Trigonometric Identities, Linear Algebra, and Computer Algebra

Ian Doust, Michael D. Hirschhorn, and Jocelyn Ho

1 INTRODUCTION.

This is a story that interweaves several different elements:

- Some surprising-looking trigonometric and combinatorial sums
- Some nice applications of elementary linear algebra
- The way that computer algebra packages can change the way that mathematics is done

Before progressing we invite the reader to try to establish the following facts:

1. $\displaystyle\sum_{m=1}^{99} \frac{\sin\left(\frac{17m\pi}{100}\right)\sin\left(\frac{39m\pi}{100}\right)}{1+\cos\left(\frac{m\pi}{100}\right)} = 1037.$

2. If $n \equiv 0 \pmod 2$ and $1 \leq j \leq k \leq n$, then

$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{\cos\left(\frac{m\pi}{n+1}\right)} = (n+1)\sin\left(\frac{j\pi}{2}\right)\sin\left(\frac{(k-1)\pi}{2}\right).$$

3. If $1 \leq j \leq k \leq n$ and β is a rational number not equal to $2\cos\left(\frac{m\pi}{n+1}\right)$ for any integer m, then

$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right)+\beta}$$

is rational.

4. If $n \not\equiv 0 \pmod 7$, then

$$\frac{7}{n}\sum_{m=1}^{n} \frac{\cos\left(\frac{2m\pi}{n}\right)}{8\cos^3\left(\frac{2m\pi}{n}\right)+4\cos^2\left(\frac{2m\pi}{n}\right)-4\cos\left(\frac{2m\pi}{n}\right)-1} \equiv n^5 \pmod 7.$$

We came across these types of identities while doing some research in Banach space geometry. In looking at certain vector space bases, it became necessary to consider the $n \times n$ matrix

$$T_n = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 0 & \ddots & \\ 0 & 1 & 1 & 1 & \ddots & \\ 0 & 0 & 1 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ 0 & & & & & 1 \end{pmatrix}.$$

MAPLE checked the first few of these for invertibility and produced the inverses when they existed. The evidence was pretty convincing that

$$\det(T_n) = \begin{cases} -1 & \text{if } n \equiv 0 \pmod 3, \\ 1 & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

and that when $n \not\equiv 2 \pmod 3$ each entry of T_n^{-1} is 0, 1, or -1. Actually, it isn't too hard to write a recursion formula for $\det(T_n)$. The point, however, is that it is much easier to check that a matrix S is the inverse of T_n than it is to calculate S algorithmically. Here the computer algebra package is critical. Given T_3^{-1}, T_6^{-1}, T_9^{-1}, and T_{12}^{-1}, it is easy to guess the general form for T_n^{-1} when $n \equiv 0 \pmod 3$. Doing the calculations by hand is possible when n is small, but in practice most of us would not have the patience to persist long enough to see the patterns forming. In this particular case, if one lets

$$D = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

and defines S_{3k} to be the $3k \times 3k$ matrix given by

$$S_{3k} = \begin{pmatrix} D & U & U & \dots & U \\ U^T & D & U & \dots & U \\ U^T & U^T & D & & U \\ \vdots & \vdots & & \ddots & \vdots \\ U^T & U^T & U^T & \dots & D \end{pmatrix}, \tag{1}$$

then it is easy to check that $S_{3k}T_{3k} = I$, so $S_{3k} = T_{3k}^{-1}$.

2 TOEPLITZ MATRICES AND TRIGONOMETRIC IDENTITIES.

Matrices like T_n that are constant on all their diagonals are called *Toeplitz matrices.* Toeplitz matrices and their infinite dimensional operator analogues appear in many areas of mathematics and in many applications (such as signal processing, communications engineering, and statistics). The links between these matrices and trigonometric series are well known, so it should have come as no surprise that various trigonometric functions soon entered the picture.

One might also try to find the inverses of such matrices using elementary linear algebra. Clearly T_n is self-adjoint for all n. This ensures the existence of an orthogonal matrix P_n and a diagonal matrix $E = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ such that $T_n = P_n^* E P_n$. If T_n is invertible, then $T_n^{-1} = P_n^* \operatorname{diag}(\lambda_1^{-1}, \dots, \lambda_n^{-1})\, P_n$.

Again, with a little enthusiasm, it is possible to find a recurrence relation for the characteristic polynomials of the T_n and use this to find their eigenvalues and eigenvectors.[1] What we actually did was to get the computer to calculate a few cases numerically, and then stared at the results! After recognising that the eigenvalues had something to do with $\cos\left(\frac{j\pi}{n+1}\right)$ it didn't take us long to guess that the eigenvalues of T_n are

$$\lambda_j = 1 + 2\cos\left(\frac{j\pi}{n+1}\right) \qquad (j = 1, \dots, n),$$

with corresponding eigenvectors

$$v_j = \left(\sin\left(\frac{j\pi}{n+1}\right), \sin\left(\frac{2j\pi}{n+1}\right), \dots, \sin\left(\frac{nj\pi}{n+1}\right)\right)^T.$$

[1]We have tested this statement on some bright undergraduates.

Checking that our guesses were right was easy, since the verification depends only on the identity

$$\sin((k-1)\theta) + \sin(k\theta) + \sin((k+1)\theta) = (1 + 2\cos\theta)\sin(k\theta). \tag{2}$$

Having identified n distinct eigenvalues, we had, of course, found them all. A small calculation shows that $||v_j|| = \sqrt{(n+1)/2}$ for all j. We now had a quite different way of expressing T_n^{-1}. As long as $n \not\equiv 2 \pmod 3$ (in which case one of the eigenvalues is zero),

$$T_n^{-1} = \frac{2}{n+1} P \operatorname{diag}\left(\frac{1}{1+2\cos\left(\frac{\pi}{n+1}\right)}, \ldots, \frac{1}{1+2\cos\left(\frac{n\pi}{n+1}\right)} \right) P, \tag{3}$$

where

$$P = P^* = \begin{pmatrix} \sin\left(\frac{\pi}{n+1}\right) & \sin\left(\frac{2\pi}{n+1}\right) & \ldots & \sin\left(\frac{n\pi}{n+1}\right) \\ \sin\left(\frac{2\pi}{n+1}\right) & \sin\left(\frac{4\pi}{n+1}\right) & \ldots & \sin\left(\frac{2n\pi}{n+1}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sin\left(\frac{n\pi}{n+1}\right) & \sin\left(\frac{2n\pi}{n+1}\right) & \ldots & \sin\left(\frac{n^2\pi}{n+1}\right) \end{pmatrix}.$$

Multiplying out the expression in equation (3) shows that the (j,k)th entry of T_n^{-1} is

$$a_{jk} = \frac{2}{n+1} \sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right)+1}.$$

With the aid of (1) it is not difficult to check that if $n \equiv 0 \pmod 3$ and we focus on the top half of the matrix (that is, if $j \le k$), then $a_{jk} = \frac{4}{3}\sin\left(\frac{2j\pi}{3}\right)\sin\left(\frac{2(k-1)\pi}{3}\right)$. This leads to

$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right)+1} = \frac{2n+2}{3}\sin\left(\frac{2j\pi}{3}\right)\sin\left(\frac{2(k-1)\pi}{3}\right).$$

At this stage we tried to prove this using other techniques, but this seems, to us at least, to be quite difficult.

Having gotten this far, we looked to see what other sorts of identities might be proved in this way. For example, given any $\beta \in \mathbb{R}$, one can alter equation (2) slightly to get

$$\sin((k-1)\theta) + \beta\sin(k\theta) + \sin((k+1)\theta) = (\beta + 2\cos\theta)\sin(k\theta).$$

As earlier, one can use this identity to check that the vectors $\{v_j\}_{j=1}^n$ are eigenvectors of the Toeplitz matrices

$$T_{\beta,n} = \begin{pmatrix} \beta & 1 & 0 & \dots & 0 \\ 1 & \beta & 1 & \ddots & \\ 0 & 1 & \beta & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \\ 0 & & & & \beta \end{pmatrix},$$

with $T_{\beta,n} v_j = (\beta + 2\cos(\frac{j\pi}{n+1}))v_j$ for $j = 1, \dots, n$.

To get the process to work, one needs to be able to determine a different expression for the entries of $T_{\beta,n}^{-1}$ than the one that the diagonalization generates. For many values of β (such as -2, -1, 0, or 2) a few lines of MAPLE code were sufficient to generate an educated guess. As before, checking that the guess is right is easy. Some of the identities that arise from these values of β are the following.

- ($\beta = 0$) If $j \le k \le n$ and $n \equiv 0 \pmod 2$, then

$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{\cos\left(\frac{m\pi}{n+1}\right)} = (n+1)\sin\left(\frac{j\pi}{2}\right)\sin\left(\frac{(k-1)\pi}{2}\right).$$

- ($\beta = -2$) If $j \le k \le n$, then

$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right) - 2} = \frac{-j(n+1-k)}{2}.$$

- ($\beta = -1$) If $j \le k \le n$ and $n \equiv 0 \pmod 3$, then

$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right) - 1} = \frac{2(n+1)}{3}\sin\left(\frac{j\pi}{3}\right)\sin\left(\frac{(k-1)\pi}{3}\right).$$

- ($\beta = 2$) If $j \le k \le n$, then

$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{\cos\left(\frac{m\pi}{n+1}\right) + 1} = (-1)^{j+k} j(n+1-k).$$

$$
\begin{aligned}
T_{\beta,1}^{-1} &= \begin{bmatrix} \frac{1}{\beta} \end{bmatrix} \\
T_{\beta,2}^{-1} &= \begin{bmatrix} \frac{\beta}{\beta^2-1} & -\frac{1}{\beta^2-1} \\ -\frac{1}{\beta^2-1} & \frac{\beta}{\beta^2-1} \end{bmatrix} \\
T_{\beta,3}^{-1} &= \begin{bmatrix} \frac{\beta^2-1}{\beta\,(\beta^2-2)} & -\frac{1}{\beta^2-2} & \frac{1}{\beta\,(\beta^2-2)} \\ -\frac{1}{\beta^2-2} & \frac{\beta}{\beta^2-2} & -\frac{1}{\beta^2-2} \\ \frac{1}{\beta\,(\beta^2-2)} & -\frac{1}{\beta^2-2} & \frac{\beta^2-1}{\beta\,(\beta^2-2)} \end{bmatrix} \\
T_{\beta,4}^{-1} &= \begin{bmatrix} \frac{\beta\,(\beta^2-2)}{\beta^4-3\,\beta^2+1} & -\frac{\beta^2-1}{\beta^4-3\,\beta^2+1} & \frac{\beta}{\beta^4-3\,\beta^2+1} & -\frac{1}{\beta^4-3\,\beta^2+1} \\ -\frac{\beta^2-1}{\beta^4-3\,\beta^2+1} & \frac{\beta\,(\beta^2-1)}{\beta^4-3\,\beta^2+1} & -\frac{\beta^2}{\beta^4-3\,\beta^2+1} & \frac{\beta}{\beta^4-3\,\beta^2+1} \\ \frac{\beta}{\beta^4-3\,\beta^2+1} & -\frac{\beta^2}{\beta^4-3\,\beta^2+1} & \frac{\beta\,(\beta^2-1)}{\beta^4-3\,\beta^2+1} & -\frac{\beta^2-1}{\beta^4-3\,\beta^2+1} \\ -\frac{1}{\beta^4-3\,\beta^2+1} & \frac{\beta}{\beta^4-3\,\beta^2+1} & -\frac{\beta^2-1}{\beta^4-3\,\beta^2+1} & \frac{\beta\,(\beta^2-2)}{\beta^4-3\,\beta^2+1} \end{bmatrix}
\end{aligned}
$$

Figure 1: Inverses of $T_{\beta,n}$ for $n = 1, 2, 3, 4$.

Note that if β is rational and $T_{\beta,n}$ is invertible, then the Gaussian elimination algorithm implies that all the entries of $T_{\beta,n}^{-1}$ are rational. This proves statement 3 at the beginning of the paper.

Being more optimistic, one might even try to find a formula for $T_{\beta,n}^{-1}$ for arbitrary real β. MAPLE is quite happy to do all the algebra to give the first few cases of $T_{\beta,n}^{-1}$. Figure 1 shows these matrices for $n \leq 4$. It is clear from examining these matrices that certain special polynomials occur in these formulas, the first six of which are:

$$\begin{aligned} p_1(x) &= x, \\ p_2(x) &= x^2 - 1, \\ p_3(x) &= x^3 - 2x, \\ p_4(x) &= x^4 - 3x^2 + 1, \\ p_5(x) &= x^5 - 4x^3 + 3x, \\ p_6(x) &= x^6 - 5x^4 + 6x^2 - 1. \end{aligned}$$

Motivated by the fact that the coefficients appearing here are binomial coefficients, we define the polynomial p_n $(n = 1, 2, 3, \dots)$ by

$$p_n(x) = \sum_{\ell=0}^{\lfloor n/2 \rfloor} (-1)^\ell \binom{n-\ell}{\ell} x^{n-2\ell},$$

where $\lfloor x \rfloor$ signifies the greatest integer that does not exceed x. The evidence from the first few cases suggests that if $1 \leq j \leq k \leq n$, then the (j,k)th entry of $T_{\beta,n}^{-1}$ is

$$\frac{(-1)^{j+k} p_{j-1}(\beta) p_{n-k}(\beta)}{p_n(\beta)}.$$

Since $T_{\beta,n}^{-1}$ must clearly be symmetric, this provides a candidate formula for this matrix. Again, it is straightforward to check that this candidate actually does the job. The argument depends on identities such as

$$\beta(p_{n-1}(\beta) - p_{n-2}(\beta)) = p_n(\beta)$$

that follow easily from properties of binomial coefficients. It is not surprising that the roots of p_n are $\left\{2\cos\left(\frac{m\pi}{n+1}\right)\right\}_{m=1}^{n}$. Thus, if β is not an element of this set of roots and if $1 \leq j \leq k \leq n$, then

$$\sum_{m=1}^{n} \frac{\sin\left(\frac{jm\pi}{n+1}\right)\sin\left(\frac{km\pi}{n+1}\right)}{2\cos\left(\frac{m\pi}{n+1}\right) + \beta} = \frac{(-1)^{j+k}(n+1)p_{j-1}(\beta)p_{n-k}(\beta)}{2p_n(\beta)}.$$

The general technique here is to find two different expressions for calculating $f(T)$ for some given matrix T and some given function f. If the eigenvalues and (generalized) eigenvectors of T can be identified, the Jordan canonical form gives one way of calculating $f(T)$.

Consider, for example, functions of the form $f(x) = x^\ell$. It is not hard (at least for small integers ℓ) to write down the entries of $T_{1,n}^\ell$. It is obvious that the entries are all nonnegative integers. Thus, when $n \geq 1$, $\ell \geq 0$, and $1 \leq j, k \leq n$, it becomes apparent that

$$\frac{2}{n+1}\sum_{m=1}^{n} \sin\left(\frac{jm\pi}{n+1}\right) \sin\left(\frac{km\pi}{n+1}\right) \left(2\cos\left(\frac{m\pi}{n+1}\right)+1\right)^\ell \in \mathbb{N}$$

and consequently that

$$\frac{2^{\ell+1}}{n+1}\sum_{m=1}^{n} \sin\left(\frac{jm\pi}{n+1}\right) \sin\left(\frac{km\pi}{n+1}\right) \cos^\ell\left(\frac{m\pi}{n+1}\right) \in \mathbb{Z}.$$

Part of the challenge here is to find matrices with nice families of eigenvalues and eigenvectors. To simplify notation, for an $(2n+1)$-tuple of real numbers

$$\mathbf{c} = (c_{1-n}, \ldots, c_{-1}, c_0, c_1, \ldots, c_{n-1}),$$

we define the Toeplitz matrix $T_\mathbf{c}$ to be

$$T_\mathbf{c} = \begin{pmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{1-n} \\ c_1 & c_0 & c_{-1} & \ddots & \\ c_2 & c_1 & c_0 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \\ c_{n-1} & \cdots & & & c_0 \end{pmatrix}.$$

Let $\mathbf{c} = (1, 0, \ldots, 0, 1, 1, 1, 0, \ldots, 0, 1)$, the vector in $\mathbb{R}^{2n-1}$ with a 1 in the first, $(n-1)$th, nth, $(n+1)$th, and last coordinates and with all remaining coordinates 0. Knowing that

$$e^{i(k-1)\theta} + e^{ik\theta} + e^{(ik+1)\theta} = (1 + 2\cos\theta)e^{ik\theta}$$

allows the eigenvalues of $T_\mathbf{c}$ to be identified as

$$\lambda_j = 1 + 2\cos\left(\frac{2j\pi}{n}\right) \qquad (j = 1, \ldots, n)$$

with corresponding eigenvectors $v_j = \left(e^{2ij\pi/n}, e^{4ij\pi/n}, \ldots, e^{2nij\pi/n}\right)^T$. Here $T_{\mathbf{c}}$ is invertible whenever $n \not\equiv 0 \pmod 3$. If $S = (s_{jk})$ is the inverse matrix, then (by observing, guessing, and checking) one can show that

$$s_{jk} = \begin{cases} \dfrac{2(-1)^{n\,(\mathrm{mod}\,3)}}{3} & \text{if } |j-k| + n \equiv 0 \pmod 3, \\ \dfrac{(-1)^{n\,(\mathrm{mod}\,3)+1}}{3} & \text{otherwise.} \end{cases}$$

(In these formulas $n\,(\mathrm{mod}\,3)$ must be taken from the set $\{0, 1, 2\}$.) Using the diagonalization of $T_{\mathbf{c}}$ gives

$$s_{jk} = \frac{1}{n} \sum_{m=1}^{n} \frac{e^{i(k-j)m\pi/n}}{1 + 2\cos\left(\frac{2m\pi}{n}\right)}.$$

Upon taking the real part of s_{jk} and writing ℓ for $k - j$, one finds that

$$\frac{3}{n} \sum_{m=1}^{n} \frac{\cos(2\ell m\pi/n)}{1 + 2\cos\left(\frac{2m\pi}{n}\right)} = \begin{cases} 2(-1)^{n\,(\mathrm{mod}\,3)} & \text{if } |\ell| + n \equiv 0 \pmod 3, \\ (-1)^{n\,(\mathrm{mod}\,3)+1} & \text{otherwise.} \end{cases}$$

In particular (taking $\ell = 1$), it follows that if $n \not\equiv 0 \pmod 3$ then

$$\frac{3}{n} \sum_{m=1}^{n} \frac{\cos(2m\pi/n)}{1 + 2\cos\left(\frac{2m\pi}{n}\right)} = n\,(\mathrm{mod}\,3).$$

Since $e^{i(k-\ell)\theta} + e^{i(k+\ell)\theta} = 2\cos(\ell\theta)e^{ik\theta}$, it is easy to show that this same set of eigenvectors will diagonalize any Toeplitz matrix $T_{\mathbf{c}}$ with $\mathbf{c}$ of the form

$$\mathbf{c} = (c_1, \ldots, c_\ell, 0, \ldots, 0, c_\ell, \ldots, c_1, c_0, c_1, \ldots, c_\ell, 0, \ldots, 0, c_\ell, \ldots, c_1).$$

Choosing $\ell = 2$ or 3 and taking $c_j = 1$ for $j = 1, 2, \ldots, \ell$ then yields identities such as

$$\frac{5}{n} \sum_{m=1}^{n} \frac{\cos(2m\pi/n)}{4\cos^2\left(\frac{2m\pi}{n}\right) + 2\cos\left(\frac{2m\pi}{n}\right) - 1} \equiv n^3 \pmod 5$$

if $n \not\equiv 0 \pmod 5$, and identity (4) at the start of this paper. (The danger of guessing formulas is shown by the fact that the obvious guess as to what happens when $\ell = 4$ isn't true!)

3 STIRLING NUMBERS AND BINOMIAL COEFFICIENTS.

Moving away from trigonometric functions, one can try one's luck with other types of matrices. For example, consider the $n \times n$ lower-triangular matrix $P_n(a)$ whose (j,k)th entry is $\binom{j-1}{k-1} a^{j-k}$. The powers of these so-called Pascal matrices were studied in [1]. Let $B_n = P_n(1)$ be the lower-triangular matrix whose nonzero entries are binomial coefficients. This matrix is not diagonalizable, but MAPLE quickly finds its Jordan canonical form:

$$B_n = U_n \begin{pmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & 1 \\ 0 & & \dots & 0 & 1 \end{pmatrix} U_n^{-1}.$$

for appropriate matrices U_n. When $n = 6$, the matrices U_6 and U_6^{-1} are

$$U_6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 6 & 6 & 1 & 0 \\ 0 & 24 & 36 & 14 & 1 & 0 \\ 120 & 240 & 150 & 30 & 1 & 0 \end{pmatrix}, \quad U_6^{-1} = \begin{pmatrix} 0 & \frac{1}{5} & \frac{-5}{12} & \frac{7}{124} & \frac{-1}{12} & \frac{1}{120} \\ 0 & \frac{-1}{4} & \frac{11}{24} & \frac{-1}{4} & \frac{1}{24} & 0 \\ 0 & \frac{1}{3} & \frac{-1}{2} & \frac{1}{6} & 0 & 0 \\ 0 & \frac{-1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In fact, staring at the entries of these matrices didn't immediately lead us to any guesses as to the general formulas. Let u_{jk} denote the (j,k)th entry of U_n. It appears that $(n-k)!$ is a factor of u_{jk}, so we looked at what is left. At this point we consulted Neil Sloane's wonderful On-Line Encyclopedia of Integer Sequences [3], which quickly identified the missing factors as Stirling numbers.

Let $s(j,k)$ and $S(j,k)$ denote the Stirling numbers of the first and second kind. There are many fine references on these numbers (see, for example, [2] or [4] and [5]). We shall just give a quick definition. For $k \geq 0$ let $(x)_k$

denote the degree k polynomial $x(x-1)\cdots(x-k+1)$. The Stirling numbers can be defined in terms of the following generating functions:

$$(x)_j = \sum_{k=0}^{j} s(j,k)x^k, \qquad x^j = \sum_{k=0}^{k} S(j,k)(x)_k.$$

The conjecture suggested by our computer experimentation is that

$$u_{jk} = (n-k)!\ S(j-1, n-k). \tag{4}$$

Similar explorations concerning v_{jk}, the (j,k)th element of U_n^{-1} led us to the belief that

$$v_{jk} = \frac{s(n-j, k-1)}{(n-j)!}.$$

Indeed, with the aid of standard identities satisfied by Stirling numbers, it is not difficult to verify that the two matrices with these entries are inverses.

Since B_n has only a one-dimensional eigenspace, it is actually not hard to generate U_n. As $B_n - I$ is a lower-triangular matrix with positive entries below its diagonal, $(B_n - I)^{n-1}$ has just a single nonzero entry in the bottom left position. Then $(B_n - I)^n = 0$, so the vector $(1, 0, \ldots, 0)$ is in $\ker((B_n - I)^n) \backslash \ker((B_n - I)^{n-1})$. The images of this vector under the powers of the matrix $B_n - I$ will then furnish a Jordan basis for B_n, and hence give the transition matrix U_n. Let $\mathbf{u}_k$ denote the vector whose jth entry is give by the formula for u_{jk} in (4). To establish our conjecture, one therefore needs to check that $\mathbf{u}_{k-1} = (B_n - I)\mathbf{u}_k$ for $k = 1, 2, \ldots, n$. In terms of the entries this amounts to checking that

$$(n-k+1)!\ S(j-1, n-k+1) = \sum_{\ell=1}^{j-1} \binom{j-1}{\ell-1} (n-k)!\ S(\ell-1, n-k)$$

holds for all j,k, and n, or, upon changing variables to make things look neater, that

$$(k+1)S(j,k+1) = \sum_{\ell=0}^{j-1} \binom{j}{\ell} S(\ell,k).$$

This is an easy consequence of the following two standard identities [2, sec 6.1]:

$$\begin{aligned} S(n,m) &= n\,S(n-1,m) + S(n-1,m-1), \\ S(n+1,m+1) &= \sum_{\ell=1}^{n} \binom{n}{\ell} S(\ell,m). \end{aligned}$$

We have therefore established concrete formulas for all the matrices in the Jordan canonical representation $B_n = U_n J U_n^{-1}$.

Using the fact that $S(j, \ell) = 0$ when $\ell > j$ and multiplying out $U_n J U_n^{-1}$ leads (after a small amount of simplification) to the identity

$$\binom{j}{k} = \sum_{\ell=0}^{j} (s(\ell, k) + \ell s(\ell - 1, k))\ S(j, \ell).$$

This is surely already known to those in the field of combinatorics, but it is interesting to see what other identities derive from it. Formulas for powers of B_n were given in [1]. It was shown there that B_n^m equals the Pascal matrix $P_n(m)$. In particular, $B_n^{-1} = P_n(-1)$, so comparing the (j, k)th entry of $P_n(-1)$ with the corresponding entry of $U_n J^{-1} U_n^{-1}$ gives that if $0 \le j, k \le n - 1$ then

$$\binom{j}{k}(-1)^{j-k} = \sum_{t=1}^{n} \sum_{\ell=t}^{n} (-1)^{\ell-t} \frac{(n-t)!}{(n-\ell)!}\ s(n-\ell, k) S(j, n-t).$$

Taking mth powers of B_n then yields (with suitable interpretation of the factorials)

$$\binom{j}{k} m^{j-k} = \sum_{\ell=0}^{j} \sum_{t=0}^{\ell} \binom{m}{t} \frac{\ell!}{(\ell-t)!} s(\ell - t, k) S(j, \ell)$$

for all j, k, and m. Summing over k and putting $m = 1$ or 2 tells us, for example, that

$$\begin{aligned} 2^j &= \sum_{k=1}^{j} \sum_{\ell=0}^{j} (s(\ell, k) + \ell s(\ell - 1, k)) S(j, \ell), \\ 3^j &= \sum_{k=0}^{j} \sum_{\ell=0}^{j} (s(\ell, k) + 2\ell s(\ell - 1, k) + \ell(\ell - 1) s(\ell - 2, k)) S(j, \ell). \end{aligned}$$

4 Conclusion

As the examples in this paper demonstrate, many standard identities can be interpreted as statements about the eigenvalues (or generalized eigenvalues) of matrices. Once one has a suitable matrix identity, then elementary linear algebra provides a powerful technique for extracting new and more complicated identities from old ones. Our little journey of discovery heavily underlines the power of a modern computer algebra package in providing inspiration in mathematical investigations.

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IAN DOUST completed his mathematical education at the Universities of Western Australia, Toronto, and Edinburgh. Although his main research interests are in functional analysis, he is also very interested in applications of computing to mathematical research and teaching. His recent activities include being Director of the Australian Mathematical Sciences Institute Summer School for senior mathematics students.
School of Mathematics, University of New South Wales, UNSW Sydney NSW 2036, Australia
i.doust@unsw.edu.au

MIKE HIRSCHHORN has been at UNSW for most of the last thirty years. He devotes a good deal of his time to trying to get inside the mind of Ramanujan. He is proud to say that he is a (mathematical) descendant of Lagrange, Dirichlet, Rademacher, and Andrews, among other luminaries, and a relative of the grandmaster Tartakover. Also, he has Erdős number 2 (while Ramanujan has Erdős number 3).

School of Mathematics, University of New South Wales, UNSW Sydney NSW 2036, Australia
M.Hirschhorn@unsw.edu.au

JOCELYN HO completed her Bachelor of Science in mathematics and computer science at the University of New South Wales. Her areas of interest include differential geometry and algebra. She has also completed a Diploma of Arts specializing in music, and her recent activities include research in the modelling of musical compositions into low-dimensional manifolds. Jocelyn will be starting a graduate program in music at the University of Missouri at Kansas City this year.
School of Mathematics, University of New South Wales, UNSW Sydney NSW 2036, Australia