

COMPACT $AC(\sigma)$ OPERATORS

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ABSTRACT. All compact $AC(\sigma)$ operators have a representation analogous to that for compact normal operators. As a partial converse we obtain conditions which allow one to construct a large number of such operators. Using the results in the paper, we answer a number of questions about the decomposition of a compact $AC(\sigma)$ into real and imaginary parts.

1. INTRODUCTION

The class of well-bounded operators was introduced to provide a theory which would allow many of the results which apply to self-adjoint operators to be extended to the Banach space setting. Since many operators which are self-adjoint on L^2 have only conditionally convergent spectral expansions on the other L^p spaces, the theory needed to allow more general types of representation theorems than those available in the theory of spectral operators.

The issue of the conditional convergence of spectral expansions arises most explicitly when considering compact well-bounded operators. In [CD] it was shown that every compact well-bounded operator T has an expansion as

$$(1) \quad T = \sum_{j=1}^{\infty} \mu_j P_j$$

where P_j is the Riesz projection onto the eigenspace corresponding to the eigenvalue λ_j , and where the terms are ordered so that $|\mu_j|$ is decreasing. Indeed, necessary and sufficient conditions were given which ensure that any sum of the form appearing in (1) is a compact well-bounded operator.

Even in the earliest papers (see, for example, [Ri]) the fact that the spectrum of a well-bounded operator is necessarily real was seen as an undesirable restriction, and various attempts at addressing this have appeared. In [BG] Berkson and Gillespie introduced the concept of an AC operator, which is one which can be written in the form $T = A + iB$ where A and B are commuting well-bounded operators. Doust and Walden [DW] showed that, as long as one takes the eigenvalues in an appropriate order, every compact AC operator has a representation in the form given in (1).

The theory of AC operators had certain drawbacks however (see [BDG]) and a smaller class of operators, known as $AC(\sigma)$ operators, was introduced in [AD1]. Whereas the functional calculus associated to theory of AC operators was restricted to functions defined on a rectangle in the plane, the theory of $AC(\sigma)$ operators is

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based on functions of bounded variation which are defined on an arbitrary nonempty compact subset $\sigma \subseteq \mathbb{C}$.

The results of [DW] clearly also apply to compact $AC(\sigma)$ operators, but even in the case of AC operators, there has not been a characterization of the compact operators in terms of properties of the eigenvalues and the corresponding projections. One of the main applications of the characterization result in [CD] has been to enable the construction of well-bounded operators with specific properties. See, for example, [CD, DG, DL]. The main aim of this paper, then, is to prove Theorem 5.1 which gives sufficient conditions to ensure that an operator of the form (1) is a compact $AC(\sigma)$ operator. To prove this theorem one has to show that (under the hypotheses of the theorem) one may sensibly define $f(T)$ for $f \in AC(\sigma)$, with a norm bound $\|f(T)\| \leq K \|f\|_{BV(\sigma)}$. A significant challenge in working with $AC(\sigma)$ operators is begin able to calculate $\|f\|_{BV(\sigma)}$ for $f \in AC(\sigma)$. In Section 4 we shall show that for certain sets σ , the $AC(\sigma)$ norm is equivalent to a norm which is much easier to calculate. Although we will not need the full force of this result to prove Theorem 5.1, we feel that this equivalence is of independent interest.

In Section 6 we show that there are compact $AC(\sigma)$ operators which are not of the form constructed in Theorem 5.1. The final section includes a discussion of the properties of the splitting of an $AC(\sigma)$ operator T into real and imaginary parts $T = A + iB$. There are many open questions regarding these splittings. In the case that T is compact however, it is possible to resolve these questions. In obtaining these results we prove a new result about rearrangements of the sum representation of a compact well-bounded operator (Corollary 7.2) which may also be of independent interest.

2. PRELIMINARIES

Throughout this paper let $\sigma \subset \mathbb{C}$ be compact and non-empty. For a Banach space X we shall denote the bounded linear operators on X by $B(X)$ and the bounded linear projections on X by $\text{Proj}(X)$. Summations over empty sets of indices should always be interpreted as having value zero.

The Banach algebra of functions of bounded variation on σ , denoted $BV(\sigma)$, was defined in [AD1]. The norm in this space is given by an expression of the form

$$(2) \quad \|f\|_{BV(\sigma)} = \sup_{\gamma} \text{cvar}(f, \gamma) \rho(\gamma).$$

In (2) the supremum is taken over all piecewise linear curves $\gamma : [0, 1] \rightarrow \mathbb{C}$ in the plane. The term $\text{cvar}(f, \gamma)$ measures the variation of f as one travels along the curve γ , and $\rho(\gamma)^{-1}$ measures how ‘sinuous’ the curve γ is. More precisely, the variation factor $\text{vf}(\gamma) \equiv \rho(\gamma)^{-1}$ is defined as the maximum number of entry points of the curve γ on any line in the plane. (Heuristically one should think of this as the maximum number of times any line intersects γ .) We refer the reader to [AD1] for the full definitions. The affine invariance of the BV norm will be used repeatedly (with little comment) to pass between estimates for functions on \mathbb{R} to those for functions defined on other lines.

The closure of the polynomials in z and \bar{z} is the subalgebra $AC(\sigma)$ of absolutely continuous functions on σ . An operator $T \in B(X)$ which admits an $AC(\sigma)$ functional calculus is said to be an $AC(\sigma)$ operator. This class includes all well-bounded operators, as well as every normal operator on a Hilbert space H .

Theorem 2.1. *Suppose that $T \in B(X)$ is a compact $AC(\sigma)$ operator. Then*

- (1) T is an AC operator (in the sense of Berkson and Gillespie [BG]);
- (2) there exist unique commuting well-bounded operators $A, B \in B(X)$ such that $T = A + iB$;
- (3) A and B are compact.

Proof. Statement (1) is Theorem 5.3 of [AD1]. Statements (2) and (3) therefore follow from [DW, Theorem 6.1]. \square

For a complex number $\mu = x + iy$ with $x, y \in \mathbb{R}$, let $|\mu|_\infty = \max\{x, y\}$. We shall now define an order \prec on \mathbb{C} by setting $\mu_1 \prec \mu_2$ if

- (i) $|\mu_1|_\infty > |\mu_2|_\infty$, or,
- (ii) if $|\mu_1|_\infty = |\mu_2|_\infty = \alpha$ and μ_2 lies on the that part of the square $|z|_\infty = \alpha$ between $-\alpha + i\alpha$ and μ_1 going from in a clockwise direction.

Theorem 2.1 has as an immediate corollary that compact $AC(\sigma)$ operators have a spectral diagonalization analogous to that for compact normal operators, but where the sum in the representation might only converge conditionally.

Corollary 2.2. [DW, Theorem 4.5] *Suppose that T is a compact $AC(\sigma)$ operator with spectrum $\{0\} \cup \{\mu_j\}_{j=1}^\infty$ and that $\{\mu_j\}$ is ordered by \prec . Then there exists a uniformly bounded sequence of disjoint projections $P_j \in B(X)$ such that*

$$T = \sum_{j=1}^{\infty} \mu_j P_j,$$

where the sum converges in the norm topology of $B(X)$.

This includes, for example, the fact that the range of the Riesz projection associated with a nonzero eigenvalue μ is exactly the corresponding eigenspace. We refer the reader to [DW] for a fuller discussion of properties of compact AC operators.

3. APPROXIMATION IN $AC(\sigma)$

An important step in proving Corollary 2.2 is that the identity function $\lambda(z) = z$ can be approximated in BV norm by functions whose support intersects $\sigma(T)$ at only a finite number of points. It follows from the results in [AD1] and [AD2] that this is still true if one uses the $BV(\sigma)$ norm. We provide here a more direct proof, the results of which we will use later.

Let σ be a nonempty compact subset of \mathbb{C} . Given $r > 0$ and $\epsilon > 0$, define $g_{r,\epsilon} : \sigma \rightarrow \mathbb{C}$ by

$$g_{r,\epsilon}(z) = \begin{cases} 0, & \text{if } |z|_\infty \leq r, \\ (|z|_\infty - r)/\epsilon, & \text{if } r < |z|_\infty \leq r + \epsilon, \\ 1, & \text{if } |z|_\infty > r + \epsilon. \end{cases}$$

Lemma 3.1. *For all $r > 0$ and $\epsilon > 0$, $g_{r,\epsilon} \in AC(\sigma)$ and $\|g_{r,\epsilon}\|_{BV(\sigma)} \leq 6$.*

Proof. Let $\sigma_R = \text{Re}(\sigma) = \{x : x + iy \in \sigma\}$ and let $\sigma_I = \text{Im}(\sigma) = \{y : x + iy \in \sigma\}$. Define $u : \sigma_R \rightarrow \mathbb{C}$ and $v : \sigma_I \rightarrow \mathbb{C}$ by $u(x) = g_{r,\epsilon}(x)$ and $v(y) = g_{r,\epsilon}(iy)$. Clearly $u \in AC(\sigma_R)$ with $\|u\|_{BV(\sigma_R)} \leq 3$ (and similarly for v). Let $\hat{u}(x + iy) = u(x)$, and $\hat{v}(x + iy) = v(y)$. By [AD1, Proposition 4.4], $\hat{u}, \hat{v} \in AC(\sigma)$ with $\|\hat{u}\|_{BV(\sigma)} \leq 3$ and $\|\hat{v}\|_{BV(\sigma)} \leq 3$. Now it is easy to check that $g_{r,\epsilon} = \hat{u} \wedge \hat{v}$, and hence, by [AD3, Proposition 2.10], $g_{r,\epsilon} \in AC(\sigma)$ and $\|g_{r,\epsilon}\|_{BV(\sigma)} \leq 6$. \square

Let $\mathbf{x}(z) = \text{Re}(z)$ and let $\mathbf{y}(z) = \text{Im}(z)$, so that $\boldsymbol{\lambda} = \mathbf{x} + i\mathbf{y}$.

Lemma 3.2. *Suppose that $\{r_n\}$ and $\{\epsilon_n\}$ are sequences of positive numbers which converge to 0. For $n = 1, 2, \dots$, let $g_n = g_{r_n, \epsilon_n}$, let $\mathbf{x}_n = g_n \mathbf{x}$ and let $\mathbf{y}_n = g_n \mathbf{y}$. Then $\mathbf{x}_n \rightarrow \mathbf{x}$ and $\mathbf{y}_n \rightarrow \mathbf{y}$ in $AC(\sigma)$. Consequently $g_n \boldsymbol{\lambda}_n \rightarrow \boldsymbol{\lambda}$ in $AC(\sigma)$.*

Proof. It suffices to show that $\mathbf{x}_n \rightarrow \mathbf{x}$. Now $\mathbf{x}_n - \mathbf{x} = (1 - g_n) \mathbf{x} = (1 - g_n) \tilde{x}_n$ where

$$\tilde{x}_n(z) = \begin{cases} \text{Re}(z), & \text{if } |\text{Re}(z)| \leq r_n + \epsilon_n, \\ 0 & \text{if } |\text{Re}(z)| > r_n + \epsilon_n. \end{cases}$$

Now $\|\tilde{x}_n\|_{BV(\sigma)} \leq 5(r_n + \epsilon_n)$ and so

$$\|\mathbf{x}_n - \mathbf{x}\|_{BV(\sigma)} \leq \|1 - g_n\|_{BV(\sigma)} \|\tilde{x}_n\|_{BV(\sigma)} \leq 30(r_n + \epsilon_n) \rightarrow 0$$

as $n \rightarrow \infty$. \square

Remark 3.3. It is clear that one could replace g_n in the above proof with many other families of ‘cut-off’ functions. In the proof of [DW, Theorem 4.5], for example, the cut-off functions are based on L-shaped regions rather than squares.

4. NORM ESTIMATES IN $AC(\sigma)$

In order to show that an operator T admits an $AC(\sigma)$ functional calculus, one often needs to find estimates for both $\|f(T)\|$ and $\|f\|_{BV(\sigma)}$. This can be difficult, even for quite simple functions.

If σ lies inside the union of a finite number of lines through the origin, then we shall show that it is possible to decompose $f \in AC(\sigma)$ into a sum of simpler functions in a way that allows good estimation of the norms. The main issue is the following. Suppose that $\text{supp } f \subseteq \sigma_0 \subseteq \sigma$ for some compact set σ_0 . One always has that $\|f|_{\sigma_0}\|_{BV(\sigma_0)} \leq \|f\|_{BV(\sigma)}$. The challenge is to prove an estimate of the form $\|f\|_{BV(\sigma)} \leq C \|f|_{\sigma_0}\|_{BV(\sigma_0)}$. Even if $\sigma \subseteq \mathbb{R}$ such an estimate need not exist, so any results need to rely on geometric properties of σ_0 and σ .

We begin with a technical lemma. As in [AD1], given points $z_1, z_2, \dots, z_k \in \mathbb{C}$, let $\Pi(z_1, z_2, \dots, z_k)$ denote the piecewise linear path with these points as vertices.

Lemma 4.1. *Suppose that $k \geq 2$ and that $S = z_1, z_2, \dots, z_k$ is a list of complex numbers such that no two consecutive numbers lie in the complement of the real*

axis. Let

$$\begin{aligned} J_1 &= \{j : z_j, z_{j+1} \in \mathbb{R}\}, \\ J_2 &= \{j : z_j \in \mathbb{R}, z_{j+1} \notin \mathbb{R}\}, \\ J_3 &= \{j : z_j \notin \mathbb{R}, z_{j+1} \in \mathbb{R}\} \end{aligned}$$

have cardinalities k_1, k_2 and k_3 respectively. Let $\gamma_S = \Pi(z_1, z_2, \dots, z_k)$. Then

$$(k_2 + k_3)\rho(\gamma_S) \leq 2.$$

Proof. The conditions on S imply that $|k_2 - k_3| \leq 1$. The bound claimed obviously holds if $k_2 = k_3 = 0$, so we shall assume that at least one of these values is nonzero.

Suppose first that $k_2 \leq k_3$. If $j \in J_3$, then z_{j+1} is an entry point of γ_S on the real axis. Thus $\rho(\gamma_S) \leq 1/k_3$ and so $(k_2 + k_3)\rho(\gamma_S) \leq 2k_3/k_3 = 2$.

If, on the other hand, $k_2 = k_3 + 1$, then the smallest element of J_2 is less than the smallest element of J_3 . Thus, in addition to the entry points associated with the elements of J_3 (as in the previous paragraph), γ_S must have an earlier entry point on the real line. Thus $\rho(\gamma_S) \leq 1/(k+1)$, and so $(k_2 + k_3)\rho(S) \leq (2k_3 + 1)/(k_3 + 1) \leq 2$. \square

In general, if $\sigma_0 \subseteq \sigma$, then $\|f|_{\sigma_0}\|_{BV(\sigma_0)} \leq \|f\|_{BV(\sigma)}$, but no reverse inequality is available, even if $\text{supp } f \subseteq \sigma_0$. Such an inequality does hold however if σ_0 is a line inside σ .

Suppose then that σ is a compact subset of \mathbb{C} and that $\sigma_0 = \sigma \cap \mathbb{R} \neq \emptyset$.

Lemma 4.2. *Suppose that $f \in BV(\sigma)$ and that $\text{supp } f \subseteq \sigma_0$. Then*

$$\|f\|_{BV(\sigma)} \leq 3 \left\{ \|f\|_{\infty} + \text{var}(f, \sigma_0) \right\} = 3 \|f|_{\sigma_0}\|_{BV(\sigma_0)}.$$

Proof. For an ordered finite subset $S = \{z_1, \dots, z_k\}$ of σ (allowing repetitions), let $\gamma_S = \Pi(z_1, \dots, z_k)$. Lemma 3.5 of [AD1] shows that

$$\text{var}(f, \sigma) = \sup_S \text{cvar}(f, \gamma_S)\rho(\gamma_S)$$

where the supremum is taken over all such finite subsets. Indeed, by adding extra points as necessary, one sees that

$$\text{var}(f, \sigma) = \sup_S \left(\rho(\gamma_S) \sum_{j=1}^{k-1} |f(z_j) - f(z_{j+1})| \right).$$

Fix such a subset S , and let $v(f, S) = \sum_{j=1}^{k-1} |f(z_j) - f(z_{j+1})|$. Clearly $v(f, S)$ is unchanged if we omit any consecutive elements of S which are both in $\sigma \setminus \sigma_0$. Note that omitting points never decreases the value of $\rho(\gamma_S)$, so we shall assume that no two consecutive elements of S are both in $\sigma \setminus \sigma_0$. We may also assume that $S \cap \mathbb{R} \neq \emptyset$ (or else $v(f, S) = 0$).

Partition $J = \{1, 2, \dots, k-1\}$ into sets J_1, J_2, J_3 as in Lemma 4.1. Clearly then

$$\sum_{j=1}^{k-1} |f(z_j) - f(z_{j+1})| = \sum_{i=1}^3 \sum_{j \in J_i} |f(z_j) - f(z_{j+1})|.$$

Let $S_0 = \{w_1, \dots, w_{k_0}\}$ be the sublist of S containing the elements that lie on the real axis, and let γ_{S_0} denote the corresponding piecewise linear curve. As noted above, $\rho(\gamma_S) \leq \rho(\gamma_{S_0})$. Thus, using [AD1, Proposition 3.6],

$$\rho(\gamma_S) \sum_{j \in J_1} |f(z_j) - f(z_{j+1})| \leq \rho(\gamma_{S_0}) \sum_{j=1}^{k_0-1} |f(w_j) - f(w_{j+1})| \leq \text{var}(f, \sigma_0).$$

On the other hand, if $j \in J_2 \cup J_3$, then $|f(z_j) - f(z_{j+1})| \leq \|f\|_\infty$ and so, by Lemma 4.1,

$$\rho(\gamma_S) \sum_{j \in J_2 \cup J_3} |f(z_j) - f(z_{j+1})| \leq (k_2 + k_3) \|f\|_\infty \rho(\gamma_S) \leq 2 \|f\|_\infty.$$

Thus

$$\text{var}(f, \sigma) \leq \text{var}(f, \sigma_0) + 2 \|f\|_\infty$$

and hence

$$\|f\|_{BV(\sigma)} \leq 3 \|f\|_\infty + \text{var}(f, \sigma_0).$$

□

Note that the factor 3 in the above inequality is sharp. If $\sigma = \{i, 0, -i\}$ and $f = \chi_{\{0\}}$ then $\|f\|_{BV(\sigma)} = 3$, and $\text{var}(f, \{0\}) = 0$.

For $0 \leq \theta < 2\pi$, let R_θ denote the ray $\{r \cos \theta, r \sin \theta : r \geq 0\}$. We shall say that $\sigma \subseteq \mathbb{C}$ is a *spoke set* if it is a subset of a finite union of such rays.

Suppose then that σ is a nonempty compact spoke set with $\sigma \subseteq \cup_{n=1}^N R_{\theta_n}$. (We shall assume that the angles θ_n are distinct.) For $n = 1, \dots, N$, let $\sigma_n = \sigma \cap R_{\theta_n}$. Given $f \in BV(\sigma)$, we shall define $f_0, f_1, \dots, f_N \in BV(\sigma)$ by setting $f_0(z) = f(0)$ and, for $1 \leq n \leq N$,

$$f_n(z) = \begin{cases} f(z) - f(0), & \text{if } z \in R_{\theta_n}, \\ 0, & \text{otherwise.} \end{cases}$$

Then $f = \sum_{n=0}^N f_n$. Also, if $f \in AC(\sigma)$, then a short limiting argument can be used to show that each f_n is also in $AC(\sigma)$. Define the spoke norm

$$\|f\|_{Sp} = |f(0)| + \sum_{n=1}^N \|f_n|_{\sigma_n}\|_{BV(\sigma_n)}.$$

Since σ_n lies in a line, the affine invariance of these norms and [AD1, Proposition 3.6] means that calculating $\|f_n|_{\sigma_n}\|_{BV(\sigma_n)}$ is relatively easy, since this just requires an estimation of the usual variation of f along the line. Thus $\|f\|_{Sp}$ is much easier to calculate than $\|f\|_{BV(\sigma)}$.

The following result shows that $\|\cdot\|_{Sp}$ and $\|\cdot\|_{BV(\sigma)}$ are equivalent. Although we do not need the full strength of this result in the next section, it does provide a useful tool in working with sets of this sort.

Proposition 4.3. *Suppose that σ is a nonempty compact spoke set. Then for all $f \in BV(\sigma)$*

$$\frac{1}{2N+1} \|f\|_{Sp} \leq \|f\|_{BV(\sigma)} \leq 3 \|f\|_{Sp}.$$

Proof. Suppose that $1 \leq n \leq N$. Then

$$\begin{aligned} \|f_n\|_{BV(\sigma_n)} &= \sup_{\sigma_n} |f - f_0| + \text{var}(f - f_0, \sigma_n) \\ &\leq 2 \|f\|_\infty + \text{var}(f, \sigma_n) \\ &\leq 2 \|f\|_{BV(\sigma)}. \end{aligned}$$

The left hand inequality then follows from the triangle inequality. On the other hand, Lemma 4.2 (and affine invariance) shows that

$$\|f\|_{BV(\sigma)} \leq |f(0)| + \sum_{n=1}^n \|f_n\|_{BV(\sigma)} \leq |f(0)| + 3 \sum_{n=1}^n \|f_n\|_{BV(\sigma_n)} \leq 3 \|f\|_{S_p}.$$

□

Let bv_0 denote the Banach space of sequences of bounded variation and limit 0. The following lemma is elementary.

Lemma 4.4. *Suppose that $\{Q_j\}_{j=0}^\infty$ is a uniformly bounded increasing family of projections on X . That is, $Q_i Q_j = Q_j Q_i = Q_i$ whenever $i \leq j$ and $\sup_j \|Q_j\| \leq K$. Suppose that $\{\mu_j\}_{j=1}^\infty \in bv_0$. Then*

- (1) $\left\| \sum_{j=n}^m \mu_j (Q_j - Q_{j-1}) \right\| \leq K \left(|\mu_n| + |\mu_m| + \sum_{j=n}^{m-1} |\mu_j - \mu_{j+1}| \right)$.
- (2) $\sum_{j=1}^\infty \mu_j (Q_j - Q_{j-1})$ converges in norm.

5. CONSTRUCTING COMPACT $AC(\sigma)$ OPERATORS

In [CD3], Cheng and Doust showed that certain combinations of disjoint projections of the form $\sum \lambda_j E_j$ always converge and define compact real $AC(\sigma)$ operators. In this section we shall provide some sufficient conditions for an operator of this form to be a compact $AC(\sigma)$ operator. Theorem 5.1 below will allow the construction of compact $AC(\sigma)$ operators which are neither scalar-type spectral, nor real $AC(\sigma)$ operators, via a given conditional decomposition of the Banach space X .

Suppose that $N \geq 1$ and that $\theta_1, \dots, \theta_n$ are distinct angles. Given

- scalars $\{\lambda_{n,m} : n = 1, \dots, N, m = 1, 2, \dots\} \subset \mathbb{C}$, and
- projections $\{E_{n,m} : n = 1, \dots, N, m = 1, 2, \dots\} \subset B(X)$

consider the following three conditions:

- (H1) For each $n = 1, \dots, N$, $\{\lambda_{n,m}\}_{m=1}^\infty \subset R_{\theta_n}$.
- (H2) For each $n = 1, \dots, N$, $|\lambda_{n,1}| > |\lambda_{n,2}| > |\lambda_{n,3}| > \dots$, and $\lambda_{n,m} \rightarrow 0$ as $m \rightarrow \infty$.
- (H3) The operators $E_{n,m}$ are pairwise disjoint, finite rank projections and there exists a constant K such that, for each $n = 1, \dots, N$, and $M = 1, 2, \dots$,

$$\left\| \sum_{m=1}^M E_{n,m} \right\| \leq K.$$

The set of indices $\mathbb{I} = \{(n, m) : n = 1, \dots, N, m = 1, 2, \dots\}$ can be ordered by declaring that $(n, m) \succ (s, t)$ if $|\lambda_{n,m}| < |\lambda_{s,t}|$, or if $|\lambda_{n,m}| = |\lambda_{s,t}|$ and $\theta_n > \theta_s$.

Let $\sigma = \{0\} \cup \{\lambda_{n,m}\}_{(n,m) \in \mathbb{I}}$, so that σ is a spoke set.

Theorem 5.1. *Suppose that $\{\lambda_{n,m} : n = 1, \dots, N, m = 1, 2, \dots\} \subset \mathbb{C}$ and $\{E_{n,m} : n = 1, \dots, N, m = 1, 2, \dots\} \subset B(X)$ satisfy (H1), (H2) and (H3). Then*

$$T = \sum_{n,m} \lambda_{n,m} E_{n,m}$$

converges in operator norm (in the order \succ) to a compact $AC(\sigma)$ operator.

Proof. Define $\Psi : AC(\sigma) \rightarrow B(X)$ by

$$(3) \quad \Psi(f) = f(0)I + \sum_{n,m} (f(\lambda_{n,m}) - f(0))E_{n,m}.$$

The first thing to verify is that Ψ is well-defined, that is, that the sum on the right-hand side of (3) converges for all $f \in AC(\sigma)$.

Suppose then that $f \in AC(\sigma)$. Let $\mu_{n,m} = f(\lambda_{n,m}) - f(0)$. Fix $\epsilon > 0$. As f is continuous at 0, if (n_0, m_0) is large enough, then

$$(4) \quad |\mu_{n,m}| < \epsilon/4N, \quad \text{for all } (n, m) \succeq (n_0, m_0).$$

As f is absolutely continuous, for every $n = 1, 2, \dots, N$, the variation along R_θ , $\text{var}(f|\sigma_n)$, is finite. Hence, if m_0 is large enough,

$$(5) \quad \sum_{m=s}^t |\mu_{n,m} - \mu_{n,m+1}| < \epsilon/2N, \quad \text{whenever } m_0 \leq s \leq t.$$

As N is finite we can choose $n_0 \in \{1, \dots, N\}$ and $m_0 \geq 1$ such that (4) holds and such that (5) holds for all n at once. Suppose then that $(n_1, m_1) \succ (n_0, m_0)$. For each n , let I_n be the (possibly empty) set $I_n = \{m : (n_0, m_0) \preceq (n, m) \prec (n_1, m_1)\}$. If $I_n \neq \emptyset$, let $s_n = \min I_n$ and $t_n = \max I_n$. The difference in the partial sums from index (n_0, m_0) to index (n_1, m_1) is therefore given by

$$\Delta = \sum_{\substack{n \\ I_n \neq \emptyset}} \sum_{m=s_n}^{t_n} \mu_{n,m} E_{n,m}.$$

Thus, by the lemma,

$$\begin{aligned} \|\Delta\| &\leq \sum_{I_n \neq \emptyset} \left\| \sum_{m=s_n}^{t_n} \mu_{n,m} E_{n,m} \right\| \\ &\leq \sum_{I_n \neq \emptyset} K \left(|\mu_{n,s_n}| + |\mu_{n,t_n}| + \sum_{m=s_n}^{t_n-1} |\mu_{n,m} - \mu_{n,m+1}| \right) \\ &< K\epsilon \end{aligned}$$

by (4) and (5). It follows that the partial sums are Cauchy and hence the series converges. Note that in particular, this implies that the sum defining $T = \Psi(\boldsymbol{\lambda})$ converges. Since each $E_{n,m}$ is finite rank, T is compact.

It is clear that Ψ is linear. For $1 \leq n \leq N$, let $\sigma_n = \sigma \cap R_{\theta_n} = \{0\} \cup \{\lambda_{n,m}\}_{m=1}^\infty$ as in Section 4, and define f_0, f_1, \dots, f_N as before Proposition 4.3. Note that, using the affine invariance of $AC(\sigma)$ and Lemma 3.2 of [CD3],

$$\|\Psi(f_n)\| \leq K \|f_n|_{\sigma_n}\|_{BV(\sigma_n)}, \quad \text{for } 1 \leq n \leq N.$$

Then, by Proposition 4.3,

$$\begin{aligned} \|\Psi(f)\| &\leq \sum_{n=0}^N \|\Psi(f_n)\| \\ &\leq |f(0)| + K \sum_{n=1}^N \|f_n|_{\sigma_n}\|_{BV(\sigma_n)} \\ &\leq K \|f\|_{S_p} \\ &\leq (2N + 1)K \|f\|_{BV(\sigma)}. \end{aligned}$$

It is easy to verify that $\Psi(fg) = \Psi(f)\Psi(g)$ if f and g are constant on a disk around 0. The continuity of Ψ then implies that Ψ is multiplicative on $AC(\sigma)$.

Finally, since $T = \Psi(\lambda)$, it follows that T has an $AC(\sigma)$ functional calculus, and hence that T is compact $AC(\sigma)$ operator. \square

6. EXAMPLES

As one might expect, Theorem 5.1 is far from giving a characterization of compact $AC(\sigma)$ operators. Here we shall give some examples which show that there are many ways of producing compact $AC(\sigma)$ operators whose spectra do not lie in a finite number of lines through the origin.

Proposition 6.1. *Let $\sigma = \{0, \lambda_1, \lambda_2, \dots\}$ be a countable set of complex number whose only limit point is 0. Define T on $AC(\sigma)$ by $Tf(z) = zf(z)$. Then T is a compact $AC(\sigma)$ operator.*

Proof. That T has the required functional calculus is an immediate consequence of that fact that $AC(\sigma)$ is a Banach algebra. Let $r_n = \epsilon_n = \frac{1}{n}$ and define $\lambda_n = g_n \lambda$ as in Lemma 3.2. It follows that $T = \lim_{n \rightarrow \infty} \lambda_n(T)$. But $\lambda_n(T)$ is a finite rank operator, and hence T is compact. \square

An important class of examples is given [AD3, Example 3.9].

Proposition 6.2. *Let A be a closed operator on a Banach space X , and suppose that for some $x \in \rho(A)$, the resolvent $(xI - A)^{-1}$ is compact and well-bounded. Then $(\mu I - A)^{-1}$ is a compact $AC(\sigma_\mu)$ operator for all $\mu \in \rho(A)$.*

Proof. Let $R(\mu, A) = (\mu I - A)^{-1}$. The resolvent identity clearly implies that if one resolvent is compact then every resolvent is compact. If we fix $\mu \notin \sigma(T)$, then $R(\mu, A) = f(R(x, A))$ where $f(t) = t/(1 + (\mu - x)t)$ is a Möbius transformation. If J is any compact interval containing $\sigma(R(x, A))$ then $\rho(f(J)) = \frac{1}{2}$. Thus $R(\mu, A)$ is an $AC(f(J))$ operator. \square

It was shown in [DG] that there exist compact AC operators (in the sense of Berkson and Gillespie) for which the sum (1) fails to converge if the eigenvalues are listed in order of decreasing modulus. It is not clear however whether that construction always produces an $AC(\sigma)$ operator. The following adaption of that construction does produce an $AC(\sigma)$ operator example. Certain aspects require more care here however due to the nature of the $BV(\sigma)$ norm.

Example 6.3. Let $\theta = \tan^{-1}(1/6)$. For $k = 1, 2, \dots$, let

$$\begin{aligned}\lambda_{k,j} &= \frac{e^{ij\theta/k}}{k}, & j &= 0, 1, \dots, k, \\ \mu_{k,j} &= \frac{\lambda_{k,j} + \lambda_{k,j-1}}{2}, & j &= 1, 2, \dots, k.\end{aligned}$$

Thus $d_k = |\mu_{k,j}|$ is independent of j , and $d_k < |\lambda_{k,j}| = \frac{1}{k}$ for all k, j .

Let $\sigma = \{\lambda_{k,j}\}_{k,j} \cup \{\mu_{k,j}\}_{k,j} \cup \{0\}$. Then σ is compact, and hence by Proposition 6.1, the operator $T \in B(AC(\sigma))$, $Tf(z) = zf(z)$ is a compact $AC(\sigma)$ operator. Thus $T = \sum_{\lambda_j=1}^{\infty} \lambda_j P(\lambda_j)$ where $\{\lambda_j\}$ is a listing of the nonzero elements of σ according to the order \prec defined in Section 2, and $P(\lambda_j)$ is the projection $P(\lambda_j)f = \chi_{\{\lambda_j\}}f$.

For $r > 0$, let $S_r = \sum_{|\lambda_j| \geq r} \lambda_j P(\lambda_j)$, so that S_r is a partial sum of the above series for T when the terms are ordered according to modulus. We shall show that the series does not converge in this order by showing that this sequence of partial sums is not Cauchy.

Fix k . Then

$$\begin{aligned}S_{d_k} - S_{1/k} &= \sum_{j=0}^k \lambda_{k,j} P(\lambda_{k,j}) \\ &= \lambda_{k,0} \sum_{j=0}^k P(\lambda_{k,j}) + \sum_{j=1}^k (\lambda_{k,j} - \lambda_{k,0}) P(\lambda_{k,j}).\end{aligned}$$

Now $\lambda_{k,0} = 1/k$. Elementary trigonometry ensures that for all j ,

$$|\lambda_{k,j} - \lambda_{k,0}| \leq |\lambda_{k,k} - \lambda_{k,0}| \leq \frac{1}{k} \tan \theta = \frac{1}{6k}.$$

As $AC(\sigma)$ is a Banach algebra, $\|P(\lambda_{k,j})\| = \|\chi_{\{\lambda_{k,j}\}}\|_{BV(\sigma)} \leq 3$ for all j . Thus

$$(6) \quad \left\| \sum_{j=1}^k (\lambda_{k,j} - \lambda_{k,0}) P(\lambda_{k,j}) \right\| \leq \frac{1}{2}.$$

Now $\sum_{j=0}^k P(\lambda_{k,j})$ is the projection of multiplication by the characteristic function of the set $\Lambda_k = \{\lambda_{k,0}, \dots, \lambda_{k,k}\}$ and so

$$\left\| \sum_{j=0}^k P(\lambda_{k,j}) \right\| = \|\chi_{\Lambda_k}\|_{BV(\sigma)}.$$

Let γ_k denote the piecewise linear curve in \mathbb{C} joining the elements of Λ_k in order. Note that γ_k passes through each of the points $\mu_{k,j}$. Clearly any line in the plane has at most two entry points on γ_k and so $\rho(\gamma_k) = 1/2$. Thus

$$\text{cvar}(\chi_{\Lambda_k}, \gamma_k) \rho(\gamma_k) = 2(k-1) \frac{1}{2} = k-1$$

and so

$$\|\chi_{\Lambda_k}\|_{BV(\sigma)} = \|\chi_{\Lambda_k}\|_{\infty} + \sup_{\gamma} \text{cvar}(\chi_{\Lambda_k}, \gamma) \rho(\gamma) \geq 1 + (k-1) = k.$$

Thus, using (6),

$$\begin{aligned} \|S_{d_k} - S_{1/k}\| &\geq \frac{1}{k} \left\| \sum_{j=0}^k P(\lambda_{k,j}) \right\| - \left\| \sum_{j=1}^k (\lambda_{k,j} - \lambda_{k,0}) P(\lambda_{k,j}) \right\| \\ &\geq 1 - \frac{1}{2} = \frac{1}{2}. \end{aligned}$$

It follows that the partial sum sequence is not Cauchy and hence the infinite sum does not converge.

7. OTHER PROPERTIES

As was noted in Section 2, every $AC(\sigma)$ operator T admits a splitting into real and imaginary parts $T = A + iB$, where A and B are commuting well-bounded operators. On nonreflexive spaces this splitting might not be unique [AD3, Example 4.5]. Even on nonreflexive spaces however, one does get a unique splitting when T is compact. As is shown in the proof of [BDG, Theorem 6.1], this is because in this case the real and imaginary parts are determined by the family of Riesz projections associated with the nonzero eigenvalues of T . That is, if $T = \sum \mu_j P_j$, then

$$A = \sum_i x_i \left(\sum_{\operatorname{Re}(\mu_j)=x_i} P_j \right)$$

where $\{x_i\}$ is the set of nonzero real parts of eigenvalues of T , ordered so that $|x_1| \geq |x_2| \geq \dots$.

Given an $AC(\sigma)$ operator T and $\omega = \alpha + i\beta \in \mathbb{C}$, the operator ωT is an $AC(\omega\sigma)$ operator. A longstanding open question is whether a splitting of αT must be given by

$$(7) \quad \omega T = (\alpha A - \beta B) + i(\alpha B + \beta A).$$

Of course, if every real linear combination of A and B is again well-bounded, then this must be a splitting. It is well-known however that the sum of two commuting well-bounded operators A and B need not be well-bounded (see [G]).

If T is compact, then ωT is also obviously compact, and hence has a unique splitting as $\alpha T = U + iV$. The main issue in showing that (7) holds is in rearranging the conditionally convergent sums that arise. The following lemma shows that while rearrangements of the sum in (1) may fail to converge, they cannot converge to a different limit.

Lemma 7.1. *Suppose that $\{c_j\}_{j=1}^{\infty}$ is a sequence of real numbers whose only limit point is 0. Suppose that $\{P_j\}$ is a sequence of disjoint finite rank projections and that there is a constant K such that*

- for all $t > 0$, $\left\| \sum_{c_j \geq t} P_j \right\| \leq K$,
- for all $t < 0$, $\left\| \sum_{c_j \leq t} P_j \right\| \leq K$.

Then $\sum_{j=1}^{\infty} c_j P_j$ is well-bounded if the sum converges.

Proof. Without loss we may assume that no c_j is zero. Suppose that $U = \sum_{j=1}^{\infty} c_j P_j$ converges. Let $\sigma = \{0\} \cup \{c_j\}$. Then U is clearly a compact operator with $\sigma \subseteq \sigma(U)$. If $\beta \in \sigma(U) \setminus \sigma$ then it is an isolated eigenvalue with corresponding Riesz projection P_β . But then $UP_\beta = \beta P_\beta = \sum c_j P_j P_\beta = 0$ which is impossible. Thus $\sigma(U) = \sigma$. It is easy to confirm that the Riesz projection corresponding to c_j is P_j .

Let π be a permutation of the positive integers so that $|c_{\pi(j)}|$ is nonincreasing. It follows from [CD, Theorem 3.3] that $V = \sum_{j=1}^{\infty} c_{\pi(j)} P_{\pi(j)}$ converges to a well-bounded operator (with $\sigma(V) = \sigma$).

Let $AC_c(\sigma) = \{f \in AC(\sigma) : f \text{ is constant on a neighbourhood of } 0\}$. Then $AC_c(\sigma)$ is dense in $AC(\sigma)$. Let \mathcal{A} denote the algebra of functions f which are analytic on a neighbourhood of σ , and for which the restriction of f to σ lies in $AC_c(\sigma)$. Note that every $f \in AC_c(\sigma)$ has an extension to a locally constant element of \mathcal{A} . Suppose then that $f \in \mathcal{A}$. Write $\sigma = \sigma_1 \cup \sigma_2$, where σ_2 is the component of the spectrum containing 0 on which f is constant, and where its complement σ_1 is finite. The Riesz functional calculus for U and V gives that

$$f(U) = \sum_{c_j \in \sigma_1} f(c_j) P_j + f(0) \left(I - \sum_{c_j \in \sigma_1} P_j \right) = f(V).$$

But V is well-bounded and so $\|f(U)\| = \|f(V)\| \leq K \|f\|_{BV(\sigma)}$ for some K . The density of $AC_c(\sigma)$ now implies that U is well-bounded. \square

Note in particular that in the above proof, if $f_n \in AC_c(\sigma)$, and $f_n \rightarrow \lambda$ in $AC(\sigma)$, then $U = \lim_n f_n(U) = \lim_n f_n(V) = V$. This proves the following result.

Corollary 7.2. *Suppose that T is a compact well-bounded operator with sum representation $\sum_j \mu_j P_j$ with $|\mu_1| \geq |\mu_2| \geq \dots$. Let π be a permutation of the positive integers. Then*

$$T = \sum_j \mu_{\pi(j)} P_{\pi(j)}$$

if the sum on the right-hand side converges.

It might be noted that we have been unable to prove the corresponding result for compact $AC(\sigma)$ operators.

We return now to the question raised at the beginning of this section.

Theorem 7.3. *Let T be a compact $AC(\sigma)$ operator with splitting $T = A + iB$, and let $\omega = \alpha + i\beta \in \mathbb{C}$. The unique splitting of ωT is*

$$\omega T = (\alpha A - \beta B) + i(\alpha B + \beta A).$$

Proof. Write $T = \sum_{\succ} \mu_j P_j$ via Corollary 2.2. Let $\mathbf{x} = \operatorname{Re}(\boldsymbol{\lambda})$ and $\mathbf{y} = \operatorname{Im}(\boldsymbol{\lambda})$. The proof of Corollary 2.2 (see Section 3) shows that the sums $\mathbf{x}(T) = \sum_{\succ} \operatorname{Re}(\mu_j) P_j$ and $\mathbf{y}(T) = \sum_{\succ} \operatorname{Im}(\mu_j) P_j$ both converge. Thus

$$T = A + iB = \sum_{\succ} \operatorname{Re}(\mu_j) P_j + i \sum_{\succ} \operatorname{Im}(\mu_j) P_j$$

and so

$$\begin{aligned}
 \omega T &= \alpha \sum_{\succ} \operatorname{Re}(\mu_j) P_j + i\beta \sum_{\succ} \operatorname{Re}(\mu_j) P_j + i\alpha \sum_{\succ} \operatorname{Im}(\mu_j) P_j - \beta \sum_{\succ} \operatorname{Im}(\mu_j) P_j \\
 (8) \quad &= \sum_{\succ} \operatorname{Re}(\omega\mu_j) P_j + i \sum_{\succ} \operatorname{Im}(\omega\mu_j) P_j.
 \end{aligned}$$

The $AC(\sigma)$ functional calculus for T now provides the bounds on the norms of sums of the Riesz projections needed so that we may apply Lemma 7.1 and deduce that $\sum_{\succ} \operatorname{Re}(\omega\mu_j) P_j$ and $\sum_{\succ} \operatorname{Im}(\omega\mu_j) P_j$ are well-bounded. Since these operators clearly commute, Equation (8) gives the unique splitting of ωT . But $\sum_{\succ} \operatorname{Re}(\omega\mu_j) P_j = \alpha A - \beta B$ and $\sum_{\succ} \operatorname{Im}(\omega\mu_j) P_j = \alpha B + \beta A$ so the proof is complete. \square

The known examples of AC operators which are not $AC(\sigma)$ operators share the property that they can be written as $T = A + iB$ where A and B are commuting well-bounded operators whose sum is not well-bounded. The previous theorem shows that, at least for compact operators, the well-boundedness of $A + B$ is necessary for T to be an $AC(\sigma)$ operator. It would of course be interesting to know whether it is sufficient.

Corollary 7.4. *Let $T = A + iB$ be a compact $AC(\sigma)$ operator. Then $A + B$ is well-bounded.*

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