PROPERTIES OF AC-OPERATORS

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Abstract. AC-operators were introduced by Berkson and Gillespie as a generalization in the context of well-boundedness of normal operators on Hilbert space. In this paper we explore some of the properties of these operators, such as the uniqueness of their splitting into real and imaginary parts, and their interpolation properties. We also examine the interpolation properties of the important subclass consisting of the trigonometrically well-bounded operators.

1. Introduction

Well-bounded operators were introduced by Smart [23] in order to provide a theory for Banach space operators which was similar to the successful theory of selfadjoint operators on Hilbert space, but which included operators whose spectral expansions may only converge conditionally. Smart and Ringrose [21,22] showed that if an operator T on a Banach space X possesses a functional calculus for the absolutely continuous functions on some compact interval of the real line then T may be represented as an integral with respect to a family of projections. In general, this representation is a little difficult to work with as not only do the projections of the family act on X^* rather than X, but they need not be uniquely determined. Nonetheless, a very satisfactory theory which covers the subclass of well-bounded operators of type (B) (and this includes all well-bounded operators on reflexive Banach spaces) has been developed, and has found applications in other areas of analysis.

One of the main restrictions of the theory of well-bounded operators has been that it only deals with operators with real spectrum, or at least with spectrum on a simple rectifiable arc in the complex plane [22, Section 8]. In [2], Berkson and Gillespie introduced the concept of an AC-operator as the natural analogue in the context of well-boundedness of normal operators on Hilbert space. An AC-operator is one which possesses a functional calculus for the absolutely continuous functions on some rectangle in $\mathbb{C}$ (more detailed definitions will be given in section 2). Berkson and Gillespie showed that these operators can be characterized by the fact that they possess a splitting into real and imaginary parts, $T = A + iB$, where A and B are

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commuting well-bounded operators. If A and B are of type (B), it was shown that this splitting is unique. The question as to whether this is always the case was left open however. One of the main results in this paper is to show that the answer to this question is negative.

We also study other properties of AC-operators. We show in section 4, for example, that the set of AC-operators is not closed under taking scalar multiples. In sections 5 to 8 we study the interpolation properties of AC-operators. That is, under suitable conditions on the indices $p < q$ and on the measure space (Ω, μ), we show that if T is an AC-operator on $L^p(\Omega, \mu)$ and on $L^q(\Omega, \mu)$, then T is also an AC-operator on $L^r(\Omega, \mu)$ for $p < r < q$. We pursue this matter furthest when considering the important subclass of trigonometrically well-bounded operators, which are the analogues of unitary operators in this context. In this case we show that many of the proofs may be simplified, or the conditions on the measure spaces relaxed. In section 8 we include some material on interpolation of separation-preserving operators. These results, which may be of independent interest, are applied to the study of trigonometrically well-bounded operators.

2. Background and notation

In this section we shall give some of the basic definitions regarding well-bounded and AC-operators. The theory of well-bounded operators is given in more detail in [10] or [9].

Throughout X will denote a complex Banach space with dual space X^*. The Banach algebra of all bounded linear operators on X will be denoted by $B(X)$.

An operator $T \in B(X)$ is said to be *well-bounded* if there exist a compact interval $[a, b] \subset \mathbb{R}$ and a constant K such that

$$\|g(T)\| \leq K\Big\{|g(a)| + \int_a^b |g'(t)|\, dt\Big\} \equiv K\, \|g\|_{BV},$$

for all polynomials g. Since the polynomials are dense in the Banach algebra of absolutely continuous functions on $[a, b]$, this is equivalent to the statement that there is a continuous unital Banach algebra homomorphism $\Phi : AC[a, b] \to B(X)$ such that if $e_n(x) = x^n$, then $\Phi(e_n) = T^n$, $(n = 0, 1, \dots)$. The spectral theorem for well-bounded operators states that there exists a family of projections $\{E(\lambda)\}_{\lambda \in \mathbb{R}} \subset B(X^*)$ known as a decomposition of the identity such that

$$\langle Tx, x^* \rangle = b\langle x, x^* \rangle - \int_a^b \langle x, E(\lambda)x^* \rangle \, d\lambda, \qquad x \in X,\ x^* \in X^*.$$

We shall not need the properties of decompositions of the identity, so we shall refer the reader to [10] or [9] for the details of the spectral theorem.

In the case that the map $\Phi : AC[a, b] \to B(X)$ is weakly compact (that is, for all $x \in X$, the map $f \mapsto \Phi(f)x$ is a weakly compact map from $AC[a, b]$ to X), we say that T is of *type (B)*. Clearly every well-bounded operator on a reflexive Banach space is of type (B). This subclass of well-bounded operators possesses a more powerful spectral theory, in that if T is a well-bounded operator of type (B), there exists a uniquely determined *spectral family* of projections $\{E(\lambda)\}_{\lambda \in \mathbb{R}} \subset B(X)$ such that

$$T = \int_{[a,b]}^{\oplus} \lambda \, dE(\lambda),$$

where the integral is a limit of Riemann-Stieltjes sums (see [10, Chapter 17]).

One of the major complications one encounters when trying to extend this theory to operators with complex spectra is deciding upon the correct concept of absolutely continuous functions of two variables to use. In the discussion that follows we shall identify subsets of $\mathbb{R}^2$ with subsets of $\mathbb{C}$ in the usual way. Let $J = [a, b]$ and $K = [c, d]$ be two compact intervals in $\mathbb{R}$. Let Λ be a rectangular partition of $J \times K$:

$$a = s_0 < s_1 < \cdots < s_n = b, \qquad c = t_0 < t_1 < \cdots < t_m = d.$$

For a function $f : J \times K \to \mathbb{C}$, define

$$V_\Lambda = \sum_{i=1}^{n} \sum_{j=1}^{m} |f(s_i, t_j) - f(s_i, t_{j-1}) - f(s_{i-1}, t_j) + f(s_{i-1}, t_{j-1})| .$$

The *variation* of f is defined to be

$$\operatorname*{var}_{J\times K} f = \sup \{V_\Lambda \,:\, \Lambda \text{ is a rectangular partition of } J \times K\} .$$

We shall say that the function f is of *bounded variation* if $\operatorname{var}_{J\times K} f$, $\operatorname{var}_J f(\cdot, d)$, and $\operatorname{var}_K f(b, \cdot)$ are all finite. The set $BV(J \times K)$ of all functions $f : J \times K \to \mathbb{C}$ of bounded variation is a Banach algebra under the norm

$$|||f||| = |f(b, d)| + \operatorname*{var}_J f(\cdot, d) + \operatorname*{var}_K f(b, \cdot) + \operatorname*{var}_{J\times K} f.$$

As with functions of one variable, there is the concept of an absolutely continuous function. Let m denote Lebesgue measure on $\mathbb{R}^2$. A function $f : J \times K \to \mathbb{C}$ is said to be *absolutely continuous* if

(1) for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{R\in\mathcal{R}} \operatorname*{var}_R f < \varepsilon$$

whenever $\mathcal{R}$ is a finite collection of non-overlapping subrectangles of $J \times K$ with $\sum_{R\in\mathcal{R}} m(R) < \delta$;

(2) the marginal functions $f(\cdot, d)$ and $f(b, \cdot)$ are absolutely continuous functions on J and K respectively.

The set $AC(J\times K)$ of all absolutely continuous functions $f : J\times K \to \mathbb{C}$ is a Banach subalgebra of $BV(J \times K)$, and is the closure in $BV(J \times K)$ of the polynomials in two real variables on $J \times K$. Equivalently, one can consider $AC(J \times K)$ to be the closure of the polynomial functions $p(z, \overline{z})$ on $J \times K \subset \mathbb{C}$.

Define the functions $u, v \in AC(J \times K)$ by $u(x, y) = x$ and $v(x, y) = y$. An operator $T \in B(X)$ is said to be an *AC-operator* if there exists a continuous unital Banach algebra homomorphism $\Psi : AC(J \times K) \to B(X)$ for which $\Psi(u + iv) = T$. Berkson and Gillespie proved that this is equivalent to the condition that T can be written as $T = U + iV$, where U and V are commuting well-bounded operators on X.

3. Decompositions of AC-operators

The most important open question regarding AC-operators was the question of uniqueness of the representation in the form $T = A + iB$ with A and B commuting well-bounded operators. In [2], Berkson and Gillespie showed that if there exists one such representation for which A and B are well-bounded of type (B), then no other representation exists. In particular, AC-operators on reflexive Banach spaces have unique representations.

As we shall now show, this is not true for AC-operators on arbitrary Banach spaces. In a positive direction however, we shall be able to give some information on the degree of nonuniqueness possible.

Example 3.1. Let $X = L^\infty[0,1] \oplus L^1[0,1]$, with the norm

$$\|(f,g)\| = \|f\|_\infty + \|g\|_1 .$$

Define the operator $A \in B(X)$ by

$$A(f,g) = (hf, hg),$$

where h is the function $h(t) = t$, $t \in [0,1]$. Then A is clearly well-bounded, and so $T = A + iA$ is an AC-operator. Consider now the operator $Q \in B(X)$ given by

$$Q(f,g) = (0, f).$$

For any $\alpha \in \mathbb{C}$ and any non-negative integer n, a simple induction proof shows that $(A + \alpha Q)^n = A^n + nA^{n-1}\alpha Q$. Thus, for any polynomial p, $p(A + \alpha Q) = p(A) + p'(A)\alpha Q$. If $(f,g) \in X$ then

$$\begin{aligned}
\|p(A+\alpha Q)(f,g)\| &= \|pf\|_\infty + \|\alpha p' f + pg\|_1 \\
&\le \|p\|_\infty \|f\|_\infty + |\alpha| \|p'\|_1 \|f\|_\infty + \|p\|_\infty \|g\|_1 \\
&\le (1 + |\alpha|) \|p\|_{BV[0,1]} \|(f,g)\| ,
\end{aligned}$$

and so $A + \alpha Q$ is well-bounded.

Let $C = A + Q$ and let $D = A + iQ$. Then C and D are well-bounded, and since A and Q commute, C and D also commute. But $C + iD = A + iA = T$, so we have constructed a different representation for T. ■

A consequence of the lack of uniqueness in the representation of T as $A + iB$ is that we also have nonuniqueness for the $AC(J \times K)$ functional calculus for T. We do not mean of course the formal nonuniqueness that arises from choosing a different rectangle $J \times K$. It is easy to check that if T has an $AC(J \times K)$ functional calculus then it also has an $AC(J' \times K')$ functional calculus for any compact intervals $J \subset J'$ and $K \subset K'$. The nonuniqueness of the splitting into real and imaginary parts produces different functional calculi based on a single rectangle $J \times K$.

Let J_1, J_2, K_1 and K_2 be compact intervals of the real line, and suppose that for $j = 1, 2$, $\theta_j\colon AC(J_j \times K_j) \to B(X)$ is a representation of $AC(J_j \times K_j)$ for which $\theta_j(u + iv) = T$. We shall say that θ_1 and θ_2 are *consistent* if for all polynomials in two variables p, $\theta_1(p) = \theta_2(p)$.

Theorem 3.2. *Let $T \in B(X)$ be an AC-operator. Then the representation of T in the form $A + iB$ with A and B commuting well-bounded operators is unique if and only if every pair of representations θ_j of $AC(J_j \times K_j)$ for which $\theta_j(u + iv) = T$ $(j = 1, 2)$ is consistent.*

Proof. ($\Longleftarrow$) The proof of [2, Theorem 5] shows how to construct, given a representation of T as $A + iB$, compact intervals J and K, and a representation θ of $AC[J \times K]$ such that $\theta(u + iv) = T$. With this construction $\theta(u) = A$, so different splittings of T into real and imaginary parts will provide inconsistent representations.
($\Longrightarrow$) It is straightforward to prove that if θ_1 and θ_2 are an inconsistent pair of representations for which $\theta_1(u + iv) = \theta_2(u + iv) = T$ then $\theta_1(u) \neq \theta_2(u)$ and $\theta_1(v) \neq \theta_2(v)$. Now, if θ is any $AC[J \times K]$ functional calculus for T, then $\theta(u)$ and $\theta(v)$ are a pair of commuting well-bounded operators for which $T = \theta(u) + i\theta(v)$. Thus we must be able to split T in two distinct ways into real and imaginary parts. ■

Note that Example 3.1 also shows that another important property of AC-operators on reflexive Banach spaces fails to extend to all AC-operators. The following commutativity theorem was proved in [2, Lemma 4].

Theorem 3.3. *(Commutativity Lemma) Let A and B be commuting well-bounded operators of type (B) on X and let $S \in B(X)$ commute with $A + iB$. Then S commutes with A and B.*

Example 3.4. Let T be the operator constructed in Example 3.1. The operator $S(f, g) = (f, 0)$ clearly commutes with T, but it does not commute with C or D. ■

It should be noted that it is relatively simple to construct similar examples on more classical non-reflexive spaces, such L^1 or L^∞. Let $AC_0[0, 1]$ denote the Banach space $\{f \in AC[0, 1] : f(0) = 0\}$. The operator T from Example 3.1 also acts on $AC_0[0, 1] \oplus L^1[0, 1] \cong L^1[0, 1]$ so this gives us an example of an AC-operator on L^1 with more than one splitting into real and imaginary parts. Taking adjoints would give an example on L^∞.

As we shall see now, it is possible to say something about how bad this nonuniqueness can be. We shall use the term *CF-decomposable* to describe an operator that is decomposable in the sense of Colojoară and Foiaş. If F is a closed subset of $\mathbb{C}$ we shall write $\mathcal{X}_A(F)$ for the usual spectral maximal space corresponding to a CF-decomposable operator A. We refer the reader to [7] for any terms which we have not defined here.

Throughout we shall assume that T is an AC-operator which possesses distinct representations $T = A+iB = C+iD$ where $\{A, B\}$ and $\{C, D\}$ are two commuting pairs of well-bounded operators. We shall let $Q = C - A$.

It is well-known that T^* must be a generalized scalar operator (see for example [7, Theorem 4.3.3]). Since T^* is a generalized scalar operator, it is CF-decomposable [7, Theorem 3.1.19] — as of course are A^*, B^*, C^* and D^*.

Our proof requires the following lemma which is an easy exercise in Gelfand theory.

Lemma 3.5. *Let $\mathcal{A}$ be a Banach algebra and suppose that a and b are elements of $\mathcal{A}$ such that $ab = ba$, $\sigma(a) \subset \mathbb{R}$ and $\sigma(b) \subset \mathbb{R}$. Then*

(i) $\sigma(a) \subset \operatorname{Re}\sigma(a+ib)$*;*
(ii) $\sigma(a+ib) \subset \sigma(a) + i\mathbb{R}$*.*

For $\lambda \in \mathbb{R}$ define

$$G_\lambda = \{z \in \mathbb{C} \,:\, \operatorname{Re} z \le \lambda\}, \qquad H_\lambda = \{z \in \mathbb{C} \,:\, \operatorname{Re} z \ge \lambda\}.$$

Lemma 3.6. *For all* $\lambda \in \mathbb{R}$*,*
(i) $\mathcal{X}_{T^*}(G_\lambda) = \mathcal{X}_{A^*}((-\infty,\lambda]) = \mathcal{X}_{C^*}((-\infty,\lambda])$*;*
(ii) $\mathcal{X}_{T^*}(H_\lambda) = \mathcal{X}_{A^*}([\lambda,\infty)) = \mathcal{X}_{C^*}([\lambda,\infty))$*.*

Proof. (i) It clearly suffices to just prove the first equality. Since A^*, B^* and T^* commute, [7, Proposition 1.3.2] shows that $\mathcal{X}_{A^*}((-\infty,\lambda])$ is invariant under each of these operators. By [7, Proposition 1.3.8] $\sigma(A^*|\mathcal{X}_{A^*}((-\infty,\lambda])) \subset (-\infty,\lambda]$ and so $\sigma(A^*|\mathcal{X}_{A^*}((-\infty,\lambda]) + iB^*|\mathcal{X}_{A^*}((-\infty,\lambda])) = \sigma(T^*|\mathcal{X}_{A^*}((-\infty,\lambda])) \subset G_\lambda$ by Lemma 3.5(ii). It is now easy to see that $\mathcal{X}_{A^*}((-\infty,\lambda]) \subset \mathcal{X}_{T^*}(G_\lambda)$.

On the other hand, [7, Proposition 1.3.8] also shows that $\sigma(T^*|\mathcal{X}_{T^*}(G_\lambda)) \subset G_\lambda$ and so Lemma 3.5(i) implies that $\sigma(A^*|\mathcal{X}_{T^*}(G_\lambda)) \subset (-\infty,\lambda]$. Arguing as above, we have that $\mathcal{X}_{T^*}(G_\lambda) \subset \mathcal{X}_{A^*}((-\infty,\lambda])$ and hence $\mathcal{X}_{T^*}(G_\lambda) = \mathcal{X}_{A^*}((-\infty,\lambda])$.

The proof for (ii) is similar. ■

Theorem 3.7. *If T is an AC-operator which possesses representations $T = A + iB = C + iD$ where $\{A,B\}$ and $\{C,D\}$ are two pairs of commuting well-bounded operators, then $Q = C - A$ (and hence also $D - B$) is a nilpotent operator of order 2.*

Proof. Let $E(\cdot)$ and $F(\cdot)$ be decompositions of the identity for A and C respectively. By [24, Theorem 3.6(iv)], $E(\lambda)X^* = \mathcal{X}_{A^*}((-\infty,\lambda])$ and $F(\lambda)X^* = \mathcal{X}_{C^*}((-\infty,\lambda])$. By Lemma 3.6 then, $E(\lambda)X^* = F(\lambda)X^*$. Similarly, $(I-E(\lambda))X^*$ and $(I-F(\lambda))X^*$ are both subspaces of $\mathcal{X}_{A^*}([\lambda,\infty)) = \mathcal{X}_{C^*}([\lambda,\infty))$. Now, using the fact that

$$E(\lambda) - F(\lambda) = (I - F(\lambda)) - (I - E(\lambda)),$$

we see that

$$(E(\lambda) - F(\lambda))X^* \subset \mathcal{X}_{A^*}((-\infty,\lambda]) \cap \mathcal{X}_{A^*}([\lambda,\infty)).$$

But by [24, Theorem 3.6(iv)] and [10, Theorem 15.8(iii)], this means that

$$(E(\lambda) - F(\lambda))X^* \subset \{x^* \in X^* \,:\, A^*x^* = \lambda x^*\}.$$

Of course we also have $(E(\lambda) - F(\lambda))X^* \subset \{x^* \in X^* \,:\, C^*x^* = \lambda x^*\}$.

It clearly follows then that

$$Q^*(E(\lambda) - F(\lambda)) = (C^* - A^*)(E(\lambda) - F(\lambda)) = 0, \qquad \text{for all } \lambda \in \mathbb{R}.$$

Thus, if $x \in X$ and $x^* \in X^*$,

$$\begin{aligned}
\langle Q^2x, x^*\rangle &= \langle CQx, x^*\rangle - \langle AQx, x^*\rangle \\
&= -\int_a^b \langle Qx, F(\lambda)x^*\rangle \, d\lambda + \int_a^b \langle Qx, E(\lambda)x^*\rangle \, d\lambda \\
&= \int_a^b \langle x, Q^*(E(\lambda) - F(\lambda))x^*\rangle \, d\lambda \\
&= 0.
\end{aligned}$$

Thus $Q^2 = 0$. ■

The following questions remain unanswered:

1. If T has two distinct splitting into real and imaginary parts $T = A+iB = C+iD$, must A, B, C, and D commute?
2. Does there always exist a representation of T as $A+iB$ such that if S commutes with T, it also commutes with A and B?
3. If such a representation exists, must it be unique?

4. Elementary properties of AC-operators

In this section we shall show that two further properties that one might hope AC-operators to possess also fail. Suppose that T is a normal operator on a Hilbert space. It is immediate from the definition of normality that for any $\alpha \in \mathbb{C}$, αT is also normal. Even on a Hilbert space however, the class of AC-operators fails to be closed under scalar multiplication.

Example 4.1. The construction uses the fact that on a Hilbert space, the sum and product of two commuting well-bounded operators need not be well-bounded. Let A, $B \in B(\ell^2)$ denote the commuting well-bounded operators constructed in [13] whose sum is not well-bounded, and let $T = A + iB$. Let $\alpha = 1 - i$. We shall show that if αT were an AC-operator then its representation would have to be given by

$$\alpha T = (A + B) + i(B - A)$$

which is impossible. We refer the reader to [13] for the details of the construction of A and B. For our purposes it suffices to know that $A = \text{SOT-}\sum_{n=1}^{\infty} \lambda_n P_n$ and $B = \text{SOT-}\sum_{n=1}^{\infty} \mu_n P_n$ where $\{\lambda_n\}$ and $\{\mu_n\}$ are given sequences of real numbers, and where P_n is the projection onto the span of the nth element of a certain conditional basis of ℓ^2.

Suppose now that αT is an AC-operator with representation $\alpha T = C+iD$. Since P_n commutes with A and B, it also commutes with αT. By the Commutativity Lemma then, P_n commutes with C and D. Now for each n,

$$\begin{aligned}\alpha T P_n &= C P_n + i D P_n \\ &= (A + B) P_n + i(B - A) P_n \\ &= ((\lambda_n + \mu_n) + i(\mu_n - \lambda_n)) P_n,\end{aligned}$$

and so $C + iD|P_n X = ((\lambda_n + \mu_n) + i(\mu_n - \lambda_n))I$. Since $\sigma(C|P_n X) \subset \sigma(C) \subset \mathbb{R}$ and the range of P_n is one-dimensional, this implies that $C|P_n X = (\lambda_n + \mu_n)I$ and $D|P_n X = (\mu_n - \lambda_n)I$. Thus $C = \text{SOT-}\sum_{n=1}^{\infty} C P_n = A + B$ and $D = B - A$. ■

Remark. The above example suggests that something a little more general may be true. Let X be reflexive and let $T = A + iB$ be an AC-operator on X. If $(x + iy)T$ is also an AC-operator then its representation should be given by $(x + iy)T = (xA - yB) + i(yA + xB)$. We have been unable to decide whether this is the case.

If T is normal, there exist (unique) selfadjoint operators A_T and B_T such that $T = A_T + iB_T$. Indeed $A_T = (T + T^*)/2$ and $B_T = (T - T^*)/2i$. It is clear then that for such operators, the map $T \mapsto A_T$ is contractive. A simple adjustment to Example 3.1 shows that it is possible to find an AC-operator of norm 1 which has a real part of arbitrarily large norm. Unfortunately, this behaviour can happen even on finite dimensional spaces where every AC-operator has a unique splitting into its real and imaginary parts.

Example 4.2. Let $X = \mathbb{C}^3$. Fix $x > 0$ and define $E_1, E_2 \in B(X)$ by the matrices

$$E_1 = \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & (1+i)\sqrt{\frac{x}{2}} \\ 0 & 0 & 0 \end{pmatrix}, \qquad E_2 = \begin{pmatrix} 0 & \sqrt{2x} & ix \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then E_1 and E_2 are commuting projections. In particular they are both well-bounded operators. Thus

$$T = E_1 + iE_2 = \begin{pmatrix} 1 & i\sqrt{2x} & 0 \\ 0 & 1+i & (1+i)\sqrt{\frac{x}{2}} \\ 0 & 0 & i \end{pmatrix}$$

is an AC-operator (and this gives the only such representation of T). Clearly, we have that $\|E_1\| / \|T\|$ is of the order of $\sqrt{x}$. It is a simple matter then to give a sequence of AC-operators T_n with real and imaginary parts A_n and B_n such that $T_n \to 0$, but $A_n \not\to 0$. Thus, even on this simple space, the function which maps an AC-operator to its real part is discontinuous. ■

5. Interpolation for AC-operators

When studying operators defined on a range of L^p spaces, it is common to ask questions of the type "If T has a certain property as an operator on L^{p_1} must it also have that property as an operator on L^{p_2}?". For the property of being well-bounded, these questions were discussed in [8]. There it was shown that if $1 \le p \le q \le \infty$ and if T defines a well-bounded operator on L^p and L^q, then T is well-bounded on L^r for all $p < r < q$. It was also shown [8, Example 5.1] that there exists T which is well-bounded on $\ell^q = L^q(\mathbb{N})$ for $q > 1$ and bounded on ℓ^1 (with real spectrum), but which is not well-bounded on ℓ^1. A simple modification of this example provides an operator which is well-bounded on ℓ^q for all q greater than some fixed index p, but which is not well-bounded on ℓ^p. This of course gives an example of a transformation which is an AC-operator on ℓ^q and which is bounded on ℓ^p, but which is not an AC-operator on ℓ^p (we leave the very minor details to the reader).

Before proceeding we shall fix some notation. Throughout this section T will be a linear transformation defined on a suitable vector space $\mathcal{D}(T)$ of (equivalence classes of) functions on some measure space $(\Omega, \mathcal{M}, \mu)$. We shall usually write L^p rather than $L^p(\Omega, \mathcal{M}, \mu)$ where there is no risk of confusion. Suppose that $1 \le p < q \le \infty$. If we say that T *defines an operator on* L^p we mean that $\mathcal{D}(T) \cap L^p$ is dense in L^p and that there exists a (necessarily unique) bounded operator $T_p \in B(L^p)$ such that $T_p f = Tf$ for all $f \in \mathcal{D}(T) \cap L^p$. The statement that T *defines operators on* L^p *and* L^q shall include the assumption that $T_p f = T_q f$ for all $f \in L^p \cap L^q$.

The interpolation question turns out to be a little delicate. Even for the property of being scalar-type spectral it seems to be difficult to prove interpolation results unless the L^p spaces are nested (see [19]). Similar problems arise when proving interpolation results about normal operators acting on a scale of Hilbert spaces [18]. The difficulty here lies in showing that if T defines an AC operator on L^p and on L^q, then the real and imaginary parts of T agree on the two spaces. If one knows this already, then the interpolation result is easy.

Theorem 5.1. *Suppose that $1 \le p \le q \le \infty$ and that A and B are linear transformations which define commuting well-bounded operators on L^p and on L^q. Then $T = A + iB$ is an AC-operator on L^r for $p \le r \le q$.*

Suppose now that our underlying measure space is such that the L^p spaces are nested. Important examples where this occurs are when the measure space is finite or purely atomic. We shall assume that $L^q \subset L^p$ for $1 \le p \le q \le \infty$, though this is only to make the exposition easier.

Theorem 5.2. *Suppose that $1 < p < q < \infty$ and that T is a linear transformation which defines AC-operators on L^p and L^q. Then T defines an AC-operator on L^r for $p < r < q$.*

Proof. Define $J : L^q \to L^p$ by $Jf = f$. The nesting assumption above ensures that J is a bounded operator. Write the representations of T on L^p and L^q as $T_p = A_p + iB_p$ and $T_q = A_q + iB_q$ respectively. The fact that T agrees on L^p and L^q can now be written as $T_p J = J T_q$. Consider the operator on $L^q \oplus L^p$ given by

$$\tilde{T} = \begin{pmatrix} T_q & 0 \\ 0 & T_p \end{pmatrix} = \begin{pmatrix} A_q & 0 \\ 0 & A_p \end{pmatrix} + i \begin{pmatrix} B_q & 0 \\ 0 & B_p \end{pmatrix} = \tilde{A} + i\tilde{B}.$$

Clearly $\tilde{T}$ is an AC-operator on $L^q \oplus L^p$. Let

$$\tilde{J} = \begin{pmatrix} 0 & 0 \\ J & 0 \end{pmatrix}.$$

As is easily checked, $\tilde{T}\tilde{J} = \tilde{J}\tilde{T}$. Since $L^q \oplus L^p$ is reflexive, the Commutativity Lemma implies that $\tilde{A}\tilde{J} = \tilde{J}\tilde{A}$ (and $\tilde{B}\tilde{J} = \tilde{J}\tilde{B}$). In other words $A_p J = J A_q$ and so $A_p f = A_q f$ for all $f \in L^p \cap L^q = L^q$. Similarly $B_p f = B_q f$ for all $f \in L^p \cap L^q = L^q$. The result now follows from Theorem 5.1. ■

As was shown in section 3, an AC-operator on L^1 need not have a unique splitting into real and imaginary parts. Under the additional assumption that the measure space is finite, we are able to prove some consistency results about linear transformations which define AC-operators on L^1 and some other L^p.

Theorem 5.3. *Suppose that $\mu(\Omega) < \infty$ and that $1 < p < \infty$. Suppose also that T is a linear transformation which defines AC-operators on L^1 and on L^p. Denote the unique representation of T on L^p by $T_p = A_p + iB_p$. Then the following are equivalent:*

- *(i) There exists some representation of $T_1 = A_1 + iB_1$ such that for all $f \in L^p$, $A_1 f = A_p f$ and $B_1 f = B_p f$.*
- *(ii) T_1 has a unique representation as $T_1 = A_1 + iB_1$ and for all $f \in L^p$, $A_1 f = A_p f$ and $B_1 f = B_p f$.*
- *(iii) There exists some representation of $T_1 = A_1 + iB_1$ with A_1 and B_1 of type (B).*

If any of these conditions hold then T is an AC-operator on L^r for $1 < r < p$.

Proof. (i) ⇒ (iii). This follows immediately from [8, Corollary 4.6]. Note that this requires that the measure space be finite.
(iii) ⇒ (ii). This is similar to the proof of Theorem 5.2. First note that since A_1 and B_1 are of type (B), T_1 has a unique splitting into real and imaginary parts.

Define $\tilde{T} = T_p \oplus T_1$ on $L^p \oplus L^1$. As before, setting $\tilde{A} = A_p \oplus A_1$ and $\tilde{B} = B_p \oplus B_1$ shows that $\tilde{T} = \tilde{A} + i\tilde{B}$ is an AC-operator. Furthermore, it is easily verified that since A_p, A_1, B_p and B_1 are all of type (B), so are $\tilde{A}$ and $\tilde{B}$. Proceeding as before we see that $\tilde{J}$ commutes with $\tilde{A}$ and $\tilde{B}$ and hence, for all $f \in L^p$, $A_1 f = A_p f$ and $B_1 f = B_p f$.
(ii) $\Rightarrow$ (i). Trivial.
The final statement follows from Theorem 5.1. ■

6. Interpolation of trigonometrically well-bounded operators

For the smaller, but important, class of trigonometrically well-bounded operators a shorter proof is available. Recall that an operator $U \in B(X)$ is said to be *trigonometrically well-bounded* is there exists a well-bounded operator A of type (B) such that $U = e^{iA}$. Given a trigonometrically well-bounded operator U there exists a unique such A called the *argument of* U, $\arg U$, such that $\sigma(A) \subset [0, 2\pi]$ and 2π is not an eigenvalue of A. Every trigonometrically well-bounded operator is an AC-operator. Indeed, an operator U is trigonometrically well-bounded if and only if there exist commuting well-bounded operators C and S of type (B) such that $U = C + iS$ and $C^2 + S^2 = I$ (see [3, Theorem 3.4]).

Suppose that $(\Omega, \mathcal{M}, \mu)$ is an arbitrary measure space, that $1 \le p < q \le \infty$, and that a linear transformation U defines invertible operators U_p and U_q on L^p and L^q respectively. We shall say that U has *consistent inverses* if $U_p^{-1} f = U_q^{-1} f$ for all $f \in L^p \cap L^q$. Clearly if the L^p spaces are nested, then an operator which is invertible on L^p and on L^q necessarily has consistent inverses. Examples (one of which we shall present in the next section) show that there are operators which lack consistent inverses. The following statements are easy consequences of the definition.

Proposition 6.1. *Suppose that $1 \le p < q \le \infty$ and that U defines invertible operators U_p and U_q on L^p and L^q respectively.*

(i) U has consistent inverses if and only if U maps $L^p \cap L^q$ onto $L^p \cap L^q$.
(ii) If U has consistent inverses then U_r is invertible on L^r for $p < r < q$ with $U_r^{-1} f = U_p^{-1} f = U_q^{-1} f$ for all $f \in L^p \cap L^q$.

Theorem 6.2. *Suppose that $1 \le p < q \le \infty$ and that U is a linear transformation which defines trigonometrically well-bounded operators on L^p and L^q. Suppose also that U has consistent inverses. Then U defines a trigonometrically well-bounded operator on L^r for $p < r < q$.*

Proof. Write the representations of U as an AC-operator on L^p and L^q as $U_p = C_p + iS_p$ and $U_q = C_q + iS_q$ respectively. Note that U is invertible on both these spaces and that $U_p^{-1} = C_p - iS_p$ and $U_q^{-1} = C_q - iS_q$. Thus, for $f \in L^p \cap L^q$,

$$C_p f = \frac{U_p + U_p^{-1}}{2} f = \frac{U_q + U_q^{-1}}{2} f = C_q f.$$

Similarly, $S_p = S_q$ on $L^p \cap L^q$. The result now follows easily from the results about well-bounded operators, and [3, Theorem 3.4]. ■

The spectral theorem for trigonometrically well-bounded operators says that if $U = e^{iA}$ is such an operator then $U = \int_{[0,2\pi]}^{\oplus} e^{i\lambda} \, dE(\lambda)$, where $\{E(\lambda)\}$ is the spectral family for the argument A. Suppose that U is trigonometrically well-bounded on L^p and L^q with

$$U_p = e^{iA_p} = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE_p(\lambda)$$

$$U_q = e^{iA_q} = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE_q(\lambda).$$

One would hope that both the arguments and the spectral families agree on the two spaces. In general we can only prove this under the assumption that U has consistent inverses.

Theorem 6.3. *Suppose that $1 \le p < q \le \infty$ and that U is a linear transformation which defines trigonometrically well-bounded operators on L^p and L^q as above. Suppose also that U has consistent inverses. Then*

(i) $A_p f = A_q f$ for all $f \in L^p \cap L^q$;
(ii) $E_p(\lambda) f = E_q(\lambda) f$ for all $\lambda \in \mathbb{R}$ and all $f \in L^p \cap L^q$.

Proof. Statement (i) follows immediately from statement (ii) and [8, Lemma 3.4]. The proof of statement (ii) is virtually identical to that of [8, Lemma 3.3]. In this case we use the fact that a trigonometrically well-bounded operator has a $BV(\mathbb{T})$ functional calculus. Since U has consistent inverses and the trigonometric polynomials are dense in $AC(\mathbb{T})$, it is clear that $g(U_p)f = g(U_q)f$ for all $g \in AC(\mathbb{T})$ and all $f \in L^p \cap L^q$. As in [8, Lemma 3.3], approximating characteristic functions pointwise by absolutely continuous functions and then using standard convergence theorems completes the proof. ∎

Two theorems in the converse direction are true, and easy to prove (see, for example [8, Lemma 3.4]).

Proposition 6.4. *Let $1 \le p < q \le \infty$. Suppose that $\{E_p(\lambda)\}$ and $\{E_q(\lambda)\}$ are spectral families on L^p and L^q respectively, and that $E_p(\lambda)f = E_q(\lambda)f$ for all $\lambda \in \mathbb{R}$ and all $f \in L^p \cap L^q$. Suppose further that both these spectral families are concentrated on $[0, 2\pi]$ and have strong operator topology limit I as $\lambda \to 2\pi^-$. Let U_p and U_q denote the corresponding trigonometrically well-bounded operators, $U_p = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE_p(\lambda)$ and $U_q = \int_{[0,2\pi]}^{\oplus} e^{i\lambda}\, dE_q(\lambda)$. Then $U_p f = U_q f$ for all $f \in L^p \cap L^q$.*

Proposition 6.5. *Let $1 \le p < q \le \infty$. Suppose that A_p and A_q are well-bounded operators of type (B) on L^p and L^q respectively, and that $A_p f = A_q f$ for all $f \in L^p \cap L^q$. Let U_p and U_q denote the corresponding trigonometrically well-bounded operators, $U_p = e^{iA_p}$ and $U_q = e^{iA_q}$. Then $U_p f = U_q f$ for all $f \in L^p \cap L^q$.*

7. Consistent inverses

One of the major hypotheses of the previous section was that U should have consistent inverses on the different L^p spaces. We do not know whether this hypothesis can be removed for trigonometrically well-bounded operators. It is known that in general an operator need not have consistent inverses. The following example is due to Ransford [20, pp. 450-451].

Example 7.1. This example is based on the Cesàro operator $\mathcal{C}$, which is defined for every locally integrable function $f : (0, \infty) \to \mathbb{C}$ by writing

$$(\mathcal{C}f)(x) = \frac{1}{x} \int_0^x f(t)\, dt.$$

The operator $\mathcal{C}$ was introduced by G.H. Hardy in [14], where it was demonstrated that for $1 < p < \infty$, the restriction $\mathcal{C}_p$ of $\mathcal{C}$ to $L^p(0,\infty)$ is a bounded linear mapping of $L^p(0,\infty)$ into $L^p(0,\infty)$ such that $\|\mathcal{C}_p\| = \dfrac{p}{p-1}$ (see, e.g., [15, pp. 240-243]). The spectrum $\sigma(\mathcal{C}_p)$ of the operator $\mathcal{C}_p$ was investigated in [6], where it was shown that for $1 < p < \infty$,

$$\sigma(\mathcal{C}_p) = \left\{ z \in \mathbb{C} : \left| z - \frac{p}{2(p-1)} \right| = \frac{p}{2(p-1)} \right\}. \tag{7-1}$$

Suppose now that $1 < p_1 < p_2 < p_3 < \infty$, and define $T : L^{p_1}(0,\infty) \cap L^{p_3}(0,\infty) \to L^{p_1}(0,\infty) \cap L^{p_3}(0,\infty)$ by writing

$$Tf = \mathcal{C}f - \frac{p_2}{p_2 - 1} f.$$

For $1 \le j \le 3$, Let $T_j : L^{p_j}(0,\infty) \to L^{p_j}(0,\infty)$ be the corresponding continuous linear extension of T, $T_j = \mathcal{C}_{p_j} - \frac{p_2}{p_2-1} I$. It is clear from (7-1) and spectral mapping that for $1 \le j \le 3$,

$$\sigma(T_j) = \left\{ z \in \mathbb{C} : \left| z - \frac{p_j}{2(p_j - 1)} + \frac{p_2}{p_2 - 1} \right| = \frac{p_j}{2(p_j - 1)} \right\}.$$

Consequently, it is easy to see that $0 \notin \sigma(T_1) \cup \sigma(T_3)$, whereas $0 \in \sigma(T_2)$. This implies that T cannot map $L^{p_1}(0,\infty) \cap L^{p_3}(0,\infty)$ onto itself, since otherwise T_1^{-1} and T_3^{-1} would agree on $L^{p_1}(0,\infty) \cap L^{p_3}(0,\infty)$ and provide, by interpolation, a bounded inverse of T_2 on $L^{p_2}(0,\infty)$. ■

In the remainder of this paper we shall examine a number of important situations where it is necessarily the case that an operator invertible on two L^p spaces has consistent inverses.

One sufficient condition is the *uniqueness of resolvent condition* (as described in [20, Definition I and Proposition 2.3-I(b), pg. 451]).

Theorem 7.2. *Suppose that μ is an arbitrary measure, that $1 \le p < q < \infty$, and that U defines trigonometrically well-bounded operators U_p and U_q on L^p and L^q respectively. Let us further suppose that $\sigma(U_p) \cup \sigma(U_q)$ does not separate the plane. Then U has consistent inverses on L^p and L^q.*

Proof. As indicated above, this is an immediate consequence of [20, pg. 451]. Nevertheless, we shall include here the following elementary proof which is available for the present context.

Let $g \in L^p \bigcap L^q$. For $z \in \mathbb{C}$ such that $|z|$ is sufficiently large, we have (with convergence in the norm topology of L^p):

$$(z - U_p)^{-1} g = \sum_{j=0}^{\infty} z^{-j-1} U_p^j g.$$

The analogous statement holds for $(z - U_q)^{-1} g$, and hence for $|z|$ sufficiently large,

$$(z - U_p)^{-1} g = (z - U_q)^{-1} g.$$

Suppose that M is an arbitrary subset of finite measure and that χ_M denotes the characteristic function of M. Then the complex-valued analytic functions

$$z \mapsto \int \chi_M (z - U_p)^{-1} g \, d\mu \quad \text{and} \quad z \mapsto \int \chi_M (z - U_q)^{-1} g \, d\mu$$

coincide on the connected open set $\mathbb{C} \setminus [\sigma(U_p) \cup \sigma(U_q)]$. Specializing to the value $z = 0$, we infer that $U_p^{-1} g = U_q^{-1} g$. ■

Another sufficient condition applies to translation invariant linear transformations. In what follows let G be a locally compact abelian group with given Haar measure m and dual group Γ. For $1 \le p < \infty$, let $M_p(\Gamma)$ denote the space of Fourier multipliers for $L^p(G)$. For $\phi \in M_p(\Gamma)$, we shall symbolize by $T_\phi^{(p)}$ the corresponding multiplier transform on $L^p(G)$.

Lemma 7.3. *Suppose that $1 \le p < \infty$, $\phi \in M_p(\Gamma)$, and $T_\phi^{(p)}$ is trigonometrically well-bounded. Then $|\phi| = 1$ locally almost everywhere on Γ.*

Proof. Clearly the inverse of $T_\phi^{(p)}$ is translation-invariant, and so can be written in the form $T_\psi^{(p)}$, where $\psi \in M_p(\Gamma)$. Notice that $T_\phi^{(p)}$ has spectrum contained in $\mathbb{T}$, and, by spectral mapping, so does its inverse $T_\psi^{(p)}$. For $n \in \mathbb{N}$, we have

$$\left\| \left(T_\phi^{(p)} \right)^n \right\|^{1/n} = \|\phi^n\|_{M_p(\Gamma)}^{1/n} \ge \|\phi^n\|_\infty^{1/n} = \|\phi\|_\infty .$$

Consequently $\|\phi\|_\infty$ does not exceed the spectral radius of $T_\phi^{(p)}$, which is equal to one. Similarly, $\|\psi\|_\infty \le 1$. Since $T_{\psi\phi}^{(p)}$ is the identity operator, we have $\psi\phi = 1$ locally almost everywhere, and the conclusion of the lemma is now evident. ■

Theorem 7.4. *Suppose that $1 \le p < q < \infty$, and that U defines translation invariant trigonometrically well-bounded operators U_p and U_q on $L^p(G)$ and $L^q(G)$ respectively. Then U has consistent inverses on $L^p(G)$ and $L^q(G)$.*

Proof. Let $U_p = T_\phi^{(p)}$, $U_q = T_\eta^{(q)}$, where $\phi \in M_p(\Gamma)$, $\eta \in M_q(\Gamma)$. By applying these operators to integrable simple functions, and taking Fourier transforms, it is easy to see that $\phi = \eta$ locally almost everywhere on Γ. Let ψ be the complex conjugate of ϕ. Then $\psi \in M_p(\Gamma) \bigcap M_q(\Gamma)$, and by Lemma 7.3, $\psi\phi = 1$ locally almost everywhere on Γ. Fix $f \in L^p(G) \bigcap L^q(G)$. We can choose a sequence of integrable simple functions $\{f_k\}_{k=1}^\infty$ such that $\|f - f_k\|_p \to 0$ and $\|f - f_k\|_q \to 0$. It follows that $T_\psi^{(p)} f = T_\psi^{(q)} f \in L^p(G) \bigcap L^q(G)$. Consequently,

$$U_p T_\psi^{(p)} f = U_q T_\psi^{(q)} f = f.$$

Multiplying by U_p^{-1} gives that $U_p^{-1} f = T_\psi^{(p)} f$. Similarly $U_q^{-1} f = T_\psi^{(q)} f$. Thus $U_p^{-1} f = U_q^{-1} f$. ■

8. Interpolation of Separation-Preserving Operators

Definition 8.1. Suppose that $(\Omega, \mathcal{M}, \mu)$ is a measure space, $1 \le p < \infty$, and X is a subspace of L^p. A linear mapping $T : X \to X$ is said to be *separation-preserving* provided that whenever $f \in X$, $g \in X$, and $fg = 0$ μ-a.e. on Ω, the pointwise product $(Tf)(Tg)$ vanishes μ-a.e. on Ω.

We begin this section by recalling some standard facts about separation-preserving operators.

Scholium 8.2 (see, e.g., [16, Section 3] or [1, §2, Scholium B]). *A bounded operator T on L^p is separation-preserving if and only if there is a bounded positivitypreserving operator $|T|$ on L^p satisfying the following condition:*

$$\text{for every } f \in L^p, \ \ |Tf| = |T|\,(|f|)\,, \ \mu\text{-a.e. on } \Omega.$$

If this is the case, then the latter condition uniquely characterizes $|T|$ among the operators on L^p. Moreover, the operator $|T|$ has the property that

$$\text{for every } f \in L^p, \ \ |Tf| = |\,|T|\,(f)\,|\,, \ \mu\text{-a.e. on } \Omega.$$

(Hence, in particular, $\|\,|T|\,(f)\,\|_p = \|Tf\|_p$.)

The next item is a well-known part of the folklore.

Scholium 8.3. *Suppose that T is a bounded invertible separation-preserving operator on L^p. Then T^{-1} is separation-preserving, $|T|$ is an invertible operator on L^p, and*

$$|T|^{-1} = \left|T^{-1}\right|.$$

Proof. Suppose $f \in L^p$, $g \in L^p$, and $fg = 0$ μ-a.e. Put $F = T^{-1}f$, $G = T^{-1}g$, $h = \min\{|F|, |G|\}$. Since $0 \le h \le |F|, |G|$, and since $|T|$ is positivity-preserving, we have

$$0 \le |T|\,(h) \le \min\{|T|\,(|F|)\,, |T|\,(|G|)\}\,. \tag{8-1}$$

But

$$\begin{aligned} &|T|\,(|F|) = |T(F)| = \left|TT^{-1}f\right| = |f|\,; \\ &|T|\,(|G|) = |g|\,. \end{aligned}$$

Hence we infer with the aid of (8-1) that

$$0 = |T|\,(h) = |\,T(h)\,|\,, \ \mu\text{-a.e.}$$

Since T is injective, this shows that $h = 0$ μ-a.e. In other words, $FG = 0$ μ-a.e., and hence T^{-1} is separation-preserving.

If $f \in L^p$, and $f \ge 0$, then

$$f = \left|T^{-1}Tf\right| = \left|TT^{-1}f\right| = \left|T^{-1}\right|(|T|\,f) = |T|\left(\left|T^{-1}\right|f\right).$$

Hence $\left|T^{-1}\right|\ |T| = |T|\ \left|T^{-1}\right| = I$. ∎

Henceforth we shall assume that the measure space $(\Omega, \mathcal{M}, \mu)$ is sigma-finite, and we shall employ the following terminology and notation from [16].

Definition 8.4. A *σ-endomorphism* of the measure algebra $(\Omega, \mathcal{M}, \mu)$ is a mapping $\Phi : \mathcal{M} \to \mathcal{M}$ (modulo μ-null sets) such that:

(i) $\Phi\left(\cup_{n=1}^{\infty} E_n\right) = \cup_{n=1}^{\infty} \Phi E_n$, for every disjoint sequence $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$;
(ii) $\Phi(\Omega \setminus E) = \Phi(\Omega) \setminus \Phi(E)$, for all $E \in \mathcal{M}$;
(iii) whenever $E \in \mathcal{M}$ and $\mu(E) = 0$, then $\mu(\Phi E) = 0$.

As described in [16, §4], a σ-endomomorphism Φ induces a unique linear operator (also denoted by Φ) from the space of all complex-valued μ-measurable functions into itself such that $\Phi(\chi_E) = \chi_{\Phi(E)}$ for all $E \in \mathcal{M}$, and such that whenever a sequence $\{f_n\}_{n=1}^{\infty}$ of μ-measurable functions converges μ-a.e. to a μ-measurable function f, we have $\Phi(f_n) \to \Phi(f)$ μ-a.e. Clearly this linear operator Φ is positive. One further property which Φ is known to enjoy as an operator on the complex-valued μ-measurable functions is multiplicativity: $\Phi(fg) = \Phi(f)\Phi(g)$. We shall require the following two results from [16].

Proposition 8.5 ([16, Theorem 4.1]). *Suppose that $(\Omega, \mathcal{M}, \mu)$ is a sigma-finite measure space, $1 \le p < \infty$, and T is a bounded separation-preserving operator on L^p. For each $E \in \mathcal{M}$ such that $\mu(E) < \infty$, let $\Phi_0 E = \{\omega \in \Omega : (T(\chi_E))(\omega) \neq 0\}$. Then Φ_0 extends to a unique σ-endomorphism Φ of $(\Omega, \mathcal{M}, \mu)$. There is a unique μ-measurable function $h : \Omega \to \mathbb{C}$ such that $h = 0$ a.e. in $\Omega \setminus \Phi(\Omega)$, and, for each $f \in L^p$, Tf is the pointwise product on Ω of h and Φf.*

Proposition 8.6 ([16, Proposition 4.1]). *Suppose that $(\Omega, \mathcal{M}, \mu)$ is a sigma-finite measure space, $1 \le p < \infty$, and T is a bounded invertible separation-preserving linear mapping of L^p onto L^p. Then, in the notation of Proposition 8.5, the following assertions hold.*

(i) *For μ-almost all $\omega \in \Omega$, $h(\omega) \neq 0$.*
(ii) *Φ is one-to-one on the space of all complex-valued μ-measurable functions defined on Ω (hence, as a σ-endomorphism, Φ is one-to-one on the class $\mathcal{M}$ of all μ-measurable sets).*
(iii) *Φ maps the space of all complex-valued μ-measurable functions defined on Ω onto itself, and, as a σ-endomorphism, Φ maps $\mathcal{M}$ onto $\mathcal{M}$. In particular, $\Phi(\Omega) = \Omega$.*

We are now in a position to state our central result on the interpolation of separation-preserving operators (compare [25, Proposition 4.1]).

Theorem 8.7. *Suppose that $(\Omega, \mathcal{M}, \mu)$ is a sigma-finite measure space, $1 \le p < q < \infty$, and T is a linear mapping which defines invertible separation-preserving operators T_p and T_q on L^p and L^q respectively. Then T has consistent inverses.*

Proof. By Scholium 8.3, T_p^{-1} and T_q^{-1} are separation-preserving. Let h_p (respectively, h_q) and Φ_p (respectively, Φ_q) correspond to T_p (respectively, T_q) as in Proposition 8.5. Likewise, let $\widetilde{h}_p$ (respectively, $\widetilde{h}_q$) and $\widetilde{\Phi}_p$ (respectively, $\widetilde{\Phi}_q$) correspond to T_p^{-1} (respectively, T_q^{-1}). Notice in particular that by Proposition 8.6-(i), we have, pointwise on Ω, $\left|h_p\right| > 0$, $\left|h_q\right| > 0$, $\left|\widetilde{h}_p\right| > 0$, and $\left|\widetilde{h}_q\right| > 0$, μ-a.e.

Suppose now that $A \in \mathcal{M}$ with $\mu(A) < \infty$. Then

$$T_p \chi_A = T_q \chi_A,$$

and consequently

$$h_p \chi_{\Phi_p(A)} = h_q \chi_{\Phi_q(A)}. \tag{8-2}$$

It follows immediately from (8-2) that the zeroes of $\chi_{\Phi_p(A)}$ and $\chi_{\Phi_q(A)}$ are identical. Hence $\Phi_p(A) = \Phi_q(A)$, whenever $A \in \mathcal{M}$ with $\mu(A) < \infty$. Proposition 8.5 now provides that Φ_p and Φ_q coincide as σ-endomorphisms on $\mathcal{M}$. Consequently, Φ_p and Φ_q are identical on the space of complex-valued μ-measurable functions. Another application of (8-2) now shows that $h_p = h_q$ μ-a.e. on $\Phi_p(\Omega)$. By Proposition 8.6-(iii), $\Phi_p(\Omega) = \Omega$, and so we have established that $h_p = h_q$, μ-a.e. on Ω.

For $f \in L^p$, we have

$$f = T_p T_p^{-1} f = T_p \left(\widetilde{h}_p \widetilde{\Phi}_p f \right) = h_p \cdot \left(\Phi_p \left(\widetilde{h}_p \right) \right) \Phi_p \left(\widetilde{\Phi}_p f \right).$$

Specializing f to be χ_A, where $\mu(A) < \infty$, we see that

$$\chi_A = h_p \cdot \left(\Phi_p \left(\widetilde{h}_p \right) \right) \chi_{\Phi_p \left(\widetilde{\Phi}_p (A) \right)}. \tag{8-3}$$

Since $\Phi_p \left(\widetilde{h}_p \right) \Phi_p \left(1/\widetilde{h}_p \right) = \Phi_p(1) = 1$ on Ω, we infer that $\left| \Phi_p \left(\widetilde{h}_p \right) \right| > 0$ on Ω. Equating the sets where the two sides of (8-3) vanish, we now obtain

$$A = \Phi_p \left(\widetilde{\Phi}_p (A) \right). \tag{8-4}$$

By Proposition 8.6-(ii),(iii), Φ_p is an injective mapping of $\mathcal{M}$ onto $\mathcal{M}$, and consequently we have just shown that whenever $\mu(A) < \infty$ we have $\Phi_p^{-1}(A) = \widetilde{\Phi}_p(A)$. Similar reasoning applies to the index q, and since $\Phi_p = \Phi_q$ on $\mathcal{M}$, we infer that $\widetilde{\Phi}_p(A) = \widetilde{\Phi}_q(A)$ whenever $\mu(A) < \infty$. Using Definition 8.4-(i) together with the sigma-finiteness of μ, we now deduce that

$$\widetilde{\Phi}_p(A) = \widetilde{\Phi}_q(A), \quad \text{for every } A \in \mathcal{M}.$$

Consequently,

$$\widetilde{\Phi}_p = \widetilde{\Phi}_q, \quad \text{on the space of all complex-valued } \mu\text{-measurable functions.} \tag{8-5}$$

Using (8-4) in (8-3), we utilize the sigma-finiteness of μ to infer that the equality $\Phi_p \left(\widetilde{h}_p \right) = 1/h_p$ holds μ-a.e. on Ω. A corresponding result holds for the index q. Since $h_p = h_q$, we see that $\Phi_p \left(\widetilde{h}_p \right) = \Phi_q \left(\widetilde{h}_q \right)$. However, it has already been shown that Φ_p and Φ_q are identical. It now follows that $\Phi_p \left(\widetilde{h}_p \right) = \Phi_p \left(\widetilde{h}_q \right)$. By Proposition 8-6-(ii) applied to Φ_p, we infer that $\widetilde{h}_p = \widetilde{h}_q$ on Ω. This fact, taken together with (8-5), shows that T_p^{-1} and T_q^{-1} coincide on $L^p \cap L^q$. ■

Corollary 8.8. *Suppose that $(\Omega, \mathcal{M}, \mu)$ is a sigma-finite measure space, $1 \le p < q < \infty$, and U is a linear mapping which defines separation-preserving trigonometrically well-bounded operators U_p and U_q on L^p and L^q respectively. Then U_r is trigonometrically well-bounded on L^r for all $r \in [p, q]$.*

We shall end by giving an example and an application of these theorems.

Example 8.9. Suppose that $0 < \alpha < 1$, and let $w \equiv \{w_k\}_{k=-\infty}^{\infty}$ be the sequence defined by putting $w_k = |k|^\alpha$ for $k \in \mathbb{Z} \setminus \{0\}$, $w_0 = 1$. Let μ be the measure defined on the subsets of $\mathbb{Z}$ which is specified by writing, for each $k \in \mathbb{Z}$, $\mu(\{k\}) = w_k$. It is well-known (see, e.g., [12, p. 407]) that the function $x \in \mathbb{R} \mapsto |x|^\alpha$ is an A_p weight in $\mathbb{R}$ provided that $1+\alpha<p$. Hence by [5, Proposition (2.4)-(ii),(iv)], the weight sequence w satisfies the A_p condition in $\mathbb{Z}$ for $1+\alpha<p$. Let U be the bilateral right shift defined on all sequences of complex numbers: $U\left(\{x_k\}_{k=-\infty}^{\infty}\right)=\{x_{k-1}\}_{k=-\infty}^{\infty}$. By [5, Theorem (4.2)], the restriction of U to L^p is a trigonometrically well-bounded operator on L^p for $1+\alpha<p$. Since U is clearly separation-preserving, it provides a further example of Corollary 8.8 when $1+\alpha<p<q<\infty$. Of course, in this situation, the interpolation of U^{-1} follows readily from that of U, since U^{-1} is the bilateral left shift. For any p satisfying $1+\alpha<p<\infty$, we denote by U_p the restriction of U to L^p, and we remark that U_p is not a power-bounded operator. In fact, it is is readily seen from [5, (1.3)] that $\left\|U_p^n\right\|=(1+|n|)^{\alpha/p}$, for all $n\in\mathbb{Z}$. ■

Theorem 8.10. *Suppose that μ is an arbitrary sigma-finite measure, and that $1<p<q<\infty$. Suppose also that U defines surjective linear isometries U_p and U_q of L^p and L^q respectively. Then U_r is trigonometrically well-bounded on L^r for all $r\in[p,q]$.*

Proof. Since $1<p<q<\infty$ and U_p, U_q are, in particular, invertible power-bounded operators, it follows by [4, Theorem (4.8)-(ii)] that U_p and U_q are trigonometrically well-bounded. Since $p<q$, at least one of the indexes p,q is distinct from 2. Denoting this index distinct from 2 by s we infer from [17, Corollary 2.1] that U_s is separation-preserving. Since U_p and U_q agree on $L^p\cap L^q$, it follows that they are both separation-preserving. By Theorem 8.7, U has consistent inverses, and so the result follows from Theorem 6.2. ■

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