A MAXIMAL THEOREM FOR HOLOMORPHIC SEMIGROUPS

by GORDON BLOWER[†]

(*Department of Mathematics and Statistics, Lancaster University, Lancaster,* LA1 4YF)

and IAN DOUST[§]

(*School of Mathematics, The University of New South Wales,* NSW, 2052, *Australia*)

[3 April 2004]

Abstract

Let X be a closed linear subspace of the Lebesgue space $L^p(\Omega,\mu)$ for some $1 < p < \infty$, and let $-A$ be an invertible operator that is the generator of a bounded holomorphic semigroup T_t on X. Then for each $0 < \alpha < 1$ the maximal function $\sup_{t>0} |T_t f(x)|$ belongs to $L^p(\Omega,\mu)$ for each f in the domain of A^α. If moreover iA generates a bounded C_0–group and A has spectrum contained in $(0,\infty)$, then A has a bounded H^∞ functional calculus.

1. Introduction and main result

Let Ω be a complete and separable metric space, and μ be a σ-finite and positive Radon measure on Ω. Let X be a closed linear subspace of the Lebesgue space $L^p(\Omega;\mu)$, where $1 < p < \infty$. For a closed and densely defined linear operator A in X, we are concerned with the Cauchy initial value problem for $v(x,t)$ and initial datum $f \in X$,

$$\frac{\partial v}{\partial t} = -Av, \qquad v(x,0) = f(x), \qquad (t > 0; x \in \Omega). \tag{1.1}$$

This has a unique solution $v(x,t) = T_t f(x)$ in the sense of Hille and Phillips [**14**, p. 622] whenever T_t is a C_0–semigroup on X with generator $-A$; one writes $T_t = e^{-tA}$ [**14**, p. 321]. In order to ensure that $v(x,t)$ converges μ-almost everywhere to $f(x)$ as $t \to 0+$, it is often necesary to impose further conditions on f. For any closed linear operator V in X, we recall that the domain of V is the Banach space $D(V) = \{f \in X : Vf \in X\}$ with the graph norm $\|f\|_{D(V)} = \|f\|_{L^p} + \|Vf\|_{L^p}$. For $0 < \alpha < 1$, we can define a fractional power A^α such that $D(A^\alpha)$ contains $D(A)$, and $A^{-\alpha}$ is bounded whenever A has bounded inverse; see [**20**, Theorem 2.3.1].

In many cases of interest, T_t extends to a bounded holomorphic semigroup on the

[†] Corresponding author; email: g.blower@lancaster.ac.uk

[§] Email: i.doust@unsw.edu.au

MSC 2000 Classification: 47D60, 47D06

sector $\Sigma_\theta = \{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$ for some θ with $0 < \theta \leq \pi/2$. We write $\arg \lambda$ for the principal value of the argument.

A bounded holomorphic semigroup [**12**, p. 59] on Σ_θ is a family of bounded linear operators T_z on X for each $z \in \Sigma_\theta$, that satisfies:–
(i) $T_z T_\zeta = T_{z+\zeta}$ for all $z, \zeta \in \Sigma_\theta$;
(ii) for each θ' with $0 \leq \theta' < \theta$ there exists $K(\theta')$ such that $\|T_z\| \leq K(\theta')$ for all $z \in \bar{\Sigma}_{\theta'}$;
(iii) T_z is a holomorphic function of $z \in \Sigma_\theta$;
(iv) $T_z f \to f$ for each $f \in X$ as $z \to 0$ with $z \in \Sigma_\theta$.

THEOREM 1.1. *Suppose that $-A$ generates a bounded holomorphic semigroup $T_z = e^{-zA}$ on Σ_θ for some θ with $0 < \theta \leq \pi/2$. Suppose further that A is invertible and that $0 < \alpha < 1$. Then for each $f \in D(A^\alpha)$ the maximal function $M(f(x)) = \sup_{0<t<\infty} |T_t f(x)|$ belongs to $L^p(\Omega; \mu)$; further, there exists $K(A, \alpha, p)$ independent of f such that*

$$\int_\Omega M(f(x))^p \mu(dx) \leq K(A, \alpha, p) \int_\Omega \big(|A^\alpha f(x)|^p + |A^{-\alpha} f(x)|^p\big)\mu(dx). \tag{1.2}$$

If $-A$ is the generator of a bounded holomorphic semigoup, then for each $s > 0$ the operator $-(sI + A)$ is invertible and also generates a bounded holomorphic semigroup; so the invertibility assumption in Theorem 1.1 does not restrict the usefulness of the result.

In many examples there is a dense subspace Y of smooth functions such that $T_t f(x) \to f(x)$ as $t \to 0+$ holds μ-almost everywhere for $f \in Y$. One can then use the Theorem to show by approximation that almost everywhere convergence holds for all $f \in D(A^{-\alpha}) \cap D(A^\alpha)$. The hypotheses are easy to verify in applications. Whereas we do not claim that the result is optimal in any particular case, it covers cases outwith the scope of the special results of Stein [**18**, p. 65] for symmetric diffusion semigroups and the results that require growth conditions on the imaginary powers A^{iu} as in [**9**,**5**,**6**].

The proof of Theorem 1.1 depends upon a functional calculus argument which we present in section 2. This was suggested by the H^∞ functional calculus of McIntosh and his co-authors [**15**, **10**], but we present a self-contained proof for the sake of completeness. In section 3 we conclude the proof of the Theorem and in the final section 4 we consider examples.

2. Functional calculus

In this section we work with the space of bounded linear operators on X with the operator norm $\| \; \|$. A closed and densely defined linear operator A is said to be *sectorial of type* φ

if its spectrum is contained in $\bar{\Sigma}_\varphi$ for some $\varphi \in (0,\pi)$ and for each $\omega \in (\varphi,\pi)$ there exists $\kappa(\omega)$ such that $\|\lambda(\lambda I - A)^{-1}\| \le \kappa(\omega)$ holds for all $\lambda \in \mathbb{C} \setminus \bar{\Sigma}_\omega$; see [**10**].

For the remainder of this section, we suppose that A is as in Theorem 1.1. By [**12**, p. 2] A is densely defined and closed. We recall from [**13**, p.44] that, for a Banach space Y, a function $G : \mathbb{R} \to Y$ is *Bochner–Lebesgue integrable* if G is strongly measurable and if $\|G(x)\|_Y$ is integrable.

LEMMA 2.1. *Let $0 < \theta' < \theta$ and let $\pi/2 - \theta' < \omega < \pi/2$.*
(i) Then A is sectorial of type $\pi/2 - \theta'$; so that, each $\lambda \ne 0$ with $|\arg\lambda| \ge \omega$ lies in the resolvent set of A and satisfies

$$\big\|\lambda(\lambda I - A)^{-1}\big\| \le K(\theta')\mathrm{cosec}(\omega + \theta' - \pi/2). \tag{2.1}$$

(ii) Similarly the Yosida approximants $A_s = (sI + A)^{-1}sA$ where $s > 0$ are sectorial of type $\pi/2 - \theta'$; so that,

$$\|\lambda(\lambda I - A_s)^{-1}\| \le (\mathrm{cosec}\,\omega/2)\big(1 + K(\theta')\,\mathrm{cosec}\,(\omega + \theta' - \pi/2)\big) \tag{2.2}$$

holds for each $\lambda \ne 0$ such that $|\arg\lambda| \ge \omega$.

Proof. (i) A standard estimate on the resolvent: one turns the line of integration in the Bochner–Lebesgue integral [**12**, p. 61]

$$\lambda(\lambda I - A)^{-1}f = -\int_0^\infty \lambda e^{z\lambda - zA} f\,dz \qquad (f \in X) \tag{2.3}$$

so that $\pi > |\arg\lambda + \arg z| \ge \omega + \theta' > \pi/2$ and then one bounds the integral using the triangle inequality.

(ii) By (i), $-s$ lies in the resolvent of A for each $s > 0$ and one can check that A_s is a bounded linear operator which satisfies

$$\lambda(\lambda I - A_s)^{-1} = \frac{\lambda}{\lambda - s} + \frac{s}{s-\lambda}\frac{\lambda s}{s-\lambda}\Big(\frac{\lambda s}{s-\lambda} I - A\Big)^{-1}. \tag{2.4}$$

The linear fractional transformation $\varphi(\lambda) = \lambda s/(s-\lambda)$ takes $\{\lambda \ne 0 : |\arg\lambda| \ge \omega\}$ into itself and by (2.4) we have

$$\big\|\lambda(\lambda I - A_s)^{-1}\big\| \le \Big|\frac{\lambda}{\lambda - s}\Big| + \frac{s}{|\lambda - s|}\big\|\varphi(\lambda)(\varphi(\lambda)I - A)^{-1}\big\|, \tag{2.5}$$

and the stated result follows from (2.1) since $|\lambda - s| \ge (\sin\omega/2)(|\lambda| + s)$.

For $0 < \alpha < 1$ we can define

$$A^{-\alpha} f = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{-\alpha} (tI + A)^{-1} f\, dt \qquad (f \in X) \tag{2.6}$$

and

$$A^{\alpha} f = \frac{\sin \pi\alpha}{\pi} \int_0^\infty t^{\alpha-1} A(tI + A)^{-1} f\, dt \qquad (f \in D(A)). \tag{2.7}$$

One can easily check that these integrals converge in the sense of Bochner–Lebesgue on the stated linear subspaces as a consequence of Lemma 2.1.

We define the fractional power $z^\alpha = \exp(\alpha \log |z| + i\alpha \arg z)$. Then for $u \in \mathbb{R} \setminus \{0\}$ we introduce the functions

$$\Phi_u(z) = \frac{z^{1-\alpha}}{u^2 + z^2} \qquad \text{and} \qquad \Psi_u(z) = \frac{z^{1+\alpha}}{u^2 + z^2}, \tag{2.8}$$

which are holomorphic on $\{z : \Re z > 0\}$. The curve γ with parametric form

$$\gamma(t) = \begin{cases} -\, te^{-i\omega}, & -\infty < t < 0; \\ te^{i\omega}, & 0 < t < \infty \end{cases} \tag{2.9}$$

lies in the resolvent set of A by Lemma 2.1.

LEMMA 2.2. *Let*

$$K_1 = 2^{-1} K(\theta') \operatorname{cosec}(\omega + \theta' - \pi/2) \operatorname{cosec}\omega \ \sec(\pi\alpha/2), \tag{2.10}$$
$$K_2 = 2^{-1}\big(1 + K(\theta')\operatorname{cosec}(\omega + \theta' - \pi/2)\big)\operatorname{cosec}\omega/2 \operatorname{cosec}\omega \ \sec(\pi\alpha/2).$$

(i) Then the Bochner–Lebesgue integral

$$\Phi_u(A) = \frac{1}{2\pi i} \int_\gamma \Phi_u(z)(zI - A)^{-1} dz \tag{2.11}$$

defines a bounded linear operator on X such that

$$\|\Phi_u(A)\| \le K_1 |u|^{-1-\alpha} \qquad (u \in \mathbb{R} \setminus \{0\}). \tag{2.12}$$

Likewise $\Phi_u(A_s)$ may be defined by such an integral to give a bounded linear operator such that $\|\Phi_u(A_s)\| \le K_2 |u|^{-1-\alpha}$ for all $u \in \mathbb{R} \setminus \{0\}$.
(ii) Similarly the Bochner–Lebesgue integral

$$\Psi_u(A) = \frac{1}{2\pi i} \int_\gamma \Psi_u(z)(zI - A)^{-1} dz$$

defines a bounded linear operator on X such that

$$\|\Psi_u(A)\| \le K_1|u|^{-1+\alpha} \qquad (u \in \mathbb{R} \setminus \{0\}). \tag{2.13}$$

Likewise $\Psi_u(A_s)$ may be defined by such an integral to give a bounded linear operator such that $\|\Psi_u(A_s)\| \le K_2|u|^{-1+\alpha}$ for all $u \in \mathbb{R} \setminus \{0\}$.
(iii) The above definitions of $\Phi_u(A_s)$ and $\Psi_u(A_s)$ are consistent with the Riesz functional calculus for bounded operators.
(iv) For all $f \in X$,

$$\Phi_u(A_s)f \to \Phi_u(A)f \quad \text{and} \quad \Psi_u(A_s)f \to \Psi_u(A)f \quad \text{as} \quad s \to \infty.$$

Proof. (i) We shall verify that the integral is absolutely convergent. The integrand is continuous in $B(X)$, hence is measurable in the sense of Bochner, and by the triangle inequality we have

$$\Big\|\frac{1}{2\pi i}\int_\gamma \frac{z^{1-\alpha}}{u^2+z^2}(zI-A)^{-1}dz\Big\| \le \frac{1}{2\pi}\int_0^\infty \frac{t^{1-\alpha}}{|t^2e^{2i\omega}+u^2|}\big(\|(te^{i\omega}I-A)^{-1}\| + \|(te^{-i\omega}I-A)^{-1}\|\big)dt, \tag{2.14}$$

so we can use Lemma 2.1(i) and plane trigonometry to obtain

$$\le \frac{K(\theta')}{\pi}\operatorname{cosec}(\theta'+\omega-\pi/2)\ \operatorname{cosec}\omega \int_0^\infty \frac{t^{1-\alpha}}{u^2+t^2}\frac{dt}{t}.$$

The latest expression involves a beta integral and reduces to

$$\frac{K(\theta')}{2\pi}\operatorname{cosec}(\theta'+\omega-\pi/2)\ \operatorname{cosec}\omega\ \Gamma\Big(\frac{1+\alpha}{2}\Big)\Gamma\Big(\frac{1-\alpha}{2}\Big)u^{-1-\alpha}. \tag{2.15}$$

The bound (2.12) follows directly when we compare (2.15) with (2.10). The proof of the bound on $\Phi_u(A_s)$ is similar, except that one uses Lemma 2.1(ii) to bound the resolvents.

(ii) This is similar to (i).

(iii) The spectrum of A_s, denoted $\sigma(A_s)$, is contained in Σ_ω by Lemma 2.1(ii). We shall prove that $\Phi_u(A_s)$, defined as in (2.11), equals the operator $A_s^{1-\alpha}(u^2I+A_s^2)^{-1}$ as defined by the Riesz functional calculus. The Riesz functional calculus for bounded operators is essentially unique as a consequence of Runge's theorem.

Now A_s is a bounded linear operator with $\|A_s\| \le s(1+K(0))$. For small $\delta > 0$ and $R > (1+K(0))s$, we introduce the simple contour $\gamma(\delta, R)$ that consists of the circular arcs $\{\delta e^{i\psi} : \omega \le \psi \le 2\pi - \omega\}$ and $\{Re^{i\psi} : -\omega \le \psi \le \omega\}$ connected by the intervals $[\delta e^{-i\omega}, Re^{-i\omega}]$ and $[Re^{i\omega}, \delta e^{i\omega}]$. This contour winds round $\sigma(A_s)$; hence

$$A_s^{1-\alpha}(u^2 I + A_s^2)^{-1} = \frac{1}{2\pi i}\int_{\gamma(\delta,R)} \Phi_u(z)(zI - A_s)^{-1} dz \tag{2.16}$$

holds by the Riesz functional calculus. Using simple estimates, one can let $\delta \to 0+$ and $R \to \infty$ to replace $\gamma(\delta, R)$ by γ in (2.16), so we deduce that

$$\Phi_u(A_s) = \frac{1}{2\pi i}\int_{\gamma} \Phi_u(z)(zI - A_s)^{-1}\, dz = A_s^{1-\alpha}(u^2 I + A_s^2)^{-1}. \tag{2.17}$$

(iv) We shall now prove that $\Phi_u(A_s)$ converges strongly to the operator $\Phi_u(A)$ as $s \to \infty$; the case of Ψ_u is similar. The Yosida approximants A_s satisfy $\|A_s f\| \le K(0)\|Af\|$ for all $s > 0$, and $A_s f \to Af$ as $s \to \infty$ for $f \in D(A)$ since $s(sI + A)^{-1} \to I$ strongly as $s \to \infty$ by Lemma 2.1(ii) and the density of $D(A)$.

Since $D(A)$ is a dense linear subspace, and $\Phi_u(A_s)$ $(s > 0)$ are uniformly bounded as in (i), it now suffices to show that $\Phi_u(A_s)f \to \Phi_u(A)f$ as $s \to \infty$ for $f \in D(A)$. By (2.2), we can take the limit of (2.17) and deduce from the dominated convergence theorem [**13**, p.45] that $\Phi_u(A_s)f \to \Phi_u(A)f$ as $s \to \infty$. One can deal with Ψ_u similarly.

REMARK. Lemma 2.2(iv) justifies the formula (2.11) as a means of defining $A^{1-\alpha}(u^2 I + A^2)^{-1}$.

3. Proof of the maximal theorem

PROPOSITION 3.1. *Let A, α and f be as in Theorem 1.1. Then*

$$e^{-tA}f = \frac{1}{\pi}\int_{-1}^{1} e^{itu}\Psi_u(A)A^{-\alpha} f du + \frac{1}{\pi}\Big(\int_{-\infty}^{-1} + \int_{1}^{\infty}\Big) e^{itu}\Phi_u(A)A^{\alpha} f du \tag{3.1}$$

holds, where the integrals converge in X in the sense of Bochner–Lebesgue.

Proof. We begin by proving a version of (3.1) for A_s. For $R > s(1 + K(0))$ we let Γ_R be the semicircular contour $[-R, R] \oplus S_R$, where S_R is the semicircular arc with centre 0 that goes from R to $-R$ in the upper half-plane. Since Γ_R winds round $\sigma(iA_s)$, but does not wind round any point in $\sigma(-iA_s)$, we have by the Riesz functional calculus

$$e^{-tA_s} f = \frac{1}{2\pi i}\int_{\Gamma_R} e^{itu}\big((uI - iA_s)^{-1} - (uI + iA_s)^{-1}\big) f\, du \qquad (f \in X). \tag{3.2}$$

For $t > 0$, we can let $R \to \infty$ and use Jordan's lemma to deduce that

$$e^{-tA_s} f = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{itu}(u^2 I + A_s^2)^{-1} A_s f\, du \qquad (f \in X), \tag{3.3}$$

and hence

$$e^{-tA_s}f = \frac{1}{\pi}\int_{-1}^{1} e^{itu}\Psi_u(A_s)A_s^{-\alpha}f\,du + \frac{1}{\pi}\Big(\int_{-\infty}^{-1} + \int_{1}^{\infty}\Big)e^{itu}\Phi_u(A_s)A_s^{\alpha}f\,du \tag{3.4}$$

where the integrals are absolutely convergent in X.

Next we shall check that we can take limits of each term in (3.4) as $s \to \infty$, and thus deduce (3.1). One can use the identity

$$(tI + A_s)^{-1} - \frac{s^2}{(s+t)^2}(tI + A)^{-1} = \frac{1}{s+t} + \frac{s}{(s+t)^2}\Big(\frac{st}{s+t}\Big)\Big(\frac{st}{s+t}I + A\Big)^{-1}t(tI + A)^{-1}$$

and Lemma 2.1(i) to obtain

$$\Big\|(tI + A_s)^{-1} - \frac{s^2}{(s+t)^2}(tI + A)^{-1}\Big\| \le \frac{1}{s+t} + \frac{s}{(s+t)^2}K(0)^2 \qquad (s, t > 0). \tag{3.5}$$

As a consequence of this bound, it follows from (2.7) and the dominated convergence theorem that $A_s^{\alpha}f \to A^{\alpha}f$ as $s \to \infty$ for $f \in D(A)$; see [**13**, p. 45]. Further, we have $A_s^{\alpha} = (s(sI+A)^{-1})^{\alpha}A^{\alpha}$ and hence $\|A_s^{\alpha}f\| \le K_3\|A^{\alpha}f\|$ for all $f \in D(A)$ and some constant K_3. Now we can let $s \to \infty$ and obtain the limits of the terms in the integrand of (3.4):

$$(u^2I + A_s^2)^{-1}A_sf = \Psi_u(A_s)A_s^{-\alpha}f \to \Psi_u(A)A^{-\alpha}f \qquad (f \in D(A^{-1})), \tag{3.6}$$

$$(u^2I + A_s^2)^{-1}A_sf = \Phi_u(A_s)A_s^{\alpha}f \to \Phi_u(A)A^{\alpha}f \qquad (f \in D(A)). \tag{3.7}$$

The final step is to apply the dominated convergence theorem to (3.4) and thus deduce that (3.1) holds for $f \in D(A) \cap D(A^{-1}) = D(A)$. Let K_1 be as in (2.10) and observe that

$$\int_{-1}^{1}\|\Psi_u(A)\|\|A^{-\alpha}f\|du + \Big(\int_{-\infty}^{-1} + \int_{1}^{\infty}\Big)\|\Phi_u(A)\|\|A^{\alpha}f\|du$$
$$\le 2K_1\int_0^1 u^{\alpha-1}du\|A^{-\alpha}f\| + 2K_1\int_1^{\infty}u^{-1-\alpha}du\|A^{\alpha}f\|. \tag{3.8}$$

Hence the right-hand side of (3.1) converges in X, for $f \in D(A^{\alpha}) \cap D(A^{-\alpha}) = D(A^{\alpha})$. The left-hand side of (3.4) converges to $e^{-tA}f$ by [**12**, p. 80].

Conclusion of the proof of Theorem 1.1. For $1 < p < \infty$, let $q = p/(p-1)$. Now let $L^q(\Omega, \mu; L^1((0,\infty), dt))$ (or briefly $L^q(L^1)$) be the space of measurable real functions $g(x,t)$ such that the norm

$$\|g\|_{L^q(L^1)} = \Big\{\int_{\Omega}\Big(\int_0^{\infty}|g(x,t)|dt\Big)^q\mu(dx)\Big\}^{1/q} \tag{3.9}$$

is finite. Let $L^p(\Omega,\mu; L^\infty((0,\infty),dt))$ be the space of equivalence classes of measurable real functions $v(x,t)$ such that the norm

$$\|v\|_{L^p(L^\infty)} = \Big\{\int_\Omega \operatorname{ess\,sup}_{t>0} |v(x,t)|^p \mu(dx)\Big\}^{1/p} \tag{3.10}$$

is finite. Then by [**13**, p.97] $L^p(L^\infty)$ is linearly isometrically isomorphic to a closed linear subspace of the dual of $L^q(L^1)$, and hence

$$\|v\|_{L^p(L^\infty)} = \sup_g \Big\{\Big|\int_\Omega\int_0^\infty v(x,t)g(x,t)\,dt\mu(dx)\Big| : g \in L^q(L^1); \|g\|_{L^q(L^1)} \le 1\Big\}. \tag{3.11}$$

We shall prove that, for f as in Theorem 1.1, $v(x,t) = T_t f(x)$ defines a bounded linear functional on $L^q(L^1)$.

The preceding estimates justify the various applications of Fubini's Theorem in the following calculations, where we use (3.1) to write

$$\begin{aligned}\int_\Omega\int_0^\infty g(x,t)T_t f(x)dt\,\mu(dx) &= \frac{1}{\pi}\int_{-1}^1\int_\Omega\Big(\int_0^\infty g(x,t)e^{itu}dt\Big)\Psi_u(A)A^{-\alpha}f(x)\mu(dx)\,du \\ &+ \frac{1}{\pi}\Big(\int_{-\infty}^{-1} + \int_1^\infty\Big)\int_\Omega\Big(\int_0^\infty g(x,t)e^{itu}dt\Big)\Phi_u(A)A^\alpha f(x)\mu(dx)\,du. \end{aligned} \tag{3.12}$$

The modulus of the first term on the right-hand side of (3.12) is

$$\le \frac{1}{\pi}\int_{-1}^1\int_\Omega\Big(\int_0^\infty |g(x,t)|dt\Big)|\Psi_u(A)A^{-\alpha}f(x)|\mu(dx)\,du, \tag{3.13}$$

and we can apply Hölder's inequality to the μ-integral to show this is

$$\le \frac{1}{\pi}\int_{-1}^1 \|g\|_{L^q(L^1)}\|\Psi_u(A)A^{-\alpha}f\|_{L^p}du.$$

By Lemma 2.2(ii), we can conclude that the latest expression is

$$\le \frac{2K_1}{\pi}\|g\|_{L^q(L^1)}\|A^{-\alpha}f\|_{L^p}\int_0^1 u^{\alpha-1}du. \tag{3.14}$$

One can deal with the other terms in (3.12) likewise, and obtain the bound

$$\Big|\int_\Omega\int_0^\infty g(x,t)T_t f(x)\,dt\,\mu(dx)\Big| \le \frac{2K_1}{\alpha\pi}\|g\|_{L^q(L^1)}\big(\|A^\alpha f\|_{L^p} + \|A^{-\alpha}f\|_{L^p}\big). \tag{3.15}$$

By the duality formula (3.11), $\sup_{t>0}|T_t f(x)|$ belongs to L^p and its norm satisfies the stated inequality (1.2).

4. Examples and bounded H^∞ functional calculus

In this section we consider which semigroups lie within the scope of Theorem 1.1.

1. When $p = 2$ and X is a Hilbert space, there is a natural source of examples given by normal operators N that are densely defined and have spectrum in the sector $\bar{\Sigma}_{\pi/2-\theta'}$. Then for any bounded and invertible linear operator S on X, the operator $A = S^{-1}NS$ satisfies the hypotheses of Theorem 1.1. See also [**8**].

2. Let $-A$ be the generator of a symmetric diffusion semigroup T_t on a probability space (Ω, μ) in the sense of Stein [**18**, p. 67] so that:–
(a) T_t $(t > 0)$ is a contraction semigroup on $L^p(\Omega, \mu)$ for $1 \leq p \leq \infty$;
(b) T_t $(t \geq 0)$ is a C_0–semigroup of self-adjoint contraction operators on $L^2(\Omega, \mu)$.
Then T_t is a C_0–semigroup on L^p for $1 \leq p < \infty$ by [**17**, Theorem X.55], and a weak-$*$ continuous semigroup on L^∞; the example of the Hermite semigroup shows that the semigroup of $T_t : L^\infty \to L^\infty$ need not be strongly continuous at 0. Further, T_t extends to define a holomorphic semigroup T_z on L^p for $1 < p < \infty$ and $z \in \Sigma_\theta$ for $\theta < \pi(1-|(2-p)/p|)/2$; so that, T_z satisfies axioms (i), (iii) and (iv) of section 1. Moreover, the imaginary powers A^{-iu} generate a C_0–group on $L^p(\Omega, \mu)$ for $1 < p < \infty$ with $\|A^{iu}\| \leq C_p e^{\pi|u|/2}$ for all $u \in \mathbb{R}$.

For such semigroups the maximal operator is bounded $L^p(\Omega, \mu) \to L^p(\Omega, \mu)$ for $1 < p < \infty$. Stein's proof makes use of the spectral theorem for self-adjoint operators, and thus of axiom (b). Instead of self-adjointness, the maximal theorems of [**1**] are obtained by transference and involve a hypothesis similar to: $f \geq 0 \Rightarrow T_t f \geq 0$.

The prototypical example of a symmetric diffusion semigroup is the heat semigroup $e^{-t\Delta}$ generated by the Laplace operator $-\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$ on $\mathbb{R}^n$; even in this case the Schrödinger operator $e^{it\Delta}$ is unbounded on $L^p(\mathbb{R}^n, dx)$ for $p \neq 2$ and $t \neq 0$ by [**16**, p. 107]. For the Schrödinger group $e^{it\Delta}$ $(t \in \mathbb{R})$ on $L^2(\mathbb{R}, dx)$, Carleson showed that the maximal operator is bounded $D(\Delta^\alpha) \to L^2$ for $\alpha = 1/8$, and Dahlberg and Kenig proved that this exponent is optimal [**7**, **11**].

3. Let $H^\infty(\Sigma_\phi)$ be the Banach algebra of bounded and holomorphic functions $F : \Sigma_\phi \to \mathbb{C}$ with the supremum norm $\| \ \|_\infty$, and let $H_0^\infty(\Sigma_\phi)$ be the subalgebra of all F such that $|z|^{-s}(1 + |z|)^{2s}|F(z)|$ is bounded on Σ_ϕ for some $s > 0$. We say that a sectorial operator A of type $\omega < \phi$ admits a *bounded $H^\infty(\Sigma_\phi)$ functional calculus* if there exists a constant κ such that

$$F(A) = \frac{1}{2\pi i} \int_\gamma F(z)(zI - A)^{-1} dz \qquad (F \in H_0^\infty(\Sigma_\phi)) \tag{4.1}$$

defines a bounded linear operator on X with $\|F(A)\| \leq \kappa\|F\|_\infty$ and the functional calculus map $F \mapsto F(A)$ extends to define a bounded homomorphism $H^\infty(\Sigma_\phi) \to B(X)$. See [**10**].

If A has a bounded $H^\infty(\Sigma_\phi)$ functional calculus for some ϕ such that $0 < \phi < \pi/2$, then the imaginary powers A^{iu} form a C_0–group with $\|A^{iu}\| \leq \kappa e^{|u|\phi}$. In this case the maximal function $M(f)$ belongs to L^p for each $f \in D(\log A)$ by Theorem 3.1 of [**6**] – a significantly weaker hypothesis on f than that required by Theorem 1.1 above.

Unfortunately, it is a stringent condition for A to have a bounded $H^\infty(\Sigma_\phi)$ functional calculus with $\phi < \pi/2$; Theorem 4.1 below gives a sufficient condition for an even stronger property to hold. Baillon and Clement have exhibited, for each $0 < \omega < \pi$, a sectorial operator A in Hilbert space of type ω such that A^{it} is unbounded for each $t \in \mathbb{R} \setminus \{0\}$; see [**2**, **15**].

THEOREM 4.1. *Let iA be the generator of a bounded C_0–group on a closed linear subspace X of $L^p(\Omega, \mu)$ for some p such that $1 < p < \infty$. Suppose further that the spectrum of A is contained in $[0, \infty)$. Then A has a bounded $H^\infty(\Sigma_\phi)$ functional calculus for each ϕ such that $0 < \phi < \pi$. Further, e^{-tA} and e^{-tA^2} define bounded holomorphic semigroups on Σ_θ for all θ such that $0 < \theta < \pi/2$.*

Proof. Berkson, Gillespie and Muhly have shown by transference that A is associated with a resolution of the identity with respect to an increasing family $E(s)$ $(0 \leq s < \infty)$ of uniformly bounded projections in $B(X)$ that is strongly right-continuous and has strong left-hand limits; see [**4**]. Further there is an integral with respect to this spectral family, and a bounded functional calculus map which we now describe.

Let $\Delta_k = [2^k, 2^{k+1})$ $(k \in \mathbb{Z})$ be the standard dyadic decomposition of $(0, \infty)$ and for a complex function F let

$$\operatorname{var}(F; \Delta_k) = \sup\Big\{ \sum_{j=1}^{N-1} |F(s_{j+1}) - F(s_j)| : 2^k \leq s_1 < \cdots < s_N \leq 2^{k+1} \Big\} \tag{4.2}$$

be the total variation of F over Δ_k. Then the Marcinkiewicz 1-multipliers are the functions F such that the norm

$$\|F\|_{\mathbb{M}^1(0,\infty)} = \sup_{s>0} |F(s)| + \sup_{k\in\mathbb{Z}} \operatorname{var}(F; \Delta_k) \tag{4.3}$$

is finite. By Theorem 1.2 of [**3**] there is a bounded functional calculus map $\mathbb{M}^1(0, \infty) \to B(X)$:

$$F \mapsto F(A) = \int_0^\infty F(s) E(ds). \tag{4.4}$$

If F is bounded and holomorphic in a sector Σ_ϕ for some $\phi \leq \pi/2$, then $x|F'(x)| \leq \|F\|_\infty \operatorname{cosec} \phi$ holds for $x > 0$ by the Cauchy integral formula, and hence the restriction of F to $(0, \infty)$ belongs to $\mathbb{M}^1(0, \infty)$. We deduce that $\|F(A)\| \leq \kappa(A, \phi)\|F\|_\infty$ for some $\kappa(A, \phi)$ which is independent of F; hence A has a bounded $H^\infty(\Sigma_\phi)$ functional calculus.

When t lies in the open right half-plane, the functions $F(x) = e^{-tx}$ and $G(x) = e^{-tx^2}$ have total variation on $(0, \infty)$ less than or equal to $|t|/\Re t$; hence the uniform bounds

$$\left\|e^{-tA}\right\| \leq \kappa(A, \theta), \quad \left\|e^{-tA^2}\right\| \leq \kappa(A, \theta) \qquad (t \in \Sigma_\theta) \tag{4.5}$$

hold for $\theta < \pi/2$.

4. Theorem 4.1 applies to the Poisson and heat semigroups on $L^p(\mathbb{R})$ $(1 < p < \infty)$, since we can take for A the operator that is determined by the Fourier multiplier $\widehat{(Af)}(\xi) = |\xi|\hat{f}(\xi)$ for $f \in C_c^\infty(\mathbb{R})$. To see that this defines a C_0–group, we write

$$e^{-iAt} = U_t P_+ + U_{-t}(I - P_+) \qquad (t \in \mathbb{R}), \tag{4.6}$$

where U_t $(t \in \mathbb{R})$ is the translation group $U_t f(x) = f(x - t)$ and P_+ is the Riesz projection onto the translation-invariant subspace $\{f \in L^p(\mathbb{R}) : \hat{f}(\xi) = 0, \xi \leq 0\}$. Since P_+ is bounded on L^p for $1 < p < \infty$, [**19**, Chapter 2] one can now easily check that e^{iAt} is uniformly bounded and strongly continuous.

Acknowledgements
GB acknowledges the hospitality of the University of New South Wales, where this work was carried out. The authors thank the referee for comments that improved the presentation of this paper.

References

1. N. Asmar, E. Berkson, and T.A. Gillespie, Transference of strong type maximal inequalities by separation–preserving representations, *Amer. J. Math.* **113** (1991), 47–74.

2. J.-B. Baillon and P. Clement, Examples of unbounded imaginary powers of operators, *J. Funct. Anal.* **100** (1991), 419–434.

3. E. Berkson and T.A. Gillespie, The q-variation of functions and spectral integration of Fourier multipliers, *Duke Math. J.* **88** (1997), 103–132.

4. E. Berkson, T.A. Gillespie, and P.S. Muhly, Abstract spectral decompositions guaranteed by the Hilbert transform, *Proc. London Math. Soc. (3)* **53** (1986), 489–517.

5. G. Blower, Maximal functions and transference for groups of operators, *Proc.*

Edinb. Math. Soc. (2) **43** (2000), 57–71.

6. G. Blower, Maximal functions and subordination for operator groups, *Proc. Edinb. Math. Soc. (2)* **45** (2002), 27–42.

7. L. Carleson, Some analytic problems related to statistical mechanics, pp. 5–45 in: *Euclidean Harmonic Analysis*, (ed. J.J. Benedetto) Lecture Notes in Mathematics, **779** Springer Berlin, 1980.

8. J.A. van Casteren, Operators similar to unitary or selfadjoint ones, *Pacific J. Math.* **104** (1983), 241-255.

9. M.G. Cowling, Harmonic analysis on semigroups, *Ann. Math.* **117** (1983), 267-283.

10. M. Cowling, I. Doust, A. McIntosh, and A. Yagi, Banach space operators with a bounded H^∞ functional calculus, *J. Austral. Math. Soc. Ser. A* **60** (1996), 51–89.

11. B.E.J. Dahlberg and C.E. Kenig, A note on almost everywhere behaviour of solutions of the Schrödinger operator, pp. 205-209, in *Harmonic Analysis*, (eds. F. Ricci and G. Weiss) Lecture Notes in Math., **908**, Springer, Berlin, 1982.

12. E.B. Davies, *One-Parameter Semigroups*, Academic Press, London 1980.

13. J. Diestel and J.J. Uhl, *Vector Measures,* American Mathematical Society, Providence RI, 1977.

14. E. Hille and R.S. Phillips, *Functional Analysis and Semigroups,* Colloquium Pubications, vol. XXXI, AMS, Providence, RI, 1957.

15. A. McIntosh and A.Yagi, Operators of type ω without a bounded H^∞ functional calculus, *Proc. Centre Math. Anal. Austral. Nat. Univ.* **24** (1990), 159–172. Canberra.

16. J. Rauch, *Partial Differential Equations*, Springer, New York, 1991.

17. M. Reed and B. Simon, *Methods of Modern Mathematical Physics II: Fourier Analysis, Self-Adjointness,* Academic Press, London, 1975.

18. E.M. Stein, *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory,* Princeton University Press, Princeton, N.J., 1970.

19. E.M. Stein, *Singular Integrals and Differentiablity Properties of Functions*, Princeton University Press, Princeton, 1970.

20. H. Tanabe, *Equations of Evolution*, Pitman, London, 1979.