REVERSING AND EXTENDED SYMMETRIES OF SHIFT SPACES

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ABSTRACT. The reversing symmetry group is considered in the setting of symbolic dynamics. While this group is generally too big to be analysed in detail, there are interesting cases with some form of rigidity where one can determine all symmetries and reversing symmetries explicitly. They include Sturmian shifts as well as classic examples such as the Thue–Morse system with various generalisations or the Rudin–Shapiro system. We also look at generalisations of the reversing symmetry group to higher-dimensional shift spaces, then called the group of *extended symmetries*. We develop their basic theory for faithful $\mathbb{Z}^d$-actions, and determine the extended symmetry group of the chair tiling shift, which can be described as a model set, and of Ledrappier's shift, which is an example of algebraic origin.

1. INTRODUCTION

The symmetries of a (topological) dynamical system always have been an important tool to analyse the structure of the system. Here, we mention the structure of its periodic or closed orbits or, more generally, the problem to assess or reject topological conjugacy of two dynamical systems. In the setting of dynamical systems defined by the action of a homeomorphism T on some (compact) topological space $\mathbb{X}$, it had been realised early on that *time reversal symmetry* has powerful consequences on the structure of dynamics as well; see [49] and references therein. This development was certainly motivated by the importance of time reversal in fundamental equations of physics.

Much later, this approach was re-analysed and extended in a more group-theoretic setting [35, 27], via the introduction of the (topological) symmetry group $\mathcal{S}(\mathbb{X})$ of $(\mathbb{X}, T)$ and its extension to the reversing symmetry group $\mathcal{R}(\mathbb{X})$. The latter, which is the group of all self conjugacies and *flip* conjugacies of $(\mathbb{X}, T)$, need not be mediated by an involution, as in the original setting; see [8, 7, 38] for examples and [36, 9, 41] for some general and systematic results. Previous studies have often been restricted to concrete systems such as trace maps [44], toral automorphisms [8] or polynomial automorphisms of the plane [7]; see [41] for a comprehensive overview from an algebraic perspective.

It seems natural to also consider invertible *shift spaces* from symbolic dynamics, where shifts of finite type (also known as topological Markov shifts) have been studied quite extensively; see [38, 32] and references given there as well as [33, Ch. 3]. However, the symmetry groups of general shift spaces are typically huge, with arbitrary finite groups and even free groups appearing as subgroups; compare [33, Thm. 3.3.18] and [39, p. 437]. In particular, they are then not even amenable, and thus perhaps of limited interest. Nevertheless, the existence of reversing symmetries can be (and has been) studied, with interesting consequences for the isomorphism problem [28] or the nature of dynamical zeta functions [32].

The situation changes significantly in the presence of algebraic, geometric or topological constraints that lead to symmetry rigidity, as present in Sturmian sequences [42] or certain ergodic substitution dynamical systems [16, 18], where the symmetry group is minimal in the sense that it just consists of the group generated by the shift itself. More generally, we speak of *symmetry rigidity* when the symmetry group coincides with the group generated by the shift or is just a finite index extension of it. The relevance of this scenario was recently realised and investigated by several groups [18, 20, 23, 21]. This led to interesting deterministic cases (hence with zero topological entropy), where one can now also determine $\mathcal{R}(\mathbb{X})$. As we shall see, symmetry rigidity is neither restricted to minimal shifts, nor to shifts with zero entropy. In general, the shift space $\mathbb{X}$ (with its Cantor structure) is important and complicated, while the shift map itself is simple — in contrast to many previously studied cases of reversibility, where the space is simple, while the mapping requires more attention. Typically, the latter situation leads to the former via symbolic codings.

Several higher-dimensional systems are also known to display symmetry rigidity, as can be seen for the chair tiling [42] or extracted from the general theory in [34, 12] for certain dynamical systems of algebraic origin. In particular, there are examples from the latter class that are not minimal. Here, due to the action of $\mathbb{Z}^d$ with $d > 1$, one first needs to generalise the concept of a reversing symmetry group. Since one now has more than one shift generator, it seems more natural to speak of *extended symmetries* rather than reversing symmetries. Extended symmetries are self-homeomorphisms which give an orbit equivalence with constant orbit cocycles. This setting will also be considered, where we develop some basic theory. In view of [47], it is perhaps not surprising that various new phenomena become possible for $\mathbb{Z}^d$-shifts. In Theorems 6 and 7, we show that the extended symmetry groups of two classic higher dimensional shifts, namely the chair tiling and Ledrappier's shift, are rather limited, reflecting the rigidity that already exists in the case of their symmetry groups.

The article is organised as follows. First, we discuss possible structures of reversing and extended symmetry groups of shift spaces through a variety of tools, which are illustrated with a substantial selection of examples. This way, we put some emphasis on how to actually determine the reversing or extended symmetries while, at the same time, preparing the way for a more systematic treatment in the future. To this end, we first recall some basic notions and tools for the standard situation (Section 2), which is followed by results on shifts in one dimension in Sections 3 and 4. The examples are selected to cover the 'usual suspects' and to develop some intuition in parallel to the general (and sometimes rather formal) tools. Next, we introduce and develop an appropriate extension of the (reversing) symmetry group for higher-dimensional shifts spaces (Section 5). This also establishes a connection with classic methods and results from higher-dimensional crystallography, through the group $\mathrm{GL}(d,\mathbb{Z})$ of invertible integer matrices. Our two paradigmatic examples, one of geometric and one of algebraic origin, are treated in Section 6.

2. Setting and general tools

The main concepts around reversibility are well known, and have recently appeared in textbook form in [41]. In the context of ergodic theory and symbolic dynamics, we also refer to [38, 28] and references therein for related work and results. In this section, we consider symbolic dynamics in one dimension, with some emphasis on *deterministic* systems such as those defined by primitive substitutions or by repetitive (or uniformly recurrent) sequences. The situation for symbolic dynamics in higher dimensions is more complex, both conceptually and mathematically, and thus postponed to Section 5.

Let $\mathcal{A} = \{a_1, \dots, a_n\}$ be a finite alphabet, and $\mathbb{X}_{\mathcal{A}} := \mathcal{A}^{\mathbb{Z}}$ the full shift space, or full shift for short, which is compact in the standard product topology. Elements are written as bi-infinite (or two-sided) sequences, $x = (x_i)_{i\in\mathbb{Z}}$. On $\mathbb{X}_{\mathcal{A}}$, we have a continuous action of the group $\mathbb{Z}$ via the *shift operator* S (simply called shift from now on) as its generator, where $(Sx)_i := x_{i+1}$. We denote the group generated by S as $\langle S\rangle$, and similarly for other groups. Clearly, $(\mathbb{X}_{\mathcal{A}}, \mathbb{Z})$ is a topological dynamical system, and $(S^{-1}x)_i = x_{i-1}$ defines the inverse of S; we refer to [39] for general background. Let $\mathbb{X}$ be a closed (hence compact) and shift invariant subspace of $\mathbb{X}_{\mathcal{A}}$. Below, we will always assume that the shift action on $\mathbb{X}$ is *faithful*, meaning that $G = \langle S|_{\mathbb{X}}\rangle \simeq \mathbb{Z}$. This implies $\mathbb{X}$ to be an infinite set. In particular, it excludes orbit closures of periodic sequences. The corresponding dynamical system $(\mathbb{X}, \mathbb{Z})$, also written as $(\mathbb{X}, S)$, is referred to as a *shift with faithful action*. When the context is clear, we will often simply write $\mathbb{X}$, and drop the word faithful.

The group of homeomorphisms of $\mathbb{X}$, the latter viewed as a topological space, is denoted by $\mathrm{Aut}(\mathbb{X})$. Clearly, one always has $S \in \mathrm{Aut}(\mathbb{X})$. The *symmetry group* of $(\mathbb{X}, S)$ is the (topological) centraliser of $\langle S\rangle$ in $\mathrm{Aut}(\mathbb{X})$, hence

$$\mathcal{S}(\mathbb{X}) \, = \, \{H \in \mathrm{Aut}(\mathbb{X}) \mid H \circ S = S \circ H\} \, = \, \mathrm{cent}_{\mathrm{Aut}(\mathbb{X})}\langle S\rangle. \tag{1}$$

This is often also called the automorphism group of a shift space [18, 23], written as $\mathrm{Aut}(\mathbb{X}, S)$, but we shall not use this terminology below to avoid misunderstandings with $\mathrm{Aut}(\mathbb{X})$. It is obvious that $\mathcal{S}(\mathbb{X})$ will contain $\langle S\rangle = \{S^n \mid n \in \mathbb{Z}\}$ as a subgroup, where we generally assume $\langle S\rangle \simeq \mathbb{Z}$ as explained earlier. However, $\mathcal{S}(\mathbb{X})$ will typically be much larger than its normal subgroup $\langle S\rangle$. In fact, $\mathcal{S}(\mathbb{X})$ will generally not even be amenable, which is the main reason why $\mathcal{S}(\mathbb{X})$ has not attracted much attention in the past. Yet, under certain geometric or algebraic constraints, $\mathcal{S}(\mathbb{X})$ becomes tractable, which are the cases of most interest to us; see Remark 1 below for more.

Similarly, one defines

$$\mathcal{R}(\mathbb{X}) \, = \, \{H \in \mathrm{Aut}(\mathbb{X}) \mid H \circ S \circ H^{-1} = S^{\pm 1}\}. \tag{2}$$

This is also a subgroup of $\mathrm{Aut}(\mathbb{X})$, called the *reversing symmetry group* of $(\mathbb{X}, S)$, which either satisfies $\mathcal{R}(\mathbb{X}) = \mathcal{S}(\mathbb{X})$ or is an index-2 extension of $\mathcal{S}(\mathbb{X})$; see [35], or [41] and references therein. Due to our general assumption on the faithful action of S, we know that S cannot act as an involution on $\mathbb{X}$. Then, an element of $\mathcal{R}(\mathbb{X})$ that conjugates S into its inverse is called a *reversor*, and $(\mathbb{X}, S)$ is called *reversible*; see [9, 41] for general background. Reversors are always elements of even (or infinite) order, and for one particular class of shifts that we study, they are always involutions; see Corollary 2. Note that the reversing symmetry group is a subgroup of the normaliser of $\mathcal{S}(\mathbb{X})$ in $\mathrm{Aut}(\mathbb{X})$, and agrees with it in the simplest situations. In general, however, it will be a true subgroup, because we do not allow for situations where S is conjugated into some other symmetry of infinite order, say. Nevertheless, for shifts $(\mathbb{X}, S)$ with faithful action, one has $\mathcal{R}(\mathbb{X}) = \mathrm{norm}_{\mathrm{Aut}(\mathbb{X})}\langle S\rangle$; see [9, 41] for more complicated situations.

Remark 1. Recent results on the symmetry groups of rigid shifts extend, in a straightforward manner, to similar results for the reversing symmetry groups. For example, Cyr and Kra [19] show that, for a transitive shift with complexity of subquadratic growth, the (possibly infinite) factor group $\mathcal{S}(\mathbb{X})/\langle S\rangle$ is *periodic*, which means that each element of it is of finite order. If H is a reversor, H^2 is a symmetry, so that, for any transitive subquadratic shift, there exist $n \in \mathbb{N}$ and $k \in \mathbb{Z}$ such that $(H^2)^n = S^k$. Hence, $\mathcal{R}(\mathbb{X})/\langle S\rangle$ is periodic as well. When the complexity of a minimal shift grows at most linearly, $\mathcal{S}(\mathbb{X})/\langle S\rangle$ is finite; see [20, 18, 23]. The key common point in all three proofs is that the set of nontrivial right-asymptotic orbit equivalence classes for shifts of at most linear complexity is *finite*; see below for definitions. These classes are then mapped to each other by a symmetry. In the same vein, a reversor maps nontrivial right-asymptotic orbit equivalence classes to nontrivial left-asymptotic classes, of which there are also only finitely many. $\Diamond$

From now on, we will usually drop the composition symbol for simplicity. For the full shift $\mathbb{X}_{\mathcal{A}}$, the *reflection* R defined by

$$(Rx)_n \, = \, x_{-n} \tag{3}$$

is an involution in $\mathrm{Aut}(\mathbb{X}_{\mathcal{A}})$, and it is easy to check that $RSR = S^{-1}$. Consequently,

$$\mathcal{R}(\mathbb{X}_{\mathcal{A}}) \, = \, \mathcal{S}(\mathbb{X}_{\mathcal{A}}) \rtimes C_2$$

is a semi-direct product by standard arguments [9, 41], where $C_2 = \langle R\rangle$ is the cyclic group of order 2 that is generated by R. This will also be the situation for all shift spaces $\mathbb{X} \subseteq \mathbb{X}_{\mathcal{A}}$ that are invariant under R. Following [4], we call a shift space $\mathbb{X}$ reflection symmetric or *reflection invariant* if $R(\mathbb{X}) = \mathbb{X}$. Note that this property only refers to $\mathbb{X}$ as a (topological) space, and makes no reference to S. The following standard result is an elementary consequence of [9, Lemma 1] and the structure of semi-direct products.

Fact 1. *If $(\mathbb{X}, S)$ is a reflection invariant shift such that S does not act as an involution, the reversing symmetry group satisfies*

$$\mathcal{R}(\mathbb{X}) \, = \, \mathcal{S}(\mathbb{X}) \rtimes C_2$$

with $C_2 = \langle R\rangle$ and R as defined in Eq. (3), which is thus an involutory reversor.

More generally, irrespective of reflection invariance of $\mathbb{X}$, the same structure of $\mathcal{R}(\mathbb{X})$ emerges if $\mathrm{Aut}(\mathbb{X})$ contains an involution that conjugates S into its inverse.

If, in addition, one has $\mathcal{S}(\mathbb{X}) = \langle S \rangle \simeq \mathbb{Z}$, any reversor must be of the form RS^m for some $m \in \mathbb{Z}$, and is thus an involution. $\square$

Let us return to the full shift and ask for other reversors. Clearly, we could also consider another reflection, R', defined by

$$(R'x)_n = x_{-n-1}. \tag{4}$$

This is another involution, related to the previous one by $R' = SR$, which will show up many times later on. More generally, any reversor can be written as GR with R from Eq. (3) and $G \in \mathcal{S}(\mathbb{X}_{\mathcal{A}})$; this is described in the proof of Proposition 1. Now, the full shift is certainly invariant under any map F_α that is induced by applying the same permutation $\alpha \in \varSigma_{\mathcal{A}}$ to each entry of a sequence x,

$$(F_\alpha(x))_n := \alpha(x_n), \tag{5}$$

where $\varSigma_{\mathcal{A}}$ denotes the permutation group of $\mathcal{A}$. We call such a map a *letter exchange map*, or LEM for short. Now, for $\mathbb{X}_{\mathcal{A}}$, we clearly also have the reversors R_α, defined by

$$(R_\alpha(x))_n := \alpha(x_{-n}), \tag{6}$$

which includes our original reversor via $R = R_e$ with e the identity in $\varSigma_{\mathcal{A}}$. Similarly, one may consider $R'_\alpha = F_\alpha R' = SR_\alpha$. Clearly, F_α commutes with both R and R'. Note also that R_α need no longer be an involution, though it is always an element of even order, which is

$$\mathrm{ord}(R_\alpha) = \mathrm{lcm}(2, \mathrm{ord}(\alpha)).$$

These letter permuting reversors form an interesting class of candidates for shifts that fail to be reflection invariant, as we shall see later. Clearly, if R_α is a reversor, then so is $R'_\alpha = SR_\alpha$ as well as $S^m R_\alpha$ for any $m \in \mathbb{Z}$, which all have the same order as R_α.

Fact 1 reveals nothing about the structure of $\mathcal{S}(\mathbb{X})$, but some classic results [16] together with recent progress [42, 18] allow the determination of this group for many interesting cases. Also, Fact 1 indicates that particularly noteworthy instances of reversibility will only emerge for shifts that *fail* to be reflection invariant. So, let us expand a little on reflection invariance.

Recall that a shift $\mathbb{X}$ is called *minimal* when every (two-sided) shift orbit in $\mathbb{X}$ is dense. A minimal shift $\mathbb{X}$ is called *palindromic* when it contains an element x that satisfies $R(x) = x$ (odd core) or $R'(x) = x$ (even core). Palindromicity of $\mathbb{X}$ then implies its reflection invariance. The converse is also true for substitution shift spaces over binary alphabets [50], but false for general shifts [11], and seems still open for substitution shifts [46] on a larger alphabet.

Consider a shift $\mathbb{X}$ that is defined by a primitive substitution rule θ, as the orbit closure of a fixed point of θ^n for some $n \in \mathbb{N}$, say. Note that the notion of a two-sided fixed point x includes the condition that the core or *seed* of x, which is the word $x_{-1}|x_0$, is legal for θ. Since any such fixed point is repetitive, $\mathbb{X}$ is minimal, which is a consequence of Gottschalk's theorem; see [43, 4] for details. Now, we call θ *palindromic* if the shift $\mathbb{X}$ defined by θ is; compare [4, Sec. 4.3]. For the palindromicity of θ, a useful sufficient criterion was formulated in [30, Lemma 3.1]; see also [4, Lemma 4.5].

Fact 2. *Let θ be a primitive substitution on $\mathcal{A} = \{a_1, \dots, a_n\}$, with $\theta(a_i) = pq_i$ for $1 \leqslant i \leqslant n$, where p and all q_i are palindromes, possibly empty. Then, θ is palindromic. Likewise, θ is palindromic if $\theta(a_i) = q_i p$ with p and all q_i being palindromes, possible empty.* □

Let us now recall the notion of the maximal equicontinuous factor, followed by the introduction of some important tools connected with it. To this end, let $(\mathbb{X}, S)$ be a topological dynamical system. The dynamical system $(\mathbb{A}, T)$ is then called the *maximal equicontinuous factor* (MEF) of $(\mathbb{X}, S)$ if $(\mathbb{A}, T)$ is an equicontinuous factor of $(\mathbb{X}, S)$, which means that it is a factor with the action of T being equicontinuous on $\mathbb{A}$, with the property that any other equicontinuous factor $(\mathbb{Y}, R)$ of $(\mathbb{X}, S)$ is also a factor of $(\mathbb{A}, T)$. The MEF of a minimal transformation is a rotation on a compact monothetic topological group [43, Thm. 2.11], which is a group $\mathbb{A}$ for which there exists an element a such that the subgroup generated by a is dense. Such a group is always Abelian and we will thus write the group operation additively.

The setting is that of a commuting diagram,

$$
\begin{array}{ccc}
\mathbb{X} & \xrightarrow{\;S\;} & \mathbb{X} \\
{\scriptstyle\phi}\big\downarrow & & \big\downarrow{\scriptstyle\phi} \\
\mathbb{A} & \xrightarrow{\;T\;} & \mathbb{A}
\end{array}
\tag{7}
$$

with ϕ a continuous surjection that satisfies $\phi \circ S = T \circ \phi$ on $\mathbb{X}$. Also, T is realised as a translation on $\mathbb{A}$, so there is a unique $a \in \mathbb{A}$ such that $T(y) = y + a$ for all $y \in \mathbb{A}$. The MEF can be trivial, but many shifts possess a nontrivial MEF, and if so, we can glean information from it about the symmetry and reversing symmetry groups. This is described by the following theorem, the proof of which is an extension of [18, Thm. 8] to include reversing symmetries.

Theorem 1. *Let $(\mathbb{X}, S)$ be a shift with faithful shift action and at least one dense orbit. Suppose further that the group rotation $(\mathbb{A}, T)$ with dense range is its MEF, with $\phi\colon \mathbb{X} \longrightarrow \mathbb{A}$ the corresponding factor map.*

Then, there is a group homomorphism $\kappa\colon \mathcal{S}(\mathbb{X}) \longrightarrow \mathbb{A}$ such that

$$\phi(G(x)) \, = \, \kappa(G) + \phi(x)$$

holds for all $x \in \mathbb{X}$ and $G \in \mathcal{S}(\mathbb{X})$, so any symmetry G induces a unique mapping $G_{\mathbb{A}}$ on the Abelian group $\mathbb{A}$ that acts as an addition by $\kappa(G)$.

Moreover, if $(\mathbb{X}, S)$ is a reversible shift, there is an extension of κ to a 1-cocycle of the action of $\mathcal{R}(\mathbb{X})$ on $\mathbb{A}$ such that

$$\kappa(GH) \, = \, \kappa(G) + \varepsilon(G)\, \kappa(H) \tag{8}$$

for all $G, H \in \mathcal{R}(\mathbb{X})$, where $\varepsilon : \mathcal{R}(\mathbb{X}) \longrightarrow \{\pm 1\}$ is a group homomorphism with values $\varepsilon(G) = 1$ for symmetries and $\varepsilon(G) = -1$ for reversing symmetries. Thus, ε can be viewed as a homomorphism into $\mathrm{Aut}(\mathbb{A})$ with kernel $\mathcal{S}(\mathbb{X})$.

As a consequence, one has $\kappa(H^2) = 0$ for all $H \in \mathcal{R}(\mathbb{X}) \setminus \mathcal{S}(\mathbb{X})$. Any such reversor H induces a mapping $H_{\mathbb{A}}$ on $\mathbb{A}$ that acts as $z \mapsto \kappa(H) - z$, hence

$$\phi(H(x)) \, = \, \kappa(H) - \phi(x),$$

for all $H \in \mathcal{R}(\mathbb{X}) \setminus \mathcal{S}(\mathbb{X})$ *and all* $x \in \mathbb{X}$.

Proof. Let $x \in \mathbb{X}$ be a point with dense shift orbit, which exists by assumption. Due to the property of the MEF, we know that T in Eq. (7) acts as addition with dense range, so there is an $a \in \mathbb{A}$ such that, for all $z \in \mathbb{A}$, $T(z) = z + a$ with $\{z + na \mid n \in \mathbb{Z}\}$ dense in $\mathbb{A}$.

Fix an arbitrary symmetry $G \in \mathcal{S}(\mathbb{X})$. We need to show that there is a unique induced mapping $G_{\mathbb{A}}$ on the MEF, via the corresponding commuting diagram, and determine what this mapping is. To this end, consider

$$f(x) \,:=\, \phi(Gx) - \phi(x),$$

which is an element of $\mathbb{A}$. Since $\phi(S^n x) = \phi(x) + na$ and $\phi(GS^n x) = \phi(S^n Gx) = \phi(Gx) + na$, one sees that $f(S^n x) = f(x)$ for $n \in \mathbb{Z}$ is the natural extension of f to a function on the orbit of x. For any $y \in \mathbb{X}$, there is a sequence $\bigl(x^{(i)}\bigr)_{i\in\mathbb{N}}$ with $x^{(i)} = S^{n_i} x$ and $x^{(i)} \xrightarrow{i\to\infty} y$, so that $f(y) := f(x)$ is the unique way to extend f to a continuous function on $\mathbb{X}$ with values in $\mathbb{A}$, which is constant in this case. Consequently, the mapping $G_{\mathbb{A}} \colon \mathbb{A} \longrightarrow \mathbb{A}$ induced on the factor by G acts as a translation by

$$\kappa(G) \,:=\, \phi(Gx) - \phi(x).$$

This defines a mapping $\kappa \colon \mathcal{S}(\mathbb{X}) \longrightarrow \mathbb{A}$, with $\kappa(S^n) = na$. The homomorphism property $\kappa(GH) = \kappa(G) + \kappa(H)$ then follows from the concatenation of two commuting diagrams.

Now, if $R \in \mathcal{R}(\mathbb{X}) \setminus \mathcal{S}(\mathbb{X})$ is a reversor, one can modify the derivation, this time leading to

$$\kappa(R) \,:=\, \phi(Rx) + \phi(x)$$

because $\phi(RSx) = \phi(S^{-1}Rx) = \phi(Rx) - \kappa(S) = \phi(Rx) - a$, hence $\phi(RS^n x) + \phi(S^n x) = \phi(Rx) + \phi(x)$ for all $n \in \mathbb{Z}$. As a result, one has $\phi(Ry) = \kappa(R) - \phi(y)$ for all $y \in \mathbb{X}$ as claimed, which means that the action of $R_{\mathbb{A}} : \mathbb{A} \longrightarrow \mathbb{A}$ induced by R is consistently given by $z \mapsto -z + \kappa(R)$. The cocycle property is then a consequence of another concatenation of commuting diagrams for the different possibilities, and $\kappa(R^2) = 0$ follows immediately. □

A closer inspection of the proof shows that Theorem 1 also holds for general equicontinuous factors. However, the MEF is the most useful one, so we will always work with it.

Let us also formulate one important consequence.

Corollary 1. *Let the setting for a shift* $(\mathbb{X}, S)$ *be that of Theorem 1, and define the covering number* $c := \min\{\mathrm{card}(\phi^{-1}(z)) \mid z \in \mathbb{A}\}$. *If* c *is finite, one has that*

(1) *for each* $d > c$, $\{z \in \mathbb{A} \mid \mathrm{card}(\phi^{-1}(z)) = d\} = \{z \in \mathbb{A} \mid \mathrm{card}(\phi^{-1}(z)) = d\} + \kappa(G)$ *holds for any* $G \in \mathcal{S}(\mathbb{X})$,

(2) *for each* $d > c$, $\{z \in \mathbb{A} \mid \mathrm{card}(\phi^{-1}(z)) = d\} = -\{z \in \mathbb{A} \mid \mathrm{card}(\phi^{-1}(z)) = d\} + \kappa(G)$ *holds for each* $G \in \mathcal{R}(\mathbb{X}) \setminus \mathcal{S}(\mathbb{X})$, *and that*

(3) $\kappa \colon \mathcal{S}(\mathbb{X}) \longrightarrow \mathbb{A}$ *and* $\kappa \colon \mathcal{R}(\mathbb{X}) \setminus \mathcal{S}(\mathbb{X}) \longrightarrow \mathbb{A}$ *are each at most* c*-to-one.*

Proof. These claims are an obvious generalisation of [18, Thm. 8]. □

We refer to [18] for details on how to compute c and $\{z \in \mathbb{A} \mid \operatorname{card}(\phi^{-1}(z)) > c\}$ for the important class of shifts that are generated by constant-length substitutions.

Let us next discuss some general tools that we will need in our further considerations. We first recall the classic Curtis–Hedlund–Lyndon (CHL) theorem [39, Thm. 6.2.9] in a version for symmetries of shift spaces, as well as the corresponding one for reversors, whose proof is similar to that of the classical version; see [39] for background.

Proposition 1. *Let $\mathbb{X}$ be a shift over the finite alphabet $\mathcal{A}$, with faithful shift action. For any $G \in \mathcal{S}(\mathbb{X})$, there exist non-negative integers ℓ, r together with a map $h\colon \mathcal{A}^{\ell+r+1} \longrightarrow \mathcal{A}$ such that $(G(x))_n = h(x_{n-\ell}, \dots, x_n, \dots, x_{n+r})$ holds for all $x \in \mathbb{X}$ and $n \in \mathbb{Z}$. Likewise, for any reversor $H \in \mathcal{R}(\mathbb{X}) \setminus \mathcal{S}(\mathbb{X})$, there are non-negative integers ℓ, r and a map h such that $(H(x))_n = h(x_{-n-r}, \dots, x_{-n}, \dots, x_{-n+\ell})$.*

Proof. If $G \in \mathcal{S}(\mathbb{X})$, we know that $G \in \operatorname{Aut}(\mathbb{X})$ and the (horizontally written) diagram

$$\begin{tikzcd} \mathbb{X} \arrow[r, "G"] \arrow[d, "S"'] & \mathbb{X} \arrow[d, "S"] \\ \mathbb{X} \arrow[r, "G"] & \mathbb{X} \end{tikzcd}$$

commutes, so the classic CHL theorem applies and asserts that G is a sliding block map with *local rule*, or *local derivation rule* h.

Now, let H (and hence also H^{-1}) be a reversor. Since the involution R from Eq. (3) need not be in $\operatorname{Aut}(\mathbb{X})$, we define $\mathbb{Y} = R(\mathbb{X})$ and consider the commutative diagram

$$\begin{tikzcd} \mathbb{X} \arrow[r, "R"] \arrow[d, "S^{-1}=H^{-1}SH"'] & \mathbb{Y} \arrow[r, "HR"] \arrow[d, "T"'] & \mathbb{X} \arrow[d, "S"] \\ \mathbb{X} \arrow[r, "R"] & \mathbb{Y} \arrow[r, "HR"] & \mathbb{X} \end{tikzcd}$$

where T is the shift on $\mathbb{Y}$, defined the same way as S. Applying the CHL theorem to $H' = HR$ gives us two integers $\ell, r \geqslant 0$ and some function h' such that $(H'y)_n = h'(y_{n-\ell}, \dots, y_{n+r})$ for all $y \in \mathbb{Y}$. Now, set $h(z_1, \dots, z_{\ell+r+1}) = h'(z_{\ell+r+1}, \dots, z_1)$ and observe that each $y \in \mathbb{Y}$ can be written as $y = Rx$ for some $x \in \mathbb{X}$, wherefore we get

$$\begin{aligned} (Hx)_n \;&=\; \bigl(HR(Rx)\bigr)_n \;=\; \bigl(H'y\bigr)_n \;=\; h'(y_{n-\ell}, \dots, y_{n+r}) \\ &=\; h'(x_{\ell-n}, \dots, x_{-r-n}) \;=\; h(x_{-n-r}, \dots, x_{-n+\ell}) \end{aligned}$$

as claimed. □

In line with [4, Sec. 5.2], if $G \in \mathcal{S}(\mathbb{X})$ has a local derivation rule h that is specified on $(\ell{+}r{+}1)$-tuples of letters in $\mathcal{A}$, we call $\max\{\ell, r\}$ the *radius* of G. Further, if H is a reversor, it can be written as $H = H'R$ with $H' : R(\mathbb{X}) \longrightarrow \mathbb{X}$ some sliding block map. We will frequently use the local rule of H' to investigate H. In the proof of Proposition 1, any involution could have been used, and indeed, sometimes it will be more convenient for us to write H as $H = (HR')R'$ and to work with the the local rule of $H' = HR'$ instead. These two approaches reflect the two possible types of (infinite) palindromes, namely those with odd or with even core.

Let $(\mathbb{X}, S)$ be a minimal two-sided shift, and let $\mathrm{d}_{\mathbb{X}}$ be a metric on $\mathbb{X}$ that induces the local topology on it. Now, we define equivalence relations on orbits in $\mathbb{X}$ as follows, where we write orbits as $\mathcal{O}(x) = \{S^n(x) \mid n \in \mathbb{Z}\}$ for $x \in \mathbb{X}$. Two orbits $\mathcal{O}_x$ and $\mathcal{O}_y$ are *right-asymptotic*, denoted $\mathcal{O}_x \overset{r}{\sim} \mathcal{O}_y$, if there exists an $m \in \mathbb{Z}$ such that $\lim_{n\to\infty} \mathrm{d}_{\mathbb{X}}(S^{m+n}x, S^n y) = 0$. The right asymptotic equivalence class of $\mathcal{O}_x$ is denoted $[x]_r$. A right-asymptotic equivalence class will be called *non-trivial* if it does not consist of a single orbit. Analogously, we define the notion of *left-asymptotic* orbits, with accompanying equivalence relation $\overset{\ell}{\sim}$ and class $[x]_\ell$. The following connection with reversibility is clear.

Fact 3. *Every reversor of a minimal two-sided shift $(\mathbb{X}, S)$ must map any non-trivial right-asymptotic class to a left-asymptotic one of the same cardinality, and vice versa.* $\square$

We know from Theorem 1 that $\kappa(H^2) = 0$ for any reversor H. Moreover, one has the following result.

Corollary 2. *Under the assumptions of Theorem 1, one has the following properties.*

(1) *Any symmetry that is the square of a reversor lies in the kernel of κ.*
(2) *If κ is injective on $\mathcal{S}(\mathbb{X})$, every reversor is an involution.*
(3) *If the MEF of $(\mathbb{X}, S)$ is non-trivial and if $\mathcal{S}(\mathbb{X}) = \langle S \rangle \simeq \mathbb{Z}$, every reversor is an involution.*

Proof. If H is a reversor, H^2 is a symmetry, with $\kappa(H^2) = 0$ by Theorem 1. Any $G \in \mathcal{S}(\mathbb{X})$ with $G = H^2$ for some reversor H must then satisfy $\kappa(G) = 0$.

Now, if κ is injective on $\mathcal{S}(\mathbb{X})$, its kernel is trivial, so $H^2 = \mathrm{Id}$ for any reversor H. Similarly, if $\mathcal{S}(\mathbb{X}) = \langle S \rangle$, one has $H^2 = S^m$ for some $m \in \mathbb{Z}$, hence $0 = \kappa(H^2) = m\kappa(S)$. When the MEF is non-trivial, we know that $\kappa(S) \neq 0$, hence $m = 0$ and $H^2 = \mathrm{Id}$. $\square$

3. Binary shifts

Let us begin with the class of Sturmian shifts over the binary alphabet $\mathcal{A} = \{a, b\}$. Following [17] and [4, Def. 4.16], we call a two-sided sequence $x \in \mathcal{A}^{\mathbb{Z}}$ *Sturmian* if it is nonperiodic, repetitive and of minimal complexity, where the latter means that x has word complexity $p(n) = n + 1$ for all $n \in \mathbb{N}_0$. The corresponding shift $\mathbb{X}_x$ is the orbit closure of x under the shift action, so

$$\mathbb{X}_x = \overline{\{S^n(x) \mid n \in \mathbb{Z}\}}$$

with the closure being taken in the product topology, which is also known as the local topology; compare [4, p. 70]. It is clear from the definition that $\mathbb{X}_x$ is minimal and aperiodic, hence the shift action on it is faithful.

Theorem 2. *Let x be an arbitrary two-sided Sturmian sequence, and $\mathbb{X}_x$ the corresponding shift. Then, $\mathcal{S}(\mathbb{X}_x) = \langle S \rangle \simeq \mathbb{Z}$ is its symmetry group. Moreover, the shift is reversible, with*

$\mathcal{R}(\mathbb{X}_x) = \mathcal{S}(\mathbb{X}_x) \rtimes \langle R \rangle \simeq \mathbb{Z} \rtimes C_2$ and R the reflection from Eq. (3). In particular, all reversors are involutions.

Proof. For any Sturmian sequence x, we know $\mathcal{S}(\mathbb{X}_x) \simeq \mathbb{Z}$ from [42, Sec. 4] or [20, Cor. 5.7]. Moreover, $\mathbb{X}_x$ is *always* palindromic as a consequence of [24, Prop. 6], so we have reflection invariance in the form $R(\mathbb{X}_x) = \mathbb{X}_x$ with R from Eq. (3), and thus, by Fact 1, $\mathcal{R}(\mathbb{X}_x) \simeq \mathbb{Z} \rtimes C_2$ as stated. The last claim also follows from Fact 1. □

Next, we look into binary shifts that are generated by an aperiodic primitive substitution. Recall that a shift is called *aperiodic* if it does not contain any element with a non-trivial period under the shift action. A primitive substitution rule is aperiodic when the shift defined by it is; compare [4, Def. 4.13].

Example 1. Consider the primitive substitution

$$\theta_{\mathrm{F}}: \quad a \mapsto ab,\, b \mapsto a,$$

which is known as the *Fibonacci substitution*; see [4, Ex. 4.6]. More generally, let $m \in \mathbb{N}$ and consider the primitive substitution

$$\theta_m: \quad a \mapsto a^m b,\, b \mapsto a,$$

which is a *noble means substitution*, with $\theta_1 = \theta_{\mathrm{F}}$; compare [4, Rem. 4.7]. Each θ_m is aperiodic and defines a minimal shift $\mathbb{X}_m$, hence with faithful shift action. In fact, any such shift is Sturmian, so Theorem 2 gives us

$$\mathcal{S}(\mathbb{X}_m) \,=\, \langle S \rangle \,\simeq\, \mathbb{Z} \quad \text{and} \quad \mathcal{R}(\mathbb{X}_{\mathrm{F}}) \,=\, \langle S \rangle \rtimes \langle R \rangle \,\simeq\, \mathbb{Z} \rtimes C_2,$$

with R the reversor from Eq. (3). This is the simplest possible situation with reversibility; compare Fact 1 and [9, Thm. 1]. In particular, *all* reversors are involutions.

Let us mention an independent way to establish reversibility via palindromicity. This is clear for θ_{F}, which is palindromic by Fact 2. More generally, one can use [4, Prop. 4.6] and observe that, for each m, the shift $\mathbb{X}_m$ can also be defined by a different substitution θ'_m that is conjugate to θ_m. If $m = 2\ell$, we use $a \mapsto a^{\ell} b a^{\ell}$ and $b \mapsto a$, which satisfies the conditions of Fact 2 with p the empty word. If $m = 2\ell + 1$, we use $a \mapsto a^{\ell+1} b a^{\ell}$ and $b \mapsto a$, so that Fact 2 applies with $p = a$ but q_b empty. Thus, all θ'_m are palindromic and all $\mathbb{X}_m$ reflection invariant.

Finally, let us also mention that the noble means hulls can be described as *regular model sets*; see [4, Rem. 4.7, Sec. 7.1 and Ex. 7.3] for details. The corresponding cut and project schemes are Euclidean in nature, both for the direct and the internal space. As a consequence, our results can alternatively be derived by means of the corresponding *torus parametrisation* [5], which adds a more geometric interpretation of the MEF in these cases. ◊

Let us comment on an aspect of constant length substitutions that can simplify the determination of reversors. We shall use this property later on.

Remark 2. Let θ be a primitive constant-length substitution, of *height* h and *pure base* θ' (see [22, 46] for background), with corresponding shift spaces $\mathbb{X} = \mathbb{X}_\theta$ and $\mathbb{Y} = \mathbb{X}_{\theta'}$. If the height h of θ is nontrivial, which means $h > 1$, one can recover $\mathcal{R}(\mathbb{X}_\theta)$ from $\mathcal{R}(\mathbb{X}_{\theta'})$. This is so because $\mathbb{X}$ can be written as a tower of height h over $\mathbb{Y}$, so $\mathbb{X}$ is conjugate to $\{(y,i) \mid y \in \mathbb{Y}, 0 \leqslant i < h\}$. Now, the arguments of [18, Prop. 10] give the symmetries of $(\mathbb{X}, S)$ and extend to show that any reversor of $(\mathbb{X}, S)$ must be of the form

$$H_i(y,j) := \begin{cases} \big(H'(y), i-j\big), & \text{if } i-j \geqslant 0, \\ \big((T^{-1}H')(x), i-j \bmod h\big), & \text{if } i-j < 0, \end{cases}$$

for some $H' \in \mathcal{R}(\mathbb{Y}) \setminus \mathcal{S}(\mathbb{Y})$ and some $0 \leqslant i < h$. In this sense, it suffices to analyse the height-1 substitutions. $\Diamond$

A height-1 substitution θ of constant length r is said to have *column number* c_θ if, for some $k \in \mathbb{N}$ and some position numbers $(i_1, \ldots, i_k)$, one has $\operatorname{card}\big(\theta_{i_1} \circ \ldots \circ \theta_{i_k}(\mathcal{A})\big) = c_\theta$ with c_θ being the least such number.[1] Here, $\theta_i(a)$ with $1 \leqslant i \leqslant r$ and $a \in \mathcal{A}$ denotes the value of $\theta(a)$ at position i. Note that, if $\mathcal{A}$ is a binary alphabet, one can use $k = 1$ in the above definition. In particular, if θ has column number 1, we will say that θ has a *coincidence*. If θ_i acts as a permutation on $\mathcal{A}$ for each $1 \leqslant i \leqslant r$, we call θ *bijective*. In the latter case, when on a binary alphabet $\{a, b\}$, θ commutes with the letter exchange $\alpha \colon\; a \leftrightarrow b$, wherefore the shift $\mathbb{X}_\theta$ is invariant under the LEM F_α. Since F_α commutes with the shift action, we have $F_\alpha \in \mathcal{S}(\mathbb{X}_\theta)$. The following result by Coven [16] is fundamental. For convenience, we reformulate it in the modern terminology of symbolic dynamics and substitution shifts [46, 4].

Lemma 1. *Let θ be a primitive and aperiodic substitution rule of constant length over a binary alphabet, $\mathcal{A} = \{a, b\}$ say, and let $\mathbb{X}_\theta$ be the unique shift defined by it. If θ has a coincidence, the symmetry group is $\mathcal{S}(\mathbb{X}_\theta) = \langle S \rangle \simeq \mathbb{Z}$. If θ is a bijective substitution, the only additional symmetry is the LEM F_α, so $\mathcal{S}(\mathbb{X}_\theta) = \langle S \rangle \times \langle F_\alpha \rangle \simeq \mathbb{Z} \times C_2$.* $\square$

Let us now turn our attention to substitution shifts outside the Sturmian class. So far, all reversible examples were actually reflection invariant shifts. This is deceptive in the sense that one can have reversibility without reflection invariance.

Example 2. Consider the primitive substitution rule

$$\theta : \quad a \mapsto aaba, \; b \mapsto babb,$$

which is aperiodic, of constant length and height 1; see [22, 46] for background. Since θ has a coincidence (in the second and third position), it defines a strictly ergodic shift $\mathbb{X}_\theta$ with pure point dynamical spectrum. Consequently, $\mathbb{X}_\theta$ is one-to-one over its MEF almost everywhere, and we also have a description as a regular model set; compare Example 6 below. The symmetry group is minimal, $\mathcal{S}(\mathbb{X}_\theta) = \langle S \rangle \simeq \mathbb{Z}$, the latter as a consequence of Lemma 1.

[1] For shifts defined by substitutions of constant length, it follows from [18] that $c_\theta = c$, where c is the covering number defined in Corollary 1.

Now, it is easy to see that $bbaa$ is a legal word, while $aabb$ is not, so $\mathbb{X}_\theta$ fails to be reflection invariant, and neither R from Eq. (3) nor R' from Eq. (4) will be a reversor for $(\mathbb{X}_\theta, S)$. As mentioned above, F_α with α the letter exchange $a \leftrightarrow b$ is an LEM on the full shift. However, F_α does not map $\mathbb{X}_\theta$ into itself, as can be seen from the same pair of words just used to exclude R'.

Recall that R' and F_α commute within $\mathrm{Aut}(\mathbb{X}_\mathcal{A})$, and that $R'_\alpha = F_\alpha R'$ is a reversor for the full shift. Now, θ has 4 bi–infinite fixed points, with legal seeds $a|a$, $a|b$, $b|a$ and $b|b$. One can check that R'_α fixes those with seeds $a|b$ and $b|a$ individually, while permuting the other two. Via a standard orbit closure argument, this implies that R'_α maps $\mathbb{X}_\theta$ into itself, and is an involutory reversor for $(\mathbb{X}_\theta, S)$. We thus get

$$\mathcal{R}(\mathbb{X}_\theta) \, = \, \langle S \rangle \rtimes \langle R'_\alpha \rangle \, \simeq \, \mathbb{Z} \rtimes C_2,$$

which is an example of reversibility with a non-standard involutory reversor, but still with the 'standard' structure of $\mathcal{R}(\mathbb{X}_\theta)$ that we know form Fact 1. In particular, all reversor are once again involutions in this case. ◊

Quite frequently, we need to check whether an LEM generates a symmetry or contributes to a reversing symmetry in the sense of Eq. (6). To this end, the following criterion is handy. Its proof is a generalisation of [18, Props. 26 and 28], and the remark following Corollary 27 in that same article. We use the map κ defined in Theorem 1.

We say that θ is *strongly injective* if θ is injective on letters and does not have any right-infinite fixed points which differ only on their initial entry, or any left-infinite fixed points which differ only on their initial entry. A form of the following lemma for symmetries has been formulated and proved in joint preliminary work by the third author and A. Quas; the authors thank him for the permission to reproduce its proof.

Lemma 2. *Let θ be a primitive substitution of constant length with height 1 and column number c_θ. Suppose that θ is strongly injective. Then, any map $G \in \mathcal{S}(\mathbb{X}_\theta)$ with $\kappa(G) = 0$ must have radius 0, so that, for some permutation $\alpha \in \Sigma_\mathcal{A}$, one has $G = F_\alpha$. Likewise, if $H \in \mathcal{R}(\mathbb{X}_\theta)$ is a reversor with $\kappa(H) = 0$, it must be of the form $G = F_\alpha R'$.*

Moreover, with $\mathcal{A}_\theta^2$ denoting the set of θ-legal words of length 2, a permutation $\alpha \in \Sigma_\mathcal{A}$ generates an LEM $F_\alpha \in \mathcal{S}(\mathbb{X}_\theta)$ if and only if

(1) *the permutation α maps $\mathcal{A}_\theta^2$ to $\mathcal{A}_\theta^2$, and*
(2) $\bigl(\alpha \circ \theta^{c_\theta !}\bigr)(ab) = \bigl(\theta^{c_\theta !} \circ \alpha\bigr)(ab)$ *for each $ab \in \mathcal{A}_\theta^2$.*

Similarly, a permutation α generates a reversor $G = F_\alpha R' \in \mathcal{R}(\mathbb{X}_\theta) \setminus \mathcal{S}(\mathbb{X}_\theta)$ if and only if

(1) *$ab \in \mathcal{A}_\theta^2$ implies $\alpha(ba) \in \mathcal{A}_\theta^2$, and*
(2) $\bigl(\alpha \circ \theta^{c_\theta !}\bigr)(ab) = \bigl(\theta^{c_\theta !} \circ \alpha\bigr)(ba)$ *for each $ab \in \mathcal{A}_\theta^2$.*

Proof. By replacing θ by a power if necessary, we assume that all θ-periodic points are fixed, and that for any letter a, $\theta(a)$ begins (ends) with a letter p such that $\theta(p)$ begins (ends) with p. The interior of a word $w = w_0 w_1 \dots w_{n-1} w_n$, with $n > 1$ to avoid pathologies, is defined as the word $w^{\circ} = w_1 \dots w_{n-1}$. Given any substitution θ, the 'reverse' substitution $\bar{\theta}$ is defined by reflection: If $\theta(a) = p_1 \dots p_k$ for $a \in \mathcal{A}$, then $\bar{\theta}(a) = p_k \dots p_1$. If θ is a primitive, strongly injective substitution of constant length with height 1 and column number c_θ, then so is $\bar{\theta}$. Note that $R' \colon \mathbb{X}_\theta \longrightarrow \mathbb{X}_{\bar{\theta}}$ maps θ-fixed points to $\bar{\theta}$-fixed points.

Suppose that θ satisfies the stated conditions, and let $G \in \mathcal{S}(\mathbb{X}_\theta)$ with $\kappa(G) = 0$. Then, G must send θ-fixed points to θ-fixed points, as follows from [18, Prop. 26] and the remark following Corollary 27 in the same article. Now, let u, v be two distinct fixed points of θ, and suppose that $G(u) = v$. If $G \in \mathcal{S}(\mathbb{X}_\theta)$ has minimal radius > 0, there must be letters c, d and e, with $d \neq e$, and indices $i \neq j$ such that $u_i = c = u_j$ together with $v_i = d$ and $v_j = e$. Due to repetitivity, we may assume that i and j are both positive.

Since u is a fixed point, this implies that, for all $n > 0$, the word $\theta^n(c)$ appears starting at the indices $r^n i$ and $r^n j$ in u, and then the interior of $G(\theta^n(c))$ appears starting at the indices $r^n i + 1$ and $r^n j + 1$ in v. But, since v is a fixed point, $\theta^n(d)$ appears starting at index $r^n i$ in v, while $\theta^n(e)$ appears starting at index $r^n j$ in v. So, for all n, $(\theta^n(d))^{\circ} = (\theta^n(e))^{\circ}$ and we thus have two right-infinite fixed points $u' = d'x$ and $v' = e'x$ which disagree on their initial entry, $d' \neq e'$, and which agree on their right rays starting at index 1. This contradicts our assumption that θ is strongly injective.

If H is a reversor with $\kappa(H) = 0$, it must be of the form $H = H' \circ R'$ with $H' = HR'$ a sliding block map of radius at most 1. This follows by an application of [18, Prop. 32] to the sliding block map $H' \colon \mathbb{X}_{\bar{\theta}} \longrightarrow \mathbb{X}_\theta$. If $\kappa(H) = 0$, then H maps θ-fixed to θ-fixed points, so therefore $H' = HR'$ maps $\bar{\theta}$-fixed points to θ-fixed points. We can now apply our previous argument to HR', with the corresponding conclusion.

Finally, an application of the arguments from the proof of [18, Prop. 28] gives us the remaining statements. □

A minimal shift need not be reversible at all, as we show next.

Example 3. Consider the primitive constant-length substitution

$$\theta : \quad a \mapsto aba,\, b \mapsto baa,$$

which is aperiodic and has a coincidence, so $\mathcal{S}(\mathbb{X}_\theta) = \langle S \rangle \simeq \mathbb{Z}$ by Lemma 1. Since $aaabaa$ is legal while $aabaaa$ is not, we have no reflection invariance of $\mathbb{X}_\theta$, so $R \notin \mathrm{Aut}(\mathbb{X}_\theta)$. As a is twice as frequent as b in any $x \in \mathbb{X}_\theta$, which follows from the substitution matrix by standard Perron–Frobenius theory, F_α (defined as in Example 2) is not in $\mathrm{Aut}(\mathbb{X}_\theta)$, and neither can be $R_\alpha = F_\alpha R$ or $R'_\alpha = F_\alpha R'$. Since θ satisfies the conditions of the Lemma 2, any reversor with κ-value 0 must indeed be of the form R'_α. Hence, there are no reversors with κ-value zero.

Finally, the only possible κ-values for a reversor are integers. To see this, we use Part (2) of Corollary 1. To continue, we employ the MEF of $\mathbb{X}_\theta$, which is the ternary odometer $\mathbb{Z}_3$. The latter can be identified canonically with the ring of 3-adic integers, which, in turn, can be used as internal space in a model set description of $\mathbb{X}_\theta$; compare Example 6 below.

Note that $\{z \in \mathbb{Z}_3 \mid \mathrm{card}(\phi^{-1}(z)) > 1\}$ is the set of points a tail of which lies in $\{0,1\}^{\mathbb{N}}$, and it can be verified that

$$\{z \in \mathbb{Z}_3 \mid \mathrm{card}(\phi^{-1}(z)) > 1\} \,=\, t - \{z \in \mathbb{Z}_3 \mid \mathrm{card}(\phi^{-1}(z)) > 1\}$$

is only possible if t is either eventually 0, or eventually 2, that is, if t is the 3-adic expansion of an integer. So, if any reversor were to exist, there would then also be one with κ-value 0, which we have already excluded.

Likewise, by completely analogous arguments, $\theta: \ a \mapsto aabba$, $b \mapsto babbb$ has a minimal symmetry group. As in the previous example, now with $\mathbb{A} = \mathbb{Z}_5$, one can verify that no reversor R'_α satisfies the conditions of Lemma 2, and one can use Part (2) of Corollary 1 to check that no non-integer κ-value is possible. $\Diamond$

Let us next mention a simple example that is neither Sturmian nor of constant length.

Example 4. The primitive substitution

$$\theta: \quad a \mapsto aab, \; b \mapsto ba,$$

which emerges from the square of the Fibonacci substitution of Example 1 by interchanging the letters in the image of b. It is not Sturmian because $\mathcal{A}^2_\theta = \{aa, ab, ba, bb\}$, so the complexity is non-minimal. The shift has pure point spectrum, so is almost everywhere one-to-one over its MEF, which is an irrational rotation in this case. Like for the Fibonacci shift from Example 1, there is once again a description as a regular model set. An argument similar to the one used for the constant-length cases, using Corollary 1, shows that the symmetry group is the minimal one, $\mathcal{S}(\mathbb{X}_\theta) \simeq \mathbb{Z}$.

Moreover, one can also rule out reversibility, hence $\mathcal{R}(\mathbb{X}_\theta) = \mathcal{S}(\mathbb{X}_\theta)$ in this case. This also shows that the substitution matrix alone, which is the same for this example and for the square of the Fibonacci substitution, does not suffice to decide upon reversibility — the actual order of the letters matters. $\Diamond$

All examples so far are *deterministic*, hence with zero topological entropy. Since shifts of finite type usually have large symmetry groups, it might be tempting to expect the rigidity phenomenon only for shifts of low complexity. However, it is well-known [14] that also shifts with positive entropy can have minimal symmetry groups. Let us add a recent example that emerges from a natural number-theoretic setting.

Example 5. An integer n is called *square-free* if n is not divisible by a non-trivial square, so $1, 2, 3, 5, 6$ are square-free while $4, 8, 9, 12, 16, 18$ are not. Clearly, n is square-free if and only if $-n$ is. The characteristic function of the square-free integers is given by $\mu(|n|)^2$, where μ is the Möbius function from elementary number theory, with $\mu(0) := 0$. Now, take the bi-infinite 0-1-sequence $x_{\mathrm{sf}} = \bigl(\mu(|n|)^2\bigr)_{n\in\mathbb{Z}}$ and define a shift $\mathbb{X}_{\mathrm{sf}}$ via orbit closure. This is the *square-free shift* (or flow), which is known to have pure point diffraction and dynamical spectrum [6, 15]. In fact, $\mathbb{X}_{\mathrm{sf}}$ is an important example of a dynamical system that emerges from a *weak model set* [4, Sec. 10.4]. As such, it is a special case within the larger class of $\mathcal{B}$-free shifts; see [40] and references therein, and [26] for general background.

Now, it is shown in [40] that $\mathbb{X}_{\mathrm{sf}}$ has minimal symmetry group, and it is also clear that $\mathbb{X}_{\mathrm{sf}}$ is reflection invariant (because $R(x_{\mathrm{sf}}) = x_{\mathrm{sf}}$), so we are once again in the standard situation of Fact 1 with $\mathcal{R}(\mathbb{X}_{\mathrm{sf}}) = \langle S \rangle \rtimes \langle R \rangle \simeq \mathbb{Z} \rtimes C_2$. More generally, any $\mathcal{B}$-free shift $\mathbb{X}_{\mathcal{B}}$ is reflection invariant, so we always get $\mathcal{R}(\mathbb{X}_{\mathcal{B}}) = \mathcal{S}(\mathbb{X}_{\mathcal{B}}) \rtimes \langle R \rangle$. We refer to [40] for the general conditions when $\mathcal{S}(\mathbb{X}_{\mathcal{B}}) = \langle S \rangle$ holds. ◊

Let us now proceed with our analysis of some of the classic examples, including cases with mixed spectrum, hence cases which are multiple covers of their MEFs.

Example 6. Consider the primitive substitution rule

$$\theta_{\mathrm{pd}} : \quad a \mapsto ab,\, b \mapsto aa,$$

which is known as the *period doubling* substitution; see [4, Sec. 4.5.1] and references given there. It is aperiodic, has constant length (with height 1) and a coincidence in the first position. So, the corresponding shift $\mathbb{X}_{\mathrm{pd}}$ is minimal and has faithful action. Also, it has pure point dynamical spectrum; compare [46]. It can be described as a regular model set with internal space $\mathbb{Z}_2$, see [4, Ex. 7.4], and thus possesses a torus parametrisation with the compact group $\mathbb{T} = \mathbb{Z}_2/\mathbb{Z}$.

By an application of Fact 2, we know that $\mathbb{X}_{\mathrm{pd}}$ is palindromic, and hence reflection invariant. So, $\mathcal{R}(\mathbb{X}_{\mathrm{pd}}) = \mathcal{S}(\mathbb{X}_{\mathrm{pd}}) \rtimes \langle R \rangle$ by Fact 1. Moreover, also in this case, the symmetry group simply is $\mathcal{S}(\mathbb{X}_{\mathrm{pd}}) = \langle S \rangle \simeq \mathbb{Z}$, again as a result of Lemma 1. Consequently, *all* reversors are involutions. ◊

Example 7. Related to Ex. 6 is the famous Thue–Morse substitution [1, 4]

$$\theta_{\mathrm{TM}} : \quad a \mapsto ab,\, b \mapsto ba.$$

This is a primitive, bijective substitution of constant length, hence we get an extra symmetry of the shift $\mathbb{X}_{\mathrm{TM}}$, namely the LEM F_α from Example 2. Due to the aperiodicity of $\mathbb{X}_{\mathrm{TM}}$, we thus have

$$\mathcal{S}(\mathbb{X}_{\mathrm{TM}}) \;=\; \langle S \rangle \times \langle F_\alpha \rangle \;\simeq\; \mathbb{Z} \times C_2$$

by Lemma 1. Since $\mathbb{X}_{\mathrm{TM}}$ contains a palindromic fixed point of θ_{TM}^2 with seed $a|a$ (even core case), Fact 1 applies and gives $\mathcal{R}(\mathbb{X}_{\mathrm{TM}}) \simeq (\mathbb{Z} \times C_2) \rtimes C_2$, where the generating involutions, F_α and R', commute. We can thus alternatively write the reversing symmetry group as

$$\mathcal{R}(\mathbb{X}_{\mathrm{TM}}) \;\simeq\; (\mathbb{Z} \rtimes C_2) \times C_2 \;=\; D_\infty \times C_2,$$

where D_∞ denotes the infinite dihedral group. Also in this case, *all* reversors are involutions; see [9, Thm. 2 (1)].

Let us also recall that $(\mathbb{X}_{\mathrm{TM}}, S)$ has mixed spectrum [46]. In fact, $\mathbb{X}_{\mathrm{TM}}$ is a globally $2:1$ extension of $\mathbb{X}_{\mathrm{pd}}$, compare [4, Thm. 4.7], and thus a.e. $2:1$ over its MEF, which is a binary odometer in this case, denoted by $\mathbb{Z}_2$. Analogously to our earlier Example 3, it can be viewed as the ring of 2-adic integers. Concretely, the factor map $\phi \colon \mathbb{X}_{\mathrm{TM}} \longrightarrow \mathbb{X}_{\mathrm{pd}}$ is defined as a sliding block map acting on pairs of letters, sending ab and ba to a as well as aa and bb to b. This also implies local derivability in the sense of [4, Sec. 5.2]. The factor map from $\mathbb{X}_{\mathrm{TM}}$ to its MEF is obtained as ϕ followed by the usual factor map from $\mathbb{X}_{\mathrm{pd}}$ to $\mathbb{Z}_2$. $\Diamond$

Let us now discuss two ways to generalise these findings to infinite families of shifts, one over the binary alphabet $\mathcal{A}$ and another for larger alphabets.

The *generalised TM* (gTM) substitution [3] is a substitution in the spirit of [31] defined by

$$\theta_{\mathrm{gTM}}^{(k,\ell)} : \quad a \mapsto a^k b^\ell ,\; b \mapsto b^k a^\ell ,$$

where $k, \ell \in \mathbb{N}$ are arbitrary, but fixed. It is primitive, bijective and of constant length $k+\ell$. Note that $k = \ell = 1$ is the classic TM substitution from Example 7. The corresponding minimal shift $\mathbb{X}_{\mathrm{gTM}}^{(k,\ell)}$ is a topological double cover of the shift $\mathbb{X}_{\mathrm{gpd}}^{(k,\ell)}$ defined by

$$\theta_{\mathrm{gpd}}^{(k,\ell)} : \quad a \mapsto ua ,\; b \mapsto ub$$

with $u = b^{k-1} a b^{\ell-1}$, the latter being known as the *generalised period doubling* (gpd) system [3]. In fact, one may use the factor map ϕ from Example 7, and gets $\mathbb{X}_{\mathrm{gpd}}^{(k,\ell)} = \phi\bigl(\mathbb{X}_{\mathrm{gTM}}^{(k,\ell)}\bigr)$ for all $k, \ell \in \mathbb{N}$. Also, we get the same extra symmetry for gTM that we had in the TM example, namely the LEM F_α, again by Lemma 1. As a generalisation of Example 6, all generalised period doubling systems can be described as regular model sets, with $\mathbb{Z}_{k+\ell}$ as internal space.

Clearly, for $k = \ell$, the shift $\theta_{\mathrm{gpd}}^{(k,\ell)}$ is palindromic by Fact 2, so reversible in the simple form that we know from Fact 1. Since also $\theta_{\mathrm{gTM}}^{(k,\ell)}$ is palindromic in this case, which is easy to see from iterating the rule on the legal seed $a|a$, the situation is analogous to Examples 7 and 6. If $k \neq \ell$, one can apply Lemma 2 to show that $\theta_{\mathrm{gTM}}^{(k,\ell)}$ has no reversors with κ-value 0. All our substitutions $\theta_{\mathrm{gTM}}^{(k,\ell)}$ have column number 2, and since $\{z \in \mathbb{Z}_{k+\ell} \mid \mathrm{card}(\phi^{-1}(z)) > 2\} = \mathbb{Z}$, no $t \notin \mathbb{Z}$ can satisfy $\{z \in \mathbb{Z}_{k+\ell} \mid \mathrm{card}(\phi^{-1}(z)) > 2\} = t - \{z \in \mathbb{Z}_{k+\ell} \mid \mathrm{card}(\phi^{-1}(z)) > 2\}$. Hence, by Corollary 1, $\theta_{\mathrm{gTM}}^{(k,\ell)}$ has no reversors if $k \neq \ell$.

Although $\theta_{\mathrm{gpd}}^{(k,\ell)}$ does not satisfy the conditions of Lemma 2, it is still the case that any reversor with κ-value 0 must have radius 0. By the discussion in the proof of Lemma 2, the radius of a reversor H (meaning that of H' in its representation as $H = H' \circ R'$) is at most one, so that a reversor comprises a local rule that acts on words of length at most 3.

Now, if one of k, ℓ is at least 3, the only legal $\theta_{\mathrm{gpd}}^{(k,\ell)}$-words of length 3 are $\{bbb, bba, bab, abb\}$. By inspection of the shift defined by the substitution $\theta_{\mathrm{gpd}}^{(k,\ell)}$, any reversor (when viewed as a sliding block map with radius 1) must map $b\underline{a}b$ to a and each of $\{b\underline{b}b, b\underline{b}a, a\underline{b}b\}$ to b. Consequently, the reversor has radius 0. Now, we check that the conditions of Lemma 2 are not satisfied, so that there are no reversors with κ-value 0. The situation $\{k, \ell\} = \{1, 2\}$ is similar, except that bbb is not a legal word. As for the generalised Thue–Morse substitutions, we have $\{z \in \mathbb{Z}_{k+\ell} \mid \mathrm{card}(\phi^{-1}(z)) > 2\} = \mathbb{Z}$, so once again $\theta_{\mathrm{gpd}}^{(k,\ell)}$ has no reversors if $k \neq \ell$. Thus we may conclude as follows.

Theorem 3. *For any $k, \ell \in \mathbb{N}$, the generalised period doubling shift $\mathbb{X}_{\mathrm{gpd}}^{(k,\ell)}$ has minimal symmetry group, $\mathcal{S}_{\mathrm{gpd}} = \langle S \rangle \simeq \mathbb{Z}$, while $\mathbb{X}_{\mathrm{TM}}^{(k,\ell)}$ as its topological double cover has symmetry group $\mathcal{S}_{\mathrm{gTM}} = \langle S \rangle \rtimes \langle F_\alpha \rangle \simeq \mathbb{Z} \times C_2$, with the involution F_α being the LEM for $a \leftrightarrow b$.*

When $k = \ell$, both shifts are reversible, with the standard involutory reversor R', so one has $\mathcal{R} = \mathcal{S} \rtimes \langle R' \rangle \simeq \mathcal{S} \rtimes C_2$. Consequently, all reversors are involutions.

When $k \neq \ell$, neither substitution is reversible, thus $\mathcal{R} = \mathcal{S}$ in this case. □

The common feature of the entire family of extensions is that we are always dealing with a topological $2:1$ mapping between two systems. Let us now look at different extensions, for which we profit from a larger alphabet.

4. Larger alphabets

To discuss another type of extension of the TM shift, let $C_N = \{0, 1, \dots, N-1\}$ with addition modulo N denote the cyclic group of order N, and define $\mathcal{A} = \{a_i \mid i \in C_N\}$ as our alphabet. Consider the substitution

$$\theta_N : \quad a_i \mapsto a_i a_{i+1}, \quad \text{for } i \in C_N,$$

where the sum in the index is understood within C_N, thus taken modulo N. This defines a primitive substitution, which results in a periodic sequence (and shift) for $N = 1$, and gives the classic TM system for $N = 2$. We denote the shift for θ_N by $\mathbb{X}_N$, which may be viewed as a cyclic generalisation of the $N = 2$ case. Note that $N = 1$ gives a shift without faithful shift action, and will later be excluded from the symmetry considerations. All other cases are aperiodic.

Consider the cyclic permutation $\pi = (0\,1\,2\,\dots\,N-1)$ of $\mathcal{A}$, which has order N, and let $F_\pi \colon \mathbb{X}_N \longrightarrow \mathbb{X}_N$ denote the induced cyclic mapping (or LEM) defined by $\big(F_\pi(x)\big)_i = \pi(x_i)$. Since $\theta_N \circ \pi = \pi \circ \theta_N$ (with obvious meaning), it is clear that $F_\pi \in \mathrm{Aut}(\mathbb{X}_N)$.

Lemma 3. *For any fixed $N \in \mathbb{N}$, all N^2 words of length 2 are legal for θ_N. Consequently, every $a_i a_j$ with $i, j \in C_N$ occurs in any element of the shift $\mathbb{X}_N$ defined by θ_N.*

Moreover, the map F_π generates a cyclic subgroup of $\mathcal{S}(\mathbb{X}_N)$, with $\langle F_\pi \rangle \simeq C_N$.

Proof. Since $a_{i-1} \mapsto a_{i-1} a_i \mapsto a_{i-1} a_i a_i a_{i+1}$ under the action of θ_N, all $a_i a_i$ are legal. As $a_i a_i \mapsto a_i a_{i+1} a_i a_{i+1}$, also all $a_{i+1} a_i$ are legal. Iterating this argument inductively shows that all $a_{i+2} a_i$ are legal, hence also all $a_{i+k} a_i$ for any $k \in C_N$, which proves our first claim.

The second is a standard consequence of the primitivity of θ_N, which implies minimality of the shift $\mathbb{X}_N$ defined by it as well as the fact that any two elements of $\mathbb{X}_N$ are locally indistinguishable, hence have the same set of subwords.

As F_π obviously commutes with the shift action, the last claim is clear. □

Let us now define a sliding block map φ via $a_i a_j \mapsto a_{i+1-j}$. Since $a_{i+k} a_{j+k} \mapsto a_{i+1-j}$ for any pair (i, j) and every $k \in C_N$, it is not difficult to check that the factor shift $\mathbb{X}'_N := \varphi(\mathbb{X}_N)$

can be generated by the substitution

$$\theta'_N : \quad a_i \mapsto a_0 a_{i+1}, \quad \text{for } i \in C_N ,$$

which is once again primitive. Also, one has a coincidence in the first position, which gives pure point dynamical spectrum and a description as a regular model set by the usual arguments.

Observe that $\varphi \circ F_\pi = \varphi$, which implies that the mapping φ is at least $N : 1$. Now, given an element $x' \in \mathbb{X}'_N$, its entry at position 0, say, can have at most one of N possibilities as preimage under the block map, again by Lemma 3. Pick one of them, and continue to the right by one position, where the local preimage is now uniquely fixed by the previous choice. This argument applies to all further positions to the right, and analogously also to the left, so our initial choice results in a unique preimage $x \in \mathbb{X}_N$. Consequently, x' has at most N preimages, and our previous observation then tells us that φ defines a mapping that is globally $N : 1$.

Note that one has $\mathbb{X}'_1 = \mathbb{X}_1$, which is the excluded periodic system, while $\mathbb{X}'_2$ is the period doubling system from Example 6 with the new alphabet $\{a_0, a_1\}$. More interestingly, the case $N = 3$ is intimately related with the non-primitive 4-letter substitution θ_{G} known from the study of the Grigorchuk group; compare [51] and references therein. In fact, using the alphabet $\{x, a, b, c\}$, one can define the latter as

$$\theta_{\mathrm{G}} : \quad x \mapsto xax,\, a \mapsto b \mapsto c \mapsto a,$$

which is non-primitive, but has a cyclic structure built into it. The one-sided fixed point starting with x has an alternating structure that suggests to use words of length 2 as new building blocks (or alphabet), namely $a_0 = xa$, $a_1 = xb$ and $a_2 = xc$. The resulting substitution rule for the new alphabet is θ'_3, which defines a shift that is conjugate to a higher power (square) shift of the Grigorchuk substitution. This observation also provides a simple path to the pure point spectrum of θ_{G} via a suspension into an $\mathbb{R}$-action. Alternatively, one can use the method from [25] to recode θ_{G} by a 6-letter substitution (with coincidence) that is topologically conjugate to θ_{G} on the level of the $\mathbb{Z}$-action generated by S.

Theorem 4. *Let $N \in \mathbb{N}$ with $N \geqslant 2$ be fixed. The shift $\mathbb{X}'_N$ has minimal symmetry group $\mathcal{S}(\mathbb{X}'_N) = \langle S \rangle \simeq \mathbb{Z}$ and is reversible, with $\mathcal{R}(\mathbb{X}'_N) \simeq \mathbb{Z} \rtimes C_2$. Consequently, all reversors are involutions.*

The cyclic TM shift $\mathbb{X}_N$ is a globally $N : 1$ cover of $\mathbb{X}'_N$, with $\mathcal{S}(\mathbb{X}_N) \simeq \mathbb{Z} \times C_N$. Moreover, $\mathbb{X}_N$ is reflection invariant only for $N = 2$, but reversible for all $N \geqslant 2$, with an involutory reversor and reversing symmetry group $\mathcal{R}(\mathbb{X}_N) \simeq (\mathbb{Z} \times C_N) \rtimes C_2$. Furthermore, also in this case, all reversors are involutions.

Proof. Since θ'_N is of length 2, the MEF is the odometer $\mathbb{Z}_2$, with factor map ϕ. Using the techniques of [18, Sec. 3.4], we see that

$$\{z \in \mathbb{Z}_2 \mid \mathrm{card}(\phi^{-1}(z)) > 1\} \, = \, \mathbb{Z},$$

where we identify $n \in \mathbb{Z}$ with its binary expansion in $\mathbb{Z}_2$. Moreover, one has $\kappa(S) = 1$ in this setting. By part (1) of Corollary 1, we see that $\kappa(H) \in \mathbb{Z}$ must hold for any $H \in \mathcal{R}(\mathbb{X}'_N)$. Since θ'_N has a coincidence, we have $c = 1$, and part (2) of the same corollary tells us that κ is injective on $\mathcal{S}(\mathbb{X}'_N)$, so that the latter is $\langle S \rangle \simeq \mathbb{Z}$ as claimed.

Since θ'_N is palindromic by Fact 2, we have $R(\mathbb{X}'_N) = \mathbb{X}'_N$ and thus reversibility according to Fact 1 with $C_2 = \langle R \rangle$ as before, so the claim on $\mathcal{R}(\mathbb{X}'_N)$ is clear.

For θ_N, the MEF of $(\mathbb{X}_N, S)$ is still $\mathbb{Z}_2$, again with $\{z \in \mathbb{Z}_2 \mid \mathrm{card}(\phi^{-1}(z)) > 1\} = \mathbb{Z}$ and $\kappa(S) = 1$, so that $\kappa(H) \in \mathbb{Z}$ for all $H \in \mathcal{R}(\mathbb{X}_N)$. The difference to above is that $c = N$ for θ_N, and κ is at most N-to-1 on $\mathcal{S}(\mathbb{X}_N)$. As $\kappa^{-1}(0) \cap \mathcal{S}(\mathbb{X}_N)$ contains $\langle F_\pi \rangle$, and hence equals the latter, one has $\mathcal{S}(\mathbb{X}_N) \simeq \mathbb{Z} \times C_N$.

Unlike in the Thue–Morse case, which is $N = 2$, the substitution θ_N is no longer reflection invariant for $N > 2$. To show that $\mathcal{R}(\mathbb{X}_N) \simeq (\mathbb{Z} \times C_N) \rtimes C_2$, it suffices to find one reversor. One can verify that R' composed with the LEM defined by the permutation $\gamma : i \leftrightarrow -i$ satisfies the conditions of Lemma 2, so that R'_γ is a reversor (and then also R_γ). Note that $\gamma \in \Sigma_{\mathcal{A}}$ is an involution ($N > 2$) or the identity ($N = 2$, and also for $N = 1$). Consequently, R_γ maps $\mathbb{X}_N$ into itself, and is an involutory reversor for any $N > 1$. This then shows $\mathcal{R}(\mathbb{X}_N) = \mathcal{S}(\mathbb{X}_N) \rtimes C_2$.

For $N = 2$, all reversors for $(\mathbb{X}_N, S)$ are involutions, as we already saw in Example 7. This remains true for $N > 2$. In fact, with $\pi = (0\,1\,2\,\ldots\,N{-}1)$ from above, one has $\gamma \circ \pi \circ \gamma = \pi^{-1}$ within $\Sigma_{\mathcal{A}}$, which implies

$$(R_\gamma F_\pi)^2 \; = \; F_{\pi^{-1}} F_\pi \; = \; \mathrm{Id}.$$

Since all reversors are of the form $R_\gamma G$ with G a symmetry, where $G = F_\pi^m S^n$ for some $0 \leqslant m < N$ and some $n \in \mathbb{Z}$, a simple calculation now shows the claim. $\square$

So far, most cases of reversibility were based upon the reversor R of Eq. (3), or a related involutory reversor. Let us now look into a classic system that fails to be reflection invariant, but is nevertheless reversible, while displaying a more complicated phenomenon than that encountered in Example 2 or in Theorem 4.

Example 8. Consider the Rudin–Shapiro substitution, formulated as

$$\theta_{\mathrm{RS}} : \quad 0 \mapsto 02, \; 1 \mapsto 32, \; 2 \mapsto 01, \; 3 \mapsto 31$$

over the alphabet $\mathcal{A} = \{0, 1, 2, 3\}$, thus following [46]; see also [4, Sec. 4.7.1]. This substitution rule commutes with the permutation $\beta = (03)(12) \in \Sigma_{\mathcal{A}}$, which thus induces a non-trivial involutory symmetry of the shift as well, denoted by F_β. Indeed, one has

$$\mathcal{S}(\mathbb{X}_{\mathrm{RS}}) \; = \; \langle S \rangle \times \langle F_\beta \rangle \; \simeq \; \mathbb{Z} \times C_2 .$$

To see this, we use Corollary 1 again. Here, as in earlier examples, we have $c = 2$ and $\mathbb{Z}_2$ as MEF. Like before, the range of κ is contained in $\mathbb{Z}$, and κ is at most two-to-one on $\mathcal{R}(\mathbb{X}_{\mathrm{RS}})$. The LEM F_β is then the only additional symmetry, and $\mathcal{S}(\mathbb{X}_{\mathrm{RS}})$ is thus as we claim.

Now, it is known that θ_{RS} fails to be palindromic, compare [4, Rem. 4.12] and references given there, and by the same method one can check that the hull is not reflection invariant either, so that R from Eq. (3) cannot be a reversor here. However, we may consider a letter permuting reversor R_α as defined in Eq. (6), with $\alpha^2 = \beta$. As one can check, $\alpha = (0231)$ as well as $\alpha^3 = (0132)$ are possible, because R_α maps $\mathbb{X}_{\mathrm{RS}}$ into itself. This can be seen as follows. There are 8 legal words of length 2, namely 10, 20, 13 and 23 together with their reflected versions. These four words are the seeds for the four bi-infinite fixed points under θ^2_{RS}, each of which can be used to define $\mathbb{X}_{\mathrm{RS}}$ via an orbit closure. Now, one has

$$\ldots 2|0 \ldots \xmapsto{R'_\alpha} \ldots 2|3 \ldots \xmapsto{R'_\alpha} \ldots 1|3 \ldots \xmapsto{R'_\alpha} \ldots 1|0 \ldots \xmapsto{R'_\alpha} \ldots 2|0 \ldots$$

with $R'_\alpha = SR_\alpha$, which then implies that $\mathbb{X}_{\mathrm{RS}}$ is invariant under R_α, where $R^2_\alpha = F_\beta$.

Putting the pieces together, we thus have reversibility with a reversor of order 4, and by elementary group theory we get

$$\mathcal{R}(\mathbb{X}_{\mathrm{RS}}) \,=\, \langle S \rangle \rtimes \langle R_\alpha \rangle \,\simeq\, \mathbb{Z} \rtimes C_4 .$$

Note that there exists no involutory reversor in this case. In fact, *all* reversors have order 4 here, in line with [9, Thm. 2 (2)]. $\Diamond$

At this point, it should be clear that there is a lot of freedom for reversors with other (even) orders. We leave further examples to the curious reader and turn our attention to higher-dimensional shift actions.

5. Extended symmetries of $\mathbb{Z}^d$-actions

Consider a compact topological space $\mathbb{X}$ under the continuous action of d commuting and independent elements $T_1, \ldots, T_d \in \mathrm{Aut}(\mathbb{X})$, where we again assume *faithfulness* of the action, which means that $\mathcal{G} := \langle T_1, \ldots, T_d \rangle = \langle T_1 \rangle \times \cdots \times \langle T_d \rangle \simeq \mathbb{Z}^d$ as a subgroup of $\mathrm{Aut}(\mathbb{X})$. This gives a topological dynamical system $(\mathbb{X}, \mathbb{Z}^d)$. Clearly, the prime examples we have in mind are the classic $\mathbb{Z}^d$-shift over a finite alphabet $\mathcal{A}$, so $\mathbb{X} = \mathcal{A}^{\mathbb{Z}^d}$ and $T_i = S_i$, and its closed shifts with faithful $\mathbb{Z}^d$-action. Here, S_i is the shift map that acts in the ith direction, so

$$(S_i x)_{\boldsymbol{n}} \,=\, x_{\boldsymbol{n}+\boldsymbol{e}_i} , \tag{9}$$

where $\boldsymbol{n} \in \mathbb{Z}^d$ and $\boldsymbol{e}_i$ denotes the canonical unit vector for the ith coordinate direction.

In any case, with $\mathbb{Z}^d \simeq \mathcal{G} \subset \mathrm{Aut}(\mathbb{X})$ as above, we can now define

$$\mathcal{S}(\mathbb{X}) \,:=\, \mathrm{cent}_{\mathrm{Aut}(\mathbb{X})}(\mathcal{G}) \tag{10}$$

as the *symmetry group* of $(\mathbb{X}, \mathbb{Z}^d)$. As in the one-dimensional case, this group will typically be huge (and not amenable), but under certain rigidity mechanisms it can also be as small as possible, meaning $\mathcal{S}(\mathbb{X}) = \mathcal{G}$. Next, we define the *extended symmetry group* as

$$\mathcal{R}(\mathbb{X}) \,:=\, \mathrm{norm}_{\mathrm{Aut}(\mathbb{X})}(\mathcal{G}), \tag{11}$$

which gives us back the reversing symmetry group for $d = 1$ and faithful $\mathbb{Z}$-action. This is one possible and natural way to extend the concept of reversing symmetries to this multi-dimensional setting. With the definitions of Eqs. (10) and (11), both $\mathcal{G}$ and $\mathcal{S}(\mathbb{X})$ are normal subgroups of $\mathcal{R}(\mathbb{X})$, while the detailed relation between $\mathcal{S}(\mathbb{X})$ and $\mathcal{R}(\mathbb{X})$ needs to be analysed further.

When the shift action is faithful, so $\mathcal{G} \simeq \mathbb{Z}^d$, more can be said about the structure of $\mathcal{R}(\mathbb{X})$. Note that $h\mathcal{G} = \mathcal{G}h$ is only possible when the conjugation action $g \mapsto hgh^{-1}$ sends a set of generators of $\mathcal{G}$ to a (possibly different) set of generators. Since $\mathcal{G}$ is a free Abelian group of rank d by our assumption, its automorphism group is $\mathrm{Aut}(\mathcal{G}) \simeq \mathrm{GL}(d,\mathbb{Z})$, the group of integer $d\times d$-matrices M with $\det(M) \in \{\pm 1\}$. Consequently, there exists a group homomorphism

$$\psi\colon\, \mathcal{R}(\mathbb{X}) \longrightarrow \mathrm{GL}(d,\mathbb{Z}) \tag{12}$$

that is induced by the conjugation action. Clearly, the kernel of ψ is $\mathcal{S}(\mathbb{X})$, thus confirming the latter as a normal subgroup of $\mathcal{R}(\mathbb{X})$.

Let us take a closer look at the full shift, $\mathbb{X}_{\mathcal{A}} = \mathcal{A}^{\mathbb{Z}^d}$.

Lemma 4. *The extended symmetry group of the full shift $\mathbb{X}_{\mathcal{A}}$ is given by*

$$\mathcal{R}(\mathbb{X}_{\mathcal{A}}) \,=\, \mathcal{S}(\mathbb{X}_{\mathcal{A}}) \rtimes \mathcal{H},$$

where $\mathcal{H} \simeq \mathrm{GL}(d,\mathbb{Z})$. In particular, each matrix $M \in \mathrm{GL}(d,\mathbb{Z})$ corresponds to a canonical conjugation with an element from $\mathrm{Aut}(\mathbb{X}_{\mathcal{A}})$.

Proof. Let $M = (m_{ij})_{1\leqslant i,j\leqslant d}$ be a $\mathrm{GL}(d,\mathbb{Z})$-matrix and consider the mapping $x \mapsto h_M(x)$ defined by

$$\bigl(h_M(x)\bigr)_{\boldsymbol{n}} \,:=\, x_{M^{-1}\boldsymbol{n}}, \tag{13}$$

where $\boldsymbol{n} = (n_1, \dots, n_d)^T$ is a column vector and the matrix M^{-1} acts on it as usual. Clearly, h_M is a continuous mapping of $\mathbb{X}_{\mathcal{A}}$ into itself and is invertible, hence $h_M \in \mathrm{Aut}(\mathbb{X}_{\mathcal{A}})$. One also checks that this definition leads to $h_M h_{M'} = h_{MM'}$, whence $\varphi\colon\, \mathrm{GL}(d,\mathbb{Z}) \longrightarrow \mathrm{Aut}(\mathbb{X}_{\mathcal{A}})$ defined by $M \mapsto h_M$ is a group homomorphism. It is now routine to check that, given M, the induced conjugation action $g \mapsto h_M g h_M^{-1}$ on $\mathcal{G}$ sends S_i to $\prod_j S_j^{m_{ji}}$, for any $1 \leqslant i \leqslant d$, so $\varphi(M) \in \mathcal{R}(\mathbb{X}_{\mathcal{A}})$. Note that the iterated conjugation action is consistent with the above multiplication rule for the h_M.

Next, consider $\mathcal{H} := \varphi(\mathrm{GL}(d,\mathbb{Z}))$, which is a subgroup of $\mathcal{R}(\mathbb{X}_{\mathcal{A}})$. Since φ is obviously injective, we have $\mathcal{H} \simeq \mathrm{GL}(d,\mathbb{Z})$. The group $\mathcal{H}$ is canonical in the sense that it does not employ any action on the alphabet, but is constructed only via operations on the coordinates.

Any element $h \in \mathcal{R}(\mathbb{X}_{\mathcal{A}})$ acts, via the above conjugation map, on the generators of $\mathcal{G}$, and this induces a mapping $\psi\colon\, \mathcal{R}(\mathbb{X}_{\mathcal{A}}) \longrightarrow \mathrm{GL}(d,\mathbb{Z}) \simeq \mathcal{H}$, with $h \mapsto \psi(h) = M_h$, which is the group homomorphism from Eq. (12), with kernel $\mathcal{S}(\mathbb{X}_{\mathcal{A}})$. So, $\varphi \circ \psi$ is a group endomorphism of $\mathcal{R}(\mathbb{X}_{\mathcal{A}})$ with kernel $\mathcal{S}(\mathbb{X}_{\mathcal{A}})$ and image $\mathrm{GL}(d,\mathbb{Z})$. Since $\varphi \circ \psi$ acts as the identity on $\mathcal{H}$ by construction, we obtain

$$\mathcal{R}(\mathbb{X}_{\mathcal{A}}) \,=\, \mathcal{S}(\mathbb{X}_{\mathcal{A}}) \rtimes \mathcal{H} \,\simeq\, \mathcal{S}(\mathbb{X}_{\mathcal{A}}) \rtimes \mathrm{GL}(d,\mathbb{Z})$$

as claimed. $\square$

Since $\mathrm{GL}(1,\mathbb{Z}) \simeq C_2$, the case $d = 1$ gives us back our old result from Fact 1. For $d > 1$, the group $\mathrm{GL}(d,\mathbb{Z})$ is infinite, and $\mathcal{R}(\mathbb{X}_{\mathcal{A}})$ is 'big'. Clearly, we will not always see such a big group for shifts, and many cases will actually have an extended symmetry group with *finite* index over $\mathcal{S}(\mathbb{X})$, meaning $[\mathcal{R}(\mathbb{X}) : \mathcal{S}(\mathbb{X})] < \infty$. In this case, only a finite subgroup of $\mathrm{GL}(d,\mathbb{Z})$ will be relevant. This is connected with the classification of (maximal) *finite* subgroups of $\mathrm{GL}(d,\mathbb{Z})$, as considered in crystallography; compare [48]. For instance, in the planar case, one could have the symmetry group of the square, which is isomorphic with D_4, but also the one of the regular hexagon, D_6. For general d, an interesting subcase emerges when the shift $\mathbb{X}$ is such that each element of $\mathcal{R}(\mathbb{X})$ under ψ maps to a signed permutation of the generators of $\mathcal{G}$. For reasons that will become clear shortly, we call such shifts *hypercubic*.

To expand on this, let $\varSigma_d$ denote the permutation group of $\{1,\dots,d\}$ and define (π,ε) with $\pi \in \varSigma_d$ and $\varepsilon \in \{\pm 1\}^d$ as the mapping given by $S_i^{\pm 1} \mapsto S_{\pi(i)}^{\pm\varepsilon_i}$ for $1 \leqslant i \leqslant d$. If compared with our previous description, this means that (π,ε) corresponds to the matrix M with $m_{ij} := \varepsilon_j \delta_{i,\pi(j)}$, which is the standard d-dimensional matrix representation [2] of the hyperoctahedral group W_d. It is straightforward to check that we get an induced multiplication rule for the signed permutations, namely

$$(\sigma,\eta)\circ(\pi,\varepsilon) \;=\; (\sigma\circ\pi, \eta_\pi \varepsilon), \tag{14}$$

where $(\eta_\pi)_i := \eta_{\pi(i)}$ and the product of two sign vectors is componentwise (also known as the Hadamard product). The inverse is given by $(\pi,\varepsilon)^{-1} = (\pi^{-1}, \varepsilon_{\pi^{-1}})$. The following property is well-known; see [2] and references therein.

Fact 4. *For $d \in \mathbb{N}$, there are $2^d d!$ signed permutations. Under the multiplication rule of Eq. (14), they form a group, the wreath product $W_d = C_2 \wr \varSigma_d \simeq \varSigma_d \ltimes C_2^d$, which is also known as the symmetry group of the d-dimensional cube or the hyperoctahedral group.* □

Note that $W_1 \simeq C_2$ and $W_2 \simeq D_4$. The latter is the dihedral group with 8 elements, which has an obvious interpretation as the symmetry group of the square. More generally, the maximal finite subgroups of $\mathrm{GL}(d,\mathbb{Z})$ play a special role in this setting. Indeed, as a consequence of Selberg's lemma, one knows that all subgroups of $\mathrm{GL}(d,\mathbb{Z})$ are virtually torsion-free, which gives the following result.

Fact 5. *A subgroup of $\mathrm{GL}(d,\mathbb{Z})$ is finite if and only if it is periodic, i.e., if and only if every element of it is of finite order. Consequently, every infinite subgroup of $\mathrm{GL}(d,\mathbb{Z})$ contains at least one element of infinite order.* □

Let us pause to mention an important subtlety here. Though we have identified a canonical group $\mathcal{H}$ in Lemma 4 that is isomorphic with $\mathrm{GL}(d,\mathbb{Z})$, it is far from unique. In fact, we could have chosen to include suitable permutations in the alphabet to augment the group elements, thus forming another group, $\mathcal{H}'$ say, that can replace $\mathcal{H}$ in the statement of the group structure. This showed up in Example 2 and in the proof of Theorem 4. It will also become important for our later analysis.

For shifts $\mathbb{X} \subset \mathbb{X}_{\mathcal{A}}$, we are now in an analogous (though more complex) situation to the one-dimensional case. In particular, for hypercubic shifts say, we have to investigate which elements from W_d leave $\mathbb{X}$ invariant, or, if any element fails in this respect, whether some combination with a symmetry of $\mathbb{X}_{\mathcal{A}}$ steps in instead. Here, it may happen that our canonical choice of $\mathcal{H}$ as the representative of W_d fails to capture all extended symmetries, as we saw in previous examples.

In general, the extended symmetry group need *not* be a semi-direct product, as is clear from the existence of non-symmorphic space groups (due to the possibility of glide reflections already for $d = 2$). Still, the most common instance of the extension can be stated as follows, where $\psi \colon \mathcal{R}(\mathbb{X}) \longrightarrow \mathrm{GL}(d, \mathbb{Z})$ is the homomorphism from Eq. (12) for faithful shift actions.

Proposition 2. *Let $\mathbb{X} \subseteq \mathcal{A}^{\mathbb{Z}^d}$ be a closed shift with faithful $\mathbb{Z}^d$-action and symmetry group $\mathcal{S}(\mathbb{X})$. Assume further that $\mathcal{R}(\mathbb{X})$ contains a subgroup $\mathcal{H}$ that satisfies $\mathcal{H} \simeq \psi(\mathcal{H})$ together with $\psi(\mathcal{H}) = \psi(\mathcal{R}(\mathbb{X}))$. Then, the extended symmetry group of $(\mathbb{X}, \mathbb{Z}^d)$ is given as*

$$\mathcal{R}(\mathbb{X}) \, = \, \mathcal{S}(\mathbb{X}) \rtimes \mathcal{H},$$

where the semi-direct product structure is analogous to that of Lemma 4.

Proof. By assumption, we have $\mathcal{G} = \mathbb{Z}^d$, where $\mathcal{G}$ is the group generated by the shift action on $\mathbb{X}$, and $\mathcal{R}(\mathbb{X}) = \mathrm{norm}_{\mathrm{Aut}(\mathbb{X})}(\mathcal{G})$ as in Eq. (11). Clearly, ψ is well-defined, with kernel $\mathcal{S}(\mathbb{X})$.

Let $\mathcal{H}' = \psi(\mathcal{R}(\mathbb{X}))$. By assumption, we have a subgroup $\mathcal{H}$ of $\mathcal{R}(\mathbb{X})$ with $\mathcal{H} \simeq \mathcal{H}' = \psi(\mathcal{H})$. Via composition, we then know that we also have a group homomorphism $\psi' \colon \mathcal{R}(\mathbb{X}) \longrightarrow \mathcal{H}$ such that $\mathcal{S}(\mathbb{X})$ is the kernel of ψ' and that the restriction of ψ' to $\mathcal{H}$ is the identity. This implies the semi-direct product structure as claimed. $\square$

When the shift from Proposition 2 is hypercubic, $\mathcal{H}$ is also isomorphic with a subgroup of W_d. There are several obvious variants of this statement, and one would like to know more about the possible group structures. One tool will be the following generalisation of Theorem 1 to this setting.

Theorem 5. *Let $(\mathbb{X}, \mathbb{Z}^d)$ be a shift with faithful shift action and at least one dense orbit. Let $\mathbb{A}$ be its MEF, with $\phi \colon \mathbb{X} \longrightarrow \mathbb{A}$ the corresponding factor map, where the induced $\mathbb{Z}^d$-action has dense range.*

Then, there is a group homomorphism $\kappa \colon \mathcal{S}(\mathbb{X}) \longrightarrow \mathbb{A}$ such that

$$\phi(G(x)) \, = \, \kappa(G) + \phi(x)$$

holds for all $x \in \mathbb{X}$ and $G \in \mathcal{S}(\mathbb{X})$, so any symmetry G induces a unique mapping $G_{\mathbb{A}}$ on the Abelian group $\mathbb{A}$ that acts as an addition by $\kappa(G)$.

Moreover, if $\kappa(\mathbb{Z}^d)$ is a free Abelian group, there is an extension of κ to a 1-cocycle of the action of $\mathcal{R}(\mathbb{X})$ on $\mathbb{A}$ such that

$$\kappa(GH) \, = \, \kappa(G) + \zeta(G)\big(\kappa(H)\big)$$

for all $G, H \in \mathcal{R}(\mathbb{X})$, where $\zeta \colon \mathcal{R}(\mathbb{X}) \longrightarrow \mathrm{Aut}(\mathbb{A})$ is a group homomorphism. In particular, any $G \in \mathcal{R}(\mathbb{X})$ induces a unique mapping on $\mathbb{A}$ that acts as $z \mapsto \kappa(G) + \zeta(G)(z)$.

Proof. Let us assume that $\mathbb{A} \neq \{0\}$, as the entire statement is trivial otherwise. The claim on the symmetries follows from the same line of arguments employed in the proof of Theorem 1, where we use Eq. (10) with $\mathcal{G} = \mathbb{Z}^d$. If we define $a_i = \kappa(S_i)$ for $1 \leqslant i \leqslant d$, we know that, for any $z \in \mathbb{A}$, the set $\bigl\{ z + \sum_i n_i a_i \mid n_i \in \mathbb{Z} \bigr\}$ is dense in $\mathbb{A}$. In particular, the subgroup $\kappa(\mathbb{Z}^d)$ is dense in $\mathbb{A}$, where $\kappa(\mathbb{Z}^d) \simeq \mathbb{Z}^m$ for some $1 \leqslant m \leqslant d$ by our assumptions.

If R is an extended symmetry, its conjugation action on the shift S_i is once again given by $S_i \mapsto \prod_j S_j^{m_{ji}}$, where $M = \psi(R) \in \mathrm{GL}(d, \mathbb{Z})$ with ψ from Eq. (12). With $S^{\boldsymbol{n}} := \prod_i S_i^{n_i}$, one can check that $R\, S^{\boldsymbol{n}} R^{-1} = S^{M\boldsymbol{n}}$. This implies $\phi(RS^{\boldsymbol{n}}x) = \phi(Rx) + \sum_{i,j} a_i m_{ij} n_j$, which is to be compared with $\phi(S^{\boldsymbol{n}}x) = \phi(x) + \sum_i a_i n_i$.

Under our assumption, $\{a_i \mid 1 \leqslant i \leqslant d\}$ contains a group basis of $\kappa(\mathbb{Z}^d) \simeq \mathbb{Z}^m$. If $m < d$, some of the a_j are 0, without further consequences because $\kappa(\mathbb{Z}^d)$ is torsion-free. We can thus set $\zeta_R(a_i) = \sum_j a_j\, m_{ji}$, which defines an element of $\mathrm{Aut}(\kappa(\mathbb{Z}^d))$ that extends to an element of $\mathrm{Aut}(\mathbb{A})$, which is zero-dimensional (or discrete) in this case. It is easy to check that $G \mapsto \zeta_G$ defines a group homomorphism ζ as claimed.

Now, one can define $\kappa(R) = \phi(Rx) - \zeta_R\bigl(\phi(x)\bigr)$ and check that this is the required extension. The induced action of R on $\mathbb{A}$ is then given by $z \mapsto \kappa(R) + \zeta_R(z)$ as stated. □

Finally, we need a higher-dimensional analogue of Proposition 1.

Proposition 3. *Let $(\mathbb{X}, \mathbb{Z}^d)$ be a shift over a finite alphabet $\mathcal{A}$, with faithful shift action. Then, any $G \in \mathcal{S}(\mathbb{X})$ is realised as a d-dimensional sliding block map of finite radius, which is a local derivation rule in the sense of* [4, Def. 5.6].

Moreover, any $R \in \mathcal{R}(\mathbb{X})$ is of the form $R = H h_M$ with $M = \psi(R)$ and h_M as defined in Eq. (13)*, where $H \colon h_M(\mathbb{X}) \longrightarrow \mathbb{X}$ is again a sliding block map of finite radius.*

Proof. Since $\mathcal{A}$ is finite, the claim for $G \in \mathcal{S}(\mathbb{X})$ is nothing but the CHL theorem for $\mathbb{Z}^d$-shifts. Sliding block maps are a special case of local derivation rules, which can be considered as an extension to the setting of arbitrary patterns in $\mathbb{R}^d$ of finite local complexity.

Next, consider an element $R \in \mathcal{R}(\mathbb{X})$, define $H = R h_{M^{-1}}$ and set $\mathbb{Y} = h_M(\mathbb{X})$. Then, for any $1 \geqslant i \geqslant d$, one obtains the diagram

$$\begin{array}{ccccc}
\mathbb{X} & \xrightarrow{\;h_M\;} & \mathbb{Y} & \xrightarrow{\;H\;} & \mathbb{X} \\
{\scriptstyle R^{-1} S_i R}\big\downarrow & & {\scriptstyle T_i}\big\downarrow & & \big\downarrow{\scriptstyle S_i} \\
\mathbb{X} & \xrightarrow{\;h_M\;} & \mathbb{Y} & \xrightarrow{\;H\;} & \mathbb{X}
\end{array}$$

where T_i is the shift on $\mathbb{Y}$ defined as in Eq. (9). Since $R^{-1} S_i R = \prod_j S_j^{m_{ji}^{-1}}$ via our previously studied conjugation action, with $\psi(h_{M^{-1}}) = M^{-1}$, one can easily check that the entire diagram is indeed commutative. In particular, H intertwines the shift action and must then be a sliding block map as claimed. □

The map H in Proposition 3 is invertible by construction, where both H and H^{-1} are represented by local derivation rules. This means that $\mathbb{X}$ and $h_M(\mathbb{X})$ are mutually locally

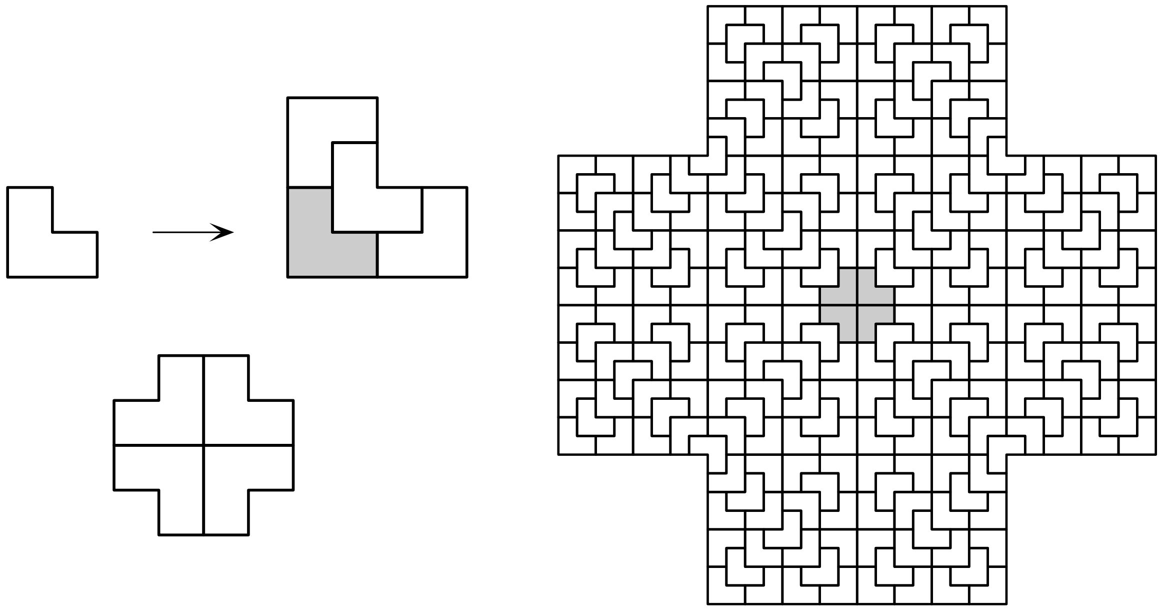


FIGURE 1. The chair inflation rule (upper left panel; rotated tiles are inflated to rotated patches), a legal patch with full D_4 symmetry (lower left) and a level-3 inflation patch generated from this legal seed (shaded; right panel). Note that this patch still has the full D_4 point symmetry (with respect to its centre), as will the infinite inflation tiling fixed point emerging from it.

derivable (MLD) from one another; see [4, Sec. 5.2] for background. This has the following consequence.

Corollary 3. *Under the assumptions of Proposition 3, a matrix $M \in \mathrm{GL}(d, \mathbb{Z})$ is an element of $\psi(\mathcal{R}(\mathbb{X}))$ if and only if $\mathbb{X}$ and $h_M(\mathbb{X})$ are MLD.* □

We are now set to consider two classic and paradigmatic examples.

6. Planar shifts

Let us begin with an example from tiling theory, namely the chair inflation (or substitution) tiling; see [4, Sec. 4.9] and references therein for background. The chair inflation rule, which is primitive and aperiodic, is illustrated in Figure 1. From the D_4-symmetric fixed point depicted there, one defines the geometric hull as the orbit closure under the translation action of $\mathbb{Z}^2$, where we assume that the short edges of the chair prototile have length 1.

To turn this tiling picture into a symbolic coding such that we can view it as a traditional $\mathbb{Z}^2$-shift, we employ the method introduced in [45]; see also [42, 4]. The coding is best

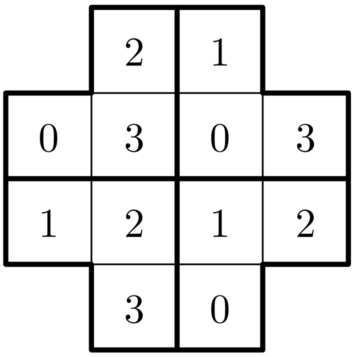
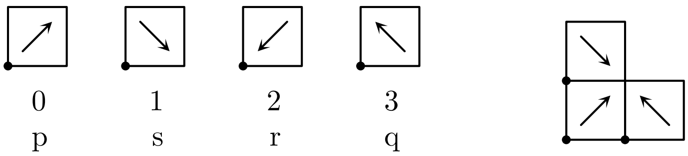


FIGURE 2. The chair tiling seed of Figure 1, now turned into a patch of its symbolic representation via the recoding of Eq. (15). The relation between the purely geometric point symmetries in the tiling picture and the corresponding combinations of point symmetries and LEMs can be seen from this seed.

summarised by an illustration, namely

(15)



where p, q, r, s are the labels from [45, 42], and $0, 1, 2, 3$ those from [4]. Using the latter, the central legal patch from Figure 1 is turned into that shown in Figure 2.

By comparison, one clearly sees that the symbolic representation of the geometric symmetries of the tiling requires the additional use of permutations. Writing the dihedral group in the usual presentation as

$$D_4 \; = \; \langle \, r, s \mid r^4 = s^2 = (rs)^2 = e \, \rangle$$

with r a counterclockwise rotation through $\pi/2$ and s the reflection in the y-axis, its counterpart for the symbolic description is generated by

$$\bigl(r, (0321)\bigr) \quad \text{and} \quad \bigl(s, (03)(12)\bigr).$$

All other pairings follow from here, or can be read off from Figure 2. In fact, if one wants to use a version of the group that fixes the configuration of this figure, one needs to add an application of S_1 from the left, both for r and for s, to make up for the shifted positions of the symbolic labels relative to the tile boundaries of the tiling representation. This is reminiscent of the 1D situation of odd core versus even core palindromes.

Theorem 6. *The $\mathbb{Z}^2$-shift $\mathbb{X}$ defined by the chair tiling is hypercubic and has pure point spectrum. Moreover, it has minimal symmetry group, $\mathcal{S}(\mathbb{X}) = \mathbb{Z}^2$, together with the maximally possible group of extended symmetries, which is $\mathcal{R}(\mathbb{X}) = \mathbb{Z}^2 \rtimes \mathcal{H}$ with $\mathcal{H} \simeq W_2 \simeq D_4$.*

Before we prove this result in the spirit of our earlier arguments based on the MEF of the system and on its description as a regular model set, let us use the geometric setting to explain why one should expect the above claims on the symmetries. Observe first that any element of $\mathcal{R}(\mathbb{X})$ must respect the tiling nature, so it must be a homeomorphism of the plane that maps tiles onto tiles and, in particular, cannot deform any of them. This is sufficiently rigid to rule out anything beyond Euclidean motions. Next, since the geometric symmetries of the fixed point tiling of Figure 1 agree with that of a perfect square, which is a maximal finite subgroup of $\mathrm{GL}(2,\mathbb{Z})$, we may conclude that the shift is hypercubic (with $d=2$).

Proof of Theorem 6. The claim on the symmetry group is [42, Thm. 3.1], and applies to both representations of the chair tiling system due to the invariance of $\mathcal{S}(\mathbb{X})$ under topological conjugacy. Let us briefly recall the argument for convenience.

Robinson shows [45] that $\mathbb{X}$ is an almost everywhere one-to-one extension of its MEF, canonically chosen as $\mathbb{A} = \mathbb{Z}_2 \times \mathbb{Z}_2$, so we have $c=1$ for the corresponding factor map ϕ. He also shows that non-singleton fibres over $\mathbb{A}$ have cardinality either 2 or 5, which is not difficult to derive from the inflation rule. The chair tiling can also be described as a regular model set with internal space $\mathbb{Z}_2 \times \mathbb{Z}_2$; compare [4, Secs. 4.9 and 7.2]. As such, it can be considered as a two-dimensional analogue of the period doubling system from Example 6.

To show that $\mathbb{X}$ has a trivial symmetry group, we will use the 5-fibres, which are the preimages of $\mathbb{Z}^2 \subset \mathbb{A}$ under ϕ. Here, we may profit from the use of the torus parametrisation, with the 'torus' $\mathbb{T} = \mathbb{A}/\mathbb{Z}^2 \simeq (\mathbb{Z}_2/\mathbb{Z})^2$, which is a compact Abelian group. The advantage of it is that we deal with entire shift orbits in $\mathbb{X}$ at once rather than with single elements. Here, any $\mathbb{Z}^2$-orbit of an element of $\mathbb{X}$ is represented by a point of $\mathbb{T}$, which defines a continuous, a.e. one-to-one mapping $\varPhi \colon \mathrm{orb}(\mathbb{X}) \longrightarrow \mathbb{T}$, with $\varPhi = \chi \circ \phi$, where $\mathrm{orb}(\mathbb{X})$ is the space of $\mathbb{Z}^2$-orbits in $\mathbb{X}$ and $\chi \colon \mathbb{A} \longrightarrow \mathbb{T}$ is the natural homomorphism to the factor group $\mathbb{T}$. In $\mathbb{X}$, we have one orbit of fibres with cardinality 5, whose representative on $\mathbb{T}$ is 0. Invoking the obvious analogue of the first claim of Corollary 1 to the two-dimensional case, we see that $\kappa(\mathcal{S}(\mathbb{X}))$ must be in the kernel of χ, which means $\kappa(\mathcal{S}(\mathbb{X})) = \mathbb{Z}^2$. Since κ is injective due to $c=1$, the symmetry claim follows.

Next, $\mathbb{X}$ is the orbit closure of a configuration (or tiling) with full D_4 symmetry, where the latter is the group discussed above. It is easy to check that this implies D_4 to be a subgroup of $\varPsi := \psi(\mathcal{R}(\mathbb{X})) \subseteq \mathrm{GL}(2,\mathbb{Z})$, and a simple calculation confirms that $\mathcal{R}(\mathbb{X})$ indeed contains a subgroup of the form $\mathbb{Z}^2 \rtimes \mathcal{H}$ with $\mathcal{H} \simeq D_4$. In the tiling representation, we can directly work with the linear maps defined by the corresponding matrices, while the symbolic representation needs the version with additional permutations on the alphabet as explained earlier.

It thus remains to show that $\varPsi$ contains no further element. Let us assume the contrary, meaning $D_4 \subsetneq \varPsi$. Since D_4 is a maximal finite subgroup of $\mathrm{GL}(2,\mathbb{Z})$, we may conclude that $\varPsi$ would then be an infinite subgroup, so must contain at least one element of infinite order by Fact 5. The latter cannot be elliptic (which means diagonalisable with eigenvalues on the unit circle), so must be parabolic or hyperbolic. Every parabolic element, up to conjugacy in $\mathrm{GL}(2,\mathbb{Z})$, is of the form $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right)$ for some $0 \neq n \in \mathbb{Z}$, while any hyperbolic element has irrational eigenvalues, with one expanding and one contracting eigenspace, neither of which is a lattice direction. When $M \in \mathrm{GL}(2,\mathbb{Z})$ is parabolic or hyperbolic, we have $h_M(\mathbb{X}) \neq \mathbb{X}$, because h_M cannot preserve (as a set) any pair of orthogonal directions (such as $(1,0)$ and $(0,1)$ or the diagonal and the anti-diagonal), and hence results in a deformation of the tiling (or a 'reshuffling' of the symbolic shift) that defines a different hull. In fact, due to minimality, one has the stronger property that $h_M(\mathbb{X}) \cap \mathbb{X} = \varnothing$.

To continue, we employ the 2-fibres. Any extended symmetry must map the set of 2-fibres onto itself. Each 2-fibre consists of a pair of tilings that agree everywhere except along one line, which is either parallel to the main diagonal or to the anti-diagonal. The only difference between the two elements in a 2-fibre is the direction of the 'stacked' chairs [45]. Now, each $\mathbb{Z}^2$-orbit of 2-fibres (of which there are uncountably many) is parametrised by a point on $\mathbb{T}$ of the form (x,x) or $(x,-x)$. They define two 'lines' (or 'directions') in $\mathbb{T}$. This parametrising set is D_4-invariant, as it must be, and D_4 is the stabiliser of this set in $\mathrm{GL}(2,\mathbb{Z})$, which we will employ shortly. Moreover, this structure remains unchanged under an arbitrary MLD rule, which means that both $\mathbb{T}$ and the directions on $\mathbb{T}$ stay the same.

In contrast, our transformations h_M will change these lines, both in real space and on $\mathbb{T}$. If M is parabolic, it can preserve one direction (but then not the other, due to the action as a shear) or it can map one direction to the other (but then the latter to a new one). This second situation can also occur for M hyperbolic. In any case, we know that at least one of our two directions is mapped to a *new* one. If we write our hypothetical extended symmetry as $H \circ h_M$, with H a local derivation rule, we thus know that there is a 2-fibre with two elements that agree anywhere off a line $\mathcal{L}$ and that get mapped by H to two tilings which agree off a line $\mathcal{L}'$ with a direction that is neither parallel to the diagonal nor to the anti-diagonal, which is not possible. This shows that $\mathbb{X}$ and $h_M(\mathbb{X})$ cannot be MLD. Now, Corollary 3 tells us that M cannot be an element of $\varPsi$, and we are done via Proposition 2. $\square$

The majority of our previous examples were minimal shifts. It follows from Example 5 and from [14, 34] that dynamical minimality is not necessary for a minimal symmetry group. In fact, also $\mathbb{Z}^d$-shifts of algebraic origin quite often display the necessary rigidity to exclude extra symmetries, irrespective of complexity or entropy; compare [12, 13].

Let $\mathbb{X}_{\mathrm{L}} \subset \{0,1\}^{\mathbb{Z}^2}$ denote Ledrappier's shift [37], which is defined by the condition that

$$x_{(m,n)} + x_{(m+1,n)} + x_{(m,n+1)} \equiv 0 \mod 2 \tag{16}$$

holds for all $x \in \mathbb{X}_{\mathrm{L}}$ and all $(m,n) \in \mathbb{Z}^2$; see Figure 3 for a geometric illustration of this condition. To expand on this, we call a lattice triangle in $\mathbb{Z}^2$ with vertices $\{(m,n),(m',n'),(m'',n'')\}$ an *L-triangle* if

$$x_{(m,n)} + x_{(m',n')} + x_{(m'',n'')} \equiv 0 \mod 2$$

holds for all $x \in \mathbb{X}_{\mathrm{L}}$. By Eq. (16), every triangle of the form $\{(m,n),(m+1,n),(m,n+1)\}$, which we call *elementary* from now on, is an L-triangle. However, there are more L-triangles. We state the well-known situation here without proof as follows.

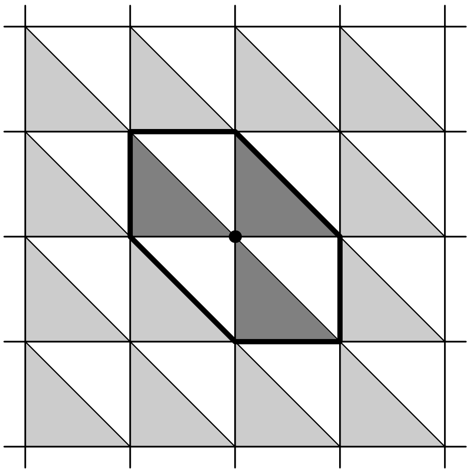


FIGURE 3. Illustration of the central configurational patch for Ledrappier's shift condition, which explains the relevance of the triangular lattice. Eq. (16) must be satisfied for the three vertices of all elementary L-triangles (shaded). The overall pattern of these triangles is preserved by all (extended) symmetries. The group D_3 from Theorem 7 can now be viewed as the colour-preserving symmetry group of the 'distorted' hexagon as indicated around the origin.

Fact 6. *A lattice triangle in $\mathbb{Z}^2$ is an L-triangle if and only if it is of the form*

$$\big\{(m,n),(m+2^i,n),(m,n+2^i)\big\}$$

for some $m,n \in \mathbb{Z}$ and $i \in \mathbb{N}_0$. In particular, the only L-triangles with area $\frac{1}{2}$ are the elementary L-triangles. $\square$

To rephrase Eq. (16), one can say that $\mathbb{X}_{\mathrm{L}}$ is the largest subset of the full shift that is annihilated by the mapping defined by $N := 1 + S_1 + S_2$, where $x + y$ means pointwise addition modulo 2. Under this addition rule, in contrast to most of our previous examples, $\mathbb{X}_{\mathrm{L}}$ becomes a compact, zero-dimensional (or discrete) Abelian group; see [47] for background. Here, N is an endomorphism of $\mathbb{X}_{\mathrm{L}}$, and we have

$$\mathbb{X}_{\mathrm{L}} = \ker(N).$$

Moreover, $\mathbb{X}_{\mathrm{L}}$ is not minimal, and has rank-1 entropy (but vanishing topological entropy). Note that $\mathbb{X}_{\mathrm{L}}$ is not aperiodic in our above (topological) sense, though it is still measure-theoretically aperiodic [4, Def. 11.1], meaning that all periodic or partially periodic elements together are still a null set for the Haar measure of $\mathbb{X}_{\mathrm{L}}$. In particular, the $\mathbb{Z}^2$-action on $\mathbb{X}_{\mathrm{L}}$ is faithful and mixing [37]; this can also be derived from [47, Thm. 6.5]. The shift action is *irreducible* in the sense that any closed and shift invariant true subgroup of $\mathbb{X}_{\mathrm{L}}$ is finite; compare [34, Sec. 4 and Ex. 4.3]. Finally, let us also mention that the spectrum is of mixed type, with pure point and absolutely continuous components; see [10] and references therein.

Theorem 7. *Ledrappier's shift $\mathbb{X}_{\mathrm{L}}$ has symmetry group $\mathcal{S}(\mathbb{X}_{\mathrm{L}}) = \mathbb{Z}^2$. The group of extended symmetries is $\mathcal{R}(\mathbb{X}_{\mathrm{L}}) = \mathbb{Z}^2 \rtimes D_3$, where $D_3 \simeq C_3 \rtimes C_2$ is the dihedral group with 6 elements.*

Geometrically, as we shall see, any extended symmetry must map elementary L-triangles to triangles of the same type. The square lattice representation is thus a 'red herring' in this case, and a more natural representation would use the triangular lattice instead. This means that the relevant maximal subgroup of $\mathrm{GL}(2,\mathbb{Z})$ for Ledrappier's shift is D_6 rather than D_4; see Figure 3 for an illustration.

Remark 3. The cyclic groups are generated by the rotation $G_3 = \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ of order 3 and by the reflection $R_{12} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ in the space diagonal. The induced conjugation action of G_3 on $\mathcal{S}(\mathbb{X}_{\mathrm{L}})$ conjugates S_1 and S_2 into $S_1^{-1} S_2$ and S_1^{-1}, respectively. R_{12} is the involution that interchanges the coordinates, and thus conjugates S_1 into S_2 and vice versa. $\Diamond$

Proof of Theorem 7. The statement on the symmetry group is a consequence of [34, Thm. 1.1 and Sec. 4], when applied to the conjugacies between $(\mathbb{X}_{\mathrm{L}}, \mathbb{Z}^2)$ and itself; see [47, Thm. 31.1 and Cor. 31.3] for a detailed derivation in this case. Indeed, any symmetry G has to be affine and has to fix $\underline{0}$, the all-zero configuration (which is also the neutral element of $\mathbb{X}_{\mathrm{L}}$ as a group), wherefore G must be a group homomorphism. The only possibility then is that $G = S^{\boldsymbol{n}} = \prod_i S_i^{n_i}$ for some $\boldsymbol{n} \in \mathbb{Z}^2$, as shown in [47, Sec. 31]. In particular, the local condition (16) is *not* invariant under the exchange $0 \leftrightarrow 1$ in a configuration.

Now, it is a straightforward to verify that the mapping h_M preserves the pattern of elementary L-triangles if and only if M is one of the six matrices

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

They form a subgroup of $\mathrm{GL}(2,\mathbb{Z})$ of order 6, namely $D_3 = C_3 \rtimes C_2$; see also Figure 3. For each such M, the mapping $h_M = \varphi(M)$ is an element of $\mathcal{R}(\mathbb{X}_{\mathrm{L}})$ with $\psi(h_M) = M$, so we see that the group $\mathbb{Z}^2 \rtimes D_3$ is certainly a subgroup of $\mathcal{R}(\mathbb{X}_{\mathrm{L}})$.

Let us next show that D_6, the unique extension of D_3 to a maximal finite subgroup of $\mathrm{GL}(2,\mathbb{Z})$, is *not* contained in $\psi(\mathcal{R}(\mathbb{X}_{\mathrm{L}}))$. Since $\left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$ has precisely two roots in $\mathrm{GL}(2,\mathbb{Z})$, namely $\pm M$ with $M = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$, it suffices to exclude M. Thus, consider $\mathbb{Y} := h_M(\mathbb{X}_{\mathrm{L}})$, which is the shift space defined by

$$y_{(m,n)} + y_{(m+1,n)} + y_{(m+1,n-1)} \equiv 0 \bmod 2$$

for all $m,n \in \mathbb{Z}$, or by $\mathbb{Y} = \ker(1 + S_1 + S_1 S_2^{-1})$. The two shift spaces differ as follows. In $\mathbb{X}_{\mathrm{L}}$, when we know $\{x_{(k,\ell)} \mid k \in \mathbb{Z}\}$ for any (fixed) $\ell \in \mathbb{Z}$, all $x_{(m,\ell+n)}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ are uniquely determined. In contrast, in $\mathbb{Y}$, the knowledge of $\{y_{(k,\ell)} \mid k \in \mathbb{Z}\}$ determines all $y_{(m,\ell-n)}$ with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. In other words, the knowledge of a configuration along a horizontal line determines everything above (in $\mathbb{X}_{\mathrm{L}}$) or below this line (in $\mathbb{Y}$).

Let us now assume that an invertible local derivation rule $H\colon \mathbb{Y} \longrightarrow \mathbb{X}_{\mathrm{L}}$ exists. Running it along a fixed horizontal line (for $y \in \mathbb{Y}$) specifies the image (in $\mathbb{X}_{\mathrm{L}}$) along this line, and thus (implicitly) also the configuration in the entire half plane above it. Due to the finite radius (r say) of H, this configuration would be the same for all $y \in \mathbb{Y}$ with the same configuration in a strip of width $2r$ around the chosen horizontal line. Moreover, the values of x below the chosen horizontal line are effectively determined by the values of y in this strip of finite width — irrespective of the values of y at places above the strip. Since there are certainly distinct elements in $\mathbb{Y}$ that agree in this strip, we see that H cannot be invertible and a local derivation rule at the same time, in contradiction to our assumption.

Next, if $\psi(\mathcal{R}(\mathbb{X}_{\mathrm{L}}))$ contains any element beyond D_3, we must also have at least one element of infinite order, by an application of Fact 5. As in our previous arguments, such an element must be parabolic or hyperbolic. A refined version of our above argument will exclude such elements. Here, one can argue with a *sector* rather than with a half plane. Corresponding to the three 'fundamental' directions defined by the sides of the elementary L-triangles, there are three types of index sectors. If the tip is in the origin, one has

$$\{(m,n) \mid m,n \geqslant 0\},\ \{(m,-n) \mid n \geqslant m \geqslant 0\},\ \text{and}\ \{(-m,n) \mid m \geqslant n \geqslant 0\}, \tag{17}$$

each with the property that the configuration inside the sector is completely determined by the configuration on one of its boundary lines.

Now, for any $M \in \mathrm{GL}(2,\mathbb{Z})$, the shift space $\mathbb{Y} = h_M(\mathbb{X}_{\mathrm{L}})$ satisfies

$$\mathbb{Y} = \ker\bigl(1 + S_1^{m_{11}} S_2^{m_{21}} + S_1^{m_{12}} S_2^{m_{22}}\bigr),$$

as one can derive from the conjugation action on the shift generators. When M is not of finite order, one has $\mathbb{Y} \neq \mathbb{X}_{\mathrm{L}}$, because the elementary triangles for $\mathbb{Y}$ are then different from the L-triangles. Indeed, they are given by the lattice translates of the triangle with vertices $\{(0,0),(m_{11},m_{21}),(m_{12},m_{22})\}$ and thus are all of the same shape. Like the elementary L-triangles, they have area $\frac{1}{2}$. Note that no lattice point other than the three vertices can lie on any of the sides of an elementary $\mathbb{Y}$-triangle, and no lattice point can lie in its interior (one way to see this employs Pick's formula for the area of a lattice triangle).

If one side of an elementary $\mathbb{Y}$-triangle is parallel to a side of an elementary L-triangle, these sides must be congruent because there cannot be a lattice point in the interior of this side. Now, we can either repeat our previous argument for a half space (if the two triangles extend to opposite directions, as above) or we can resort to one of the three sectors from Eq. (17). In the latter case, since the two elementary triangles cannot be congruent, the selected sector for $\mathbb{Y}$ will differ from that for $\mathbb{X}_{\mathrm{L}}$, but they share one boundary line. If it is smaller (larger), due to the assumed local nature of the derivation rule H, we find ourselves in the situation that the image configuration is underdetermined (overdetermined), in contradiction to the invertibility of H. We leave the details of this argument to the reader.

It remains to consider the case that no side of en elementary $\mathbb{Y}$-triangle is parallel to that of an elementary L-triangle. If a $\mathbb{Y}$-sector overlaps with an $\mathbb{X}_{\mathrm{L}}$-sector, one boundary line of each sector will (except for $(0,0)$) lie in the interior of the other, and we may argue once more that our hypothetical local derivation rule H is overdetermined and thus cannot be invertible. In the remaining case, the $\mathbb{Y}$-sectors (except for the point $(0,0)$ again) lie entirely inside the complement of the sectors from Eq. (17). In this case, the values of $y \in \mathbb{Y}$ in the positive quadrant are *not* determined from the values on $\{(m, 0) \mid m \geqslant 0\}$, while the values of $x = Hy$ must be. If H has finite radius and is invertible, this is impossible.

Put together, this rules out any additional element beyond D_3 in $\psi(\mathcal{R}(\mathbb{X}_{\mathrm{L}}))$, wherefore we are in the situation of Proposition 2, which proves our claim. $\square$

It is clear that there are many more examples with small symmetry groups and interesting extended symmetry groups. Also, for $d \geqslant 2$, one can have the situation that $\mathcal{S}(\mathbb{X}) \simeq \mathbb{Z}^d$ is paired with $\mathcal{R}(\mathbb{X}) = \mathcal{S}(\mathbb{X})$ or with $\mathcal{R}(\mathbb{X}) \simeq \mathcal{S}(\mathbb{X}) \rtimes \mathrm{GL}(d, \mathbb{Z})$, as will be shown in a forthcoming publication. It remains an interesting open question to what extent the analysis of extended symmetries can contribute to some partial structuring of the class of $\mathbb{Z}^d$-shifts.

Acknowledgements

It is our pleasure to thank David Damanik, Franz Gähler, Mariusz Lemańczyk, Klaus Schmidt, Anthony Quas and Luca Zamboni for useful discussions. This work was supported by the Australian Research Council (ARC) and also by the German Research Foundation (DFG), within the CRC 701. MB would like to thank the Erwin Schrödinger Institute (ESI) in Vienna for hospitality, where this manuscript was completed.

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