bar{y}) = \bar{\gamma}\bar{\kappa}\bar{x}^2\bar{y}^2 - \bar{\gamma}\bar{\lambda}\bar{x}^2\bar{y} - \bar{\xi}\bar{\kappa}\bar{x}\bar{y}^2 + \bar{\gamma}\bar{\mu}\bar{x}^2 + \bar{\kappa}\bar{\mu}\bar{y}^2 + \bar{\xi}\bar{\mu}\bar{x} + \bar{\lambda}\bar{\mu}\bar{y} + \bar{\mu}^2 = 0. \qquad (94)$$

(3) $$\bar{B}_a(\bar{x}, \bar{y}) = \bar{\beta}\bar{x}^2\bar{y} + \bar{\delta}\bar{x}\bar{y}^2 + \bar{\xi}\bar{x} + \bar{\lambda}\bar{y} = 0. \qquad (95)$$

It is clear that case (1) above is case (VI) from appendix B and case (3) is case (IIb). It is assumed that the coefficients in (3) are in *general position*, i.e. with no non-trivial relation between them, otherwise rational biquadratics can occur (in particular, this happens if any of the coefficients in (95) are 0). Case (1) in general position or with $\bar{\epsilon} = 0$ is elliptic. Taking any one of the remaining coefficients $\bar{\alpha}$, $\bar{\gamma}$, $\bar{\kappa}$ or $\bar{\mu}$ in (93) equal to 0 results in a degenerate elliptic biquadratic which can be parametrized by trigonometric or hyperbolic functions. In fact, it suffices in this degenerate situation to consider only case (1) with $\bar{\alpha} = 0$ since $\bar{\gamma} = 0$ or $\bar{\kappa} = 0$ or $\bar{\mu} = 0$ can be transformed into it by using, respectively, the transformations $(\bar{x}, \bar{y}) = (\hat{x}, 1/\hat{y})$, $(\bar{x}, \bar{y}) = (1/\hat{x}, \hat{y})$, $(\bar{x}, \bar{y}) = (1/\hat{x}, 1/\hat{y})$.

Case (2) contains cases (IIa) and (IIIb) of appendix B rewritten and regrouped in a consistent (and, for later purposes, more convenient) way. Specifically, taking (140) of appendix B, using $\bar{\gamma} = -\bar{\beta}\bar{\mu}/\bar{\lambda}$ to replace the combination $\bar{\beta}\bar{\mu}$ in the coefficients of $\bar{x}^2\bar{y}^2$, $\bar{x}^2\bar{y}$ and $\bar{x}^2$ by $-\bar{\gamma}\bar{\lambda}$ and then dividing both sides by $-\bar{\lambda}$ gives (94). The subcase of (94) with $\bar{\xi} = 0$ is also an elliptic biquadratic. So is the subcase of (94) if we take $\bar{\lambda} = 0$, and this is equivalent to case (IIIb) of appendix B (this is seen by multiplying (144) by $\bar{\mu}/\bar{\delta}$ and then using $\bar{\mu}\bar{\alpha} = \bar{\gamma}\bar{\kappa}$, $\bar{\mu}\bar{\delta} = -\bar{\xi}\bar{\kappa}$, etc to obtain (94) with $\bar{\lambda} = 0$). Setting any of the other coefficients in (94) to 0 leads to rational biquadratics.

It proves useful to notice that case (1) of proposition 7 with $\bar{\alpha} = 1$ (as we can always divide through by $\bar{\alpha} \neq 0$) or with $\bar{\alpha} = 0$ can be simplified further. The possible cases are listed in table 1 for $\bar{\alpha} = 1$ and table 2 for $\bar{\alpha} = 0$ ($\bar{\gamma}$, $\bar{\kappa}$ and $\bar{\mu}$ are assumed to be non-zero).

Cases A, D, E, H, A′ and D′ with a rescaling $(\bar{x}, \bar{y}) = (\sqrt{\bar{\kappa}/\bar{\gamma}}\hat{x}, \hat{y})$ and cases B and C with the transformation $(\bar{x}, \bar{y}) = (1/\hat{x}, \sqrt{\bar{\mu}}\hat{y})$ become symmetric, so it follows from proposition 5 and notes (ii) and (iii) thereafter that we can achieve

$$\tilde{x}^2\tilde{y}^2 + \tilde{\gamma}(\tilde{x}^2 + \tilde{y}^2) + \tilde{\epsilon}\tilde{x}\tilde{y} \pm 1 = 0 \qquad (96)$$

**Table 1.** Possible cases of (93) with $\bar{\alpha} = 1$.

| Case | $\bar{\gamma}$ | $\bar{\kappa}$ | $\bar{\mu}$ |
|---|---|---|---|
| A | >0 | >0 | >0 |
| B | >0 | <0 | >0 |
| C | <0 | >0 | >0 |
| D | <0 | <0 | >0 |
| E | >0 | >0 | <0 |
| F | >0 | <0 | <0 |
| G | <0 | >0 | <0 |
| H | <0 | <0 | <0 |

**Table 2.** Possible cases of (93) with $\bar{\alpha} = 0$.

| Case | $\bar{\gamma}$ | $\bar{\kappa}$ |
|---|---|---|
| A′ | >0 | >0 |
| B′ | >0 | <0 |
| C′ | <0 | >0 |
| D′ | <0 | <0 |

if $\bar{\alpha} = 1$, and

$$(\tilde{x}^2 + \tilde{y}^2) + \tilde{\epsilon}\tilde{x}\tilde{y} \pm 1 = 0 \tag{97}$$

if $\bar{\alpha} = 0$. For cases F, G, B′ and C′, a rescaling $(\bar{x}, \bar{y}) = (\sqrt{|\bar{\kappa}/\bar{\gamma}|}\hat{x}, \hat{y})$ can be used to make the coefficient of $x^2$ and $y^2$ equal and opposite. Finally, for cases F and G a further rescaling $(\hat{x}, \hat{y}) = |\bar{\gamma}\bar{\mu}/\bar{\kappa}|^{1/4}(\tilde{x}, \tilde{y})$ can be used to make the coefficient of $\hat{x}^2\hat{y}^2$ equal but opposite to the constant term, i.e.

$$\tilde{x}^2\tilde{y}^2 + \tilde{\gamma}(\tilde{x}^2 - \tilde{y}^2) + \tilde{\epsilon}\tilde{x}\tilde{y} - 1 = 0, \tag{98}$$

and for B′ and C′ a further rescaling $(\hat{x}, \hat{y}) = \sqrt{|\bar{\mu}/\bar{\kappa}|}(\tilde{x}, \tilde{y})$ results in

$$(\tilde{x}^2 - \tilde{y}^2) + \tilde{\epsilon}\tilde{x}\tilde{y} \pm 1 = 0. \tag{99}$$

We will illustrate the application of method II to an asymmetric biquadratic which results in the transformed biquadratic taking the form (96).

**Example 4.**

$$B_4(x, y, K) = x^2y^2 - 2x^2y + xy^2 - 4x^2 + y^2 + 3xy + 3x - y + K = 0. \tag{100}$$

The asymmetric McMillan mapping preserving this foliation is

$$x' = -x - \frac{y^2 + 3y + 3}{y^2 - 2y - 4}, \qquad y' = -y + \frac{2x'^2 - 3x' + 1}{x'^2 + x' + 1}. \tag{101}$$

Using method II on (100) with $K = 16$ involves using a transformation $T_1 : (x, y) \mapsto ((p - x)/(x - q), (m - y)/(y - n))$, where $(p, q, m, n) = (-3.1095, -0.064\,04, -17.433, 1.5454)$, and produces the curve

$$\bar{B}_4(\bar{x}, \bar{y}, 16) = 16.182\bar{x}^2\bar{y}^2 + 322.42\bar{x}^2 - 59.796\bar{y}^2 + 279.24\bar{x}\bar{y} + 2782.6 = 0. \tag{102}$$

As the coefficients of $\bar{x}^2$ and $\bar{y}^2$ are of opposite sign we cannot use a rescaling to symmetrize (102), but instead we use the transformation $T_2 : (\bar{x}, \bar{y}) \mapsto (1/\bar{x}, \bar{y})$ to produce the curve

$$\hat{B}_4(\hat{x}, \hat{y}, 16) = -59.796\hat{x}^2\hat{y}^2 + 2782.6\hat{x}^2 + 16.182\hat{y}^2 + 279.24\hat{x}\hat{y} + 322.42 = 0. \tag{103}$$

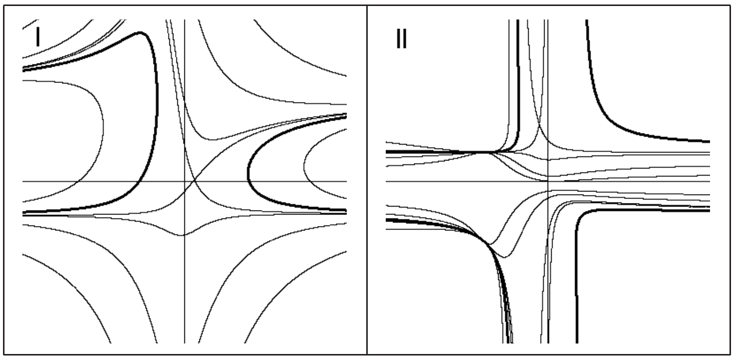



**Figure 6.** Picture I shows the phase portrait of the original foliation (100) with the asymmetric curve to be reduced to canonical form in bold. Picture II shows the phase portrait of the transformed foliation with the transformation needed to transform the bold curve given in picture I to (symmetric) canonical form applied globally. The transformed curve, in bold, is the only curve with a symmetry, in fact a double symmetry. Both phase portraits are on the square $[-6, 6] \times [-6, 6]$.

Next we can use two further transformations, $T_3 : (\hat{x}, \hat{y}) \mapsto (a\hat{x}, \hat{y})$ and $T_4 : (\tilde{x}, \tilde{y}) \mapsto (b\tilde{x}, b\tilde{y})$ where $(a, b) = (13.113, 0.181\,22)$. $T_3$ symmetrizes the curve and $T_4$ makes the coefficient of $\tilde{x}^2\tilde{y}^2$ and the constant equal but of opposite sign, resulting in the curve

$$\breve{B}_4(\breve{x}, \breve{y}, 16) = \breve{x}^2\breve{y}^2 - 1.5282(\breve{x}^2 + \breve{y}^2) - 2.0111\breve{x}\breve{y} - 1 = 0. \tag{104}$$

In summary, the biquadratic (100) with $K = 16$ becomes (104) with the transformation $T = T_4 \circ T_3 \circ T_2 \circ T_1 : (x, y) \mapsto (ab(q-x)/(x-p), b(m-y)/(y-n))$. The map preserving (104) is the symmetric McMillan given by

$$\breve{x}' = \breve{y}, \qquad \breve{y}' = -\breve{x} + 2.0111\,\frac{\breve{y}}{\breve{y}^2 - 1.5282}. \tag{105}$$

See figure 6.

It remains to investigate whether the three cases of elliptic biquadratics isolated in proposition 7 are related by modular transformations (we discard henceforth the degenerate elliptic biquadratics given by proposition 7(1) with any of $\bar{\alpha}, \bar{\gamma}, \bar{\kappa}, \bar{\mu}$ equal to 0).

**Proposition 8.** *Suppose $(x, y)$ satisfies $B_a(x, y) = 0$, with $B_a(x, y)$ any of the elliptic asymmetric biquadratics (1)–(3) of proposition 7. Then there exists an invertible transformation $T(x, y) : (x, y) \mapsto (\bar{x}, \bar{y}) = (f(x), g(y))$, with $f$ and $g$ modular, such that the biquadratic becomes*

$$\bar{B}(\bar{x}, \bar{y}) = \bar{x}^2\bar{y}^2 + \bar{\gamma}(\bar{x}^2 + \bar{y}^2) + \bar{\epsilon}\bar{x}\bar{y} \pm 1 = 0 \tag{106}$$

*or*

$$\bar{B}(\bar{x}, \bar{y}) = \bar{x}^2\bar{y}^2 + \bar{\gamma}(\bar{x}^2 - \bar{y}^2) + \bar{\epsilon}\bar{x}\bar{y} - 1 = 0. \tag{107}$$

**Proof.** The (non-degenerate) elliptic biquadratic given in proposition 7(1) was reduced in the discussion following proposition 7 to the form (106) or (107).

Consider the elliptic biquadratics (2) and (3) of proposition 7 with the bars dropped for convenience, and assume $\gamma\mu < 0$ and $\kappa\mu < 0$ in the former case and $\beta\lambda > 0$ and $\delta\xi > 0$ in the latter case. Using the respective transformations $(x, y) = (\sqrt{-\mu/\gamma}\,\bar{x}, \sqrt{-\mu/\kappa}\,\bar{y})$ and $(x, y) = (\sqrt{\lambda/\beta}\bar{x}, \sqrt{\xi/\delta}\bar{y})$, both transformed biquadratics now assume the form

$$\bar{B}_a(\bar{x}, \bar{y}) = A(\bar{x}^2\bar{y}^2 - \bar{x}^2 - \bar{y}^2 + 1) + B\bar{y}(\bar{x}^2 + 1) + C\bar{x}(\bar{y}^2 + 1) = 0, \tag{108}$$

where

$$(A, B, C) = \left(\mu, \lambda\sqrt{-\frac{\mu}{\kappa}}, \xi\sqrt{-\frac{\mu}{\gamma}}\right) \tag{109}$$

for the transformed version of proposition 7(2) and

$$(A, B, C) = \left(0, \lambda\sqrt{\frac{\xi}{\delta}}, \xi\sqrt{\frac{\lambda}{\beta}}\right) \tag{110}$$

for the transformed version of proposition 7(3). Applying the modular transformation $(\bar{x}, \bar{y}) = ((p + q\hat{x})/(1 + \hat{x}), (m + n\hat{y})/(1 + \hat{y}))$ to (108) and multiplying by $(1 + x^2)(1 + y^2)$, the coefficients of $\hat{x}^2\hat{y}$, $\hat{x}\hat{y}^2$, $\hat{x}$ and $\hat{y}$ are, respectively,

$$\begin{aligned}
&2A(mn - 1)(q^2 - 1) + B(m + n)(q^2 + 1) + 2C(mn + 1)q,\\
&2A(pq - 1)(n^2 - 1) + C(p + q)(n^2 + 1) + 2B(pq + 1)n,\\
&2A(pq - 1)(m^2 - 1) + C(p + q)(m^2 + 1) + 2B(pq + 1)m,\\
&2A(mn - 1)(p^2 - 1) + B(m + n)(p^2 + 1) + 2C(mn + 1)p.
\end{aligned}$$

These are all zero when $p = -q = 1$ and $m = -n = 1$ (irrespective of $A, B, C$), in which case the transformed version of (108) is equivalent to

$$\hat{B}_a(\hat{x}, \hat{y}) = (B + C)(\hat{x}^2\hat{y}^2 - 1) + (C - B)(\hat{x}^2 - \hat{y}^2) - 8A\hat{x}\hat{y} = 0. \tag{111}$$

Finally, dividing through by $B + C$ gives

$$\tilde{B}_a(\tilde{x}, \tilde{y}) = \tilde{x}^2\tilde{y}^2 + \tilde{\gamma}(\tilde{x}^2 - \tilde{y}^2) + \tilde{\epsilon}\tilde{x}\tilde{y} - 1 = 0, \tag{112}$$

which is of the form (107) as required. Note that the coefficient $B + C$ is necessarily non-zero. This follows since $B + C = 0$ in (111) implies this biquadratic, and the initial curve from which it is transformed, is rational, contradicting the elliptic assumption. It is clear from (110) and (111) that case (3) of proposition 7 when $\bar{\beta}\bar{\lambda} > 0$ and $\bar{\delta}\bar{\xi} > 0$ leads to (112) with $\tilde{\epsilon} = 0$.

In a similar way to above, transformation to either (106) or (107) via modular transformations can be achieved for all the remaining possible sign combinations of the non-zero coefficients $(\bar{\gamma}, \bar{\mu}, \bar{\kappa})$ in the elliptic biquadratic proposition 7(2) or the non-zero coefficients $(\bar{\beta}, \bar{\lambda}, \bar{\xi}, \bar{\delta})$ in the elliptic biquadratic proposition 7(3). The explicit details are given in appendix C. □

Summarizing this section, we find that (107) represents a fundamentally asymmetric normal form for many possibilities of cases (1)–(3) of the asymmetric elliptic biquadratics of proposition 7. On the other hand, (106) is the normal form for a symmetric (elliptic or rational) biquadratic curve (cf proposition 5) and can also be obtained from transforming some elliptic asymmetric biquadratics.

Finally, we note that if we allow complex variables and parameters, asymmetry in elliptic biquadratics can always be removed since (107) can be transformed to (106)[5].

**Proposition 9.** *Suppose* $(x, y)$ *satisfies* $B(x, y) = 0$*, where x and y are complex, with* $B(x, y)$ *either of the biquadratics of proposition 8. Then there exists a rescaling transformation* $T(x, y) : (x, y) \mapsto (\bar{x}, \bar{y}) = (ax, by)$ *such that the biquadratic becomes*

$$\bar{B}(\bar{x}, \bar{y}) = \bar{x}^2\bar{y}^2 + \bar{\gamma}(\bar{x}^2 + \bar{y}^2) + \bar{\epsilon}\bar{x}\bar{y} + 1 = 0. \tag{113}$$

**Proof.** The biquadratic (106) given in proposition 8 with the + sign is already in the form (113), while for (106) with the − sign, rescaling both variables by $\sqrt{\mathrm{i}}$ and multiplying by $-1$ gives the required form. For (107), use of the transformation $(x, y) = (\mathrm{i}\bar{x}, \bar{y})$ followed by multiplication by $-1$ also leads to (113). □

[5] This complex normal form for biquadratics has recently been found independently in [17].

## 6. Parametrization of biquadratics by elliptic functions

In this section we discuss the possibility of parametrizing the canonical biquadratic curves discussed in the previous section, and consequently the symmetric and asymmetric biquadratic curves (15) and (6), respectively.

In [21] the following result is attributed to Euler (where the variables and parameters are taken as complex).

**Theorem 10.** *For the general correspondence*

$$B(x,y) = a_0x^2y^2 + a_1(x^2y + xy^2) + a_2(x^2 + y^2) + a_3xy + a_4(x + y) + a_5 = 0, \tag{114}$$

*there exists an even elliptic function of the second order* $f(z)$ *such that if* $x = f(z)$*, then* $y = f(z \pm a)$ *for some constant shift* $a$*. (In degenerate cases a trigonometric function appears instead of an elliptic function.)*

In [2, pp 471–3], Baxter states that the generic symmetric complex biquadratic (in the form given in theorem 10) can be reduced to (113). Baxter then proceeds to show that (113), bars removed, can be parametrized in terms of elliptic functions (Baxter considers a complex parametrization). The parametrization given is $(x, y) = (\sqrt{k}\,\mathrm{sn}(u, k), \sqrt{k}\,\mathrm{sn}(u \pm \eta, k))$, where $\mathrm{sn}(u, k)$ is the Jacobian elliptic function sn, $u$ is the argument, $k$ the modulus and $\eta$ is a function of the coefficients in (113) (we refer the reader throughout this section to e.g. [3, pp 18–31, 284–5] for definitions and properties of Jacobian elliptic functions).

Proposition 9 of the previous section shows that every elliptic asymmetric biquadratic and every symmetric biquadratic can be reduced to (113) when the variables are complex. Theorem 10 and the parametrization given by Baxter show that all these (asymmetric or symmetric) biquadratic curves have an elliptic parametrization (cf also [17]).

When the variables are real, the above discussion no longer applies. We have been able to parametrize most cases of elliptic biquadratics that can be reduced to (106) of proposition 8, which is equivalent to $B_{\mathrm{s}}^{\mathrm{can}}$ of proposition 5. Consider (106) after dropping the bars and making the replacement $\epsilon \to 2\epsilon$, that is

$$B(x,y) = x^2y^2 + \gamma(x^2 + y^2) + 2\epsilon xy \pm 1 = 0. \tag{115}$$

Writing $y$ as a function of $x$, we get

$$y = \frac{-\epsilon x \pm \sqrt{-\gamma x^4 + (\epsilon^2 - \gamma^2 \mp 1)x^2 \mp \gamma}}{x^2 + \gamma}. \tag{116}$$

The nature of the curves given by (115), e.g. whether they are bounded or unbounded, can be determined by looking at the quartic

$$\Delta(x) = -\gamma x^4 + (\epsilon^2 - \gamma^2 \mp 1)x^2 \mp \gamma. \tag{117}$$

The non-trivial possibilities are listed in figure 7, where we introduce

$$\mathcal{B} = \frac{(\epsilon^2 - \gamma^2 \mp 1)}{\gamma}. \tag{118}$$

Note that $\Delta(x)$ has three extrema if $\mathcal{B} > 0$ which occur at $x = 0, \pm\sqrt{\mathcal{B}/2}$ and one if $\mathcal{B} \leqslant 0$ at $x = 0$. Furthermore, note $\Delta(0) = \mp\gamma$ (so that the sign of $\Delta(0)/\gamma$ is the opposite of the sign of the constant term in (115)) and $\Delta(\pm\sqrt{\mathcal{B}/2}) = (\gamma/4)(\mathcal{B}^2 \mp 4)$. We recall that necessarily $\gamma \neq 0$ in (115) for it to be an elliptic biquadratic.

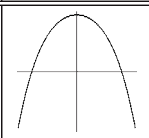
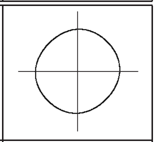
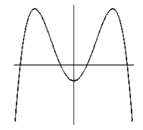
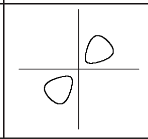
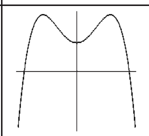
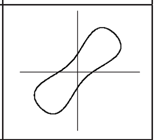
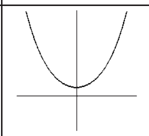
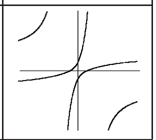
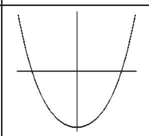
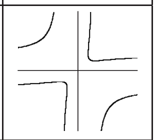
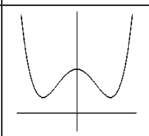
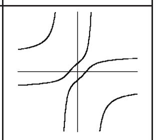
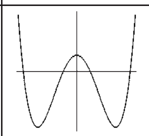
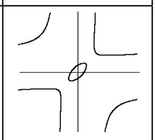
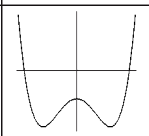
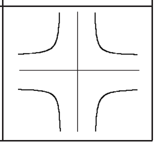
| Case | $\gamma$ | $\mathcal{B}$ | $\Delta(0)$ | $\Delta(x)$ | $B(x,y)=0$ |
|---|---|---|---|---|---|
| 1 | >0 | ≤0 | $\gamma$ |  |  |
| 2 | >0 | >2 | $-\gamma$ |  |  |
| 3 | >0 | >0 | $\gamma$ |  |  |
| 4 | <0 | ≤0 | $-\gamma$ |  |  |
| 5 | <0 | ≤0 | $\gamma$ |  |  |
| 6 | <0 | >0, <2 | $-\gamma$ |  |  |
| 7 | <0 | >2 | $-\gamma$ |  |  |
| 8 | <0 | >0 | $\gamma$ |  |  |

**Figure 7.** Possible forms of $\Delta(x)$ (see (117)), where $\gamma \neq 0$, and corresponding representative symmetric biquadratics $B(x, y) = 0$ of (115).

The function $\Delta(x)$ takes four different possible forms:

$$-\gamma\left(x^4 + \frac{\epsilon^2 - \gamma^2 - 1}{-\gamma}x^2 + 1\right), \qquad -\gamma > 0, \tag{119}$$

$$-\gamma\left(x^4 + \frac{\epsilon^2 - \gamma^2 + 1}{-\gamma}x^2 - 1\right), \qquad -\gamma > 0, \tag{120}$$

$$\gamma\left(-x^4 + \frac{\epsilon^2 - \gamma^2 - 1}{\gamma}x^2 - 1\right), \qquad \gamma > 0, \tag{121}$$

$$\gamma\left(-x^4 + \frac{\epsilon^2 - \gamma^2 + 1}{\gamma}x^2 + 1\right), \qquad \gamma > 0. \tag{122}$$

What we have in mind is to use these forms as a starting point to find appropriate Jacobian elliptic functions to parametrize the cases listed in figure 7. We will illustrate with cases (1) and (3) from figure 7 and then list the other cases in table 3.

Consider the differential equation that the Jacobian elliptic function $\mathrm{cn}(u,k)$ satisfies (see [3, p 25]):

$$\left[\frac{\mathrm{d}}{\mathrm{d}u}\mathrm{cn}(u,k)\right]^2 = (1-\mathrm{cn}^2(u,k))\,(k'^2+k^2\,\mathrm{cn}^2(u,k)). \tag{123}$$

In (123), $k \in [0,1]$ and $k' = \sqrt{1-k^2}$ is the complementary modulus. Since also $(\mathrm{d}/\mathrm{d}u)\mathrm{cn}(u,k) = -\mathrm{sn}(u,k)\mathrm{dn}(u,k)$, (123) can be rewritten as

$$\begin{aligned}\left[\frac{\mathrm{sn}(u,k)\mathrm{dn}(u,k)}{k'}\right]^2 &= (1-\mathrm{cn}^2(u,k))\left(1+\frac{k^2}{k'^2}\mathrm{cn}^2(u,k)\right)\\ &= -\frac{k^2}{k'^2}\mathrm{cn}^4(u,k)+\left(\frac{k}{k'}-\frac{k'}{k}\right)\frac{k}{k'}\,\mathrm{cn}^2(u,k)+1.\end{aligned} \tag{124}$$

We can use (124) to write (122) as a perfect square by making the substitutions

$$x = \sqrt{\frac{k}{k'}}\,\mathrm{cn}(u,k) \tag{125}$$

and

$$\mathcal{B} = \frac{\epsilon^2-\gamma^2+1}{\gamma} = \frac{k}{k'}-\frac{k'}{k}. \tag{126}$$

Note in (126) that the range of the monotonic function $k/k'-k'/k$ on the domain $k \in [0,1]$ is $\mathbb{R}$ so that all values of $\mathcal{B}$ can be accommodated. Finally, by defining a parameter $\eta$ such that

$$\gamma = \frac{\mathrm{dn}^2(\eta,k)}{kk'\,\mathrm{sn}^2(\eta,k)} \tag{127}$$

and setting

$$\epsilon = -\frac{\mathrm{cn}(\eta,k)}{kk'\,\mathrm{sn}^2(\eta,k)} \tag{128}$$

where the sign of $\eta$ is appropriately chosen, it follows, from substituting the parametrizations for $x$, $\gamma$ and $\epsilon$ in (116) and using (124), that we find

$$y = \sqrt{\frac{k}{k'}}\,\mathrm{cn}(u\pm\eta,k). \tag{129}$$

This and the remaining parametrizations are presented in table 3, where we use the standard conventions for the ratio of two Jacobian elliptic functions and the reciprocal of the functions: e.g. ds = dn/sn, sc = sn/cn, ns = 1/sn, etc. In table 3 there are two possible parametrizations of $x$, $y$ for each case. The second is related to the first by a shift of the argument $u \to u+\mathbf{K}(k)$, where

$$\mathbf{K}(k) = \int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

is the complete elliptic integral of the first kind (and equals one quarter of the period of $\mathrm{sn}(u,k)$ and $\mathrm{cn}(u,k)$). The real version of Baxter's complex parametrization occurs in case 7**. From the ranges of the functions $\mathcal{B}$ in table 3, we observe that there are two cases of $B(x,y)=0$ which we have not been able to parametrize using (ratios of) Jacobian elliptic functions. These are case 6 of figure 7 and case 4 of figure 7 when $-2 < \mathcal{B} < 0$.

**Table 3.** Elliptic parametrization of the cases given in figure 7.

| Case | $x$ | $y$ | $\gamma$ | $\epsilon$ | $\mathcal{B}$ |
|---|---|---|---|---|---|
| 1, 3 | $\sqrt{k/k'}\,\mathrm{cn}(u)$<br>$\sqrt{kk'}\,\mathrm{sd}(u)$ | $\sqrt{k/k'}\,\mathrm{cn}(u \pm \eta)$<br>$\sqrt{kk'}\,\mathrm{sd}(u \pm \eta)$ | $\mathrm{ds}^2(\eta)/kk'$ | $-\mathrm{cs}(\eta)\mathrm{ns}(\eta)/kk'$ | $k/k' - k'/k \in \mathbb{R}$ |
| 2 | $\mathrm{dn}(u)/\sqrt{k'}$<br>$\sqrt{k'}\,\mathrm{nd}(u)$ | $\mathrm{dn}(u \pm \eta)/\sqrt{k'}$<br>$\sqrt{k'}\,\mathrm{nd}(u \pm \eta)$ | $\mathrm{cs}^2(\eta)/k'$ | $-\mathrm{ds}(\eta)\mathrm{ns}(\eta)/k'$ | $1/k' + k' \geqslant 2$ |
| 4 | $\sqrt{k'}\,\mathrm{sc}(u)$<br>$\mathrm{cs}(u)/\sqrt{k'}$ | $\sqrt{k'}\,\mathrm{sc}(u \pm \eta)$<br>$\mathrm{cs}(u \pm \eta)/\sqrt{k'}$ | $-\mathrm{cs}^2(\eta)/k'$ | $\mathrm{ds}(\eta)\mathrm{ns}(\eta)/k'$ | $-1/k' - k' \leqslant -2$ |
| 5, 8 | $\sqrt{k'/k}\,\mathrm{nc}(u)$<br>$\mathrm{ds}(u)/\sqrt{kk'}$ | $\sqrt{k'/k}\,\mathrm{nc}(u \pm \eta)$<br>$\mathrm{ds}(u \pm \eta)/\sqrt{kk'}$ | $-\mathrm{ds}^2(\eta)/kk'$ | $\mathrm{cs}(\eta)\mathrm{ns}(\eta)/kk'$ | $k'/k - k/k' \in \mathbb{R}$ |
| 7* | $\mathrm{ns}(u)/\sqrt{k}$<br>$\mathrm{dc}(u)/\sqrt{k}$ | $\mathrm{ns}(u \pm \eta)/\sqrt{k}$<br>$\mathrm{dc}(u \pm \eta)/\sqrt{k}$ | $-\mathrm{ns}^2(\eta)/k$ | $\mathrm{cs}(\eta)\mathrm{ds}(\eta)/k$ | $1/k + k \geqslant 2$ |
| 7** | $\sqrt{k}\,\mathrm{sn}(u)$<br>$\sqrt{k}\,\mathrm{cd}(u)$ | $\sqrt{k}\,\mathrm{sn}(u \pm \eta)$<br>$\sqrt{k}\,\mathrm{cd}(u \pm \eta)$ | $-\mathrm{ns}^2(\eta)/k$ | $\mathrm{cs}(\eta)\mathrm{ds}(\eta)/k$ | $1/k + k \geqslant 2$ |

*Indicates the unbounded part of the curve and **the bounded (the explicit dependence of functions on the modulus is suppressed for convenience).

Excepting these cases, the parametrizations of table 3 allow an action-angle variable description of the dynamics on each of the symmetric biquadratics of figure 7 under their corresponding McMillan maps $M_{\mathrm{s}}^{\mathrm{can}}$:

$$M_{\mathrm{s}}^{\mathrm{can}}: \qquad x' = y, \quad y' = -x - \frac{2\epsilon y}{y^2 + \gamma}. \tag{130}$$

Identifying $(x, y) \to (x_{n-1}, x_n)$ and $(x', y') \to (x_n, x_{n+1})$, $n \in \mathbb{Z}$ being discrete time, (115) becomes

$$x_n^2 x_{n+1}^2 + \gamma(x_n^2 + x_{n+1}^2) + 2\epsilon x_n x_{n+1} \pm 1 = 0, \tag{131}$$

whereas (130) becomes

$$x_{n+1} + x_{n-1} = -\frac{2\epsilon x_n}{{x_n}^2 + \gamma}. \tag{132}$$

Via the parametrizations in table 3, in each case we can solve (132) using

$$x_n = p(u_n, k) = p(n\,\eta + u_0, k), \tag{133}$$

where $p$ is the appropriate elliptic function from the table, the modulus $k(\mathcal{B})$ is dependent on $\gamma$ and $\epsilon$ in the combination (118), $\eta$ is chosen from the corresponding parametrizations for $\gamma$ and $\epsilon$, and $u_0$ is fixed by the initial value of $x_0$. As a result, the dynamics on each canonical symmetric biquadratic curve can be described in terms of the modulus $k$ and argument $u$ of the elliptic function:

$$k_{n+1} = k_n,$$
$$u_{n+1} = u_n + \eta.$$

Finally, recall from proposition 8 that (115), equivalently (131), is a normal form for elliptic symmetric biquadratics *as well as* certain elliptic asymmetric biquadratics. In this case we know that the original elliptic asymmetric (symmetric) biquadratic, written in terms of $(x_n, y_n)$, can be related to (131), now considered with bars added, using an asymmetric (symmetric) modular transformation

$$(x_n, y_n) = (f(\bar{x}_n), g(\bar{y}_n)), \tag{134}$$

where $f$ and $g$ are modular with $f = g$ in the symmetric case. It now follows that successive points on the original biquadratic can be written:

$$\begin{aligned}(x_n, y_n) &= (f(\bar{x}_n), g(\bar{y}_n)) = (f(\bar{x}_n), g(\bar{x}_{n+1})) \\ &= ((f \circ p)(n\,\eta + u_0, k), (g \circ p)((n+1)\,\eta + u_0, k)).\end{aligned} \tag{135}$$

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## Appendix A. Fixed points of maps possessing an integral

This appendix generalizes our discussion in section 4 concerning the relationship between fixed points of the maps $L_a$ or $L_s$ and critical points of their integral $k(x, y)$. Although our interest here is the planar case ($n = 2$), the results are given for arbitrary dimension.

Suppose $L : \boldsymbol{x} \mapsto \boldsymbol{x}'$ with $\boldsymbol{x} \in \mathbb{R}^n$ is a diffeomorphism with a smooth integral (or invariant) $K(\boldsymbol{x})$ satisfying $K(\boldsymbol{x}') = K(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$. Then we also have

$$K(\boldsymbol{x}^{(m)}) = K(\boldsymbol{x}) \tag{136}$$

for any integer $m \geqslant 1$, where $\boldsymbol{x}^{(m)} = L^m \boldsymbol{x}$. Differentiating (136) gives the matrix equation

$$\mathrm{d}L^m(\boldsymbol{x})^{\mathrm{T}} \frac{\partial K}{\partial \boldsymbol{x}}(\boldsymbol{x}^{(m)}) = \frac{\partial K}{\partial \boldsymbol{x}}(\boldsymbol{x}), \tag{137}$$

where $\mathrm{d}L^m(\boldsymbol{x})^{\mathrm{T}}$ is the transpose of the $n \times n$ Jacobian matrix of $L^m$ and $(\partial K/\partial \boldsymbol{x})(\boldsymbol{x})$ is $\nabla K$ written as a column vector. By definition, critical points $\boldsymbol{x}^*$ satisfy $(\partial K/\partial \boldsymbol{x})(\boldsymbol{x}^*) = \boldsymbol{0}$.

From (137), we learn the following:

(1) a critical point is mapped to a critical point (since $L$ is a diffeomorphism so that $\mathrm{d}L(\boldsymbol{x})$ is non-singular);

(2) isolated critical points always belong to $n$-cycles of $L$, $n \geqslant 1$ ($n = 1$ giving fixed points); in particular, if there are a finite number of critical points of $K$, they all belong to $n$-cycles of $L$;

(3) if $x$ is a point of an $n$-cycle (i.e. $\boldsymbol{x}^{(n)} = \boldsymbol{x}$), we find

$$(\mathrm{d}L^n(\boldsymbol{x})^{\mathrm{T}} - \mathbb{1}) \frac{\partial K}{\partial \boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{0}, \tag{138}$$

so if $(\mathrm{d}L^n(\boldsymbol{x})^{\mathrm{T}} - \mathbb{1})$ is non-singular, then the point is also a critical point. The non-singularity condition is equivalent to saying $\mathrm{d}L^n(\boldsymbol{x})$ has no eigenvalue equal to 1, which in turn means that the $n$-cycle containing $\boldsymbol{x}$ is isolated from other $n$-cycles.

One way to summarize this is as follows: isolated critical points of the integral belong to (isolated) cycles of the map and the points of isolated cycles of the map are (isolated) critical points of the integral. However, it should be noted that in integrable maps (e.g. $n = 2$ when existence of one integral suffices) $n$-cycles generically are *not* isolated and come in one-parameter families (the points of which are not critical points of the integral).

## Appendix B. A partition of transformed asymmetric biquadratics

The following list is an exhaustive partition of all the transformed asymmetric biquadratics (6) and (7) (with bars added to $x$, $y$ and parameters, and suppressing $K$ which is fixed) that satisfy equations (92). Satisfying the latter means in general that four of the transformed coefficients can be eliminated in favour of the five remaining transformed coefficients. Where a coefficient is not explicitly specified below, it can be taken as arbitrary.

(I) $(\bar\alpha = \bar\delta^2\bar\mu/\bar\lambda^2, \bar\beta = \bar\delta\bar\xi/\bar\lambda, \bar\gamma = \bar\xi^2\bar\kappa/\bar\lambda^2, \bar\epsilon = 2(\bar\delta\bar\mu + \bar\xi\bar\kappa)/\bar\lambda;\ \bar\lambda \neq 0, \bar\epsilon \neq 0)$

$$\begin{aligned}\bar B_{\rm a}(\bar x, \bar y) &= \bar\delta^2\bar\mu\bar x^2\bar y^2 + \bar\delta\bar\xi\bar\lambda\bar x^2\bar y + \bar\delta\bar\lambda^2\bar x\bar y^2 + \bar\xi^2\bar\kappa\bar x^2 + \bar\lambda^2\bar\kappa\bar y^2\\ &\quad + 2\bar\lambda(\bar\delta\bar\mu + \bar\xi\bar\kappa)\bar x\bar y + \bar\lambda^2\bar\xi\bar x + \bar\lambda^3\bar y + \bar\lambda^2\bar\mu = 0. \end{aligned} \tag{139}$$

(IIa) $(\bar\alpha = -\bar\beta\bar\kappa/\bar\lambda, \bar\gamma = -\bar\beta\bar\mu/\bar\lambda, \bar\epsilon = 0, \bar\delta = -\bar\xi\bar\kappa/\bar\mu;\ \bar\lambda \neq 0, \bar\mu \neq 0)$

$$\begin{aligned}\bar B_{\rm a}(\bar x, \bar y) &= \bar\mu\bar\beta\bar\kappa\bar x^2\bar y^2 - \bar\mu\bar\beta\bar\lambda\bar x^2\bar y + \bar\xi\bar\kappa\bar\lambda\bar x\bar y^2 + \bar\beta\bar\mu^2\bar x^2 - \bar\mu\bar\kappa\bar\lambda\bar y^2\\ &\quad - \bar\mu\bar\lambda\bar\xi\bar x - \bar\mu\bar\lambda^2\bar y - \bar\mu^2\bar\lambda = 0. \end{aligned} \tag{140}$$

(IIb) $(\bar\alpha = 0, \bar\gamma = 0, \bar\epsilon = 0, \bar\kappa = 0, \bar\mu = 0;\ \bar\lambda \neq 0, \bar\xi \neq 0)$

$$\bar B_{\rm a}(\bar x, \bar y) = \bar\beta\bar x^2\bar y + \bar\delta\bar x\bar y^2 + \bar\xi\bar x + \bar\lambda\bar y = 0. \tag{141}$$

(IIc) $(\bar\alpha = -\bar\beta\bar\kappa/\bar\lambda, \bar\gamma = 0, \bar\epsilon = 0, \bar\xi = 0, \bar\mu = 0;\ \bar\lambda \neq 0, \bar\kappa \neq 0)$

$$\bar B_{\rm a}(\bar x, \bar y) = \bar\beta\bar\kappa\bar x^2\bar y^2 - \bar\beta\bar\lambda\bar x^2\bar y - \bar\lambda\bar\delta\bar x\bar y^2 - \bar\kappa\bar\lambda\bar y^2 - \bar\lambda^2\bar y = 0. \tag{142}$$

(IIIa) $(\bar\lambda = 0, \bar\kappa = 0, \bar\epsilon = 0, \bar\gamma = -\bar\alpha\bar\xi/\bar\delta, \bar\mu = 0;\ \ \bar\delta \neq 0)$

$$\bar B_{\rm a}(\bar x, \bar y) = \bar\alpha\bar\delta\bar x^2\bar y^2 + \bar\beta\bar\delta\bar x^2\bar y + \bar\delta^2\bar x\bar y^2 - \bar\alpha\bar\xi\bar x^2 + \bar\xi\bar\delta\bar x = 0. \tag{143}$$

(IIIb) $(\bar\lambda = 0, \bar\beta = 0, \bar\epsilon = 0, \bar\gamma = -\bar\alpha\bar\xi/\bar\delta, \bar\mu = -\bar\xi\bar\kappa/\bar\delta; \bar\delta \neq 0, \bar\kappa \neq 0, \bar\alpha \neq 0)$

$$\bar B_{\rm a}(\bar x, \bar y) = \bar\alpha\bar\delta\bar x^2\bar y^2 + \bar\delta^2\bar x\bar y^2 - \bar\alpha\bar\xi\bar x^2 + \bar\kappa\bar\delta\bar y^2 + \bar\xi\bar\delta\bar x - \bar\xi\bar\kappa = 0. \tag{144}$$

(IIIc) $(\bar\lambda = 0, \bar\beta = 0, \bar\epsilon = 0, \bar\alpha = 0, \bar\gamma = 0, \bar\mu = -\bar\xi\bar\kappa/\bar\delta;\ \bar\delta \neq 0, \bar\kappa \neq 0)$

$$\bar B_{\rm a}(\bar x, \bar y) = \bar\delta^2\bar x\bar y^2 + \bar\kappa\bar\delta\bar y^2 + \bar\xi\bar\delta\bar x - \bar\xi\bar\kappa = 0. \tag{145}$$

(IV) $(\bar\lambda = 0, \bar\gamma = \bar\beta^2\bar\kappa/\bar\delta^2, \bar\epsilon = 2\bar\beta\bar\kappa/\bar\delta, \bar\xi = 0, \bar\mu = 0;\ \bar\delta \neq 0)$

$$\bar B_{\rm a}(\bar x, \bar y) = \bar\alpha\bar\delta^2\bar x^2\bar y^2 + \bar\beta\bar\delta^2\bar x^2\bar y + \bar\delta^3\bar x\bar y^2 + \bar\beta^2\bar\kappa\bar x^2 + \bar\delta^2\bar\kappa\bar y^2 + 2\bar\beta\bar\delta\bar\kappa\bar x\bar y = 0. \tag{146}$$

(V) $(\bar\lambda = 0, \bar\delta = 0, \bar\kappa = 0, \bar\epsilon = 2\bar\alpha\bar\xi/\bar\beta, \bar\mu = \bar\alpha\bar\xi^2/\bar\beta^2;\ \beta \neq 0)$

$$\bar B_{\rm a}(\bar x, \bar y) = \bar\beta^2\bar\alpha\bar x^2\bar y^2 + \bar\beta^3\bar x^2\bar y + \bar\gamma\bar\beta^2\bar x^2 + 2\bar\alpha\bar\beta\bar\xi\bar x\bar y + \bar\xi\bar\beta^2\bar x + \bar\alpha\bar\xi^2 = 0. \tag{147}$$

(VI) $(\bar\lambda = 0, \bar\delta = 0, \bar\beta = 0, \bar\xi = 0)$

$$\bar B_{\rm a}(\bar x, \bar y) = \bar\alpha\bar x^2\bar y^2 + \bar\gamma\bar x^2 + \bar\kappa\bar y^2 + \bar\epsilon\bar x\bar y + \bar\mu = 0. \tag{148}$$

(VII) $(\bar\lambda = 0, \bar\delta = 0, \bar\beta = 0, \bar\kappa = 0, \bar\alpha = 0, \bar\epsilon = 0;\ \bar\xi \neq 0)$

$$\bar B_{\rm a}(\bar x, \bar y) = \bar\gamma\bar x^2 + \bar\xi\bar x + \bar\mu = 0. \tag{149}$$

## Appendix C. Reduction to canonical form of elliptic biquadratics

In this appendix, we complete the proof of proposition 8 by considering the transformation of the remaining cases of elliptic biquadratics (2) and (3) of proposition 7 (we start with (94) and (95) with bars dropped for convenience). Consider the following transformations:

(2)(i) $(x, y) = (\sqrt{\mu/\gamma}\,\bar x, \sqrt{\mu/\kappa}\,\bar y)$ when $\gamma\mu > 0$ and $\kappa\mu > 0$,
(2)(ii) $(x, y) = (\sqrt{\mu/\gamma}\,\bar x, \sqrt{-\mu/\kappa}\,\bar y)$ when $\gamma\mu > 0$ and $\kappa\mu < 0$,
(2)(iii) $(x, y) = (\sqrt{-\mu/\gamma}\,\bar x, \sqrt{\mu/\kappa}\,\bar y)$ when $\gamma\mu < 0$ and $\kappa\mu > 0$,
(3)(i) $(x, y) = (\sqrt{-\lambda/\beta}\,\bar x, \sqrt{-\xi/\delta}\,\bar y)$ when $\beta\lambda < 0$ and $\delta\xi < 0$,
(3)(ii) $(x, y) = (\sqrt{-\lambda/\beta}\,\bar x, \sqrt{\xi/\delta}\,\bar y)$ when $\beta\lambda < 0$ and $\delta\xi > 0$,
(3)(iii) $(x, y) = (\sqrt{\lambda/\beta}\,\bar x, \sqrt{-\xi/\delta}\,\bar y)$ when $\beta\lambda > 0$ and $\delta\xi < 0$.

If these transformations are applied, respectively, to the biquadratics (2) and (3) of proposition 7 with the corresponding parameter sign combinations, we obtain

(2*)(i) $\bar{B}_{\rm a}(\bar{x},\bar{y}) = A(\bar{x}^2\bar{y}^2+\bar{x}^2+\bar{y}^2+1) - B\bar{y}(\bar{x}^2-1) - C\bar{x}(\bar{y}^2-1) = 0,$
(2*)(ii) $\bar{B}_{\rm a}(\bar{x},\bar{y}) = A(-\bar{x}^2\bar{y}^2+\bar{x}^2-\bar{y}^2+1) - B\bar{y}(\bar{x}^2-1) + C\bar{x}(\bar{y}^2+1) = 0,$
(2*)(iii) $\bar{B}_{\rm a}(\bar{x},\bar{y}) = A(-\bar{x}^2\bar{y}^2-\bar{x}^2+\bar{y}^2+1) + B\bar{y}(\bar{x}^2+1) - C\bar{x}(\bar{y}^2-1) = 0,$
(3*)(i) $\bar{B}_{\rm a}(\bar{x},\bar{y}) = -B\bar{y}(\bar{x}^2-1) - C\bar{x}(\bar{y}^2-1) = 0,$
(3*)(ii) $\bar{B}_{\rm a}(\bar{x},\bar{y}) = -B\bar{y}(\bar{x}^2-1) + C\bar{x}(\bar{y}^2+1) = 0,$
(3*)(iii) $\bar{B}_{\rm a}(\bar{x},\bar{y}) = B\bar{y}(\bar{x}^2+1) - C\bar{x}(\bar{y}^2-1) = 0,$

where

(2*)(i) $(A,B,C) = (\mu, \lambda\sqrt{\mu/\kappa}, \xi\sqrt{\mu/\gamma}),$
(2*)(ii) $(A,B,C) = (\mu, \lambda\sqrt{-\mu/\kappa}, \xi\sqrt{\mu/\gamma}),$
(2*)(iii) $(A,B,C) = (\mu, \lambda\sqrt{\mu/\kappa}, \xi\sqrt{-\mu/\gamma}),$
(3*)(i) $(B,C) = (\lambda\sqrt{-\xi/\delta}, \xi\sqrt{-\lambda/\beta}),$
(3*)(ii) $(B,C) = (\lambda\sqrt{\xi/\delta}, \xi\sqrt{-\lambda/\beta}),$
(3*)(iii) $(B,C) = (\lambda\sqrt{-\xi/\delta}, \xi\sqrt{\lambda/\beta}).$

Applying to the above transformed biquadratics the respective modular transformations of the form $(\bar{x},\bar{y}) = ((p+q_0\hat{x})/(1+q_1\hat{x}), (m+n_0\hat{y})/(1+n_1\hat{y})),$

(2*)(i) $(\bar{x},\bar{y}) = (\hat{x}, (1-\hat{y})/(1+\hat{y})),$
(2*)(ii) $(\bar{x},\bar{y}) = ((1-\hat{x})/(1+\hat{x}), \hat{y}),$
(2*)(iii) $(\bar{x},\bar{y}) = (\hat{x}, (1-\hat{y})/(1+\hat{y})),$
(3*)(i) $(\bar{x},\bar{y}) = (\hat{x}, (1-\hat{y})/(1+\hat{y})),$
(3*)(ii) $(\bar{x},\bar{y}) = ((1-\hat{x})/(1+\hat{x}), \hat{y}),$
(3*)(iii) $(\bar{x},\bar{y}) = (\hat{x}, (1-\hat{y})/(1+\hat{y})),$

the biquadratics $\bar{B}_{\rm a}(\bar{x},\bar{y})$ of (2*) and (3*) above become equivalent to

(2**)(i) $\hat{B}_{\rm a}(\hat{x},\hat{y}) = (2A+B)(\hat{x}^2\hat{y}^2+1) + (2A-B)(\hat{x}^2+\hat{y}^2) + 4C\hat{x}\hat{y} = 0,$
(2**)(ii) $\hat{B}_{\rm a}(\hat{x},\hat{y}) = -(2A+C)(\hat{x}^2\hat{y}^2-1) + (2A-C)(\hat{x}^2-\hat{y}^2) + 4B\hat{x}\hat{y} = 0,$
(2**)(iii) $\hat{B}_{\rm a}(\hat{x},\hat{y}) = -(2A+B)(\hat{x}^2\hat{y}^2-1) - (2A-B)(\hat{x}^2-\hat{y}^2) + 4C\hat{x}\hat{y} = 0,$
(3**)(i) $\hat{B}_{\rm a}(\hat{x},\hat{y}) = B(\hat{x}^2\hat{y}^2-\hat{x}^2-\hat{y}^2+1) + 4C\hat{x}\hat{y} = 0,$
(3**)(ii) $\hat{B}_{\rm a}(\hat{x},\hat{y}) = -C(\hat{x}^2\hat{y}^2+\hat{x}^2-\hat{y}^2-1) + 4B\hat{x}\hat{y} = 0,$
(3**)(iii) $\hat{B}_{\rm a}(\hat{x},\hat{y}) = -B(\hat{x}^2\hat{y}^2-\hat{x}^2+\hat{y}^2-1) + 4C\hat{x}\hat{y} = 0.$

Finally, dividing through by the coefficient of $\hat{x}^2\hat{y}^2$ we get the required result of proposition 8. In doing the division, note that the coefficient of $\hat{x}^2\hat{y}^2$ in each case is necessarily non-zero. This follows since the coefficient of $\hat{x}^2\hat{y}^2$ being zero implies the biquadratic, and the initial curve from which it is transformed, is rational, as can easily be checked. This contradicts the assumption that the original biquadratic was elliptic.

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