

Symmetries and reversing symmetries of polynomial automorphisms of the plane

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Abstract

The polynomial automorphisms of the affine plane over a field K form a group which has the structure of an amalgamated free product. This well-known algebraic structure can be used to determine some key results about the symmetry and reversing symmetry groups of a given polynomial automorphism.

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1. Introduction

In a series of recent articles [5, 6, 27, 32], the symmetries and reversing symmetries of some dynamical systems (automorphisms) have been investigated systematically by means of algebraic methods. An automorphism L of some space is said to have a *symmetry* if there exists an automorphism S that satisfies

$$S \circ L \circ S^{-1} = L \quad (1)$$

and a *reversing symmetry*, or *reversor*, if there exists an automorphism R so that

$$R \circ L \circ R^{-1} = L^{-1}. \quad (2)$$

The set of symmetries is non-empty (it certainly contains all powers of L) and this set is actually a group, the *symmetry group* $\mathcal{S}(L)$. On the other hand, the existence *a priori* of any reversing symmetries for a particular L is unclear. When L has a reversing symmetry, we call it *reversible*, and *irreversible* otherwise. The set $\mathcal{R}(L)$ of all symmetries and reversing symmetries of L is a group, too, called the *reversing symmetry group* [21] of L (see also [14]). In particular, $\mathcal{R}(L)$ admits a binary grading: the composition of two reversing symmetries is a symmetry, whereas the composition of a symmetry and a reversing symmetry is a reversing symmetry. If L is irreversible or if L is the identity or an involution (i.e. if $L^2 = 1$), one has $\mathcal{R}(L) = \mathcal{S}(L)$; otherwise, $\mathcal{R}(L)$ is a group extension of $\mathcal{S}(L)$ of index 2.

The simultaneous consideration of symmetries and reversing symmetries of reversible automorphisms (which may arise as the time-one maps of reversible flows) is now known

to provide some powerful algebraic insights. For example, the results of [14, 21] illustrate that much can be said about the nature of possible reversing symmetries in $\mathcal{R}(L)$ given the knowledge of the structure of $\mathcal{S}(L)$. For example, if L (with $L^2 \neq 1$) has an involutory reversor R (i.e. $R^2 = 1 \neq R$), one has $\mathcal{R}(L) \simeq \mathcal{S}(L) \rtimes C_2$, where C_n denotes the cyclic group of order n and $N \rtimes H$ is the semi-direct product of N and H , with N the normal subgroup. In many cases of reversible automorphisms (and also in the analogous continuous-time case of reversible flows), it is in fact found that all reversing symmetries R that satisfy (2) are involutions. In this case, the automorphism L can be written as the composition of two involutions, e.g. $L \circ R$ and R , or R and $R \circ L$. References [22, 29] include reviews of the properties and applications of reversible automorphisms and flows.

The programme followed in the papers [5, 6, 27, 32] can be summarized as follows. The nature of $\mathcal{S}(L)$ and $\mathcal{R}(L)$ has been investigated for some well-known groups of automorphisms where the group structure admits an algebraic investigation of the relations (1) and (2). This necessitates restricting the search for S and R to some suitable group that contains the automorphism L (which might be argued to be a natural first step). Dynamical systems considered in this programme have included toral automorphisms in two and higher dimensions and polynomial automorphism of \mathbb{R}^3 that are closely related (by semi-conjugacies) to two-dimensional toral automorphisms and arise as *trace maps* in the study of quasi-periodic phenomena and the theory of aperiodic order.

In [28], we considered the group of planar polynomial automorphisms. This group comprises ‘maps’ of the form

$$x' = P(x, y), \quad y' = Q(x, y), \quad (3)$$

where $P(x, y)$ and $Q(x, y)$ are polynomials with coefficients in some field K , and there is an inverse that is also polynomial (so the polynomial map $x' = x^3, y' = x + y$, although a bijection over \mathbb{R}^2 , is *not* in the group since its inverse involves cube roots; see [33] for the contrasting complex case). The term ‘map’ in this context is actually a slight abuse of language. Over finite fields, different polynomials (such as $P_1(x) = x$ and $P_2(x) = x^2$ over \mathbb{F}_2 , the finite field with two elements) can define the same mapping. If we use the term ‘map’ or ‘polynomial map’ in this paper, we actually mean to distinguish them according to their polynomial structure.

The group of polynomial automorphisms of the plane K^2 , denoted $\text{GA}_2(K)$, has been studied in some detail because it has the structure of an amalgamated free product (compare [9, 10] and references therein, and section 2 below for more details). Obviously, polynomial maps are much-studied as dynamical systems. In particular, $\text{GA}_2(\mathbb{R})$ and $\text{GA}_2(\mathbb{C})$ have received considerable attention. They include, for example, the Hénon quadratic map family,

$$x' = y, \quad y' = -\delta x + y^2 + c \quad (4)$$

with constants $c, \delta \in \mathbb{C}$ and $\delta \neq 0$. This is one of the more famous ‘toy models’ of discrete dynamics. Exploitation of the group structure of $\text{GA}_2(\mathbb{R})$ and $\text{GA}_2(\mathbb{C})$ has been used to a great extent to investigate various properties of their elements, e.g. their roots [2] or their dynamical entropy [11]. The same idea was used in [28] to give a description of possible $\mathcal{S}(L)$ and $\mathcal{R}(L)$ structures for the subset of $\text{GA}_2(\mathbb{R})$ of maps in the so-called *generalized standard form*

$$x' = x + P_1(y), \quad y' = y + P_2(x') \quad (5)$$

with polynomials P_1 and P_2 (and inverse: $y = y' - P_2(x'), x = x' - P_1(y)$). The form (5) is a common one for area-preserving maps in the dynamics literature. In [28], we also provided normal forms for maps of the form (5) with the various possible symmetries or reversing symmetries. Subsequently, Gómez and Meiss [12] have given normal forms for general elements of $\text{GA}_2(\mathbb{R})$ and $\text{GA}_2(\mathbb{C})$ that possess involutory reversing symmetries.

In this paper, we return to symmetries and reversing symmetries of general elements of $\text{GA}_2(K)$. As compared to [12, 28], our approach will significantly be more algebraic. This is possible due to $\text{GA}_2(K)$ being an amalgamated free product of two well understood groups, so that combinatorial group theory can be used very effectively. Unfortunately, since no such structure is at hand for more than two dimensions, compare [19, example 2.4], our approach is presently restricted to the planar case.

Particular goals of this paper are: (i) to make maximal use of the algebraic consequences of the amalgamated free product structure of the group; (ii) to concentrate on characterizing $S(L)$ before moving onto the study of reversing symmetries, in view of the benefits that can flow algebraically in this direction and (iii) to carry through some of the results for a general field K , before specializing to \mathbb{R} or \mathbb{C} . Of course, the real and complex cases would seem to be the most interesting ones historically. However, dynamical systems over finite fields are becoming more topical (see [30, 31] and references therein). In particular, [31] studies the cycle statistics of permutations associated with reductions to finite fields of planar polynomial automorphisms. It turns out that application to the finite fields case of proposition 7 of section 4 below helps to understand how the possession of orientation-reversing involutory polynomial reversors leads to more cycles of shorter average length than would otherwise occur in, for example, random permutations.

As an indication of the results we obtain through this algebraic approach, we mention some of them for a ‘typical’ infinite order element L of $\text{GA}_2(\mathbb{R})$ (here, ‘typical’ means a *CR element* f , cf section 2):

- any nontrivial symmetry of L of finite order is an involution conjugate to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $S(L)$ contains at most one nontrivial finite subgroup (then isomorphic with the cyclic group C_2) (theorem 2, section 4); a strong characterization can be given both for the involutory symmetry and for L (theorem 3 and corollary 2, section 4);
- any reversor of L is of finite order, being an involution, conjugate to $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, or an element of order 4, conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (theorems 4 and 5, section 5);
- if L has a reversor, a normal form to which L is conjugate in $\text{GA}_2(\mathbb{R})$ can be found (propositions 11 and 12, section 5).

The plan of this paper is as follows. In section 2, we recall key results about the group structure of the planar polynomial automorphisms. In section 3, we summarize results from [37] and various other sources [24–26] concerning Abelian subgroups of $\text{GA}_2(K)$. This is exploited in section 4, where we characterize the symmetry groups and symmetries of typical elements. Finally, in section 5, we employ knowledge of the symmetries to characterize the possible reversing symmetries.

In the final preparation of this manuscript, we became aware of related results by Gómez and Meiss in a preprint that has now appeared [13]. They concentrate on the cases $K = \mathbb{R}$ and $K = \mathbb{C}$, using rather explicit calculations with normal forms, while our focus is more on the general setting, with stronger focus on algebraic methods. We compare their main results with ours in remarks preceding theorem 2 in section 4 and following propositions 11 and 12 in section 5, augmented by various smaller remarks throughout the paper.

2. Recollections and mathematical setting

Let us first recall a number of well-known results about the group structure of the polynomial automorphisms of the plane. We do this in some generality, and simultaneously introduce our notation. Most of what is contained in this section is classic material, and mainly relies

on [11, 18, 37] and references given therein. Still, it seems worthwhile to combine several results in a fashion that suits our purpose and makes the paper more self-contained.

Let K be a field and consider the group $\mathcal{G}_K = \text{GA}_2(K)$ of polynomial automorphisms of the affine plane over K , i.e. the set of mappings of the form (3) with $P, Q \in K[x, y]$ (the ring of polynomials in x, y with coefficients in K) such that the inverse exists and is also polynomial. Group multiplication is composition of maps, usually written as gg' rather than $g \circ g'$ in the following. The neutral element of the group will be written as 1, denoting the identity map. For mappings, in comparison with (3), we interchangeably also use the notation

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} P(x, y) \\ Q(x, y) \end{pmatrix}. \quad (6)$$

Whenever K is clear from the context, we will write \mathcal{G} rather than \mathcal{G}_K for simplicity.

Note that the Jacobian dg of any element $g \in \mathcal{G}$ (defined via the algebraic derivative of the polynomials involved, see [9, p 5]) has constant determinant $\neq 0$, i.e. $\det(dg)$ is an element of $K^* = K \setminus \{0\}$, the latter representing the only units of the ring $K[x, y]$, compare [9, proposition 1.14]. The converse question is connected with the famous Jacobian conjecture, namely whether $\det(dg) = \text{const} \neq 0$ is sufficient for a polynomial mapping to be an automorphism (see [9] for a summary and [33] for an interesting partial result) together with some comments on the influence of the field K being algebraically closed or not.

The group \mathcal{G} contains two particularly important subgroups. First, there is the group \mathcal{A} of *affine* transformations,

$$\mathcal{A} = \{(\mathbf{a}, M) \mid \mathbf{a} \in K^2, M \in \text{GL}(2, K)\},$$

where (\mathbf{a}, M) encodes the mapping $\mathbf{x} \mapsto M\mathbf{x} + \mathbf{a}$. We write \mathbf{x} for a column vector with two entries, and tacitly identify the elements of \mathcal{A} with the corresponding ones of \mathcal{G} . In particular, a matrix M is identified with the linear mapping $\mathbf{x} \mapsto M\mathbf{x}$, and a vector space element $\mathbf{a} \in K^2$ with the translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{a}$. The multiplication of two elements of \mathcal{A} reads

$$(\mathbf{a}, A)(\mathbf{b}, B) = (\mathbf{a} + A\mathbf{b}, AB),$$

which shows that \mathcal{A} is a semi-direct product, $\mathcal{A} = K^2 \rtimes \text{GL}(2, K)$, where K^2 is the normal subgroup. Note that the inverse of (\mathbf{a}, A) reads $(\mathbf{a}, A)^{-1} = (-A^{-1}\mathbf{a}, A^{-1})$.

The second group, \mathcal{E} , consists of all mappings of \mathcal{G} of the form

$$e : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha x + P(y) \\ \beta y + v \end{pmatrix} \quad (7)$$

with P a polynomial, $\alpha, \beta, v \in K$ and $\alpha\beta \neq 0$. The inverse reads

$$e^{-1} : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \frac{1}{\alpha} \left(x - P \left(\frac{y - v}{\beta} \right) \right) \\ \frac{1}{\beta} (y - v) \end{pmatrix}. \quad (8)$$

The elements of \mathcal{E} are called *elementary* transformations. They map lines with constant y -coordinate to lines of the same type. The relevance of these two subgroups comes from the following fact, which was proved by Jung [16] for $K \in \{\mathbb{R}, \mathbb{C}\}$ and later by van der Kulk [20] for arbitrary fields K , see also [37, section 1.5] and [11, p 68].

Fact 1. *The group \mathcal{G} of polynomial automorphisms of the plane K^2 is generated by the two subgroups \mathcal{A} and \mathcal{E} . \square*

The intersection of \mathcal{A} and \mathcal{E} , both seen as subgroups of \mathcal{G} , is another group, called \mathcal{B} (for *basic*) from now on. It consists of all mappings of the form

$$b : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} \quad (9)$$

with $\alpha, \beta, \gamma, u, v \in K$ and $\alpha\beta \neq 0$. If \mathcal{T} denotes the subgroup of $\text{GL}(2, K)$ which consists of all upper (invertible) triangular matrices, one can see that \mathcal{B} is again a semi-direct product,

$$\mathcal{B} = K^2 \rtimes \mathcal{T}.$$

The following result [34, 37] is important, see also [10, theorem 5.1.11 and corollary 5.3.6].

Fact 2. *The group \mathcal{G} is the free product of the groups \mathcal{A} and \mathcal{E} , called factors, amalgamated along their intersection, \mathcal{B} , abbreviated as $\mathcal{G} = \mathcal{A} *_B \mathcal{E}$. \square*

This gives access to the structure of the group \mathcal{G} , and to its subgroups in particular. To make explicit use of it later on, we need a natural way to represent group elements uniquely. This is achieved by a partition of \mathcal{G} into (right) cosets, which we will indicate below by the symbol $\dot{\cup}$ (for disjoint union). Define

$$\mathcal{I} := \left\{ \begin{pmatrix} 0 & 1 \\ 1 & \beta \end{pmatrix} \mid \beta \in K \right\} \subset \mathcal{A} \setminus \mathcal{B}, \quad (10)$$

again identified with the corresponding subset of \mathcal{G} , and

$$\mathcal{J} := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y^2 P(y) \\ y \end{pmatrix} \mid 0 \neq P \in K[y] \right\} \subset \mathcal{E} \setminus \mathcal{B}. \quad (11)$$

Note that \mathcal{J} is invariant under taking inverses, i.e. if $e \in \mathcal{J}$, so is e^{-1} . Furthermore, all elements of \mathcal{I} and \mathcal{J} fix the origin.

Now, either following [37, sections 1.6 and 1.7] and observing that we use upper triangular matrices in \mathcal{B} for consistency with [11], or verifying it by a direct computation, one obtains

Fact 3. *Let \mathcal{I} and \mathcal{J} be the sets defined in (10) and (11). Then, the subgroups \mathcal{A} and \mathcal{E} of \mathcal{G} satisfy the unique right coset decompositions $\mathcal{A} = \dot{\cup}_{a \in \mathcal{I} \cup \{1\}} \mathcal{B}a$ and $\mathcal{E} = \dot{\cup}_{e \in \mathcal{J} \cup \{1\}} \mathcal{B}e$ with respect to the subgroup $\mathcal{B} = \mathcal{A} \cap \mathcal{E}$ of \mathcal{G} . \square*

This admits the introduction of a powerful concept, the so-called *normal form* of an element $g \in \mathcal{G}$, compare [34, chapter I.1.2] and [25, section 4.2]. We also recall the slightly weaker, but sometimes more useful, result on the *reduced word* representation [7, 25].

Proposition 1. *Each element $g \in \mathcal{G} \setminus \mathcal{B}$ can be written as a reduced word*

$$g = g_n \circ g_{n-1} \circ \cdots \circ g_1, \quad (12)$$

where $n \geq 1$ and the g_i alternates between $\mathcal{A} \setminus \mathcal{B}$ and $\mathcal{E} \setminus \mathcal{B}$, starting and ending with either one. Such a product can never be in \mathcal{B} , and it cannot be in $\mathcal{A} \cup \mathcal{E}$ whenever $n > 1$.

Moreover, each element $g \in \mathcal{G}$ has a unique representation in the form

$$g = b \circ a_m \circ e_m \circ \cdots \circ a_1 \circ e_1 \quad (13)$$

for some (unique) $m \geq 1$. Here, $b \in \mathcal{B}$ (including the case that b is the neutral element 1, hence effectively missing), while $a_i \in \mathcal{I}$ and $e_i \in \mathcal{J}$ for all $1 \leq i \leq m$, except that a_m and/or e_1 are allowed to be missing. The representation (13) is called the *normal form* of g with respect to the coset representatives \mathcal{I} and \mathcal{J} .

Proof. The reduced word is a standard way to represent elements in amalgamated free products, see [7, theorem 26].

The unique normal form emerges as soon as coset representatives of the factors (\mathcal{A} and \mathcal{E} in our case) mod the amalgamation group (\mathcal{B}) are selected. This is achieved by fact 3. For details, see [7, theorem 25]. \square

Remark. All elements of \mathcal{I} (respectively \mathcal{J}) have Jacobians of determinant -1 (respectively $+1$), so that $|\det(dg)| = |\det(db)|$, if $b \in \mathcal{B}$ is the starting element according to the decomposition (13). Also, an element in normal form (13) fixes the origin if and only if the element b in it does.

For $g \in \mathcal{G}$, both the reduced word (12) and the normal form (13), which we often prefer to deal with, admit the introduction of several useful concepts, one being the *length* of an element g , written as $\text{len}(g)$. If g is given in normal form (13), $\text{len}(g)$ is the total number of factors from \mathcal{I} and \mathcal{J} , hence an integer between $2m - 2$ and $2m$, depending on which factors are absent. One has $\text{len}(1) = 0$, and, more generally, $\text{len}(b) = 0$ for all $b \in \mathcal{B}$. Moreover, the value of $\text{len}(g)$ does not depend on the choice of coset representatives of the factors, such as \mathcal{I} and \mathcal{J} above. Consequently, if $g \notin \mathcal{B}$, $\text{len}(g)$ is the same for all reduced word representations (12) obtained from (13) by inserting $b_i b_i^{-1}$, with arbitrary $b_i \in \mathcal{B}$, between any a_i and the following e_i , and rewriting it in the form (12). One thus has $\text{len}(g) = n$ for (12).

An element $g \in \mathcal{G}$ in normal form (13) is called *cyclically reduced* if $b^{-1}g$ starts with an element in \mathcal{I} and ends with one in \mathcal{J} , or vice versa (so that the word $b^{-1}g$ alternates between elements of \mathcal{I} and \mathcal{J} when wrapped on a circle). In other words, g has a cyclically reduced normal form (CRNF) iff the length of g is even and >0 . With this definition, which follows [11, 34, 37] but deviates from [25], elements from the factors or their conjugates can neither have a CRNF, nor be a conjugate to one. This will prove useful shortly.

From now on, we will call $g \in \mathcal{G}$ a *CR element* if it is conjugate to an element with a CRNF. Note that, in general, a CR element itself does not have a CRNF. As an example, consider the CR element $g = e_1^{-1} a_1^{-1} a e a_1 e_1$, with $a_1^{-1} a \notin \mathcal{B}$. Then, g is essentially in normal form (possibly with $b \neq 1$, after rewriting it with the proper representatives from \mathcal{I} and \mathcal{J}), but not cyclically reduced. Rather, g is conjugate to ae , which is cyclically reduced.

Note that CR elements of \mathcal{G} play a similar role as hyperbolic elements do in the class of toral automorphisms [6], and they are the ones we are mainly interested in dynamically. They are also the ones that can be accessed algebraically, due to the very structure of \mathcal{G} . The following result is standard, compare [34, section I.1.3] and [25, theorem 4.6].

Fact 4. *Any element $g \in \mathcal{G}$ is either a conjugate to an element of \mathcal{A} or \mathcal{E} , or is a CR element, the two cases being mutually exclusive. Moreover, no CR element is of finite order, wherefore any element of finite order is conjugate to an element in one of the factors.* \square

Remark. If g has a CRNF, one finds $\text{len}(g) \geq 2$ and can check that $\text{len}(g^n) = |n| \text{len}(g)$, for all $n \in \mathbb{Z}$ (note that $\text{len}(g^{-1}) = \text{len}(g)$). Clearly, an element g cannot be of finite order unless the set $\{\text{len}(g^m) \mid m \geq 0\}$ is bounded and contains 0. Note, however, that this is not sufficient for g to be of finite order. For example, the sequence of lengths is identically 0 for iterates of $g : x' = cx, y' = cy$ where $c \in K^*$ is an element of infinite multiplicative order (such as $c = 2$ for a field of characteristic 0). If all elements of K^* are of finite multiplicative order (e.g. if K is a finite field), all elements of \mathcal{A} and \mathcal{E} , and hence all conjugates of such elements, are of finite order [15]. In this case, since all remaining elements are CR elements, finite order and bounded length are equivalent.

For completeness, let us recall the following result of Serre [34, section I.4.3, theorem 8 and its corollary], which we formulate in our setting (though it is valid for any amalgamated free product of two groups).

Fact 5. *Any subgroup of $\mathcal{G} = \mathcal{A}_B^* \mathcal{E}$ of bounded length is conjugate to a subgroup of one of the factors. In particular, every finite subgroup of \mathcal{G} is conjugate to a subgroup of \mathcal{A} or \mathcal{E} . \square*

Note that stronger statements are possible for our special group, concerning the conjugacy of finite order elements and finite subgroups to linear ones, if K has characteristic 0 and is algebraically closed, compare [19, p 57 and theorem 2.3] and [17, theorem 4.3 and corollary 4.4]. However, this is not so for general K , see [3], wherefore we omit further details.

A related concept is that of the *degree* of a (non-zero) polynomial mapping. The degree of $P \in K[x, y]$ is the maximum of the degrees of its monomials with nonzero coefficient, where $\deg(x^m y^n) = m + n$, and the degree of the polynomial mapping (6) is then defined as the maximum of the degrees of P and Q . All affine maps have degree 1, and the degree of an elementary map (7) is $\max(1, \deg(P))$. Consequently, the degree cannot be multiplicative in general, but it is for the decomposition (13), see [11, theorem 2.1].

Fact 6. *If $g \in \mathcal{G}$ is decomposed according to (13) of proposition 1, the degree of g is the product of the degrees of the factors, i.e.*

$$\deg(g) = \prod_{i=1}^m \deg(e_i),$$

where we set $\deg(e_1) = 1$ if e_1 is missing in the product (13). \square

By analogy to before, the degree of a subgroup is defined as the maximum of the degrees of its elements. For any $g \in \mathcal{G}$, one has the relation $\deg(g) \geq 2^{\lfloor \text{len}(g)/2 \rfloor}$, in obvious modification of [37, equation (21)]. Consequently, the degree implies a bound on the length, both for elements and for groups. Note, however, that a group of bounded length need not be of bounded degree; this can be seen from \mathcal{E} , which is of length 1, but of unbounded degree.

3. Conjugacy and Abelian subgroups in \mathcal{G}

Let us first recall the following result about conjugacy, where we rephrase [25, theorem 4.6] in our terminology. To simplify the following formulations, we specify:

- unless stated otherwise, conjugate always means conjugate in \mathcal{G} .

Moreover, although ultimately we have our special group $\mathcal{G} = \mathcal{G}_K$ in mind, the statements upto theorem 1 are not restricted to this case (unless stated so explicitly), but are actually valid for the free products of two groups \mathcal{A} and \mathcal{E} with an amalgamated subgroup \mathcal{B} .

Proposition 2. *In the amalgamated free product $\mathcal{G} = \mathcal{A}_B^* \mathcal{E}$, every element of \mathcal{G} is conjugate to an element of \mathcal{A} or \mathcal{E} , or is a CR element, i.e. conjugate to an element with CRNF.*

Moreover, if g is itself an element of $\mathcal{A} \cup \mathcal{E}$ or an element with CRNF, one has the following three possibilities.

- (1) *If g is conjugate to an element $b \in \mathcal{B}$, then g lies in one of the factors, and there is a sequence $b, h_1, h_2, \dots, h_\ell, g$ where each h_i lies in \mathcal{B} and consecutive elements of the sequence are conjugate in a factor.*
- (2) *If g is conjugate to an element g' that is in some factor but not in a conjugate of \mathcal{B} , then also g lies in the same factor, and g and g' are conjugate within this factor.*

(3) If g is conjugate to an element g' in CRNF, one can obtain g from g' by a cyclic permutation of the factors of g' , followed by a conjugacy with an element from \mathcal{B} .

For general elements, these possibilities apply up to conjugacy. \square

An amalgamated free product of two factors admits some access to the structure of its subgroups, in particular the Abelian ones. Let us first consider two commuting elements.

Lemma 1. *Let $g, g' \in \mathcal{G}$ with $gg' = g'g$. Then, one of the following three cases applies.*

- (1) *The element g or g' is in a conjugate of \mathcal{B} .*
- (2) *If neither g nor g' is in a conjugate of \mathcal{B} , but g is in a conjugate of a factor, then g' is in that same conjugate, too.*
- (3) *If neither g nor g' is in a conjugate of a factor (i.e. if both g and g' are CR elements), then $g = d^k c b c^{-1}$ and $g' = d^\ell c b' c^{-1}$ for some $c, d \in \mathcal{G}$, $k, \ell \in \mathbb{Z}$ and $b, b' \in \mathcal{B}$, where $b c b^{-1}$, $c b' c^{-1}$ and d pairwise commute.*

Moreover, if \mathcal{G} is our special group $\text{GA}_2(K)$, its centre is trivial.

Proof. The first three assertions simply are a reformulation of [25, theorem 4.5] in our context. From [25, corollary 4.5], we also know that $\text{cent}(\mathcal{G}) = \text{cent}(\mathcal{A}) \cap \text{cent}(\mathcal{E}) \subset \mathcal{B}$, whenever $\mathcal{A} \neq \mathcal{B} \neq \mathcal{E}$. This is clearly the case for our special group $\mathcal{G} = \text{GA}_2(K)$. It is not difficult to verify that $\text{cent}(\mathcal{A}) = \{1\}$, which then establishes the last claim. \square

The next step is a complete characterization of the Abelian subgroups into three types, which goes back to Moldavanskii [26]. It was later put into a more general framework in [18], and a complete account is also contained in [37, section 0]. We first rephrase [37, theorem 0.3] in our terminology, but still for a general setting. We will then specialize step by step.

Theorem 1. *If \mathcal{H} is an Abelian subgroup of $\mathcal{G} = \mathcal{A}_\mathcal{B}^* \mathcal{E}$, it is precisely of one of the following three types.*

- (T1) *\mathcal{H} is conjugate to a subgroup of \mathcal{A} or to a subgroup of \mathcal{E} .*
- (T2) *\mathcal{H} is not conjugate to any subgroup of \mathcal{A} or \mathcal{E} , but there exists a nested chain of subgroups $\mathcal{H}_0 \subset \mathcal{H}_1 \subset \dots \subset \mathcal{H}_i \subset \dots$ such that $\mathcal{H} = \bigcup_{i=0}^{\infty} \mathcal{H}_i$, where each \mathcal{H}_i is conjugate to a subgroup of \mathcal{B} . This chain is inevitably infinite and non-stationary.*
- (T3) *$\mathcal{H} = \mathcal{F} \times \langle g \rangle$, where \mathcal{F} is conjugate to a subgroup of \mathcal{B} , and g is a CR element, hence not of finite order and not conjugate to any element of \mathcal{A} or \mathcal{E} .* \square

As is immediate, type 2 subgroups are the more delicate ones to deal with. We will now focus on our special group of polynomial automorphisms $\mathcal{G} = \mathcal{G}_K$, which admits further simplifications. For completeness, we will consider all three types here, even though later on we will mainly need Abelian subgroups of type 3.

Proposition 3. *Let \mathcal{H} be an Abelian subgroup of $\mathcal{G} = \mathcal{A}_\mathcal{B}^* \mathcal{E}$. Then, the following three assertions are equivalent.*

- (1) *\mathcal{H} is of type 1.*
- (2) *\mathcal{H} is conjugate to a subgroup of either \mathcal{A} or \mathcal{E} .*
- (3) *\mathcal{H} is of bounded length, i.e. $\max\{\text{len}(g) \mid g \in \mathcal{H}\} < \infty$.*

Proof. (1) \iff (2) is the definition.

(2) \implies (3): Each element of $\mathcal{A} \cup \mathcal{E}$ has length 0 or 1, which is then also true of any subgroup H of \mathcal{A} or \mathcal{E} . Since $\text{len}(ghg^{-1}) \leq \text{len}(g) + \text{len}(h) + \text{len}(g^{-1})$, any conjugate subgroup gHg^{-1} is then of bounded length, too.

(3) \implies (2): This is proved in [37, proposition 0.35], or follows from fact 5. \square

From now on, let $U = U_K$ denote the group of roots of 1 in K , so $U = \{\pm 1\}$ for $K = \mathbb{R}$ and $U = \{z \in S^1 \mid z^n = 1 \text{ for some } n \in \mathbb{N}\}$ for $K = \mathbb{C}$ (with S^1 the unit circle in \mathbb{C}). Moreover, let $U(n)$ denote the (multiplicative) subgroup of n th roots of unity of K . Note that $U(n)$ is a finite cyclic group [23, theorem IV.1.9], the order of which divides n . If n is a power of $\text{char}(K)$, one has $U(n) = \{1\}$, the trivial group. If n is not divisible by $\text{char}(K)$, and if K is algebraically closed, one has $U(n) \simeq C_n$; without algebraic closure, $U(n)$ can be a genuine subgroup of C_n , as happens in \mathbb{R} versus \mathbb{C} , compare [23, section VI.3] for more.

Following [37, theorem 1.21 and corollary 1.22], one can summarize the situation of type 2 subgroups of \mathcal{G} as the following proposition.

Proposition 4. *The group \mathcal{G} does not contain Abelian subgroups of type 2 if K is a finite field, or if K has characteristic 0 and finite U .*

Otherwise, if \mathcal{H} is a type 2 Abelian subgroup of \mathcal{G} , the necessarily non-stationary subgroup chain $(\mathcal{H}_i)_{i \geq 0}$ of theorem 1 satisfies one of the following two conditions.

- (1) *Each \mathcal{H}_i is conjugate to a finite subgroup of K^2 , viewed as a subgroup of \mathcal{A} .*
- (2) *Each \mathcal{H}_i is conjugate to a subgroup of the diagonal matrices of the form $\{\text{diag}(u, u^m) \mid u \in U(n)\}$, where m and n are coprime integers that depend on i .*

Moreover, if K has characteristic 0, only case (2) is possible. □

In general, all situations can occur, see examples 2.2 and 2.5 of [37]. If restricting to \mathbb{Q} , \mathbb{R} or \mathbb{C} , we are in the case of characteristic 0. But while for $K \in \{\mathbb{Q}, \mathbb{R}\}$ we do not have type 2 subgroups (since then $U_K = K \cap U_{\mathbb{C}} = \{\pm 1\}$, so that we cannot have non-stationary subgroup chains), this is not so for $K = \mathbb{C}$.

Finally, we recall [37, theorem 1.24].

Proposition 5. *If \mathcal{H} is a type 3 Abelian subgroup of \mathcal{G} , then $\mathcal{H} = \mathcal{F} \times \langle g \rangle$, where $g \in \mathcal{H}$ is a CR element (hence not of finite order and not conjugate to an element of \mathcal{A} or \mathcal{E}), and \mathcal{F} is a subgroup of \mathcal{H} such that one of the following two conditions holds.*

- (1) *\mathcal{F} is conjugate to a subgroup of K^2 , the latter viewed as a subgroup of \mathcal{A} .*
- (2) *\mathcal{F} is conjugate to a subgroup of the diagonal matrices of the form $\{\text{diag}(u, u^m) \mid u \in U(n)\}$, where m and n are (fixed) coprime integers.*
In particular, \mathcal{F} is a finite cyclic group.

Once again, if K has characteristic 0, only case (2) is possible. □

As follows from examples 2.9 and 2.10 of [37], both possibilities of proposition 5 can be realized in general fields. We will provide another example of this in the following section. If $\text{char}(K) = 0$ or if K is a finite field, the group \mathcal{F} is finite. Whether this is generally the case, as addressed on p 613 of [37], does not yet seem to have been resolved [38].

4. Symmetries

We now turn our attention to the symmetry group $\mathcal{S}(f) = \text{cent}_{\mathcal{G}}(f) = \{h \in \mathcal{G} \mid fh = hf\}$ for $f \in \mathcal{G}$. In particular, we would like to know its structure, e.g. whether it is Abelian. Although this need not be the case in general, the knowledge of the Abelian subgroups reviewed in the previous section will prove most useful to determine the structure of $\mathcal{S}(f)$.

We are mainly interested in the case that f is a CR element, because this is dynamically the most interesting. This is also justified by the observation that, due to lemma 1, the investigation of the symmetries of other elements can essentially be handled within the factors \mathcal{A} or \mathcal{E} . Even though this is a task in itself (note that it also includes the analysis of point and space groups

in the plane, hence cases where $\mathcal{S}(f)$ is *not* Abelian), it is more or less decoupled from \mathcal{G} , due to the very structure of \mathcal{G} as an amalgamated free product. In fact, the analysis of space groups is essentially restricted to \mathcal{A} , see [8, section 4.5] for details.

As to \mathcal{E} , consider an element $e \in \mathcal{E}$ of the form given in (7) with $\alpha = 1$ and $v = 0$. It is immediate that it always commutes with the simple translation $t : x' = x + 1, y' = y$, where $\langle t \rangle$ is isomorphic with C_∞ (respectively C_p) if $\text{char}(K) = 0$ (respectively $\text{char}(K) = p$ with p prime). Now, let the polynomial P from e be *odd* (i.e. $P(-y) = -P(y)$), and consider an arbitrary field K with $\text{char}(K) \neq 2$, so that $-1 \neq 1$. Clearly, e now also commutes with the mapping defined by $I = \text{diag}(-1, -1) \in \text{GL}(2, K)$, but t and I do *not* commute (one has $I \circ t = t^{-1} \circ I$). Though both $\langle e, t \rangle$ and $\langle e, I \rangle$ are Abelian subgroups of \mathcal{E} , hence Abelian subgroups of \mathcal{G} of type 1, $\mathcal{S}(e)$ is never Abelian in this case, as it contains $\langle t, I \rangle$, which is a dihedral group of infinite order (D_∞) or of order $2p$ (denoted by D_p).

We now specialize to investigate $\mathcal{S}(f)$ for $f \in \mathcal{G}$ a CR element. Two themes will run through our investigation of the symmetries of such an element: (i) we profit from studying the ‘local’ symmetry group $\langle f, h \rangle$ with $h \in \mathcal{S}(f)$, which is an Abelian subgroup of $\mathcal{S}(f)$, though $\mathcal{S}(f)$ itself might not be Abelian; (ii) the order of possible symmetries (and later of reversing symmetries) is driven by the nature of the roots of unity in the chosen field K .

We start with a simple observation which highlights a first difference with the earlier example of the (non-CR) element from \mathcal{E} .

Lemma 2. *If $f \in \mathcal{G}$ is a CR element of \mathcal{G} , it cannot be contained in any Abelian subgroup of \mathcal{G} of type 1 or 2.*

Proof. A CR element is not conjugate to any element of \mathcal{A} or \mathcal{E} , by fact 4, hence cannot lie in an Abelian subgroup of type 1, by proposition 3.

On the other hand, by theorem 1 and proposition 4, all type 2 Abelian subgroups \mathcal{H} are obtained as inductive limits of a sequence $(\mathcal{H}_i)_{i \geq 0}$ of nested groups, each of which is conjugate to a subgroup of \mathcal{B} . Since neither the CR element f nor any of its (finite) powers can be an element of any of these subgroups, f is not an element of \mathcal{H} either. \square

Example. Let K be an arbitrary field, and consider the mapping

$$f : \quad x' = y, \quad y' = x + Q(y)$$

with the polynomial $Q(y) = y^p - y$, where p is a prime. This is a CR element of length 2, whose square would be in the generalized standard form (5) with $P_1 = P_2 = Q$. If K has characteristic p (e.g. if $K = \mathbb{F}_p$), it is easy to check that $Q(y+1) = Q(y)$, because $\binom{p}{\ell} = 0 \pmod{p}$ for all $1 < \ell < p$ (this is equivalent to the existence of the Frobenius endomorphism in characteristic p , defined by $y \mapsto y^p$, cf [23, p 179]). As a consequence, f commutes with the translation $t \in \mathcal{B} : x' = x + 1, y' = y + 1$, the latter generating the cyclic group C_p .

If we now restrict to *odd* primes (i.e. $p \neq 2$), the polynomial Q is odd, and f also commutes with $I = \text{diag}(-1, -1)$. The latter, in turn, does *not* commute with t , and $\langle t, I \rangle \simeq D_p$. The minimal example emerges for $p = 3$, where $D_3 \simeq S_3$ (with S_n the symmetric group) is the smallest non-Abelian group. Clearly, $\mathcal{S}(f)$, which contains $\langle t, I \rangle$, is not Abelian either.

Note, however, that both $\langle f, t \rangle$ and $\langle f, I \rangle$ are Abelian subgroups of \mathcal{G} of type 3, fitting cases (1) and (2) of proposition 5, respectively.

This example shows that, in general, the symmetry group of a CR element will not be Abelian, but also that interesting new phenomena occur when one works over finite fields or over fields with characteristic $\neq 0$.

To use the knowledge of Abelian subgroups of \mathcal{G} of the previous section, it seems a reasonable strategy to restrict, as far as possible, to ‘local’ symmetries, i.e. to the groups

generated by a CR element f together with a single symmetry. Then, lemma 2 has the following consequence.

Proposition 6. *Let $f \in \mathcal{G}$ be a CR element of $\mathcal{G} = \mathcal{G}_K$ and h be a symmetry of f , i.e. $h \in \mathcal{S}(f)$. Then, $\langle f, h \rangle$ is an Abelian subgroup of \mathcal{G} of type 3.*

Moreover, if $\text{char}(K) = 0$, one has $\langle f, h \rangle \simeq C_\ell \times C_\infty$ for some $\ell \in \mathbb{N}$. This means that one either has $f^k = h^m$ for some $k, m \in \mathbb{Z} \setminus \{0\}$, or $\langle h \rangle$ is a finite cyclic group.

Proof. Even though $\mathcal{S}(f)$ itself need not be Abelian, each subgroup of the form $\langle f, h \rangle$ certainly is. Since $f \in \langle f, h \rangle$ and f is a CR element, the first claim follows from lemma 2.

If K is a field of characteristic 0, part (2) of proposition 5 says that the Abelian subgroups of type 3 are all of the form $\mathcal{F} \times C_\infty$, where \mathcal{F} is isomorphic with a subgroup of the cyclic group C_n , for a suitable n . Consequently, $\mathcal{F} \simeq C_\ell$ for some divisor ℓ of n .

So, we have $\langle f, h \rangle \simeq \langle t \rangle \times \langle g \rangle$ with $t^\ell = 1$ and g an element of infinite order. Clearly, $f = t^\epsilon g^r$ for some $0 \leq \epsilon < \ell$ and $0 \neq r \in \mathbb{Z}$, hence $f^\ell = g^{\ell r}$. By the same argument, $h = t^{\epsilon'} g^q$ and $h^\ell = g^{\ell q}$ for some $q \in \mathbb{Z}$, possibly 0. Consequently, $f^k = h^m$ with $k = \ell q$ and $m = \ell r$.

If $k \neq 0$, we must also have $m \neq 0$, since f is not of finite order. This gives the first possibility claimed, where h is not of finite order. If $k = 0$, one has $h^m = 1$, whence h is of finite order. This is only possible for $q = 0$, so $\langle h \rangle$ is isomorphic with a subgroup of C_ℓ and hence cyclic. \square

Remark. If $\text{char}(K) = 0$, proposition 6 excludes, for f a CR element, the existence of a subgroup of $\mathcal{S}(f)$ of the form $C_\infty \times C_\infty$ that contains f , i.e. the existence of an infinite order symmetry which is independent of f . This observation forms the basis of a result of Veselov [35, 36] that an (area-preserving) polynomial automorphism cannot possess a polynomial or rational integral $I(x, y)$ that is preserved under iteration of f .

The following example illustrates (e.g. when $K = \mathbb{C}$) that symmetries of arbitrarily large finite order can indeed occur.

Example. Let K be a field, with unit group U_K . Take $f \in \mathcal{G}_K$ of the generalized standard form (5) and look for a linear symmetry h defined by the matrix $\text{diag}(\lambda, \mu) \in \text{GL}(2, K)$, hence with $\lambda, \mu \in K^* = K \setminus \{0\}$. One finds that $f \circ h = h \circ f$ if and only if the polynomials P_1 and P_2 of (5) satisfy

$$P_1(\mu z) = \lambda P_1(z) \quad \text{and} \quad P_2(\lambda z) = \mu P_2(z).$$

In particular, unless $P_1 = P_2 = 0$ or one polynomial vanishes while the other is a non-constant monomial, there is no solution except when λ and μ are roots of unity, i.e. when $\lambda, \mu \in U$.

If $\lambda \in U(n)$ is a primitive n th root of 1 in K , and if $\mu = \lambda^{-1}$, the order of h is n . Moreover, the polynomial condition is satisfied if $zP_1(z)$ and $zP_2(z)$ are actually polynomials in z^n without constant term. Similarly, if $\mu = \lambda$, one needs $(1/z)P_i(z)$ to be a polynomial in z^n , for $i \in \{1, 2\}$. Consequently, if $K = \mathbb{C}$, symmetries of any finite order are possible.

Remark. Proposition 6 shows that, when $\text{char}(K) = 0$, a symmetry h of a CR element f is: either (i) of finite order, conjugate to a diagonal matrix with entries from the roots of unity; or (ii) of infinite order and h and f are both roots of a common CR element. Furthermore, from the proof of proposition 6 and part (2) of proposition 5, it follows immediately that $f = (c \text{diag}(u, u^m)c^{-1})^\epsilon (dGd^{-1})^r$ for some $u \in U(\ell)$, $0 \leq \epsilon < \ell$, $c, d, G \in \mathcal{G}$ with G having a CRNF. Moreover, the two bracketed elements commute. Equivalently, f is conjugate to $(\text{diag}(u, u^m))^\epsilon g^r$, where g is CR and commutes with the linear map defined by the diagonal matrix. For $K = \mathbb{C}$, theorem 1 (or, in more detail, theorem 7 and corollary 9) of [13] is a

stronger result in this spirit, obtained by constructive means. It is shown that f is conjugate to $(\text{diag}(u, u^m))^\epsilon H^r$, with $H = h_m \circ \dots \circ h_1$, where the Hénon maps h_i are defined by

$$h_i : \quad x' = y, \quad y' = -\delta_i x + Q_i(y). \quad (14)$$

It follows from [11] that every CR element of \mathcal{G} is conjugate to a composition $h_m \circ \dots \circ h_1$ for some $m \geq 1$ ([11] also shows that some normalization can be made to each $Q_i(y)$). With the choice of coset representatives \mathcal{I} of (10) and \mathcal{J} of (11), leading to the normal form (13), we have a similar result: namely, every CR element is conjugate to a uniquely-expressed composition $(a_m \circ e_m) \circ \dots \circ (a_1 \circ e_1)$, respectively one with an extra $b \in \mathcal{B}$ in front of it. Note that

$$a_i \circ e_i : \quad x' = y, \quad y' = x + (y^2 P_i(y) + \beta_i y) \quad (15)$$

is an orientation-reversing Hénon map.

In the cases $K = \mathbb{Q}$ and $K = \mathbb{R}$, there are more severe restrictions on the nature of symmetries and, in fact, $\mathcal{S}(f)$ turns out to be Abelian.

Theorem 2. *Let K be a field of characteristic 0 with group of roots of unity $U \simeq C_2$ (which includes the cases $K = \mathbb{Q}$ and $K = \mathbb{R}$), and let f be a CR element of \mathcal{G} . Then, any symmetry of f in \mathcal{G} of finite order must be the identity or an involution.*

Moreover, the symmetry group of f in \mathcal{G} can contain at most one nontrivial finite group, which is then of the form $\langle s \rangle$ with s an involution that is conjugate to $I = \text{diag}(-1, -1)$.

Proof. Let h be any element of $\mathcal{S}(f)$. Due to the assumptions, $\langle f, h \rangle$ is an Abelian subgroup of type 3, hence equals $\mathcal{F} \times \langle g \rangle$ for some CR element g and some finite group \mathcal{F} . Since we are in case (2) of proposition 5, \mathcal{F} is isomorphic to a subgroup of the group U of roots of unity in K , hence to the trivial group or C_2 . So, $\mathcal{F} = \langle s \rangle$ with $s^2 = 1$. Since \mathcal{F} contains all elements of $\langle f, h \rangle$ of finite order, and h was an arbitrary symmetry, this shows that any symmetry of f of finite order must be 1 or an involution.

Clearly, f itself is in $\mathcal{S}(f)$, but it is a CR element, hence not of finite order. So, we must have $f = s^\epsilon g^m$ for ϵ either 0 or 1 and some nonzero integer m , hence $f^2 = g^{2m}$. So far, we have established that f , together with any single symmetry of it, generates an Abelian group of the form $\mathcal{F} \times C_\infty$ with \mathcal{F} the trivial group or C_2 . We now need to understand better how different groups of this kind fit together as subgroups of $\mathcal{S}(f)$.

So, let us assume that $\mathcal{S}(f)$ contains two different involutions, say s_1 and s_2 . Then, the product $s = s_1 s_2$ also commutes with f . Our previous argument applies to $h = s$, so we have $\langle f, s \rangle = \mathcal{F} \times \langle g \rangle$ with \mathcal{F} the trivial group or C_2 , and g a CR element with $f^2 = g^{2m}$ for some nonzero $m \in \mathbb{Z}$. So, $\mathcal{F} = \langle t \rangle$ with $t^2 = 1$, hence $s = t^\epsilon g^k$ with $\epsilon \in \{0, 1\}$ and $k \in \mathbb{Z}$.

If s is not of finite order, one has $k \neq 0$ and $s^{2m} = (t^\epsilon g^k)^{2m} = g^{2mk} = f^{2k}$. With $s = s_1 s_2$, observe $s_i s^{2m} s_i = s^{-2m}$, for $i \in \{1, 2\}$, so that s_1 and s_2 are reversors of f^{2k} . But the s_i are also symmetries of f , hence of f^{2k} , and we obtain $f^{-2k} = s_i f^{2k} s_i^{-1} = f^{2k}$ which would imply $f^{4k} = 1$ —a contradiction.

So, $s = t^\epsilon g^k$ must be of finite order. This implies $k = 0$ (because $g^0 = 1$ is the only finite order element of $\langle g \rangle \simeq C_\infty$), hence $s = t^\epsilon$ and $s^2 = 1$. Since $s_1 \neq s_2$ by assumption, we know that $s \neq 1$, and s must be an involution. This also implies that s_1 and s_2 commute. But $s_1 \neq s_2$ now means that we have an Abelian subgroup $\langle s_1 \rangle \times \langle s_2 \rangle \times \langle f \rangle \simeq C_2 \times C_2 \times C_\infty$ of \mathcal{G} which must be of type 3. However, the finite group \mathcal{F} here is Klein's 4-group, which is not cyclic. This contradicts proposition 5.

Consequently, there can be at most one true involution which commutes with f , which shows the claim about the finite subgroup of $\mathcal{S}(f)$. In fact, part (2) of proposition 5 implies that s is conjugate to $I = \text{diag}(-1, -1)$. \square

Let us draw some further conclusions from theorem 2, under the assumptions given there. If g is an element with CRNF and $h \in \mathcal{G}$ satisfies $g^n = h^n$ for some positive integer n , h must also have CRNF (this follows from a simple argument involving the length of the elements and their powers and the fact that the power of a CR element h , after reduction to normal form, must start and end with elements of the same type, i.e. from \mathcal{I} or \mathcal{J} , as h itself). In fact, the only possibility is $h = bg$ for some $b \in \mathcal{B}$. Clearly, h commutes with g^n . This implies that $b \in \mathcal{S}(g^n)$, and b must be of finite order (since otherwise $\langle b, g^n \rangle \simeq C_\infty \times C_\infty$, which is impossible). By theorem 2, either $b = 1$ (whence $b \in \mathcal{S}(g)$) or b is the unique involution in $\mathcal{S}(g^n)$. In the latter case, also gbg^{-1} is an involution in $\mathcal{S}(g^n)$, hence $gbg^{-1} = b$ by uniqueness, and $b \in \mathcal{S}(g)$. Since this applies to general CR elements by conjugacy, we have the following.

Fact 7. *If K is a field with $\text{char}(K) = 0$ and $U_K \simeq C_2$, a CR element $f \in \mathcal{G}$ has at most one n th root in \mathcal{G} for n odd, and at most two for $n > 0$ even. If two roots exist, one is obtained from the other by multiplication with the unique involution in $\mathcal{S}(f)$. \square*

Corollary 1. *Let the assumptions be as in theorem 2, with $f \in \mathcal{G}$ a CR element. Then, $\mathcal{S}(f) \simeq \mathcal{F} \times C_\infty$, where \mathcal{F} is either the trivial group or C_2 , and C_∞ is generated by a CR element. In particular, $\mathcal{S}(f)$ is Abelian.*

Proof. If $\mathcal{S}(f)$ contains any nontrivial element of finite order at all, say s , it must be an involution and is unique, due to theorem 2. For an arbitrary $h \in \mathcal{S}(f)$, hsh^{-1} is also an involution, hence $hsh^{-1} = s$. So, s commutes with all elements of $\mathcal{S}(f)$ and is thus an element of its centre. Moreover, s is conjugate to I by theorem 2.

No element of $\mathcal{S}(f)$ other than 1 and possibly s can be of finite order. In fact, they must all be CR elements (otherwise, we would obtain an Abelian subgroup of the form $C_\infty \times C_\infty$, which is impossible). If $g \neq 1$ is such an element, we know from part (3) of lemma 1 that $f = cbc^{-1}d^k$ and $g = cb'c^{-1}d^\ell$ with $b, b' \in \mathcal{B}$, $d \in \mathcal{G}$ and suitable $k, \ell \in \mathbb{Z}$. Also, cbc^{-1} , $cb'c^{-1}$ and d pairwise commute, so all must be elements of $\mathcal{S}(f)$. Consequently, each of b and b' can only be 1 or conjugate to s , while d must be a CR element (and $k, \ell \neq 0$).

Let us now, without loss of generality, assume that f has CRNF, so $\text{len}(f) = 2n$ with $n \geq 1$. This implies that the equation $f = h^m$, with $h \in \mathcal{G}$, can at most have a solution if m divides n and if h is another element with CRNF. Clearly, h itself commutes with f . An analogous restriction applies to the equation $f = sh^m$ with s an involution from $\mathcal{S}(f)$, because then $sf = h^m$, and h commutes with $sf = fs$.

In both cases, we can invoke lemma 1 once more. Since f , by fact 7, can only have one odd root and at most two even roots, there must be a fundamental element h which, possibly together with the unique involution s , can be used for *all* symmetries g of infinite order, so that $g = s^\epsilon h^m$ for some $\epsilon \in \{0, 1\}$ and some $m \in \mathbb{Z} \setminus \{0\}$, with m even if $\epsilon = 1$. This shows that $\mathcal{S}(f) = \mathcal{F} \times \langle h \rangle$, where \mathcal{F} is the trivial group or C_2 , and $\langle h \rangle \simeq C_\infty$. \square

Remark. It would be interesting to know whether $\mathcal{S}(f)$ is always Abelian for the case $K = \mathbb{C}$.

In view of theorem 2, and also for later use as potential reversors, it is of particular interest to know the involutions in \mathcal{G} , up to conjugacy. Since for $\text{char}(K) = 2$ one has $1 = -1$, so that there are no 2^k th roots of unity except 1, usually no involutions (or elements of order 2^k) exist in $\text{GL}(2, K)$ that are of interest to us here (though new involutions in \mathcal{G} will show up, such as the elements of \mathcal{J}). Consequently, we will exclude fields of characteristic 2 in what follows.

Lemma 3. *If K is a field with $\text{char}(K) \neq 2$, the possible involutions in \mathcal{E} are*

$$e: \quad x' = -x + P(y), \quad y' = y \quad (16)$$

with arbitrary polynomial $P \in K[y]$, or

$$e : \quad x' = \alpha x + P(y), \quad y' = -y + v \quad (17)$$

with arbitrary $v \in K$, $\alpha \in \{\pm 1\}$ and a polynomial $P \in K[y]$ that satisfies $P(v-y) = -\alpha P(y)$.

Moreover, if K is a field of characteristic 0 with $U \simeq C_2$, any element of \mathcal{E} of finite order is either the identity or an involution.

Proof. Consider $e \in \mathcal{E}$, parametrized as in (7). Then, the first claim is a straightforward calculation around the equation $e^2 = 1$.

For the second claim, write e^n , for integer $n \geq 0$, as $x \mapsto x_n$ and $y \mapsto y_n$. Setting $v_0 = 0$ and $v_n = (1 + \beta + \dots + \beta^{n-1})v$ for $n \geq 1$, one has $y_n = \beta^n y + v_n$, and a direct calculation gives

$$x_n = \alpha^n x + \sum_{\ell=0}^{n-1} \alpha^{n-1-\ell} P(y_\ell).$$

Clearly, $e^n = 1$ implies $\alpha^n = \beta^n = 1$, $v_n = 0$ and $\sum_{\ell=0}^{n-1} \alpha^{n-1-\ell} P(y_\ell) = 0$. In particular, n odd is impossible due to $U \simeq C_2$, unless $\alpha = \beta = 1$.

The case $\beta = 1$ means $v_n = nv$, hence $v = 0$ because $\text{char}(K) = 0$. If also $\alpha = 1$, the polynomial must be $P = 0$, and hence $e = 1$. If $\alpha = -1$, $e^n = 1$ is true for all even n and arbitrary P , but one actually has $e^2 = 1$.

For the case $\beta = -1$, one has $v_n = 0$ for all even n , and $e^n = 1$ follows for all $P \in K[y]$ with $\alpha P(y) + P(v-y) = 0$. Clearly, one has $e^2 = 1$ in these cases, too. \square

This classifies the involutions in \mathcal{E} for $\text{char}(K) \neq 2$. The involutions in \mathcal{A} are the elements of the form $(a, M) \neq (0, 1)$ that satisfy

$$M^2 = 1 \quad \text{and} \quad Ma = -a. \quad (18)$$

Investigating the first of these requirements, we find the following lemma.

Lemma 4. *Let K be an arbitrary field with $\text{char}(K) \neq 2$. If M is an involution in $\text{GL}(2, K)$, it is either $I = \text{diag}(-1, -1)$, or it is $\text{GL}(2, K)$ -conjugate to $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or, equivalently, to $S = \text{diag}(-1, 1)$. Moreover, if an involution $M \neq I$ is not upper triangular, it is conjugate to T by a matrix $A \in \mathcal{T}$, the subgroup of $\text{GL}(2, K)$ of invertible upper triangular matrices.*

Proof. Consider the equation $M^2 = 1$ with $M \in \text{GL}(2, K)$, which is clearly solved by I . An easy direct calculation shows that all other solutions satisfy $\text{tr}(M) = 0$ and $\det(M) = -1$, hence share the characteristic polynomial $P(x) = x^2 - 1 = (x+1)(x-1)$. As this is then the minimal polynomial (we excluded $\text{char}(K) = 2$), they also share all polynomial invariants and must have the same rational canonical form, compare [1, chapter 4.4]. Consequently, they are all $\text{GL}(2, K)$ -similar to the Frobenius companion matrix of $P(x)$, which is T . In particular, S is conjugate to T in $\text{GL}(2, K)$.

If $M \neq I$ is an involution, we know that $M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a^2 + bc = 1$. If M is not upper triangular, we also know that $c \neq 0$. Then, it is easy to check that $A = \begin{pmatrix} c & -a \\ 0 & 1 \end{pmatrix}$, which lies in \mathcal{T} , satisfies $AMA^{-1} = T$. \square

Continuing the investigation of affine involutions, but now also considering how the involutions of lemma 3 are related to linear ones, we obtain (compare also [19, p 57]) the following proposition.

Proposition 7. *If K is a field with $\text{char}(K) \neq 2$, all involutions in $\mathcal{A} \cup \mathcal{E}$, and hence in \mathcal{G} , are conjugate to linear maps. More concretely, they are conjugate to either $I = \text{diag}(-1, -1)$ or to $S = \text{diag}(-1, 1)$, equivalently to $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where $T \in \mathcal{T}$ of (10).*

Proof. First, consider the affine involutions that must satisfy (18). From lemma 4, one possibility based around $M = I$ is $g \in \mathcal{B}$ defined by $x' = -x + u$, $y' = -y + v$, with arbitrary $u, v \in K$. Taking $h \in \mathcal{B}$ via $x' = x - u/2$ and $y' = y - v/2$, one finds hgh^{-1} is the linear map defined by I . The second possibility for affine involutions, from lemma 4, consists of those that are $\text{GL}(2, K)$ -conjugate to $g \in \mathcal{A}$ defined by $x' = y + u$ and $y' = x - u$ (noting from (18) that the entries of \mathbf{a} must have opposite signs when $M = T$). However, this g is itself conjugate in \mathcal{B} , via the above-mentioned h with $v = -u$, to the linear map defined by T . Finally, it is clear that T is conjugate in $\text{GL}(2, K)$ to the matrix S .

We turn now to the elementary involutions as described in lemma (3). Consider the involution $e \in \mathcal{E}$ from (16). Defining $h \in \mathcal{E}$ by $x' = x - P(y)/2$ and $y' = y$, one can easily check that heh^{-1} is the linear map defined by S . This establishes a conjugacy within \mathcal{E} .

Next, consider $e \in \mathcal{E}$ from (17), with $\alpha = \{\pm 1\}$, $v \in K$ and $P(v - y) = -\alpha P(y)$. Define $h \in \mathcal{G}$ via $x' = y - v/2$ and $y' = x + P(y)/2\alpha$ (which has the inverse h^{-1} given by $y' = x + v/2$ and $x' = y - P(y')/2\alpha$). A short calculation using the symmetry property of P confirms that heh^{-1} is the linear map defined by the matrix $\text{diag}(-1, \alpha)$, hence either I or S . \square

Comparing proposition 7 and lemma 4, it is worth pointing out that it is sometimes useful in deriving normal forms to have the freedom to use either of the $\text{GL}(2, K)$ -conjugate matrices S or T , where T is an element of \mathcal{I} , while S is not. This is particularly true when we study reversing symmetries in the following section (cf proposition 11). In the case of symmetries, I is the important involution, as will turn out shortly.

It will prove useful to define the so-called *poly-degree* of an element $g \in \mathcal{G} \setminus \mathcal{A}$. If g is in normal form (13), the poly-degree is defined by

$$\text{pol deg}(g) := (\text{deg}(e_m), \dots, \text{deg}(e_2), \text{deg}(e_1)), \quad (19)$$

where we drop the last entry if e_1 is missing in the normal form.

Theorem 3. *Let K be a field with $\text{char}(K) \neq 2$ and let $f \in \mathcal{G}$ be a CR element. If f has a symmetry in \mathcal{G} which is an involution, this symmetry is conjugate to $I = \text{diag}(-1, -1)$. Also, f is conjugate to an element with CRNF*

$$f' = b \circ a_m \circ e_m \circ \dots \circ a_1 \circ e_1. \quad (20)$$

In the expression (20), $m \geq 1$, $a_i \in \mathcal{I}$ of (10), $e_i \in \mathcal{J}$ of (11) must have $P_i(y)$ odd, e_1 and a_m must appear and b is linear of the form $b = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$. It follows that, when f has such an involutory symmetry, it is conjugate to a cyclically reduced element f' which fixes the origin and has $\text{pol deg}(f') = (n_m, \dots, n_1)$, where all n_i are odd integers ≥ 3 .

If K is a field of characteristic 0 with group of roots of unity $U \simeq C_2$ (which includes the cases $K = \mathbb{Q}$ and $K = \mathbb{R}$), this gives, up to conjugacy, the description of all finite order symmetries and the corresponding normal form of f .

Proof. Suppose f has an involutory symmetry. With proposition 7, we can write $f(h^{-1}ih) = (h^{-1}ih)f$, where $h \in \mathcal{G}$ and i is the linear map defined by I or S . Consequently, we have $(hfh^{-1})i = i(hfh^{-1})$, so that a conjugate of f , necessarily also CR, commutes with I or S . Since $\text{char}(K) \neq 2$, the equation $2k = 0$ with $k \in K$ has only the trivial solution, so that K^2 cannot contain an involution. Thus, we are in the situation of case (2) of proposition 5. Consider $M = \text{diag}(u, u^m)$ with $u \in U(n)$ and n, m coprime. M can only be an involution if $u = -1$ which implies that n must be even. Then, m must be odd and $M = I$ is the only

possibility, while S is ruled out—a result that can also be obtained by some lengthy explicit calculations with the normal forms.

So let us characterize those CR elements f' that commute with I , equivalently those that satisfy $If'I = f'$. We take for f' an expression of the form (13) and observe that I commutes with elements $a_i \in \mathcal{I}$ of (10), whereas for $e_i \in \mathcal{J}$ of (11) we have $e_i I = I e'_i$, with e'_i obtained from e_i by the replacement $P_i(y) \rightarrow -P_i(-y)$. Note that e'_i is still an element of \mathcal{J} . Also, $b' := IbI$ is still an element of \mathcal{B} .

The uniqueness of the normal form (13) for f' applied to $If'I = f'$ forces $e'_i = e_i$ and $b' = b$, hence the odd degree constraint $P_i(y) = -P_i(-y)$ in e_i together with $IbI = b$. The latter implies that b is linear, so f' fixes the origin. If the normal form for f' so found is cyclically reduced, at least one e_i and one a_i must be present by definition. Certainly, it can be brought to the form (20), possibly after a further conjugation by an element of \mathcal{I} . This conjugation leaves the symmetry I unchanged, so the leading basic element of the new normal form can remain linear. If f' is not already cyclically reduced, further conjugations by a_i and e_i with odd $P_i(y)$ can be used to obtain (20). Again, these will leave the symmetry as I because they both commute with it. Thus, these additional conjugations, if required, will preserve the linear nature of the leading basic element and the oddness of the polynomials in the elementary coset representatives.

The last statement of the theorem is simply a reminder from theorem 2 of the stronger statement that can be made under these circumstances. \square

Remark. If f is a CR element, but not cyclically reduced to begin with, a cyclically reduced element \tilde{f} conjugate to f can always be found in an algorithmic fashion. The poly-degree of any such element can be used to check the necessary condition given earlier on the odd entries in $\text{pol deg}(f')$. This follows since $\text{pol deg}(f')$ must be the same, up to a cyclic permutation, as the poly-degree of the cyclically reduced element \tilde{f} (from part (3) of proposition 2). In [28], as an illustration of theorem 3, we showed by explicit calculation that the CR elements $f \in \mathcal{G}_{\mathbb{R}}$ of the generalized standard form (5) could only have symmetries of finite order conjugate to I . This occurred when both P_1 and P_2 were odd.

However, even if a cyclically reduced element that f is conjugate to satisfies the above poly-degree requirement, a further decisive test for an involutory symmetry still follows from theorem 3 together with proposition 2.

Corollary 2. *Let K be a field with $\text{char}(K) \neq 2$ and let $f \in \mathcal{G}$ be a CR element. Then, f has a symmetry that is an involution iff any cyclically reduced word to which f is conjugate commutes with $x' = -x + u$, $y' = -y + v$, with some $u, v \in K$. If this cyclically reduced word corresponds to (3), this commutation means P and Q satisfy $P(-x + u, -y + v) + P(x, y) = u$ and $Q(-x + u, -y + v) + Q(x, y) = v$.*

Proof. Let \tilde{f} be a cyclically reduced word with $f = h\tilde{f}h^{-1}$ and let f have an involutory symmetry (take $h = 1$ if f is already cyclically reduced). From theorem 3, we also know that f is conjugate to a cyclically reduced word in normal form, i.e. f' of (20), and that f' commutes with $I = \text{diag}(-1, -1)$ by construction. It follows that \tilde{f} and f' are two cyclically reduced words that are themselves conjugate. By proposition 2, \tilde{f} differs by a cyclic permutation of the elements of f' , followed by conjugation with a basic element (9). The cyclic permutation is itself a conjugacy by elements a_i and e_i of (20). It follows that \tilde{f} commutes with a conjugate of I , indeed the same conjugacy used to derive \tilde{f} from f' . As a_i and e_i of (20) commute with I , the only conjugacy that can alter the symmetry of \tilde{f} away from I is the one by a basic element. One easily checks, for b in the form (9), that bIb^{-1} differs from I by at most a translation.

The last statement of the result follows from forcing the form (3) to commute with such an involution. \square

Remark. The previous result shows that, when one deals with a cyclically reduced element \tilde{f} of \mathcal{G} , the presence or absence of an involutory symmetry is, in some sense, obvious. If present, it must be of a very simple linear (or affine) form. Inspecting the phase portrait for the case $K = \mathbb{R}$ of \tilde{f} , one must see the invariance by a rotation through π around some fixed point as a prerequisite for the existence of any finite order symmetry (the necessity of the existence of a common unique fixed point of both \tilde{f} and the possible involutory symmetry, if present, follows from theorem 3).

Another useful result, which we will need later, concerns the conjugacy of linear maps within the group \mathcal{G} .

Lemma 5. *Let $f, g \in \mathcal{G}$ be linear maps, defined by the matrices $A_f, A_g \in \text{GL}(2, K)$. If $f = hgh^{-1}$ for some $h \in \mathcal{G}$, then A_f and A_g are already conjugate within $\text{GL}(2, K)$.*

Proof. Observe first that $(dh(\mathbf{a}))^{-1} = dh^{-1}(h(\mathbf{a}))$, for arbitrary $\mathbf{a} \in K^2$, which follows from the chain rule applied to $h^{-1}h = 1$. Since $df \equiv A_f$ and $dg \equiv A_g$, one then derives from differentiating $f = hgh^{-1}$ at the point $\mathbf{a} = h(\mathbf{0})$ that

$$A_f = dh(\mathbf{0})A_g(dh(\mathbf{0}))^{-1},$$

where $dh(\mathbf{0})$ is clearly an element of $\text{GL}(2, K)$. \square

Remark. We made use of the formal differentiation rules for polynomials here. If one is in a setting where diffeomorphisms are well defined, the claim can be extended accordingly.

5. Reversing symmetries

Recall that we denote the reversing symmetry group of an element $f \in \mathcal{G}$ by

$$\mathcal{R}(f) = \{h \in \mathcal{G} \mid hfh^{-1} = f^{\pm 1}\}.$$

This group contains the symmetry group $\mathcal{S}(f)$ as a normal subgroup, and the factor group $\mathcal{R}(f)/\mathcal{S}(f)$ is either the trivial group or C_2 . In general, it is difficult to determine these groups explicitly, but if one is in a group theoretic setting (as we are), one can at least determine the *structure* of the reversing symmetry group to some extent. This, of course, need only be done up to conjugacy, because $\mathcal{R}(hfh^{-1}) = h\mathcal{R}(f)h^{-1}$. As before, we shall focus on elements $f \in \mathcal{G}$ of infinite order, and on CR elements in particular. This means that it actually suffices to look at elements that possess a CRNF.

Let us start with a general observation, which is a rather direct consequence of a result of Goodson, see [14, proposition 2] and the generalization mentioned afterwards. We use the general group theoretic setting mentioned in the introduction.

Lemma 6. *Let f be an element of infinite order, and assume that $\mathcal{S}(f) = \mathcal{F} \times \langle g \rangle$ where \mathcal{F} is some finite group of order N (not necessarily Abelian), and g is some generator (then necessarily of infinite order). If r is a reversor of f , then r is an element of finite order. Its order is even and divides $2N$.*

Proof. If r is a reversor, r^2 is a symmetry, hence $r^2 = sg^m$, for some $s \in \mathcal{F}$ and some integer m . Note that, due to the assumption of the direct product structure, we always have $sg = gs$, even if \mathcal{F} itself is not Abelian. Since the group \mathcal{F} is finite and of order N , we know that $s^n = 1$ for some $n \neq 0$ that divides N . Clearly, we then have $r^{2n} = g^{mn}$.

As f is not of finite order, but clearly an element of $\mathcal{S}(f)$, we may assume $f^N = g^k$ for some (positive) integer k without loss of generality, modifying the argument just used (in particular, $k \neq 0$, while $k > 0$ might require the replacement of g by g^{-1}).

Since $rf = f^{-1}r$ by assumption (hence also $rf^\ell = f^{-\ell}r$, for all $\ell \in \mathbb{Z}$), we choose $\ell = mnN$ and obtain $rg^{kmn} = g^{-kmn}r$. Since $g^{kmn} = r^{2nk}$, this implies $rr^{2nk} = r^{-2nk}r$ and thus $r^{4nk} = 1$, i.e. r is of finite order. Since $r^{2n} = g^{mn}$, this is only possible for $mn = 0$, hence $m = 0$. This implies $r^{2n} = 1$, so the order of r divides $2N$. If f is not of finite order, it is not an involution, and r can then not be of odd order [21, proposition 5] (hence also $r \neq 1$). \square

Theorem 4. *Let K be a field of characteristic 0 or a finite field, and let f be a reversible CR element of \mathcal{G} , with reversor r . Then, r is an element of finite even order.*

If $\text{char}(K) = 0$ and if, in addition, the roots of unity in K are $U = \{\pm 1\} \simeq C_2$, the reversor r is an involution or an element of order 4.

Proof. If r is a reversor, r^2 is a symmetry, so $r^2 \in \mathcal{S}(f)$. Consider the group $\langle f, r^2 \rangle$ which is Abelian, hence of type 3 in this case. Consequently, $\langle f, r^2 \rangle = \mathcal{F} \times \langle g \rangle$ with $\langle g \rangle \simeq C_\infty$ and \mathcal{F} a finite group, by proposition 5. Considering this as a ‘local’ symmetry group of f , within $\langle f, r \rangle$ say, we can invoke lemma 6 and conclude that r must be of finite even order.

If the additional assumptions on K are satisfied, the finite group \mathcal{F} is the trivial group or C_2 , and we can use lemma 6 to see that $r^4 = 1$. Since $r \neq 1$, it must be an involution or an element of order 4. \square

For fields K with suitable unit group U_K , reversors of arbitrary even order ≥ 2 may exist, as the following example illustrates.

Example. Consider a field K with $\text{char}(K) \neq 2$ and unit group U_K . Take once again $f \in \mathcal{G}_K$ of the generalized standard form (5) and look for a linear reversing symmetry r as defined by the matrix $\begin{pmatrix} 0 & \mu \\ \nu & 0 \end{pmatrix} \in \text{GL}(2, K)$. Its square is $\text{diag}(\lambda, \lambda)$ with $\lambda = \mu\nu$, where we assume that $\lambda \in U_K$. One finds that $r \circ f = f^{-1} \circ r$ if and only if

$$P_1(\nu z) = -\mu P_2(z) \quad \text{and} \quad P_2(\mu z) = -\nu P_1(z),$$

which also implies that $P_i(\lambda z) = \lambda P_i(z)$ for $i \in \{1, 2\}$. Consequently, r^2 is a symmetry of the kind explained in the example preceding theorem 2.

The nontrivial solutions once again occur for λ a primitive n th root of unity, for some $n \in \mathbb{N}$, provided such a λ exists in U_K . For $K = \mathbb{C}$, solutions exist for all $n \in \mathbb{N}$. In these cases, the order of r is $2n$. If $\nu = -\mu$, the polynomial condition is satisfied if $P_1(z) = P_2(z) = zQ(z^n)$, with some polynomial Q .

Remark. For the case $K = \mathbb{C}$, [13] contains a comprehensive treatment of reversors of even order, with illustrative examples. Also, [13, theorem 11] gives a constructive proof of theorem 4 above for $K = \mathbb{R}$.

Theorem 4 motivates the benefit of knowing what possibilities there are for elements of order 2 and 4 in our group \mathcal{G} . We have discussed the situation of elements of order 2 in section 4, which we will use once more below. Let us now look into the remaining case when f is reversible with a reversor r of order 4 (note that we do not necessarily require $\text{char}(K) = 0$, although it provides an obvious motivation for this case).

Theorem 5. *Let K be a field with $\text{char}(K) \neq 2$, with a unit group U that contains $\{\pm 1\}$, but no primitive 4th root of unity (thus including the case $U \simeq C_2$). Let $f \in \mathcal{G}$ be a reversible CR element, with a reversor r of order 4. Then, r is conjugate to the linear map (from $\mathcal{A} \setminus \mathcal{B}$) defined by the matrix $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, hence r^2 is conjugate to $R^2 = I$.*

Proof. Since r^2 is an involution and commutes with the CR element f , theorem 3 shows that r^2 must be conjugate to $I = \text{diag}(-1, -1)$. Since r itself is of finite order, it must be conjugate to an element of \mathcal{A} or \mathcal{E} , by fact 4. However, using the formulae given in the proof of lemma 3 for e^n when $e \in \mathcal{E}$, one deduces that there can be no genuine order 4 element that is elementary if primitive 4th roots of unity are absent. In particular, this excludes $e \in \mathcal{B}$, as the appendix also reveals. So, r is conjugate to an element $(\mathbf{a}, M) \in \mathcal{A} \setminus \mathcal{B}$, with $M^2 = I$.

Clearly, the matrix R from the statement satisfies $R^2 = I$, so it is a root of I in $\text{GL}(2, K)$. Moreover, all other roots of I in $\text{GL}(2, K)$ are conjugate to R in $\text{GL}(2, K)$. To see this, observe first that any $M \in \text{GL}(2, K)$ with $M^2 = I$ must satisfy $\text{tr}(M) = 0$ and $\det(M) = 1$. This follows from a simple direct calculation, which uses that $x^2 = -1$ has no solution in K . So, all solutions share the characteristic polynomial $P(x) = \det(x - M) = x^2 + 1$. This polynomial is irreducible over K (by the assumption on U), but splits as $P(x) = (x - i)(x + i)$ over the algebraic closure \hat{K} of K , with i being a root of -1 in \hat{K} , which cannot be in U and hence not in K . This implies that $P(x)$ is also the minimal polynomial of all the possible solutions. Consequently, they all have the same polynomial invariants, hence are similar to one another, and also to the Frobenius companion matrix of $P(x)$, which is the matrix R (see [1, chapter 4.4] for details).

Returning now to r , we have $r = h(\mathbf{a}, M)h^{-1}$ for some $h \in \mathcal{G}$, $\mathbf{a} \in K^2$ and $M \in \text{GL}(2, K)$ with $M^2 = I$. Since 1 is not in the spectrum of M , $1 - M$ is invertible. With $\mathbf{c} = (1 - M)^{-1}\mathbf{a}$, it is easy to check that

$$(\mathbf{c}, 1)(0, M)(-\mathbf{c}, 1) = (\mathbf{a}, M),$$

so that (\mathbf{a}, M) is conjugate, in \mathcal{A} , to the linear map defined by M . Now, putting things together, r is conjugate to M within \mathcal{G} and, possibly employing one more $\text{GL}(2, K)$ -conjugation, also to the linear map defined by the matrix R , as claimed. \square

From theorem 4, we can derive the possible structures of $\mathcal{R}(f)$, e.g. for $K = \mathbb{R}$.

Corollary 3. *Let K be a field of characteristic 0, with $U_K \simeq C_2$. If $f \in \mathcal{G}$ is a reversible CR element, $\mathcal{R}(f)$ is one of the groups $D_\infty \simeq C_\infty \rtimes C_2$, $C_\infty \rtimes C_4$, or $(C_\infty \times C_2) \rtimes C_2$ (the last group comprising two different cases).*

Proof. By theorem 2 and corollary 1, we have either $\mathcal{S}(f) \simeq C_\infty$ or $\mathcal{S}(f) \simeq C_2 \times C_\infty$. If $\mathcal{S}(f) \simeq C_\infty$, a reversor r of f must be an involution, whence $\mathcal{R}(f) \simeq C_\infty \rtimes C_2 \simeq D_\infty$.

Let $\mathcal{S}(f) \simeq C_2 \times C_\infty$ with involutory symmetry s , which is then unique by theorem 2, and $C_\infty = \langle h \rangle$. If the reversor r is an involution, one has $\mathcal{R}(f) \simeq \mathcal{S}(f) \rtimes C_2$. Since rsr is also an involutory symmetry, we get $rsr = s$ by uniqueness, and r and s commute. Since $r \neq s$, this gives $\mathcal{R}(f) \simeq (C_\infty \times C_2) \rtimes C_2$, with either $rhr^{-1} = h^{-1}$ (then giving $\mathcal{R}(f) \simeq D_\infty \times C_2$) or $rhr^{-1} = sh^{-1}$ (in which case f must be an even power of h). Note that, in the latter case, $\rho = hr$ is an element of order 4 and a reversor for f .

If f has a reversor r of order 4, r^2 is an involutory symmetry of f , hence unique and conjugate to $I = R^2$ with R of theorem 5. This implies $\mathcal{S}(f) \simeq C_2 \times C_\infty$ by corollary 1, with $C_2 = \langle r^2 \rangle$, $C_\infty = \langle h \rangle$ and $f = r^{2\epsilon}h^m$ for $\epsilon \in \{0, 1\}$ and some integer $m \neq 0$. In particular, r^2 and h commute, and rhr^{-1} is a symmetry of f , so that $rhr^{-1} = r^{2k}h^\ell$ for $k \in \{0, 1\}$ and some $\ell \in \mathbb{Z}$. Clearly, in view of $rf r^{-1} = f^{-1}$, this forces $\ell = -1$.

If $k = 0$, r is also a reversor for h , and we have $\mathcal{R}(f) \simeq C_\infty \rtimes C_4$. This is the only case for m odd, while for m even $k = 1$ is also possible, i.e. $rhr^{-1} = r^2h^{-1}$. This gives a group with the presentation

$$\mathcal{R}(f) = \langle r, h \mid r^4 = 1, r h^{\pm 1} = h^{\mp 1} r^{-1} \rangle$$

which is an index 2 extension of $\mathcal{S}(f) \simeq C_\infty \times C_2$, but does not look like a simple semi-direct product. However, $\eta = h^{-1}r$ is an involution that satisfies $\eta h \eta = r^2 h^{-1}$ and is a reversor of f . This brings us back to $\mathcal{R}(f) \simeq (C_\infty \times C_2) \rtimes C_2$, where the outer C_2 is generated by η . \square

Examples of CR elements $f \in \mathcal{G}_\mathbb{R}$ of the generalized standard form (5) illustrating all except the second possibility of corollary 3 are given in [28, table 6] (in particular, one can extract examples for both subcases of the third group). To find an example of the remaining group structure (i.e. $C_\infty \rtimes C_4$), the simplest way [15] is to consider $f = r e r e^{-1}$ with $e: x' = x + y^3, y' = y$ (which commutes with I) and the linear map r defined by the matrix R of theorem 5. Then, f is reversible with reversor r , but has no root in $\mathcal{G}_\mathbb{R}$ (though it has a root in $\mathcal{G}_\mathbb{C}$, which then changes $\mathcal{S}(f)$ and $\mathcal{R}(f)$ in $\mathcal{G}_\mathbb{C}$). The structure of this example will become more transparent from fact 8 and proposition 12.

We now give various characterizations of reversible CR elements. One algebraic condition can be formulated via the poly-degree introduced in (19). If we define the *reversal* of a finite sequence of integers as $(n_1, \dots, n_k) := (n_k, \dots, n_1)$, we observe in the following lemma.

Lemma 7. *For all $g \in \mathcal{G} \setminus \mathcal{A}$, one has $\text{pol deg}(g^{-1}) = \overline{\text{pol deg}(g)}$.*

Proof. Assume g is written in normal form. Its inverse is then a word in affine and elementary mappings, potentially with an element from \mathcal{B} at the rightmost position. This gives a new sequence of degrees, noting only those of the elementary mappings. Since e and e^{-1} have the same degree (compare (7) and (8)), for all $e \in \mathcal{E}$, this new sequence is nothing but $\overline{\text{pol deg}(g)}$.

This sequence of degrees is not changed if the representation of g^{-1} is now brought to normal form, by pulling the \mathcal{B} -element to the left and replacing, position by position, the mappings by the proper representatives from \mathcal{I} and \mathcal{J} . So, $\overline{\text{pol deg}(g)}$ is actually the poly-degree of g^{-1} , which proves the claim. \square

This enables us to formulate a rather restrictive necessary condition for the reversibility of CR elements in \mathcal{G} .

Proposition 8. *Let the normal form of $g \in \mathcal{G}$ be cyclically reduced, which is then also true of the element g^{-1} . A necessary condition for the reversibility of g is that $\text{pol deg}(g^{-1})$, which is the reversal of $\text{pol deg}(g)$, is a cyclic permutation of $\text{pol deg}(g)$.*

If, more generally, g' is a CR element, it is conjugate to some element g with CRNF. The necessary condition for g' is then that the previous condition is met by g . The outcome does not depend on the choice of g .

Proof. Let g have a CRNF, which is then of length $2m$ with $m \geq 1$, so that $\text{pol deg}(g)$ is a sequence of length m (recall that the poly-degree only keeps track of the elementary maps). From lemma 7, we know that $\text{pol deg}(g^{-1}) = \overline{\text{pol deg}(g)}$, and the statement about g now follows from the result about conjugacy, see part (3) of proposition 2.

If g' is a CR element, we cannot apply the criterion directly, but we can pick any representative g of the conjugacy class of g' with CRNF. Since g' is reversible if and only if g is, the necessity of the claimed condition is obvious. It does not depend on the choice of the representative because the poly-degrees of different representatives are cyclic permutations of one another. \square

Example. Suppose g' is a CR element, conjugate to a g in CRNF. If g contains up to two elements $e_i \in \mathcal{J}$, then proposition 8 does not restrict $\text{pol deg}(g)$ for g (and g') to be reversible (because any sequence of up to two integers is a cyclic permutation of its reversal). However, restrictions generically arise when g contains three or more elements of \mathcal{J} . For instance,

if g has poly-degree $(2, 3, 4)$, it can never be reversible. This corresponds, in fact, to the lowest degree of g (i.e. $2 \cdot 3 \cdot 4 = 24$) for which $\text{pol deg}(g)$ alone can be exploited to rule out reversibility.

We now proceed to describe, in more detail, the nature of reversible elements of \mathcal{G} , which will lead ultimately to the normal forms of propositions 11 and 12 for elements with involutory and order 4 reversors, respectively.

Proposition 9. *If $g \in \mathcal{G}$ has a reversor $r \in \mathcal{G}$, then $\det(\text{dg}) = \pm 1$.*

Proof. By assumption, $g^{-1} = rgr^{-1}$ with $g, r \in \mathcal{G}$. Since the Jacobians of polynomial automorphisms have constant determinant, a simple application of the chain rule gives $\det(\text{dg}^{-1}) = \det(\text{dg})$, hence $\det(\text{dg})^2 = 1$, which gives the claim. \square

In view of the remark after proposition 1, reversibility puts an immediate restriction on the normal form.

Corollary 4. *A necessary condition for $g \in \mathcal{G}$ to be reversible is that the element $b \in \mathcal{B}$ of its normal form (13) satisfies $\det(\text{db}) = \pm 1$.* \square

Some further restrictions emerge for mappings which possess fixed points.

Proposition 10. *Let g be reversible, with reversor r . If \mathbf{a} is a fixed point of g , the Jacobian matrices $\text{dg}(\mathbf{a})$ and $\text{dg}(r\mathbf{a})$ must have reciprocal spectrum.*

Proof. Since $g^{-1} = rgr^{-1}$, the chain rule (evaluated at the point $r\mathbf{a}$) gives

$$\text{dg}^{-1}(r\mathbf{a}) = \text{dr}(\mathbf{a}) \text{dg}(\mathbf{a}) \text{dr}^{-1}(r\mathbf{a}).$$

Since $\text{dr}(\mathbf{a})$ and $\text{dr}^{-1}(r\mathbf{a})$ are the inverses of each other (visible from the chain rule applied to $r^{-1}r = 1$), $\text{dg}^{-1}(r\mathbf{a})$ and $\text{dg}(\mathbf{a})$ are isospectral.

Observing $g^{-1}r\mathbf{a} = r\mathbf{a} = r\mathbf{a}$ and applying the chain rule to $gg^{-1} = 1$, one sees that $\text{dg}^{-1}(r\mathbf{a})$ is the inverse of $\text{dg}(r\mathbf{a})$, from which the claim follows. \square

To continue, we recall the following helpful factorization property from [21], formulated within the automorphism group of some space. It will also shed more light on the examples discussed after corollary 3.

Fact 8. *An automorphism L is reversible, with reversor W , if and only if some automorphism V exists such that $L = VW^{-1}$ together with $V^2 = W^2$. In this case, V is also a reversor.*

Proof. If W is a reversor of L , define $V = LW$, which is invertible. Clearly, $L = VW^{-1}$, and $V^2 = (LW)^2 = W^2$, as a consequence of the relation $WLW^{-1} = L^{-1}$. Also, one quickly checks that $VLV^{-1} = L^{-1}$. Conversely, assuming $L = VW^{-1}$ with $V^2 = W^2$, the last two relations follow immediately. \square

We now consider a normal form for reversible elements of \mathcal{G} which have a reversing symmetry that is an involution. Via fact 8, it follows that an automorphism is reversible with an involutory reversor if and only if it is the product (i.e. composition) of two involutions (actually, this property goes back to Birkhoff [4] while fact 8 represents a generalization of it). Specializing to automorphisms in \mathcal{G} , recall that we know from proposition 7 that involutions are conjugate to one of two possibilities: $I = \text{diag}(-1, -1)$ or $S = \text{diag}(-1, 1)$, equivalently $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. There are advantages of taking T over S in normal forms since the former is in $\mathcal{A} \setminus \mathcal{B}$, indeed is in \mathcal{I} of (10). Note that I is orientation-preserving, whereas S and T are orientation-reversing. The canonical case of reversibility is that of area-preserving maps which are the composition of two orientation-reversing involutions. But the following result covers all possibilities, not just this one.

Proposition 11. *Let K be a field with $\text{char}(K) \neq 2$. A CR element $f \in \mathcal{G}$ is reversible with a reversor that is an involution if and only if f is conjugate to one of the following types of cyclically reduced normal forms:*

$$\tilde{e}_m \circ a_{m-1} \circ e_{m-1} \circ \cdots \circ a_1 \circ e_1 \circ T \circ e_1^{-1} \circ a_1^{-1} \circ \cdots \circ e_{m-1}^{-1} \circ a_{m-1}^{-1} \circ \tilde{e}_m^{-1} \circ T, \quad (21)$$

$$\tilde{e}_m \circ a_{m-1} \circ e_{m-1} \circ \cdots \circ a_1 \circ \hat{e} \circ \hat{a}_1^{-1} \circ \cdots \circ e_{m-1}^{-1} \circ a_{m-1}^{-1} \circ \tilde{e}_m^{-1} \circ T, \quad (22)$$

$$a_m \circ e_{m-1} \circ a_{m-1} \circ \cdots \circ e_1 \circ a_1 \circ \hat{e} \circ a_1^{-1} \circ e_1^{-1} \circ \cdots \circ a_{m-1}^{-1} \circ e_{m-1}^{-1} \circ a_m^{-1} \circ \bar{e}. \quad (23)$$

In these normal forms, $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $a_i \in \mathcal{I}$ of (10), $e_i \in \mathcal{J}$ of (11), \hat{e} and \bar{e} are particular cases of the involutions in $\mathcal{E} \setminus \mathcal{B}$ of the form (16) or (17) and $\tilde{e}_m = b \circ e_m$, with $e_m \in \mathcal{J}$ and $b \in \mathcal{B}$ a special case of (9) (as described further below). Each normal form has $\det = \pm 1$ depending on the involutions present. In each normal form, the only restriction on the appearance of a_i and e_i is that an e_i must occur if there is no elementary involution present, plus the form must be cyclically reduced. It follows that if f' is any cyclically reduced element conjugate to f , then $\text{pol deg}(f') = (n_m, \dots, n_1, \hat{n}, n_1, \dots, n_m, \bar{n})$, with all entries ≥ 2 and \hat{n} and \bar{n} absent or present according to the type of normal form above.

Proof. From the fact that f can be written as a composition of involutions, together with proposition 7, we have that $f \in \mathcal{G}$ is reversible with an involutory reversor if and only if $f = h_1 T_1 h_1^{-1} h_2 T_2 h_2^{-1}$ with $h_i \in \mathcal{G}$ and $T_1, T_2 \in \{T, I\}$. Hence, the conjugate of f given by $h_2^{-1} f h_2$ takes the form

$$h T_1 h^{-1} T_2 \quad (24)$$

with $h = h_2^{-1} h_1$. We can take h to be in the form (13). The cyclically reduced normal forms follow from working through the possible forms of h , and combinations of T_1 and T_2 . When T_1 and T_2 are different, it suffices to consider $T_1 = I$ and $T_2 = T$, since the reverse possibility is conjugate to it. A guiding principle, since f is assumed to be CR, is that there always remains an element of $\mathcal{E} \setminus \mathcal{B}$ and an element of $\mathcal{A} \setminus \mathcal{B}$ in $h T_1 h^{-1} T_2$ after any possible reductions into its cyclically reduced form. Furthermore, we need the following characterizations of involutions which follow from the proofs of lemma 4 and proposition 7 (with $\tilde{b} \in \mathcal{B}$ in (i)–(iii)):

- (i) An involution $a \in \mathcal{A} \setminus \mathcal{B}$ can be written $a = \tilde{b}^{-1} T \tilde{b}$ with \tilde{b} of the form $x' = \alpha x + \gamma y + u$ and $y' = y + u$.
- (ii) An involution $e \in \mathcal{E} \setminus \mathcal{B}$ conjugate to T or S can be written as $e = \tilde{b}^{-1} \tilde{e} \tilde{b}$ with $\tilde{e} \in \mathcal{E} \setminus \mathcal{B}$ of the form $x' = -x + y^2 Q(y)$, $y' = y$ or $x' = x + P(y)$, $y' = -y$ with $Q \neq 0$ and $P(y)$ odd of degree ≥ 3 .
- (iii) An involution $e \in \mathcal{E} \setminus \mathcal{B}$ conjugate to $I = \text{diag}(-1, -1)$ can be written as $e = \tilde{b}^{-1} \tilde{e} \tilde{b}$ with $\tilde{e} \in \mathcal{E} \setminus \mathcal{B}$ of the form $x' = -x + P(y)$, $y' = -y$ with $P(y)$ even of degree ≥ 2 .
- (iv) An involution in \mathcal{B} is conjugate in \mathcal{B} to one of S , $-S$ or I .

Note that in (ii)–(iv), \tilde{b} of (9) has $\alpha = \beta = 1$.

We illustrate the reduction first for $T_1 = T_2 = T \in \mathcal{I}$. First, suppose h ends in e_1 and begins with e_m , $m \geq 1$. Then, (24) is the cyclically reduced element (21) with \tilde{e}_m actually in \mathcal{J} . If h ends in e_1 and begins with $b \circ a_m$, a_m possibly missing, we conjugate (24) and consider $g = a_m^{-1} b^{-1} (h T h^{-1} T) b a_m$. This word ends with the affine involution $a = a_m^{-1} b^{-1} T b a_m$. If this involution belongs to $\mathcal{A} \setminus \mathcal{B}$, use characterization (i) to see that $\tilde{b} g \tilde{b}^{-1}$ takes the form (21), with $\tilde{e}_m = \tilde{b} e_m$. Now \tilde{e}_m is in $\mathcal{E} \setminus \mathcal{B}$, but possibly not in \mathcal{J} . Otherwise, the involution is $a = a_m^{-1} b^{-1} T b a_m \in \mathcal{B}$, conjugate in \mathcal{B} to $\pm S$ from (iv). Then, consider $g' = e_m^{-1} g e_m$ which ends with the elementary involution $e = e_m^{-1} a e_m$. If $e \in \mathcal{E} \setminus \mathcal{B}$, use characterization (ii) to see

that g' is conjugate to (22) with \hat{e} one of the forms described. Whereas, if $e \in \mathcal{B}$, one continues by considering $g'' = a_{m-1}^{-1}g'a_{m-1}$, which ends in the affine involution $a' = a_{m-1}^{-1}ea_{m-1}$ which is either in $\mathcal{A} \setminus \mathcal{B}$ or in \mathcal{B} . It is clear how this repeated process must eventually exhaust itself.

Next, suppose h in (24) takes the form $h = e_m \circ \cdots \circ e_1 \circ a_1$, i.e. h ends with an affine coset representative a_1 and begins with e_m , $m \geq 1$. Then, (24) contains the affine involution $a = a_1 T a_1^{-1}$. If $a \in \mathcal{A} \setminus \mathcal{B}$, use characterization (i) again and rewrite h in the form (13) to obtain (21). Otherwise, $a \in \mathcal{B}$ is conjugate to $\pm S$ and one moves on to study the elementary involution $e = e_1 a e_1^{-1}$. This process leads to a cyclically reduced word (22) with \hat{e} of characterization (ii) if $e \in \mathcal{E} \setminus \mathcal{B}$, or returns once more to the study of an affine involution $a_2 e a_2^{-1}$ etc.

Finally, consider the case that h ends in $a_1 \in \mathcal{I}$ but begins with $b \circ a_m$, whence we have $h = b \circ a_m \circ e_{m-1} \circ \cdots \circ e_1 \circ a_1$. Now one uses, in tandem, the combination of the earlier-mentioned procedures. One takes $g = a_m^{-1} b^{-1} h T h^{-1} T b a_m$ and sees that the processes will exhaust themselves in one of (21)–(23), with the elementary involutions occurring being those of characterization (ii).

The cases in (24) when $T_1 = T_2 = I$ and when $T_1 = I$ and $T_2 = T$ follow a similar, but simpler, path. This is because linear elements such as a_i commute with I . This leads to fewer cases that need to be considered. When $T_1 = T_2 = I$, we obtain (23) with both \hat{e} and \bar{e} of the form described in characterization (iii). When $T_1 = I$ and $T_2 = T$, we obtain (22) or (23) with \hat{e} of characterization (iii) and \bar{e} of characterization (ii).

We remark that, without loss of generality, the element $b \in \mathcal{B}$ occurring at the start of (21)–(23) can be chosen from the quotient of \mathcal{B} and the centralizer in \mathcal{B} of the last element of (21)–(23). For example, for (21) and (22), this gives an element (9) containing just three parameters instead of five. \square

Remark. The normal forms of proposition 11 are similar to those found in [12, theorem 1]. There, the authors express their cyclically reduced normal forms using compositions of Hénon maps $h_m \circ \cdots \circ h_1$ (and the inverse of such a composition) with h_i of (14), instead of our expressions earlier in terms of a_i and e_i .

Finally, we consider a normal form for reversible elements of \mathcal{G} with a reversor of order 4.

Proposition 12. *Let K be a field with $\text{char}(K) \neq 2$, with a unit group U that contains $\{\pm 1\}$, but no primitive 4th root of unity (thus including the case $U \simeq C_2$). A CR element $f \in \mathcal{G}$ is reversible with a reversor r of order 4 if and only if f is conjugate to the CRNF*

$$e_m \circ \cdots \circ a_1 \circ e_1 \circ R_1 \circ e_1^{-1} \circ a_1^{-1} \circ \cdots \circ e_m^{-1} \circ R_2. \quad (25)$$

Here, $R_1 = R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{A} \setminus \mathcal{B}$ and $R_2 = \begin{pmatrix} \alpha & -(\alpha^2+1)/\gamma \\ \gamma & -\alpha \end{pmatrix} \in \mathcal{A} \setminus \mathcal{B}$ (since $\gamma \neq 0$), the latter including $R_2 = R$ via $\alpha = 0$ and $\gamma = 1$. Moreover, $m \geq 1$, $e_i \in \mathcal{J}$ of (11) must have $P_i(y)$ odd and e_1 and e_m must appear. It follows that f necessarily has $\det(df) = 1$ and has a fixed point. Also, if f' is a cyclically reduced element conjugate to f , $\text{pol deg}(f') = (n_m, \dots, n_1, n_1, \dots, n_m)$ where n_i are odd integers ≥ 3 , and f' commutes with $x' = -x + u$, $y' = -y + v$, with some $u, v \in K$.

Proof. From fact 8, one can see that an automorphism L has a reversor W of order 4 if and only if $L = V W^{-1}$ with $V^2 = W^2$ and V also a reversor of order 4. Take in (25) $W^{-1} = R_2$ and for V the first term, conjugate to R_1 . We see that $V^2 = W^2 = \text{diag}(-1, -1)$ for either possibility of R_1 and R_2 under the assumptions given on e_m . Hence, (25) has order 4 reversors, e.g. V and $W = R_2^3 = -R_2$, and this property will be preserved under conjugacy.

Consider the converse. Since theorem 5 characterizes the order 4 reversors, we have, using fact 8, that $f = h_1 R h_1^{-1} h_2 R h_2^{-1}$ with $h_i \in \mathcal{G}$ and R the matrix in the statement (note that W

an order 4 reversor implies the same property for V and W^{-1}). This shows immediately that $\det(df) = 1$. Hence, the conjugate of f given by $h_2^{-1}fh_2$ takes the form

$$hRh^{-1}R \tag{26}$$

with $h = h_2^{-1}h_1$.

It follows from fact 8 that h (and hence (26)) commutes with $R^2 = I = \text{diag}(-1, -1)$, equivalently $IhI = h$. If we take for h an expression of the form (13), identical reasoning to that used in the proof of theorem 3 establishes that h has b linear with e_i having $P_i(y)$ odd. In the expression for h , there must be at least one e_i , otherwise (26) $\in \mathcal{A}$ and f is then conjugate to an affine element, in contradiction to f being CR.

Next, we need to consider the different possibilities for h and the reduction of (26), if necessary, to a cyclically reduced word. If h ends with an element from \mathcal{J} , we obtain (25) with $R_1 = R$. If, in addition, h starts with $b \circ a_m$ followed by an element from \mathcal{J} , a_m possibly absent and b possibly the identity, study the conjugate $a_m^{-1}b^{-1}(hRh^{-1}R)ba_m$. It is of the form (25) and ends with a linear traceless order 4 element $R_2 = a_m^{-1}b^{-1}Rba_m$ of the form indicated. A straightforward calculation shows that the entry γ is necessarily non-zero, because $x^2 = -1$ has no solution in K by assumption.

Otherwise, if the last element of h was $a \in \mathcal{I}$, we could write $hRh^{-1} = h'(aRa^{-1})h'^{-1}$, with h' ending in an elementary map and with the new linear order 4 element $R' = aRa^{-1} = \begin{pmatrix} -\beta & 1 \\ -(1+\beta^2) & \beta \end{pmatrix}$. Again, $1 + \beta^2 \neq 0$ in K by assumption, so $R' \in \mathcal{A} \setminus \mathcal{B}$. However, we can rewrite $R' = bRb^{-1}$ with $b = \begin{pmatrix} 1 & -\beta \\ 0 & -(1+\beta^2) \end{pmatrix} \in \mathcal{B}$. Hence $hRh^{-1} = (h'b)R(h'b)^{-1}$, where $h'b$ takes the form (13) ending in an element from \mathcal{J} . This returns us to the case of the previous paragraph. Considering now the start of $h'b$ as above, and possibly using a further affine conjugacy, again returns the form (25) with $R_1 = R$ and R_2 as given.

This explains the normal form given. The result for $\text{pol deg}(f')$ is a direct consequence of the odd nature of the $P_i(y)$ in e_i . The fact that f' has the symmetry indicated follows from the fact that $R_i^2 = I = \text{diag}(-1, -1)$ commutes with (25) and from corollary 2. \square

Remark. For the case $K = \mathbb{C}$, [13] presents normal forms for CR elements that possess a reversor of order $2n$.

Appendix. Elements of \mathcal{B} of finite order

Symmetries of CR elements of finite order are conjugate to elements of \mathcal{B} of finite order. As these are of particular relevance for detecting existing symmetries, we add a short classification here, for an arbitrary field K .

Recall that $\mathcal{B} = \{(\mathbf{a}, M) \mid \mathbf{a} \in K^2, M \in \mathcal{T}\} = K^2 \rtimes \mathcal{T}$, where \mathcal{T} denoted the subgroup of all upper triangular matrices of $\text{GL}(2, K)$. Since

$$(\mathbf{a}, M)^n = ((1 + M + M^2 + \dots + M^{n-1})\mathbf{a}, M^n),$$

it is clear that $(\mathbf{a}, M)^n = (\mathbf{0}, 1)$ implies $M^n = 1$ and $(1 + M + M^2 + \dots + M^{n-1})\mathbf{a} = \mathbf{0}$.

Consider a matrix $M = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$ with $\alpha, \beta, \gamma \in K$ and $\alpha\beta \neq 0$, so that M is invertible.

Lemma 8. For $n \in \mathbb{Z}$, the matrix powers of M are

$$M^n = \begin{pmatrix} \alpha^n & \gamma(n) \\ 0 & \beta^n \end{pmatrix},$$

where $\gamma(0) = 0$, and, for all $n \geq 1$, $\gamma(-n) = -\gamma(n)/(\alpha\beta)^n$ with

$$\gamma(n) = \gamma \sum_{m=0}^{n-1} \alpha^m \beta^{n-1-m}.$$

Proof. The formula for $\gamma(n)$, for positive n , is easy to check by induction, while the inversion formula for 2×2 -matrices gives the result for negative n , and $\gamma(0) = 0$ is clear. \square

If $M^n = 1$, we must have $\alpha^n = \beta^n = 1$ and $\gamma(n) = 0$. If $\alpha^n = \beta^n = 1$, but $\alpha \neq \beta$, a simple geometric series argument shows that $\gamma(n) = 0$ is automatic. On the other hand, if $\alpha = \beta$, one finds

$$\gamma(n) = n\alpha^{n-1}\gamma.$$

In characteristic 0, this can only vanish for $\gamma = 0$. Otherwise, $\gamma(n)$ vanishes also if $\text{char}(K)$ divides n . Consequently, $\text{ord}(M) = \text{lcm}(\text{ord}(\alpha), \text{char}(K))$. This gives the following proposition.

Proposition 13. Consider $M = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$ with $\alpha, \beta \in U_K$, and let $n = \text{lcm}(\text{ord}(\alpha), \text{ord}(\beta))$. If $\text{char}(K) = 0$, M is of finite order iff either $\alpha \neq \beta$ or $\alpha = \beta$ with $\gamma = 0$. In both cases, $\text{ord}(M) = n$. If $\text{char}(K) \neq 0$, M is of finite order for all $\gamma \in K$, with $\text{ord}(M) = n$ for $\alpha \neq \beta$ and $\text{ord}(M) = \text{lcm}(\text{char}(K), n)$ for $\alpha = \beta$. \square

Now, we have to extend to the affine case. Let M be a matrix of order n . If $(1 - M)$ is invertible, another geometric series argument shows that all affine extensions (\mathbf{a}, M) are also of order n . So, assume $(1 - M)$ is not invertible, i.e. 1 is an eigenvalue of M . If $M\mathbf{x} = \mathbf{x}$, with $\mathbf{x} \neq \mathbf{0}$, one has $(1 + M + \cdots + M^{m-1})\mathbf{x} = m\mathbf{x}$, which vanishes only for $\text{char}(K) | m$. This always happens if m is some suitable multiple of n , as long as $\text{char}(K) \neq 0$. In characteristic 0, however, the translational part of the affine extension has to avoid the kernel of $(M - 1)^k$, where k is the exponent of the factor $(x - 1)$ in the minimal polynomial of M .

Proposition 14. Let $M \in T$ with $\text{ord}(M) = n < \infty$. Let k be the exponent of $(x - 1)$ in the minimal polynomial of M , and set $S = \ker((M - 1)^k)$. If $\text{char}(K) = 0$, the element (\mathbf{a}, M) with $\mathbf{a} \in K^2$ is of finite order iff \mathbf{a} has no component in the generalized eigenspace S . In this case, the order is n . If $\text{char}(K) \neq 0$, (\mathbf{a}, M) is of finite order for all $\mathbf{a} \in K^2$, but the order can be a multiple of n . \square

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