A comparison of algebras of functions of bounded variation

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Abstract

Motivated by problems in the spectral theory of linear operators the authors previously introduced a new concept of variation for functions defined on a nonempty compact subset of the plane. In this paper we examine the algebras of functions of bounded variation one obtains from these new definitions for the case where the underlying compact set is either a rectangle or the unit circle, and compare these algebras with the ones previously used by Berkson and Gillespie in their theories of AC-operators and trigonometrically well-bounded operators.

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1 Introduction

For a function whose domain is a subset of the plane, there are a number of different ways of measuring its variation. For applications in operator theory, one is typically interested in looking at algebras of functions which are defined on the spectrum of some bounded operator. This leads one to seek a definition which is applicable when the domain is a compact subset of the plane.

In modelling operators which possess spectral expansions of a conditional nature, Smart [Sm] introduced well-bounded operators, which are those operators which admit a functional calculus for the absolutely continuous functions on some compact interval of the real line. Well-bounded operators can be thought of a generalization to Banach spaces of self-adjoint operators on a Hilbert space. The analogue of unitary operators in this context, the trigonometrically well-bounded operators, were introduced by Berkson and Gillespie [BG2] using the natural definition for functions of bounded variation on the unit circle. The challenge in providing a suitable analogue of general normal operators was to find a suitable concept of variation for functions defined on an arbitrary compact plane set. In [BG1], Berkson and Gillespie used a notion of variation due to Hardy [Ha] and Krause [Kr] to introduce an algebra $BV_{HK}(J \times K)$ of function of bounded variation on rectangles in the plane, and the corresponding notion of an AC-operator. Motivated by a desire to extend the theory of well-bounded and trigonometrically well-bounded operators, the authors [AD] recently introduced a new definition of variation for complex-valued functions defined on arbitrary compact subsets $\sigma \subset \mathbb{C}$. This definition leads to a Banach algebra $BV(\sigma)$, and a corresponding new classes of operators.

It is natural therefore to ask what the relationship is between these earlier concepts of variation and the one introduced in [AD]. We shall show in this paper that

(i) $BV_{HK}(J \times K) \subset BV(J \times K)$.

(ii) The inclusion map $BV_{HK}(J \times K) \hookrightarrow BV(J \times K)$ is continuous.

(iii) If J and K are nondegenerate, then $BV_{HK}(J \times K) \neq BV(J \times K)$.

We also show that for the case where σ is the unit circle $\mathbb{T}$, the new definition essentially reproduces the more classical one. We discuss the operator theoretic consequences of these results in Section 6.

2 $BV(\sigma)$ for $\sigma \subset \mathbb{C}$ Compact

Here we shall recall briefly the definition of a function of bounded variation introduced in [AD]. The reader is referred to [AD] for the full details.

By a curve in the plane we shall mean an element of the set $\Gamma = C([0,1])$. If $\gamma_1, \gamma_2 \in \Gamma$ and if there exists $h : [0,1] \to [0,1]$ where h is a continuous non-decreasing or non-increasing surjective function such that $\gamma_1(t) = \gamma_2(h(t))$ for all $t \in [0,1]$ then we write $\gamma_1 \cong \gamma_2$.

Let $\gamma \in \Gamma$. Then $t \in [0,1]$ is said to be an *entry point* for γ on a line l if either

(i) $t = 0$ and $\gamma(0) \in l$ or

(ii) $\gamma(t) \in l$ and for all $\epsilon > 0$ there exists $s \in (t - \epsilon, t) \cap [0,1]$ such that $\gamma(s) \notin l$.

Suppose $\gamma \in \Gamma$. We define $\mathrm{vf}(\gamma, l)$ to be the number of entry points of γ on l.

We set $\mathrm{vf}(\gamma)$ to be the supremum of $\mathrm{vf}(\gamma, l)$ over all lines l. We write $\mathrm{vf}_{\mathrm{H}}(\gamma)$ for the supremum of $\mathrm{vf}(\gamma, l)$ over all horizontal lines l, vf_{V} for the supremum of $\mathrm{vf}(\gamma, l)$ over all vertical lines. Clearly $\mathrm{vf} \geq \mathrm{vf}_{\mathrm{H}}$ and $\mathrm{vf} \geq \mathrm{vf}_{\mathrm{V}}$. We write ρ for $\dfrac{1}{\mathrm{vf}}$. If, for example, $\mathrm{vf}(\gamma) = \infty$ then we take the convention that $\rho(\gamma) = 0$.

Let $\sigma \subset \mathbb{C}$ be compact and let l be a line parameterized by $\mathbb{R}$. Then $t \in \mathbb{R}$ is said to be an entry point of l on σ if $l(t) \in \sigma$ and for all $\epsilon > 0$ there exists $s \in (t - \epsilon, t)$ such that $l(s) \notin \sigma$. Again set $\mathrm{vf}(\sigma, l)$ to be the number of entry points of l on σ and $\mathrm{vf}(\sigma)$ to be the supremum of $\mathrm{vf}(\sigma, l)$ over all lines l. Clearly $\mathrm{vf}(\sigma, l)$ does not depend on the choice of parameterization of the line l.

Let $z_0, \ldots, z_n \in \mathbb{C}$. Write $\Pi(z_0, z_1, \ldots, z_n)$ for the (uniform speed parameterization) of the curve consisting of line segments joining the points $z_0, z_1, \ldots z_n$. We write

$$\Gamma_L = \{\gamma \in \Gamma \,:\, \gamma \cong \Pi(z_0, \ldots, z_n) \text{ for some } z_0, \ldots z_n \in \mathbb{C}\}$$

for the set of all piecewise linear curves.

Let $\gamma \in \Gamma$. Let $S = \{s_0 < s_1 < \dots s_n\} \subset [0,1]$. Set

$$\gamma_S = \Pi(\gamma(s_1), \gamma(s_2), \dots, \gamma(s_n)) \in \Gamma_L.$$

The curve γ_S is said to be the *S approximation of γ*.

Lemma 2.1. *Let $\gamma \in \Gamma$ and suppose* $\mathrm{vf}(\gamma) < \infty$. *Then* $\lim_{S \in \Lambda([0,1])} \rho(\gamma_S) = \rho(\gamma)$.

Let $\gamma \in \Gamma$ and let $\emptyset \neq \sigma \subset \mathbb{C}$ be compact. We say that $\{z_i\}_{i=1}^n$ is a *partition of γ over σ* if $z_i \in \sigma$ for all i and if there exists $\{s_0 < s_1 < \dots s_n\} \subset [0,1]$ such that $z_i = \gamma(s_i)$ for all i. Let $\Lambda(\sigma, \gamma)$ be the lattice of partitions of γ over σ.

Let $f : \sigma \mapsto \mathbb{C}$ and let $\gamma \in \Gamma$. We define the *variation along the curve* γ by

$$\mathrm{cvar}(f, \gamma, \sigma) = \mathrm{cvar}(f, \gamma) = \sup_{\{z_i\}_{i=1}^n \in \Lambda(\sigma,\gamma)} \sum_{i=1}^{n-1} |f(z_{i+1}) - f(z_i)|$$

Lemma 2.2. *Let $f : \sigma \to \mathbb{C}$. Let $\gamma_1, \gamma_2 \in \Gamma$ and suppose that $\gamma_1 \cong \gamma_2$. Then* $\mathrm{cvar}(f, \gamma_1) = \mathrm{cvar}(f, \gamma_2)$.

Definition 2.3. *Let $f : \sigma \to \mathbb{C}$. Then* variation *of f on σ is defined to be*

$$\mathrm{var}(f, \sigma) = \sup_{\gamma \in \Gamma} \rho(\gamma)\, \mathrm{cvar}(f, \gamma). \tag{1}$$

Here we take the convention that if $\gamma \in \Gamma$ is such that $\rho(\gamma) = 0$ and if $\mathrm{cvar}(f, \gamma) = \infty$ then $\rho(\gamma)\, \mathrm{cvar}(f, \gamma) = 0$.

In practice, Γ is usually too large a set to work with. As the next lemma shows, one can replace Γ with Γ_L.

Lemma 2.4. *Let $f : \sigma \to \mathbb{C}$. Then*

$$\sup_{\gamma \in \Gamma_L} \rho(\gamma)\, \mathrm{cvar}(f, \gamma) = \sup_{\gamma \in \Gamma} \rho(\gamma)\, \mathrm{cvar}(f, \gamma).$$

Note that for subintervals of $\mathbb{R}$, this new definition agrees with the standard one.

Proposition 2.5. *Let $f \in BV([0,1])$. Then*

$$\operatorname*{var}_{[} 0,1] f = \sup_{\gamma \in \Gamma} \rho(\gamma)\, \mathrm{cvar}(f, \gamma).$$

For $f : \sigma \to \mathbb{C}$, set $\|f\|_{BV(\sigma)} = \|f\|_\infty + \mathrm{var}(f, \sigma)$. The functions of bounded variation with domain σ are defined to be

$$BV(\sigma) = \left\{ f : \sigma \mapsto \mathbb{C} : \|f\|_{BV(\sigma)} < \infty \right\}.$$

Theorem 2.6. $\left(BV(\sigma), \|\cdot\|_{BV(\sigma)}\right)$ *is a Banach algebra.*

3 A comparison of $BV(J \times K)$ and $BV_{HK}(J \times K)$

Let $J \times K = [a, b] \times [c, d] \subset \mathbb{R}^2 \cong \mathbb{C}$ be a fixed rectangle. For functions on such sets, definitions of bounded variation and absolute continuity were given by Hardy and Krause, and these definitions were used by Berkson and Gillespie [BG1] to define AC operators. The functions of bounded variation in the Hardy-Krause sense form a Banach algebra which we denote $BV_{HK}(J \times K)$. Both $BV(J \times K)$ and $BV_{HK}(J \times K)$ are isomorphic to $BV(J)$ if K is degenerate (that is $K = [c, c]$). It is natural to ask about the relationship between $BV(J \times K)$ and $BV_{HK}(J \times K)$ since this will determine how the operator theory based around these new function algebras compares with the established theory. In this section we shall show that $BV_{HK}(J \times K)$ forms a subset of $BV(J \times K)$ and that the inclusion is proper if J and K are nondegenerate. The operator theoretic consequences of this are discussed in Section 6.

We say that $\{s_i, t_j\}_{i,j=1}^{n,m}$ is a *partition of* $J \times K$ if $\{s_i\}_{i=1}^n \in \Lambda(J)$ and $\{t_i\}_{j=1}^m \in \Lambda(K)$. The set of partitions of $J \times K$ is denoted $\Lambda(J \times K)$. Let $S = \{s_i, t_j\}_{i,j=1}^{n,m} \in \Lambda(J \times K)$, $T = \left\{s'_i, t'_j\right\}_{i,j=1}^{n',m'} \in \Lambda(J \times K)$. Then T is said to be a *refinement* of S if $\{s'_i\}_{i=1}^{n'}$ is a refinement of $\{s_i\}_{i=1}^n$ and $\{t'_i\}_{i=1}^{m'}$ is a refinement of $\{t_i\}_{i=1}^m$. We then write $S \le T$. We shall write $S \vee T$ for the partition with the least number of elements which is a refinement of both S and T.

Let $f : J \times K \to \mathbb{C}$ and let $S = \{s_i, t_j\}_{i,j}^{n,m} \in \Lambda(J \times K)$. Define

$$\omega(f, S) = \sum_{i,j=1}^{n-1,m-1} |f(s_i, t_j) - f(s_{i+1}, t_j) + f(s_{i+1}, t_{j+1}) - f(s_i, t_{j+1})| .$$

The two dimensional variation in the Hardy-Krause sense is defined as

$$\mathrm{var}_{\mathrm{HK}}(f, J \times K) = \sup_{S \in \Lambda(J \times K)} \omega(f, S).$$

The norm

$$\|f\|_{BV_{HK}} = |f(a, c)| + \mathrm{var}(f(\cdot, c), J) + \mathrm{var}(f(a, \cdot), K) + \mathrm{var}_{\mathrm{HK}}(f, J \times K).$$

is equivalent to that introduced by Berkson and Gillespie [BG1]. The functions of bounded variation in the Hardy-Krause sense are defined to be

$$BV_{HK}(J \times K) = \left\{f : J \times K \to \mathbb{C} : \|f\|_{BV_{HK}} < \infty\right\} .$$

For properties of this Banach algebra see [CA] and [BG1].

It is an important fact that if $f \in BV(J)$ is real-valued then there exist $f_1, f_2 \in BV(J)$ such that $f = f_1 - f_2$, f_1 and f_2 are non-decreasing functions, $\|f_1\|_{BV(J)} \le \|f\|_{BV(J)}$ and $\|f_2\|_{BV(J)} \le \|f\|_{BV(J)}$. One of our aims in this sections is show an analogous result for $f \in BV_{HK}(J \times K)$.

It is easy to see that if $f \in BV(J)$ then $u(f) \in BV_{HK}(J \times K)$ and that $\|f\|_{BV(J)} = \|u(f)\|_{BV_{HK}}$. Similarly if $g \in BV(K)$ then $v(g) \in BV_{HK}(J \times K)$ and $\|g\|_{BV(K)} = \|v(g)\|_{BV_{HK}}$.

Let $x \in \mathbb{R}$. We define $x^+ = \max\{0, x\}$ and $x^- = \min\{0, x\}$. Let $f \in BV_{HK}(J \times K)$ be real-valued. Set

$$\omega^+(f, S) = \sum_{i,j=1}^{n-1,m-1} (f(s_i, t_j) - f(s_{i+1}, t_j) + f(s_{i+1}, t_{j+1}) - f(s_i, t_{j+1}))^+$$

and

$$\omega^-(f, S) = -\sum_{i,j=1}^{n-1,m-1} (f(s_i, t_j) - f(s_{i+1}, t_j) + f(s_{i+1}, t_{j+1}) - f(s_i, t_{j+1}))^-.$$

Clearly $\omega(f, S) = \omega^+(f, S) + \omega^-(f, S)$. For $(x, y) \in J \times K$ set

$$\begin{aligned} \upsilon_f^+(x, y) &= \sup_{S \in \Lambda([a,x] \times [c,y])} \omega^+(f, S), \\ \upsilon_f^-(x, y) &= \sup_{S \in \Lambda([a,x] \times [c,y])} \omega^-(f, S). \end{aligned}$$

The following result is immediate.

Lemma 3.1. *Let $f \in BV_{HK}(J \times K)$ be real-valued. Then $\upsilon_f^+(\cdot, c) = \upsilon_f^-(\cdot, c) = 0$ and $\upsilon_f^+(a, \cdot) = \upsilon_f^-(a, \cdot) = 0$.*

Lemma 3.2. *Let $f \in BV_{HK}(J \times K)$ be real-valued. Fix $x \in J$ and fix $y \in K$. Then the functions $\upsilon_f^+(x, \cdot)$, $\upsilon_f^-(x, \cdot)$, $\upsilon_f^+(\cdot, y)$ and $\upsilon_f^-(\cdot, y)$ are non-decreasing.*

Proof. Let $a \le x_1 \le x_2 \le b$. Then $\Lambda([a, x_1] \times [c, y]) \subset \Lambda([a, x_2] \times [c, y])$. Hence

$$\begin{aligned} \upsilon_f^+(x_1, y) &= \sup_{S \in \Lambda([a,x_1] \times [c,y])} \omega^+(f, S) \\ &\le \sup_{S \in \Lambda([a,x_2] \times [c,y])} \omega^+(f, S) \\ &= \upsilon_f^+(x_2, y). \end{aligned}$$

The other claims have similar proofs. □

Lemma 3.3. *Let $f \in BV_{HK}(J \times K)$ be real-valued and let $(x, y) \in J \times K$. Then*

$$\upsilon_f^+(x, y) + \upsilon_f^-(x, y) = \mathrm{var}_{\mathrm{HK}}(f, [a, x] \times [c, y]).$$

Proof.

$$\begin{aligned} \mathrm{var}_{\mathrm{HK}}(f, [a, x] \times [c, y]) &= \sup_{S \in \Lambda([a,x] \times [c,y])} \omega(f, S) \\ &= \sup_{S \in \Lambda([a,x] \times [c,y])} (\omega^+(f, S) + \omega^-(f, S)) \\ &\le \sup_{S \in \Lambda([a,x] \times [c,y])} \omega^+(f, S) + \sup_{S \in \Lambda([a,x] \times [c,y])} \omega^-(f, S) \\ &= \upsilon_f^+(x, y) + \upsilon_f^-(x, y). \end{aligned}$$

Fix $\epsilon > 0$. There exists $S_1, S_2 \in \Lambda([a, x] \times [c, y])$ such that $\upsilon_f^+(x, y) \leq \omega^+(f, S_1) + \frac{\epsilon}{2}$ and $\upsilon_f^-(x, y) \leq \omega^-(f, S_2) + \frac{\epsilon}{2}$. Then

$$\begin{aligned}\upsilon_f^+(x, y) + \upsilon_f^-(x, y) &\leq \omega^+(f, S_1) + \omega^-(f, S_2) + \epsilon \\ &\leq \omega^+(f, S_1 \vee S_2) + \omega^-(f, S_1 \vee S_2) + \epsilon \\ &= \omega(f, S_1 \vee S_2) + \epsilon \\ &\leq \mathrm{var}_{\mathrm{HK}}(f, [a, x] \times [c, y]) + \epsilon.\end{aligned}$$

□

Lemma 3.4. *Let $f \in BV_{HK}(J \times K)$ be real-valued and let $(x, y) \in J \times K$. Then*

$$\upsilon_f^+(x, y) - \upsilon_f^-(x, y) = f(a, c) - f(x, c) + f(x, y) - f(a, y).$$

Proof. Fix $\epsilon > 0$. There exists $S \in \Lambda([a, x] \times [c, y])$ such that

$$\begin{aligned}\omega^+(f, S) &\leq \upsilon_f^+(x, y) \leq \omega^+(f, S) + \epsilon \text{ and} \\ \omega^-(f, S) &\leq \upsilon_f^-(x, y) \leq \omega^-(f, S) + \epsilon.\end{aligned}$$

Without loss of generality we can assume that $(a, c), (x, y) \in S$. But by cancellation of terms $\omega^+(f, S) - \omega^-(f, S) = f(a, c) - f(x, c) + f(x, y) - f(a, y)$ and so the result follows. □

Lemma 3.5. *Let $f \in BV_{HK}(J \times K)$ be real-valued. Furthermore, suppose that for all $(x_1, y_1), (x_2, y_2) \in J \times K$ where $x_1 \leq x_2$ and $y_1 \leq y_2$ that $f(x_2, y_2) - f(x_1, y_2) \geq f(x_2, y_1) - f(x_1, y_1)$. Then for any $[x_1', x_2'] \times [y_1', y_2'] \subset J \times K$*

$$\mathrm{var}_{\mathrm{HK}}(f, [x_1', x_2'] \times [y_1', y_2']) = f(x_1', y_1') - f(x_2', y_1') + f(x_2', y_2') - f(x_1', y_2').$$

Proof. Let $S = \{s_i, t_j\}_{i,j=1}^{n,m} \subset \Lambda([x_1', x_2'] \times [y_1', y_2'])$. By refining if necessary we can assume that $(s_1, t_1) = (x_1', y_1')$ and $(s_n, t_m) = (x_2', y_2')$. By assumption, for each i and j, $f(s_i, t_j) - f(s_{i+1}, t_j) + f(s_{i+1}, t_{i+1}) - f(s_i, t_{i+1}) \geq 0$. Hence $\omega(f, S) = \omega^+(f, S)$. There is therefore cancellation of terms in $\omega(f, S)$ to give $\omega(f, S) = f(x_1', y_1') - f(x_2', y_1') + f(x_2', y_2') - f(x_1', y_2')$. Taking the supremum over all $S \in \Lambda(J \times K)$ on the left hand side gives the result. □

Lemma 3.6. *Let $f \in BV_{HK}(J \times K)$ be real-valued. Let $[x_1, x_2] \times [y_1, y_2] \subset J \times K$. Then*

$$\begin{aligned}&\mathrm{var}_{\mathrm{HK}}(\upsilon_f^+, [x_1, x_2] \times [y_1, y_2]) \\ &\qquad = \upsilon_f^+(x_1, y_1) - \upsilon_f^+(x_2, y_1) + \upsilon_f^+(x_2, y_2) - \upsilon_f^+(x_1, y_2), \quad \textit{and} \\ &\mathrm{var}_{\mathrm{HK}}(\upsilon_f^-, [x_1, x_2] \times [y_1, y_2]) \\ &\qquad = \upsilon_f^-(x_1, y_1) - \upsilon_f^-(x_2, y_1) + \upsilon_f^-(x_2, y_2) - \upsilon_f^-(x_1, y_2).\end{aligned}$$

Proof. To prove the first equality it suffices, by Lemma 3.5, to show that for all $(x_1, y_1), (x_2, y_2) \in J \times K$ where $x_1 \leq x_2$ and $y_1 \leq y_2$, that $\upsilon_f^+(x_2, y_2) - \upsilon_f^+(x_1, y_2) \geq \upsilon_f^+(x_2, y_1) - \upsilon_f^+(x_1, y_1)$. For $i, j \in \{1, 2\}$ let $S_{i,j} \in \Lambda([a, x_i] \times [c, y_j])$ be such that $S_{1,1} \leq S_{1,2} \leq S_{2,2}$ and $S_{1,1} \leq S_{2,1} \leq S_{2,2}$. Then cancellation gives $\omega^+(f, S_{2,2}) - \omega^+(f, S_{1,2}) \geq \omega^+(f, S_{2,1}) - \omega^+(f, S_{1,1})$. The result now follows by refining $S_{2,2}$ (and hence all $S_{i,j}$). The second equality has a similar proof. □

Lemma 3.7. *Let $f \in BV_{HK}(J \times K)$ be real-valued. Then*

$$\mathrm{var}_{\mathrm{HK}}(v_f^+, J \times K) \leq \mathrm{var}_{\mathrm{HK}}(f, J \times K)$$

and

$$\mathrm{var}_{\mathrm{HK}}(v_f^-, J \times K) \leq \mathrm{var}_{\mathrm{HK}}(f, J \times K).$$

Proof. This follows immediately from Lemmas 3.1, 3.3 and 3.6. □

Let $f \in BV_{HK}(J \times K)$ be real-valued. Then f is said to have property (UR) (or up and to the right) if it satisfies

(i) for each $x \in [a, b]$ the function $f(x, \cdot) : K \to \mathbb{R}$ is non-decreasing,

(ii) for each $y \in [c, d]$ the function $f(\cdot, y) : J \to \mathbb{R}$ is non-decreasing,

(iii) $\mathrm{var}_{\mathrm{HK}}(f, (x_1, x_2) \times (y_1, y_2)) = f(x_1, y_1) - f(x_1, y_2) + f(x_2, y_2) - f(x_2, y_1)$ for all $(x_1, x_2) \times (y_1, y_2) \subset J \times K$.

Similarly we say f has property (UL) if non-decreasing in property (ii) is replaced by non-increasing. Our aim is to show that if $f \in BV_{HK}(J \times K)$ is real-valued then $f = g - h$ where g and h are (UR), $\|g\|_{BV_{HK}} \leq 4 \|f\|_{BV_{HK}}$ and $\|h\|_{BV_{HK}} \leq 4 \|f\|_{BV_{HK}}$.

Let $l = \Pi(z_1, z_2)$. Then l is said to be (UR) if either

(i) $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$ or

(ii) $\mathrm{Re}(z_1) \geq \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \geq \mathrm{Im}(z_2)$.

Similarly l is said to be (UL) if either

(i) $\mathrm{Re}(z_1) \leq \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \geq \mathrm{Im}(z_2)$ or

(ii) $\mathrm{Re}(z_1) \geq \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$.

Line segments parameterized by $[0, 1]$ are either (UL) or (UR) or both. If $\gamma \in \Gamma_L$ then we can write $\gamma \cong l_1 \circ l_2 \circ \cdots \circ l_n$ where each l_i is a line segment parameterized by $[0, 1]$. Then γ is said to be (UL) (respectively (UR)) if each l_i is (UL) (respectively (UR)). Clearly if $\gamma_1, \gamma_2 \in \Gamma_L$ and $\gamma_1 \cong \gamma_2$ then, for example, γ_1 is (UR) if and only if γ_2 is (UR). Also, for example, if $\gamma_1, \gamma_2 \in \Gamma_L$ are both (UL) then so is $\gamma_1 \circ \gamma_2$.

Lemma 3.8. *Let $f \in BV_{HK}(J \times K)$ be real-valued. Then v_f^+ and v_f^- have property (UR).*

Proof. This follows from Lemma 3.2 and Lemma 3.6. □

Lemma 3.9. *Let $f \in BV_{HK}(J \times K)$ be real-valued and let $g \in BV(J)$. Suppose f has property (UR) and g is non-decreasing. Then $f + u(g)$ has property (UR).*

Proof. Clearly $f+u(g)$ has properties (i) and (ii). To see that it has property (iii) we note for any $(x_1',y_1'),(x_2',y_2')\in J\times K$ that $u(g)(x_1',y_1')-u(g)(x_2',y_1')+u(g)(x_2',y_2')-u(g)(x_1',y_2')=g(x_1')-g(x_2')+g(x_2')-g(x_1')=0$ and so

$$\begin{aligned}(f+u(g))(x_1',y_1')-(f+u(g))(x_2',y_1')&\\ +(f+u(g))(x_2',y_2')-(f+u(g))(x_1',y_2')&\\ =f(x_1',y_1')-f(x_2',y_1')+f(x_2',y_2')-f(x_1',y_2').&\end{aligned}$$

Hence if $[x_1,x_2]\times[y_1,y_2]\subset J\times K$ then $\omega(f+u(g),S)=\omega(f,S)$ for all $S\in\Lambda([x_1,x_2]\times[y_1,y_2])$. So $\mathrm{var}_{\mathrm{HK}}(f+u(g),[x_1,x_2]\times[y_1,y_2])=\mathrm{var}_{\mathrm{HK}}(f,[x_1,x_2]\times[y_1,y_2])$. Then

$$\begin{aligned}&\mathrm{var}_{\mathrm{HK}}(f+u(g),[x_1,x_2]\times[y_1,y_2])\\ &\quad=\mathrm{var}_{\mathrm{HK}}(f,[x_1,x_2]\times[y_1,y_2])\\ &\quad=f(x_1,y_1)-f(x_2,y_1)+f(x_2,y_2)-f(x_1,y_2)\\ &\quad=(f+u(g))(x_1,y_1)-(f+u(g))(x_2,y_1)\\ &\qquad+(f+u(g))(x_2,y_2)-(f+u(g))(x_1,y_2).\end{aligned}$$

□

Remark 3.10. *The previous lemma also holds if we take $g\in BV(K)$, g non-decreasing and use v instead of u.*

Proposition 3.11. *Let $f\in BV_{HK}(J\times K)$ be real-valued. Then there exists $g,h\in BV_{HK}(J\times K)$ such that*

(i) $f=g-h$,

(ii) g and h are (UR) and

(iii) $\|g\|_{BV_{HK}}\le\|f\|_{BV_{HK}}$ and $\|h\|_{BV_{HK}}\le\|f\|_{BV_{HK}}$.

Proof. There exists $f_1,f_2\in BV(J)$ such that $f(\cdot,c)-f(a,c)=f_1-f_2$, f_1, f_2 are non-decreasing, $\mathrm{var}(f_1,J)\le\mathrm{var}(f(\cdot,c),J)$ and $\mathrm{var}(f_2,J)\le\mathrm{var}(f(\cdot,c),J)$. There also exists $f_3,f_4\in BV(K)$ such that $f(a,\cdot)-f(a,c)=f_3-f_4$, f_3, f_4 are non-decreasing, $\mathrm{var}(f_3,K)\le\mathrm{var}(f(a,\cdot),K)$ and $\mathrm{var}(f_4,K)\le\mathrm{var}(f(a,\cdot),K)$. Set $g=\upsilon_f^++u(f_1)+v(f_3)+f(a,c)$ and $h=\upsilon_f^-+u(f_2)+v(f_4)$. Then by Lemmas 3.8 and 3.9, g and h are (UR). By Lemma 3.4

$$\begin{aligned}f(x,y)&=\upsilon_f^+(x,y)-\upsilon_f^-(x,y)+f(x,c)+f(a,y)-f(a,c)\\ &=\upsilon_f^+(x,y)-\upsilon_f^-(x,y)+(f_1(x)-f_2(x)+f(a,c))\\ &\qquad+(f_3(y)-f_4(y)+f(a,c))-f(a,c)\\ &=\upsilon_f^+(x,y)+f_1(x)+f_3(y)+f(a,c)\\ &\qquad-(\upsilon_f^-(x,y)+f_2(x)+f_4(y))\\ &=g(x,y)-h(x,y).\end{aligned}$$

Also, using Lemma 3.7,

$$\begin{aligned}\|g\|_{BV_{HK}}&=|g(a,c)|+\mathrm{var}(g(\cdot,c),J)+\mathrm{var}(g(a,\cdot),K)+\mathrm{var}_{\mathrm{HK}}(g,J\times K)\\ &=|f(a,b)|+\mathrm{var}(f_1,J)+\mathrm{var}(f_3,K)+\mathrm{var}_{\mathrm{HK}}(\upsilon_f^+,J\times K)\\ &\le|f(a,b)|+\mathrm{var}(f(\cdot,c),J)+\mathrm{var}(f(a,\cdot),K)+\mathrm{var}_{\mathrm{HK}}(f,J\times K)\\ &=\|f\|_{BV_{HK}}\end{aligned}$$

Similarly $\|h\|_{BV_{HK}} \le \|f\|_{BV_{HK}}$. □

Remark 3.12. *Note we can similarly write $f = g - h$ where g and h have the same norm inequalities but are (UL) instead of (UR).*

Lemma 3.13. *Let $f \in BV_{HK}(J \times K)$ satisfy property (UR). Let $\{h_j\}_{j=1}^m \subset \Gamma_L$ be a set of horizontal line segments. Suppose that $n = \mathrm{vf}_{\mathrm{V}}(\cup_{j=1}^m h_j)$. Then*

$$\frac{1}{n}\sum_{j=1}^{m} \mathrm{cvar}(f, h_j) \le \|f\|_{BV_{HK}}.$$

Proof. The idea of the proof is to replace each line segment h_i by another horizontal line segment h_i' where the image of each h_i' is a subset of $J \times \{d\}$ and show the inequality for the h_i's. For $1 \le j \le m$ let $x_j + iy_j$ and $x_j' + iy_j$ be the left and right endpoints of h_j. Recall that $J \times K = [a, b] \times [c, d]$. Let $h_j' = \Pi(x_j + id, x_j' + id)$ for each j. Since f is (UR) then for all $x_j \le s \le t \le x_j'$ we have that $f(t + iy_j) - f(s + iy_j) \le f(t + id) - f(s + id)$ and hence $|f(t + iy_j) - f(s + iy_j)| \le |f(t + id) - f(s + id)|$. From this we can conclude that $\mathrm{cvar}(f, h_i) \le \mathrm{cvar}(f, h_i')$ for all i. Also $\mathrm{vf}_{\mathrm{V}}(\cup_{i=1}^m h_i) = \mathrm{vf}_{\mathrm{V}}(\cup_{i=1}^m h_i')$. Using the same proof as in Proposition 2.5 we can conclude that $\rho_{\mathrm{V}}(\cup_{i=1}^m h_i') \sum_{i=1}^{m} \mathrm{cvar}(f, h_i') \le \mathrm{cvar}(f, \Pi(a + id, b + id))$. Hence

$$\begin{aligned}
\frac{1}{n}\sum_{j=1}^{m} \mathrm{cvar}(f, h_i) &\le \frac{1}{n}\sum_{j=1}^{m} \mathrm{cvar}(f, h_i') \\
&\le \mathrm{cvar}(f, \Pi(a + id, b + id)) \\
&= \mathrm{var}(f(\cdot, d), J) \\
&\le \|f\|_{BV_{HK}}.
\end{aligned}$$

The second last step follows by the monotonicity of $f(\cdot, d)$. □

Remark 3.14. *A similar lemma holds when $\{l_i\}_{i=1}^m$ are vertical line segments and/or f has property (UL).*

Lemma 3.15. *Let $f \in BV_{HK}(J \times K)$ satisfy property (UR). Let $\{\gamma_j\}_{j=1}^m \subset \Gamma_L$ be such that γ_j is (UR) for all j. Then*

$$\rho(\cup_{j=1}^m \gamma_j) \sum_{j=1}^{m} \mathrm{cvar}(f, \gamma_j) \le 2\,\|f\|_{BV_{HK}}.$$

Proof. For each j we shall replace γ_j with horizontal and vertical lines

$$\begin{aligned}
h_j &= \Pi(\gamma_j(0), \mathrm{Re}(\gamma_j(1)) + i\mathrm{Im}(\gamma_j(0))), \quad \text{and} \\
v_j &= \Pi(\mathrm{Re}(\gamma_j(1)) + i\mathrm{Im}(\gamma_j(0)), \gamma_j(1)),
\end{aligned}$$

defined so that $\gamma_j = h_j \circ v_j$. Since f is (UR) then $\mathrm{cvar}(f, \gamma_j) = \mathrm{cvar}(f, h_j) + \mathrm{cvar}(f, v_j)$. Also note that for any vertical line l, $\mathrm{vf}(\cup_{j=1}^m \gamma_j, l) = \mathrm{vf}(\cup_{j=1}^m h_j, l)$ and so $\rho(\cup_{j=1}^m \gamma_j) \leq \rho_{\mathrm{V}}(\cup_{j=1}^m \gamma_j) = \rho_{\mathrm{V}}(\cup_{j=1}^m h_j)$. Similarly $\rho(\cup_{j=1}^m \gamma_j) \leq \rho_{\mathrm{H}}(\cup_{j=1}^m v_j)$. Hence

$$\begin{aligned}
\rho(\cup_{j=1}^m \gamma_j) \sum_{j=1}^m \mathrm{cvar}(f, \gamma_j) &= \rho(\cup_{j=1}^m \gamma_j) \sum_{j=1}^m (\mathrm{cvar}(f, h_j) + \mathrm{cvar}(f, v_j)) \\
&\leq \rho_{\mathrm{V}}(\cup_{j=1}^m h_j) \sum_{j=1}^m \mathrm{cvar}(f, h_j) \\
&\quad + \rho_{\mathrm{H}}(\cup_{j=1}^m v_j) \sum_{j=1}^m \mathrm{cvar}(f, v_j) \\
&\leq 2 \|f\|_{BV_{HK}}.
\end{aligned}$$

The last step follows from Lemma 3.13. □

Theorem 3.16. *The inclusion $BV_{HK}(J \times K) \hookrightarrow BV(J \times K)$ is continuous.*

Proof. It is sufficient to show that if $f \in BV_{HK}(J \times K)$ is real-valued then $\mathrm{var}(f, J \times K) \leq 8 \|f\|_{BV_{HK}}$. Let $\gamma \in \Gamma_L$. We can write γ as the composition of curves which alternate between being (UL) and (UR). Without loss of generality assume that $\gamma \cong \lambda_1 \circ \mu_1 \circ \lambda_2 \circ \mu_2 \circ \cdots \circ \lambda_n \circ \mu_n$ where each $\lambda_i \in \Gamma_L$ is (UL) and each μ_i is (UR). By Lemma 2.2, $\mathrm{cvar}(f, \gamma) = \sum_{i=1}^n (\mathrm{cvar}(f, \lambda_i) + \mathrm{cvar}(f, \mu_i))$. By Proposition 3.11 we can write $f = f_1 - g_1 = f_2 - g_2$ where f_1, g_1 satisfy property (UR) and f_2, g_2 satisfy (UL). Then $\mathrm{vf}(\cup_{i=1}^n \lambda_i) \leq \mathrm{vf}(\gamma)$ and so $\rho(\cup_{i=1}^n \lambda_i) \geq \rho(\lambda)$. Similarly $\rho(\cup_{i=1}^n \mu_i) \geq \rho(\gamma)$. Using Lemma 3.15 and Proposition 3.11 we have

$$\begin{aligned}
\rho(\gamma)\, \mathrm{cvar}(f, \gamma) &= \rho(\gamma) \sum_{i=1}^n (\mathrm{cvar}(f, \lambda_i) + \mathrm{cvar}(f, \mu_i)) \\
&\leq \rho(\gamma) \sum_{i=1}^n (\mathrm{cvar}(f_1, \lambda_i) + \mathrm{cvar}(g_1, \lambda_i) \\
&\qquad + \mathrm{cvar}(f_2, \mu_i) + \mathrm{cvar}(g_2, \mu_i)) \\
&\leq \rho(\cup_{i=1}^n \lambda_i) \left(\sum_{i=1}^n \mathrm{cvar}(f_1, \lambda_i) + \sum_{i=1}^n \mathrm{cvar}(g_1, \lambda_i) \right) \\
&\quad + \rho(\cup_{i=1}^n \mu_i) \left(\sum_{i=1}^n \mathrm{cvar}(f_2, \mu_i) + \sum_{i=1}^n \mathrm{cvar}(g_2, \mu_i) \right) \\
&\leq 2 \|f_1\|_{BV_{HK}} + 2 \|g_1\|_{BV_{HK}} + 2 \|f_2\|_{BV_{HK}} + 2 \|g_2\|_{BV_{HK}} \\
&\leq 4 \times 2 \|f\|_{BV_{HK}} \\
&= 8 \|f\|_{BV_{HK}}.
\end{aligned}$$

The result now follows from Lemma 2.4. □

Example 3.17. *Here we show that if $J \times K$ is nondegenerate then $BV_{HK}(J \times K) \subsetneq BV(J \times K)$. Without loss of generality assume $J \times K$ is the unit square. Let $A \subset J \times K$ be the closed triangle with vertices at 0, 1 and $1 + i$. We show that $\chi_A \in BV(J \times K)$ but $\chi_A \notin BV_{HK}(J \times K)$.*

It is easy to see that if B is a half-plane then $\mathrm{var}(\chi_B, J\times K)\le 1$ *and so* $\chi_B \in BV(J\times K)$*. In particular* $B=\{x+iy\subset \mathbb{C} : x\le y\}$ *is a half-plane and so* $\chi_A = \chi_B | J\times K \in BV(J\times K)$*.*

Fix $n\in\mathbb{N}$*. Set* $t_i = \frac{i}{n}$*. Then* $S=\{t_i,t_j\}_{i,j=0}^{n,n}\in\Lambda(J\times K)$*. For each* i, $\chi_A(t_i,t_i)-\chi_A(t_{i+1},t_i)+\chi_A(t_{i+1},t_{i+1})-\chi_A(t_i,t_{i+1}) = 1-0+1-1=1$*. Hence* $\omega(\chi_A,S)\ge \sum_{i=0}^{n-1}|\chi_A(t_i,t_i)-\chi_A(t_{i+1},t_i)+\chi_A(t_{i+1},t_{i+1})-\chi_A(t_i,t_{i+1})| = n$*. So* $\mathrm{var}_{\mathrm{HK}}(\chi_A, J\times K)\ge n$*. Hence* $\chi_A\notin BV_{HK}(J\times K)$*.*

4 $AC(\sigma)$ for $\sigma\subset\mathbb{C}$ Compact

From an operator theoretic point of view, one would like to be able to deduce structural information about an operator T from bounds on $\|p(T)\|$ for p in some small algebra of functions. Let $\mathcal{P}$ denote the set of all complex polynomials. In the case that X is reflexive and $\sigma(T)\subset\mathbb{R}$, then a bound of the form $\|p(T)\|\le C\,\|p\|_{C(\sigma(T))}$ for all $p\in\mathcal{P}$ is sufficient to show that T can be written as an integral with respect to a countably additive spectral measure, whereas a weaker bound of the form $\|p(T)\|\le C\,\|p\|_{BV[a,b]}$ for $p\in\mathcal{P}$ implies that T has an integral representation with respect to a spectral family of projections. If the spectrum is not real then one would not to expect to be able to prove much unless the algebra contains at least $\mathcal{P}_2$, the polynomials in two variables (or equivalently the polynomials in z and $\bar z$). This leads to our definition of the absolutely continuous functions defined on a nonempty compact subset σ of $\mathbb{C}$.

Definition 4.1. $AC(\sigma)$*, the set of absolutely continuous functions on* σ*, is defined to be the closure in* $BV(\sigma)$ *of* $\mathcal{P}_2$*.*

Note that by [AD, Corollary 3.14] the polynomials are always of bounded variation. For any nonempty compact subset σ, $AC(\sigma)$ is a Banach subalgebra of $BV(\sigma)$. If $\sigma=[a,b]\subset\mathbb{R}$ then $AC(\sigma)$ coincides with the usual notion of absolute continuity.

We shall now compare our definition of $AC(J\times K)$ with the definition used by Berkson and Gillespie [BG1] which corresponds to the Hardy-Krause definition of variation. Let μ be Lebesgue measure on $\mathbb{C}$ and $f : J\times K\to\mathbb{C}$. We say f is *absolutely continuous in the Hardy-Krause sense* if for each $\epsilon>0$ there is a $\delta>0$ such that for any finite collection of rectangles $\{R_i\}_{i=1}^n$ whose sides are parallel to the axes and whose interiors are disjoint and such that $\sum_{i=1}^n \mu(R_i)<\delta$ then $\sum_{i=1}^n \mathrm{var}_{\mathrm{HK}}(f,R_i)<\epsilon$. We denote the set of such functions by $AC_{HK}(J\times K)$. The set $AC_{HK}(J\times K)$ forms a Banach subalgebra of $BV_{HK}(J\times K)$, and is the closure of $\mathcal{P}_2$ in the $BV_{HK}(J\times K)$ norm (see [BG1]). If K is then $BV_{HK}(J\times K)$ to the standard algebra $AC(J)$. See [BG1] and [CA] for properties of $AC_{HK}(J\times K)$. The following result is an immediate consequence of Theorem 3.16 and the density of the polynomials in both algebras.

Theorem 4.2. *The inclusion* $AC_{HK}(J\times K)\hookrightarrow AC(J\times K)$ *is continuous.*

Remark 4.3. *If $J \times K$ is nondegenerate then $AC_{HK}(J \times K) \subsetneq AC(J \times K)$. For example let $J \times K = [0,1] \times [0,1]$. Let f be the function $J \times K \to \mathbb{R}$ defined by*

$$f(x,y) = \begin{cases} 0 & \text{if } x < y \\ x - y & \text{if } x \geq y. \end{cases}$$

Then f is continuous and piecewise planar and so, by [AD, Lemma 4.10], $f \in AC(J \times K)$. For $i, n \in \mathbb{N}$ where $i \leq n$ set $R_{i,n} = \left[\frac{i-1}{n}, \frac{i}{n}\right] \times \left[\frac{i-1}{n}, \frac{i}{n}\right]$. Then $\lim_{n\to\infty} \sum_{i=1}^{n} \mu(R_{i,n}) = \lim_{n\to\infty} n \times \frac{1}{n^2} = 0$. But

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} \mathrm{var}_{\mathrm{HK}}(f, R_{i,n}) & \\ &\geq \lim_{n\to\infty} \sum_{i=1}^{n} \Bigg| f\left(\frac{i-1}{n}, \frac{i-1}{n}\right) - f\left(\frac{i}{n}, \frac{i-1}{n}\right) \\ &\qquad + f\left(\frac{i}{n}, \frac{i}{n}\right) - f\left(\frac{i-1}{n}, \frac{i}{n}\right)\Bigg| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \left| 0 - \frac{1}{n} + 0 - 0 \right| \\ &= 1. \end{aligned}$$

Hence $f \notin AC_{HK}(J \times K)$.

5 A comparison of $BV_{\mathrm{new}}(\mathbb{T})$ and $BV_{\mathrm{old}}(\mathbb{T})$

In [BG2] Berkson and Gillespie introduced the class of trigonometrically well-bounded operators. These operators, which have formed an important tool in their study of operator-valued harmonic analysis, are defined as operators which admit a weakly compact functional calculus for the absolutely continuous functions on the unit circle in $\mathbb{C}$ (or equivalently, are of the form e^{iA} where A is well-bounded of type (B)). The concepts of variation and absolute continuity used in [BG2] are just the natural extensions obtained by transferring the usual definitions for an interval in $\mathbb{R}$ onto the unit circle. In this section we show that our new definitions are equivalent to these earlier ones used in this setting.

For $f : \mathbb{T} \to \mathbb{C}$, let $\mathrm{var}_{\mathbb{T}}(f) = \mathrm{var}_{[0,2\pi]} f(e^{i(\cdot)})$. That is

$$\mathrm{var}_{\mathbb{T}}(f) = \sup_{\mathcal{P}} \sum_{j=1}^{n} |f(\omega_j) - f(\omega_{j-1})|,$$

where the supremum is taken over all partitions $\mathcal{P} = \{\omega_j = e^{i\theta_j}\}_{j=1}^{n}$ of the circle with $0 = \theta_0 \leq \theta_1 \leq \cdots \leq \theta_n = 2\pi$. Berkson and Gillespie worked with the following norm and Banach algebra:

$$\begin{aligned} \|f\|_{BV_{\mathrm{old}}} &= \sup_{z\in\mathbb{T}} |f| + \mathrm{var}_{\mathbb{T}}(f) \\ BV_{\mathrm{old}}(\mathbb{T}) &= \{f : \mathbb{T} \to \mathbb{C} : \|f\|_{BV_{\mathrm{old}}} < \infty\}. \end{aligned}$$

For comparison, we shall write $BV_{\text{new}}(\mathbb{T})$ for the algebra given by the definitions in Section 2. The subalgebras obtained by taking the closures of the trigonometric polynomials in these algebras will be denoted by $AC_{\text{old}}(\mathbb{T})$ and $AC_{\text{new}}(\mathbb{T})$.

Lemma 5.1. *For all* $f : \mathbb{T} \to \mathbb{C}$,

$$\operatorname*{var}_{\mathbb{T}}(f) \le 2 \operatorname{var}(f, \mathbb{T}).$$

Proof. Let $\mathcal{P} = \{\omega_j\}_{j=1}^n$ be a partition of $\mathbb{T}$. Let $\gamma_{\mathcal{P}} = \Pi(\omega_0, \dots, \omega_n)$. Then $\operatorname{cvar}(f, \gamma_{\mathcal{P}}) = \sum_{j=1}^{n} |f(\omega_j) - f(\omega_{j-1})|$ and $\rho(\gamma_{\mathcal{P}}) = \frac{1}{2}$. Thus

$$\begin{aligned}
\operatorname*{var}_{\mathbb{T}}(f) &= \sup_{\mathcal{P}} \operatorname{cvar}(f, \gamma_{\mathcal{P}}) \\
&= 2 \sup_{\mathcal{P}} \rho(\gamma_{\mathcal{P}} \operatorname{cvar}(f, \gamma_{\mathcal{P}}) \\
&\le 2 \sup_{\gamma \in \Gamma} \rho(\gamma) \operatorname{cvar}(f, \gamma) \\
&= 2 \operatorname{var}(f, \mathbb{T}).
\end{aligned}$$

□

We might note that the value of 2 is sharp in this inequality, since if f is the characteristic function of a single point in $\mathbb{T}$, then $\operatorname*{var}_{\mathbb{T}}(f) = 2$ and $\operatorname{var}(f, \mathbb{T}) = 1$.

Before proving a reverse inequality, we introduce some notation and terminology.

Definition 5.2. *A* reparameterization *of* $\mathbb{T}$ *is any continuous, orientation preserving bijection* $\tau : \mathbb{T} \to \mathbb{T}$.

Definition 5.3. *Let* $\Gamma_{L,\mathbb{T}} = \{\Pi(z_0, \dots, z_n) : z_0, \dots, z_n \in \mathbb{T}\}$ *denote the set of piecewise linear curves with vertices on* $\mathbb{T}$.

Given $f : \mathbb{T} \to \mathbb{C}$ and any reparameterization τ of $\mathbb{T}$, let $f_\tau = f \circ \tau$. Clearly $\operatorname*{var}_{\mathbb{T}}(f) = \operatorname*{var}_{\mathbb{T}}(f_\tau)$. The reparameterization τ also determines a map $\Gamma_{L,\mathbb{T}} \to \Gamma_{L,\mathbb{T}}$, $\gamma = \Pi(z_0, \dots, z_n) \mapsto \gamma_\tau = \Pi(\tau(z_0), \dots, \tau(z_n))$.

Lemma 5.4. *Suppose that* f *and* τ *are as above and* $\gamma \in \Gamma_{L,\mathbb{T}}$. *Then*

(i) $\operatorname{vf}(\gamma) = \operatorname{vf}(\gamma_\tau)$;

(ii) $\rho(\gamma) \operatorname{cvar}(f, \gamma) = \rho(\gamma_\tau) \operatorname{cvar}(f_\tau, \gamma_\tau)$.

(iii) $\operatorname*{var}_{\mathbb{T}} f = \operatorname*{var}_{\mathbb{T}} f_\tau$.

Proof. (i) follows immediately from the observation that if the chords $\gamma = \Pi(z_0, z_1)$ and $\ell = \Pi(w_0, w_1)$ intersect, then so do γ_τ and ℓ_τ. Clearly $\operatorname{cvar}(f, \gamma) = \operatorname{cvar}(f_\tau, \gamma_\tau)$ so this proves (ii). The final statement is obvious. □

Theorem 5.5. *For all* $f \in BV(\mathbb{T})$,

$$\operatorname{var}(f, \mathbb{T}) \le \operatorname*{var}_{\mathbb{T}}(f).$$

Proof. Suppose that $\gamma \in \Gamma_L$ intersects $\mathbb{T}$ at $\gamma(s_0), \dots, \gamma(s_n)$ with $s_0 < s_1 < \dots s_n$. (We may assume that $n \geq 1$.) Let $S = \{s_j\}_{j=0}^n$ and let $\gamma_S = \Pi(\gamma(s_0), \gamma(s_n))$ denote the S approximation to γ. As in the proof of [AD, Lemma 3.1], $\rho(\gamma_S) \geq \rho(\gamma)$, and so

$$\rho(\gamma)\,\mathrm{cvar}(f, \gamma) \leq \rho(\gamma_S)\,\mathrm{cvar}(f, \gamma_S). \tag{2}$$

Now choose a reparameterization τ such that $\mathrm{Im}(\tau(z_j)) > 0$ for $j = 1, \dots, n$. Let $\gamma_2 = (\gamma_S)_\tau$. Then by Lemma 5.4,

$$\rho(\gamma_S)\,\mathrm{cvar}(f, \gamma_S) = \rho(\gamma_2)\,\mathrm{cvar}(f_\tau, \gamma_2). \tag{3}$$

Define $f_3 : [-1, 1] \to \mathbb{C}$, $f_3(t) = f_\tau(\sqrt{1 - t^2})$, and $\gamma_3 = \mathrm{Re} \circ \gamma_2$. Thus the image of γ_3 is a subset of $[-1, 1]$. Clearly then

$$\mathrm{vf}(\gamma_3) = \mathrm{vf}_{\mathrm{V}}(\gamma_3) = \mathrm{vf}_{\mathrm{V}}(\gamma_2) \leq \mathrm{vf}(\gamma_2),$$

and so $\rho(\gamma_2) \leq \rho(\gamma_3)$. Note also that $\mathrm{cvar}(f_\tau, \gamma_2) \leq \mathrm{cvar}(f_3, \gamma_3)$. Thus

$$\begin{aligned}
\rho(\gamma)\,\mathrm{cvar}(f, \gamma) &\leq \rho(\gamma_2)\,\mathrm{cvar}(f_\tau, \gamma_2) \qquad \text{(by (2) and (3))} \\
&\leq \rho(\gamma_3)\,\mathrm{cvar}(f_3, \gamma_3) \\
&\leq \operatorname*{var}_{[-1,1]} f_3 \qquad \text{(by [AD, Proposition 3.6])} \\
&= \operatorname*{var}_{[0,\pi]} f_\tau(e^{i(\cdot)}) \\
&\leq \operatorname*{var}_{\mathbb{T}} f_\tau \\
&= \operatorname*{var}_{\mathbb{T}} f \qquad \text{(by Lemma 5.4)}
\end{aligned}$$

Taking the supremum over all $\gamma \in \Gamma_L$ gives the result. □

Corollary 5.6. *The Banach algebras $BV_{\mathrm{new}}(\mathbb{T})$ and $BV_{\mathrm{old}}(\mathbb{T})$ are isomorphic.*

Corollary 5.7. *The Banach algebras $AC_{\mathrm{new}}(\mathbb{T})$ and $AC_{\mathrm{old}}(\mathbb{T})$ are isomorphic.*

6 Operator theory

In this section we shall note some of the operator theoretic consequences of the results in this paper.

We shall say that an operator $T \in B(X)$ is an $AC(\sigma)$ operator if it admits an $AC(\sigma)$ functional calculus; that is, if there exists a continuous Banach algebra homomorphism $\Psi : AC(\sigma) \to B(X)$ such that $\Psi(1) = I$ and $\Psi(\lambda) = T$. The theory of $AC(\sigma)$ operators will be pursued more fully in [A2]. Berkson and Gillespie [BG1] defined an operator to be an AC operator if it admits an $AC_{HK}(J \times K)$ functional calculus for some rectangle $J \times K$. It is natural to ask what the relationship is between these two classes of operators.

Theorem 6.1. *If $T \in B(X)$ is an $AC(\sigma)$ operator then T is an AC operator (in the sense of Berkson and Gillespie), and hence there exist commuting well-bounded operators $A, B \in B(X)$ such that $T = A + iB$.*

Proof. Suppose that T is an $AC(\sigma)$ operators with functional calculus map $\Psi : AC(\sigma) \to B(X)$. Let $J \times K$ be any rectangle containing σ. Define $\widehat{\Psi} : AC_{HK}(J \times K) \to B(X)$ by $\widehat{\Psi}(f) = \Psi(f|\sigma)$. It follows from Theorem 4.2 and [AD, Lemma 4.5] that this map is a well-defined algebra homomorphism with

$$\left\|\widehat{\Psi}(f)\right\| \le \|\Psi\| \, \|f|\sigma\|_{BV(\sigma)} \le \|\Psi\| \, \|f\|_{BV(J\times K)} \le 8 \, \|\Psi\| \, \|f\|_{BV_H K(J\times K)} .$$

It follows that T has an $AC_{HK}(J \times K)$ functional calculus □

The splitting into real and imaginary parts is necessarily unique if X is reflexive [BG1]. There are examples on nonreflexive spaces however of AC operators with more than one splitting of this type (see [BDG] Example 3.1). Each such splitting gives a different $AC_{HK}(J \times K)$ functional calculus. The situation for $AC(\sigma)$ operators is less clear however. An $AC(\sigma)$ may still have more than one representation in the form $A + iB$ but not all such representations give rise to an $AC(\sigma)$ functional calculus. In particular, no example is known of an operator with more than one $AC(\sigma)$ functional calculus.

It is not difficult to see that the class of $AC(\sigma)$ operators is in fact strictly smaller than the class of AC operators. Lemma 4.1 of [AD] shows that if $T \in B(X)$ is an $AC(\sigma)$ operator then for all $\alpha, \beta \in \mathbb{C}$, $\alpha T + \beta I$ is an $AC(\alpha\sigma + \beta)$ operator. It follows that the example from [BDG] of an AC operator T such that $(1 + i)T$ is not an AC operator also gives an example of an AC operator which is not an $AC(\sigma)$ operator (for any σ).

One of the most important subclasses of AC operators has been the family of trigonometrically well-bounded operators. The following is a consequence of Corollary 5.7 and the definition of being trigonometrically well-bounded [BG2].

Theorem 6.2. *An operator $T \in B(X)$ is trigonometrically well-bounded operator if and only if it admits a weakly compact $AC(\mathbb{T})$ functional calculus. In particular, if X is reflexive, then T is trigonometrically well-bounded operator if and only if it is an $AC(\mathbb{T})$ operator.*

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