5 Infinite products

5.1 Interpolation

A common and classical problem in calculus is to find a function that takes specified values at certain specified points. For example, it is easy to find a polynomial p for which you have specified N values

$$p(z_j) = c_j, \qquad j = 1, \ldots, N.$$

Indeed you can choose the Lagrange interpolating polynomial,

$$p(z) = \sum_{j=1}^{N} c_j \frac{\prod_{i \neq j}(z - z_i)}{\prod_{i \neq j}(z_j - z_i)}.$$

Written another way, this expresses $p(z)$ as $\sum c_j \ell_j(z)$ where ℓ_j is a polynomial for which $\ell_j(z_i) = \delta_{ij}$.

If you wish to find an analytic function that satisfies infinitely many conditions, things are much more complicated. We have already seen, for example, that it is impossible to find an entire function f such that

$$f(1/j) = \begin{cases} 0, & \text{if } j \text{ is even} \\ 1/j, & \text{if } j \text{ is odd.} \end{cases}$$

(It is of course easy to construct a continuous function with these constraints!)

A natural question to ask is whether one can ever construct anything like the Lagrange interpolating polynomial with *infinitely* many terms. This would require us to write some form of infinite product of terms $(z - z_j)$. This obviously requires some care as such an infinite product can easily be zero or unbounded!

One of the amazing truths about analytic functions is that essentially all entire functions can be written as infinite products. This was discovered but not proved, as was so much else, by Euler in about 1750. We shall begin this section by studying one example:

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

This example introduces most of the key ideas. We shall then look more carefully at the theory of infinite products of numbers, such as

$$\frac{2}{\pi} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{4n^2}\right)$$

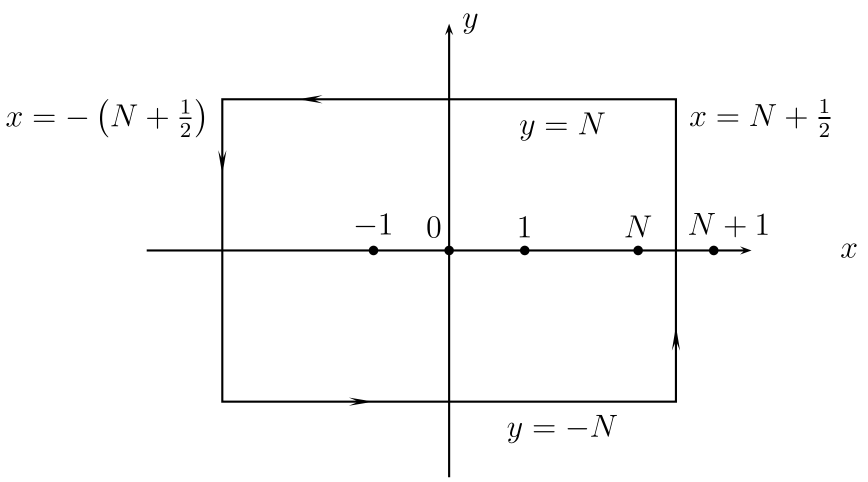
and infinite products of functions. Later we shall apply the ideas to two further examples, the gamma function and the zeta function.

5.2 A nontrivial example

The basis of our example is the following partial fraction series representation of $\cot \pi z$.

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \qquad (\text{for } z \neq 0, \pm 1, \pm 2, \ldots).$$

To prove this we consider the contour C_N which is the boundary of the rectangle bounded by the lines $y = N$, $y = -N$, $x = N + \frac{1}{2}$ and $x = -\left(N + \frac{1}{2}\right)$.



We first show that on C_N there is a bound for $|\cot \pi z|$ which is independent of N, that is, there is a B such that, for all $N = 1, 2, 3, \ldots$ $\quad |\cot \pi z| \leq B$ whenever $z \in C_N$. Note that

$$\cos \pi z = \cos \pi x \cosh \pi y - i \sin \pi x \sinh \pi y$$

$$\sin \pi z = \sin \pi x \cosh \pi y + i \cos \pi x \sinh \pi y.$$

Therefore, on $x = \pm (N + 1/2)$,

$$\frac{|\cos \pi z|}{|\sin \pi z|} = \frac{|\sinh \pi y|}{|\cosh \pi y|} = \frac{|\sinh \pi y|}{\sqrt{1 + \sinh^2 \pi y}} \leq 1$$

whilst on $y = \pm N$,

$$\begin{aligned}\frac{|\cos \pi z|}{|\sin \pi z|} &= \frac{\sqrt{\cos^2 \pi x \cosh^2 \pi N + \sin^2 \pi x \sinh^2 \pi N}}{\sqrt{\sin^2 \pi x \cosh^2 \pi N + \cos^2 \pi x \sinh^2 \pi N}} \\ &= \frac{\sqrt{\cos^2 \pi x + \sinh^2 \pi N}}{\sqrt{\sin^2 \pi x + \sinh^2 \pi N}} \\ &\leq \sqrt{\frac{1 + \sinh^2 \pi N}{\sinh^2 \pi N}} \leq 1 + \frac{1}{\sinh \pi N} \\ &\leq 1 + \frac{1}{\pi N} < 2.\end{aligned}$$

So in fact we can take $B = 2$.

Suppose now that $z \notin \mathbb{Z}$ and make sure that $N > |z|$. We will now evaluate

$$I_{N,z} = \frac{1}{2\pi i} \int_{C_N} \frac{\pi \cot \pi w}{w^2 - z^2} \, dw.$$

Note that from the above bounds, and using the fact that if $w \in C_N$, then $|w^2| \geq N^2$ and hence $|w^2 - z^2| \geq N^2 - |z|^2$,

$$|I_{N,z}| \leq \frac{1}{2\pi} \int_{C_N} \frac{\pi |\cot \pi w|}{|w^2 - z^2|} \, |dw| \leq \int_{C_N} \frac{1}{N^2 - |z|^2} \, |dw| = \frac{1}{N^2 - |z|^2} \cdot 2(4N + 1),$$

which tends to 0 as $N \to \infty$.

On the other hand, for any N, the integral is just the sum of the residues at the singularities inside C_N. The singularities occur where $\sin \pi w = 0$ and where $w^2 - z^2 = 0$, that is, at integers $w = -N, -(N-1), \ldots, (N-1), N$ and $w = \pm z$.

Let $f(w) = \dfrac{\pi \cot \pi w}{w^2 - z^2}$. Then each integer is a simple zero of $\sin \pi w$ and thus is a simple pole of f and the Laurent expansion is of the form

$$f(w) = \frac{c_{-1}}{w - n} + c_0 + c_1(w - n) + c_2(w - n)^2 + \cdots$$

Calculating the residue is easy at such a point:

$$\operatorname{Res}(f, n) = \lim_{w \to n} (w - n) f(w) = \lim_{w \to n} \frac{\pi(w - n) \cos \pi w}{(w^2 - z^2) \sin \pi w} = \frac{1}{n^2 - z^2}.$$

A similar argument shows that the residues at z and $-z$ are

$$\frac{\pi \cot \pi z}{2z} \quad \text{and} \quad \frac{\pi \cot(-\pi z)}{-2z}$$

Thus the Residue Theorem gives

$$I_{N,z} = \operatorname{Res}(f, z) + \operatorname{Res}(f, -z) + \sum_{n=-N}^{N} \operatorname{Res}(f, n) = \frac{\pi \cot \pi z}{2z} + \frac{\pi \cot(-\pi z)}{-2z} + \sum_{n=-N}^{N} \frac{1}{n^2 - z^2}$$

Thus, with z fixed, and taking a limit as $N \to \infty$,

$$\lim_{N\to\infty} I_{N,z} = 0 = \frac{\pi \cot \pi z}{2z} + \frac{\pi \cot(-\pi z)}{-2z} + \sum_{n=-\infty}^{\infty} \frac{1}{n^2 - z^2}$$

or, using symmetry and pulling out the $n = 0$ term,

$$0 = \frac{\pi \cot \pi z}{z} - \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{2}{n^2 - z^2}.$$

Therefore, multiplying through by z,

$$\pi \cot \pi z - \frac{1}{z} = \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

as claimed.

Clearly each term on the right hand side has an antiderivative related to $\log(z^2 - n^2)$ and the left hand side has an antiderivative related to $\log(\sin \pi z) - \log(\pi z)$. If no-one is looking you might write that

$$\log(\sin \pi z) - \log(\pi z) = \log\left(\frac{\sin \pi z}{\pi z}\right) = \sum_{n=1}^{\infty} \log(z^2 - n^2) = \log \prod_{n=1}^{\infty} (z^2 - n^2)$$

or

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} (z^2 - n^2)$$

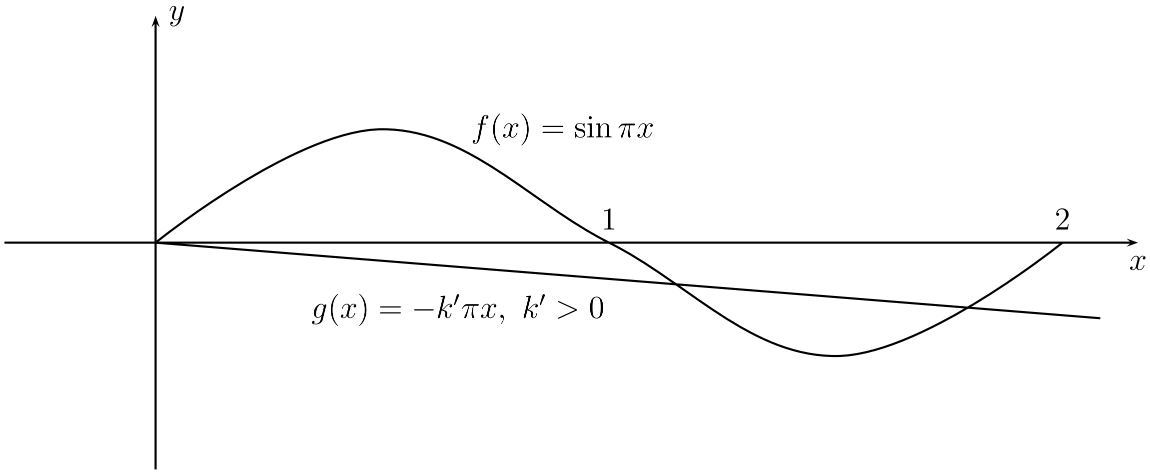
...before you realize that you have no idea what any of this means! Our task then is to turn this into something that does make sense to make this rigorous.

Let us begin by considering $\ell\sin(z) = \mathrm{Log}\frac{\sin \pi z}{\pi z}$ on the punctured disk, $0 < |z| < 1$.

Lemma 16. *$\ell\sin(z)$ is analytic on the punctured disk, $0 < |z| < 1$, where, as usual,* Log *is the principal branch of log.*

Proof. $\mathrm{Log}\, f(z)$ is analytic in any region where the analytic function $f(z)$ is neither zero nor takes a negative real value, since this is the cut for the function Log. First note that $\sin \pi z / z$ can only vanish where $\sin \pi z = 0$, and none of the zeros is inside the punctured disk. Next suppose that the function takes a negative real value, $-k$, or $\sin \pi z = -k\pi z$ where $k > 0$. Equating real parts gives $\sin \pi x \cosh \pi y = -k\pi x$ or simply $\sin \pi x = -k'\pi x$

where $k' > 0$.



From the graphs this can only occur if $x = 0$ or $|x| > 1$. So we can assume that $x = 0$ and equate the imaginary parts to obtain $\sinh \pi y = -k\pi y$. Since $k > 0$ the only solution is $y = 0$ and therefore the function $\ell\sin$ is analytic on the punctured disk. ■

In fact, the function $\sin \pi z / \pi z$ clearly has a removable singularity at $z = 0$ and so we can make $\ell\sin$ analytic at $z = 0$, provided we give $\sin \pi z/\pi z$ the value 1 there. Consequently $\ell\sin$ is analytic on the disk $|z| < 1$. Moreover, for $0 < |z| < 1$ we have

$$\frac{d}{dz}\,\ell\sin z = \frac{d}{dz}\,\mathrm{Log}\,\frac{\sin \pi z}{\pi z} = \frac{d}{dz}\left(\mathrm{Log}\sin \pi z - \mathrm{Log}\, z - \mathrm{Log}\,\pi\right) = \pi \cot \pi z - \frac{1}{z}.$$

From this we deduce that the deriviative of $\ell\sin$ at $z = 0$ must equal $\lim_{z\to 0}(\pi \cot \pi z - \frac{1}{z}) = 0$. Note also that

$$\frac{d}{dz}\mathrm{Log}\left(1 - \frac{z^2}{n^2}\right) = \frac{\frac{-2z}{n^2}}{1 - \frac{z^2}{n^2}} = \frac{2z}{z^2 - n^2}.$$

Thus, for $|z| < 1$,

$$\begin{aligned}
\frac{d}{dz}\,\ell\sin z &= \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \\
&= \sum_{n=1}^{\infty} \frac{d}{dz}\mathrm{Log}\left(1 - \frac{z^2}{n^2}\right) \\
&= \frac{d}{dz}\sum_{n=1}^{\infty} \mathrm{Log}\left(1 - \frac{z^2}{n^2}\right).
\end{aligned}$$

Here we could interchange d/dz and $\sum$ since the series of Logs converges uniformly (on any compact set not containing an integer.) One easy way to prove this is to observe that $|\text{Log}(1-w)| \le 2|w|$ when $|w| \le 1/2$. Therefore

$$\left|\text{Log}\left(1-\frac{z^2}{n^2}\right)\right| \le \frac{2|z|^2}{n^2} \le \frac{2}{n^2} \quad \text{for } |z|<1 \text{ and } n \ge 2.$$

But $\sum_{n=2}^{\infty} \frac{2}{n^2} < \infty$ and so we have the result by the Weierstrass M-test (we can clearly forget a finite number of terms in the series if we want to). Therefore, for $|z|<1$,

$$\text{Log}\left(\frac{\sin \pi z}{\pi z}\right) = \sum_{n=1}^{\infty} \text{Log}\left(1-\frac{z^2}{n^2}\right) + C$$

for some constant C. Substituting $z=0$ shows that $C=0$. That is,

$$\text{Log}\left(\frac{\sin \pi z}{\pi z}\right) = \sum_{n=1}^{\infty} \text{Log}\left(1-\frac{z^2}{n^2}\right) = \lim_{N\to\infty} \sum_{n=1}^{N} \text{Log}\left(1-\frac{z^2}{n^2}\right).$$

At this point we would like to take the sum inside the logariPi*xthm, but we need to remember that $\text{Log}\, a + \text{Log}\, b$ is not necessarily $\text{Log}\, ab$. Rather

$$\text{Log}\, a + \text{Log}\, b = \ln|a| + i\,\text{Arg}\, a + \ln b + i\,\text{Arg}\, b = \ln|ab| + i(\text{Arg}\, a + \text{Arg}\, b)$$

If $\text{Arg}\, a + \text{Arg}\, b \in (-\pi, \pi)$ then everything does work OK.

Exercise: (a) Show that $\left|\text{Arg}\left(1-\frac{z^2}{n^2}\right)\right| \le \sin^{-1}\left(\frac{|z|^2}{n^2}\right)$.

(b) Use the fact that $t \le 1.1 \sin t$ for $t \in [0, 0.25]$ to show that for $|z|<1$ and $n \ge 2$,

$$\sin^{-1}\left(\frac{|z|^2}{n^2}\right) < \frac{1.1}{n^2}.$$

(c) Deduce that for any N,

$$\sum_{n=1}^{N}\left|\text{Arg}\left(1-\frac{z^2}{n^2}\right)\right| < \frac{\pi}{2} + 1.1\sum_{n=2}^{N}\frac{1}{n^2} \le \frac{\pi}{2} + 1.1\left(\frac{\pi^2}{6}-1\right) < 2.3 < \pi.$$

Pi*x In light of this then,

$$\begin{aligned}\text{Log}\left(\frac{\sin \pi z}{\pi z}\right) = \sum_{n=1}^{\infty} \text{Log}\left(1-\frac{z^2}{n^2}\right) &= \lim_{N\to\infty} \sum_{n=1}^{N} \text{Log}\left(1-\frac{z^2}{n^2}\right) \\ &= \lim_{N\to\infty} \text{Log}\left(\prod_{n=1}^{N}\left(1-\frac{z^2}{n^2}\right)\right)\end{aligned}$$

Now recalling that $\exp(\operatorname{Log} w) = w$ for all $w \neq 0$, and, of course, that exp is continuous,

$$\begin{aligned}\frac{\sin \pi z}{\pi z} &= \exp\left(\operatorname{Log}\left(\frac{\sin \pi z}{\pi z}\right)\right)\\ &= \exp\left(\lim_{N\to\infty} \operatorname{Log}\left(\prod_{n=1}^{N}\left(1-\frac{z^2}{n^2}\right)\right)\right)\\ &= \lim_{N\to\infty} \exp\left(\operatorname{Log}\left(\prod_{n=1}^{N}\left(1-\frac{z^2}{n^2}\right)\right)\right)\\ &= \lim_{N\to\infty} \prod_{n=1}^{N}\left(1-\frac{z^2}{n^2}\right).\end{aligned}$$

We shall write this as

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

for $|z| < 1$. We will soon see that this equation holds for all z.

If we substitute $z = 1/2$ we obtain

$$\frac{1}{\frac{\pi}{2}} = \prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right) = \prod_{n=1}^{\infty}\frac{(2n-1)(2n+1)}{(2n)^2}$$

so that

$$\frac{\pi}{2} = \prod_{n=1}^{\infty}\frac{(2n)^2}{(2n-1)(2n+1)} \quad \text{or} \quad \frac{\pi}{2} = \frac{2^2}{1\cdot 3}\,\frac{4^2}{3\cdot 5}\,\frac{6^2}{5\cdot 7}\cdots$$

which is traditionally and rather unsatisfactorily written as Wallis' product

$$\frac{\pi}{2} = \frac{2^2\cdot 4^2\cdot 6^2\cdots}{1^2\cdot 3^2\cdot 5^2\cdot 7^2\cdots}.$$

Our immediate aim is to provide a general theoretical framework for what we have done, to regularize the example and to allow us to treat other examples.

5.3 Products of constants

Definition: If $\{p_n\}_{n=1}^{\infty}$ is a sequence of *non-zero* complex numbers we say that the infinite product $\prod_{n=1}^{\infty} p_n$ *converges* to P if the sequence of partial products $P_N = p_1 p_2 \cdots p_N$ converges to a *non-zero* limit P. If the infinite products converge to zero or infinity then the product is said to *diverge*. Infinite products which do not converge are said to *diverge*.

Remark: Intuitively, for the infinite product to converge we'd expect to need that $\operatorname{Log} p_n$ converges to 0, which means that $p_n \to 1$. As we'll see shortly, this turns out to be correct, so it is common to consider infinite products of the form $\prod_{n=1}^{\infty}(1+\alpha_n)$ where $\alpha_n \neq -1$ for all but a finite number of n. Clearly we then must have $\alpha_n \to 0$ for the product to exist.

Definition: More generally we agree to say that an infinite product $\displaystyle\prod_{n=1}^{\infty} p_n$ exists if

1. at most a finite number of factors are zero; and

2. the product of the non-vanishing terms exists in the above sense.

Hence a (convergent) infinite product has the value 0 if and only if one or more of its factors is 0.

Example: (a) Consider

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{n}\right) = 0\cdot\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots$$

so that the infinite product exists if and only if $\lim_{n\to\infty}\left(\frac{1}{2}\,\frac{2}{3}\,\cdots\,\frac{n-1}{n}\right)$ exists and is non-zero. But the value is $= \lim_{n\to\infty}\frac{1}{n} = 0$. Therefore the infinite product $\prod_1^{\infty}(1-\frac{1}{n})$ does not exist. It *diverges* to 0. (Ignoring the initial 0, this corresponds to the fact that

$$\sum_{n=1}^{N}\log\frac{n}{n+1} = -\log n + 1$$

diverges to $-\infty$.)

Example: (b) Consider

$$\prod_{n=1}^{\infty}\left(1+\frac{(-1)^n}{n+1}\right) = \left(1-\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots = \frac{1}{2}\cdot\frac{4}{3}\cdot\frac{3}{4}\cdot\frac{6}{5}\cdots.$$

Here $P_N = 1/2$ if N is odd and $= \frac{N+2}{2(N+1)}$ if N is even. Clearly then $\lim_{N\to\infty} P_N = 1/2$ so that the infinite product exists and equals $1/2$. Note that if 0 were prefixed as the first factor in example (b) then the product would still exist and would equal 0. We say that it converges to 0.

Example: (c) It is tempting to think that if $P = \prod_{n=1}^{\infty} p_n$ exists then $\operatorname{Log} P = \sum_{n=1}^{\infty}\operatorname{Log} p_n$. This need not be the case however.

Let $p_n = \exp(-i\pi/2^n)$, so that the partial products are given by

$$P_N = \prod_{n=1}^{N} p_n = \exp(-i\pi(1-1/2^N)).$$

In this case $P_N \to P = -1$, so we have $\displaystyle\sum_{n=1}^{\infty}\operatorname{Log} p_n = -\pi i$ while $\operatorname{Log} P = \pi i$.

Lemma 17. *If $P = \prod_{n=1}^{\infty} p_n$ exists then $p_n \to 1$.*

Proof. If P exists, but is zero, then we need only look at those terms past the last p_n that is zero, so without loss of generality, let's assume that $P \neq 0$.

With the notation as above then,

$$\lim_{n\to\infty} p_n = \lim_{n\to\infty} \frac{P_n}{P_{n-1}} = \frac{P}{P} = 1.$$

■

Theorem 18. *$P = \prod_{n=1}^{\infty} p_n$ exists and is nonzero if and only if $\sum_{n=1}^{\infty} \operatorname{Log} p_n$ converges.*

Proof. As we saw in the concrete example we did earlier, the main issue here is that we might not have $\operatorname{Log} \prod p_n = \sum \operatorname{Log} p_n$. Recall however that for any N

$$\operatorname{Arg}\left(\prod_{n=1}^{N} p_n\right) = \sum_{n=1}^{N} \operatorname{Arg} p_n + 2k_N\pi, \qquad \text{some } k_N \in \mathbb{Z},$$

and so

$$\operatorname{Log} \prod_{n=1}^{N} p_n = \sum_{n=1}^{N} \operatorname{Log} p_n + 2k_N\pi i.$$

Thus

$$P_N = \prod_{n=1}^{N} p_n = \exp\left(\operatorname{Log} \prod_{n=1}^{N} p_n\right) = \exp\left(\sum_{n=1}^{N} \operatorname{Log} p_n + 2k_N\pi i\right) = \exp\left(\sum_{n=1}^{N} \operatorname{Log} p_n\right).$$

One direction is now easy! If $\sum_{n=1}^{\infty} \operatorname{Log} p_n$ converges then $P_N \to \exp\left(\sum_{n=1}^{\infty} \operatorname{Log} p_n\right)$ as $N \to \infty$ (since exp is continuous) and so P_N is convergent with a non-zero limit.

For the converse[1], assume that $P = \lim P_N$ exists and is nonzero. Let us first assume that $P \notin (-\infty, 0)$. Then Log is continuous at P and hence $\operatorname{Log} P = \operatorname{Log} \lim_N P_N = \lim_N \operatorname{Log} P_N$ exists. As above, for all N,

$$c_N := \operatorname{Log} P_N = \sum_{n=1}^{N} \operatorname{Log} p_n + 2k_N\pi i$$

for some integer k_N. Since the sequence $\{c_N\}$ converges it is Cauchy, and in particular

$$c_N - c_{N-1} = \operatorname{Log} p_N + 2\pi i(k_N - k_{N-1}) \to 0, \qquad \text{as } N \to \infty.$$

[1]Thanks to Joel for googling the fix to this proof!

But we know that $p_N \to 1$, and hence that $\operatorname{Log} p_N \to 0$. This implies that $2\pi i(k_N - k_{N-1}) \to 0$ too, which can obviously only happen if the sequence $\{k_N\}$ is eventually constant. Thus, there exists an integer k such that for all sufficiently large N,

$$\sum_{n=1}^{N} \operatorname{Log} p_n = \operatorname{Log} P_N - 2k\pi i.$$

Since the right-hand side converges (to $\operatorname{Log} P - 2k\pi i$), the left-hand side converges too.

The remaining situation is if $P \in (-\infty, 0)$. If every p_n were real and positive, then P would be too, so there must exist at least one term, say p_m which is not real and positive. In this case $P' = \prod_{n \neq m} p_n$ converges to a nonzero point which is not on the negative real axis. The previous case would then imply that $\sum_{n \neq m} \operatorname{Log} p_n$ converges, and hence that $\sum_{n=1}^{\infty} \operatorname{Log} p_n$ converges too. ∎

Remark: It is worth noting that we actually showed that if $\prod_{n=1}^{\infty} p_n$ is non-zero then the sum $\sum_{n=1}^{\infty} \operatorname{Log} p_n$ can only differ from $\operatorname{Log} P$ by an integer multiple of $2\pi i$.

The following result provides us with a very simple test.

Theorem 19. *$\sum_{n=1}^{\infty} \operatorname{Log} p_n$ converges absolutely if and only if $\sum_{n=1}^{\infty}(1 - p_n)$ converges absolutely. Therefore if $\sum(1 - p_n)$ converges absolutely then $\prod_{n=1}^{\infty} p_n$ converges.*

Proof. Let us write $p_n = 1 + \alpha_n$. Then, for $|\alpha_n| < 1$, the Taylor series for Log gives

$$\begin{aligned} \big|\operatorname{Log}(1 + \alpha_n) - \alpha_n\big| &= \Big| -\frac{\alpha_n^2}{2} + \frac{\alpha_n^3}{3} - \frac{\alpha_n^4}{4} + \cdots \Big| \\ &\leq \frac{|\alpha_n|^2}{2} + \frac{|\alpha_n|^3}{3} + \frac{|\alpha_n|^4}{4} + \cdots \\ &\leq \frac{|\alpha_n|}{2}\left(|\alpha_n| + |\alpha_n|^2 + |\alpha_n|^3 + \cdots\right). \end{aligned}$$

If we further assume $|\alpha_n| \leq 1/2$ then the geometric series in the last line has sum at most $1/2$ and so, for $|\alpha_n| \leq 1/2$, $|\operatorname{Log}(1 + \alpha_n) - \alpha_n| \leq \frac{1}{2}|\alpha_n|$ or, equivalently,

$$\frac{1}{2}|1 - p_n| \leq \big|\operatorname{Log} p_n\big| \leq \frac{3}{2}|1 - p_n|.$$

The result now follows by the comparison test since clearly for n large enough, we can assume that $|\alpha_n| \leq 1/2$. ∎

Corollary 20. *If* $\sum_{n=1}^{\infty}|\alpha_n| < \infty$ *then* $\prod_{n=1}^{\infty}(1+\alpha_n)$ *converges.*

Exercise: Show that $\sum_{n=1}^{\infty}|\alpha_n| < \infty$ if and only if $\prod_{n=1}^{\infty}(1+|\alpha_n|)$ converges.

If either of these above conditions hold then we say that $\prod_1^\infty(1+\alpha_n)$ *converges absolutely*. It therefore follows that if $\prod_1^\infty(1+\alpha_n)$ converges absolutely then $\prod_1^\infty(1+\alpha_n)$ converges. This is, of course, because the similar result for series is true.

To complete the discussion we shall see, by two examples, that the convergence of $\sum_1^\infty \alpha_n$ is neither necessary nor sufficient for the convergence of the product $\prod_1^\infty(1+\alpha_k)$.

Example: (a) Let $\alpha_{2n-1} = n^{-1/2}$ and $\alpha_{2n} = -n^{-1/2} + n^{-1}$. Then $\sum_{n=1}^\infty \alpha_n$ is divergent (being the harmonic series). Note that

$$\left(1+\frac{1}{\sqrt{n}}\right)\left(1-\frac{1}{\sqrt{n}}+\frac{1}{n}\right) = 1+\frac{1}{n^{3/2}}$$

so $\prod_1^\infty(1+\alpha_n) = \prod_1^\infty(1+n^{-3/2})$ is convergent.

Example: (b) On the other hand if $\alpha_n = (-1)^n/n^{\frac{1}{2}}$ then $\sum_2^\infty \alpha_n$ is clearly convergent, whilst $\prod_2^\infty(1+\alpha_n)$ is divergent.

Exercise: Prove this last statement! Hint: Prove and use the fact that

$$(1+\alpha_{2n-1})(1+\alpha_{2n}) = \left(1-\frac{1}{\sqrt{2n-1}}\right)\left(1+\frac{1}{\sqrt{2n}}\right) < 1-\frac{1}{2n}.$$

5.4 Products of functions

Next suppose we are given a sequence of functions $\{\alpha_n : \Omega \to \mathbb{C}\}_{n=1}^\infty$ defined on some region Ω. As for infinite sums, for each fixed $z \in \Omega$ we can consider the convergence of the product $\prod_{n=1}^{\infty}\Big(1+\alpha_n(z)\Big)$. If, for each $z \in \Omega$, this infinite product exists then we can define the pointwise limit function

$$P(z) = \prod_{n=1}^{\infty}\Big(1+\alpha_n(z)\Big), \qquad z \in \Omega.$$

The harder questions concern the properties of the limit function P. Suppose for example that each function α_n is analytic on Ω. In this case each partial product

$$P_N(z) = \prod_{n=1}^{N}\Big(1+\alpha_n(z)\Big)$$

is clearly also analytic, but what about P? How can we guarantee that the infinite product is analytic?

Theorem 21. *Let* $\{\alpha_n\}_{n=1}^{\infty}$ *be a sequence of analytic functions defined on* Ω *such that* $\sum |\alpha_n|$ *converges uniformly on each compact subset of* Ω*. Then*

$$P(z) = \prod_{n=1}^{\infty}(1 + \alpha_n(z))$$

converges uniformly on each compact subset of Ω *and* P *is analytic on* Ω*.*

Proof. First observe that if $\sum |\alpha_n|$ converges uniformly on each compact subset of Ω then certainly for any $z \in \Omega$, $\sum |\alpha_n(z)|$ converges and hence by the previous results, $P(z) = \prod_{n=1}^{\infty}(1 + \alpha_n(z))$ exists.

As before, let

$$P_N(z) = \prod_{n=1}^{N}(1 + \alpha_n(z)), \qquad z \in \Omega$$

so that

$$P_{N+1}(z) - P_N(z) = (1 + \alpha_{N+1}(z))P_N(z) - P_N(z) = \alpha_{N+1}(z)P_N(z).$$

For $M > N$ then, using the fact that for $t \geq 0$, $1 + t \leq e^t$,

$$\begin{aligned}
\big|P_M(z) - P_N(z)\big| &= \Big|\sum_{j=N}^{M-1} \alpha_{j+1}(z)\, P_j(z)\Big| \\
&\leq \sum_{j=N}^{M-1} |\alpha_{j+1}(z)| \prod_{n=1}^{j} |(1 + \alpha_n(z))| \\
&\leq \sum_{j=N}^{M-1} |\alpha_{j+1}(z)| \prod_{n=1}^{j} (1 + |\alpha_n(z)|) \\
&\leq \sum_{j=N}^{M-1} |\alpha_{j+1}(z)| \prod_{n=1}^{j} e^{|\alpha_n(z)|} \\
&\leq \sum_{j=N}^{M-1} |\alpha_{j+1}(z)| \prod_{n=1}^{\infty} e^{|\alpha_n(z)|} \\
&= e^{\sum |\alpha_j(z)|} \sum_{j=N}^{M-1} |\alpha_{j+1}(z)|
\end{aligned}$$

Let K be a compact subset of Ω and let $\epsilon > 0$. Since $\sum_1^{\infty} |\alpha_j|$ converges uniformly on K, it has a continuous limit. In particular, the limit is bounded and so there is a constant $C > 0$ such that $\exp\left(\sum_1^{\infty} |\alpha_j(z)|\right) \leq C$ for all $z \in K$. Moreover there is a positive integer N_0 such that $\sum_N^{\infty} |\alpha_j(z)| < \epsilon/C$ whenever $N > N_0$ and $z \in K$. Therefore

$$|P_M(z) - P_N(z)| < \epsilon$$

whenever $M, N > N_0$ and $z \in K$. Thus $\{P_N\}$ forms a Cauchy sequence in $C(K)$ with the supremum norm, and hence it converges uniformly on K.

The analyticity of P on Ω now follows from Theorem 13. ■

Corollary 22. *(The Logarithmic Derivative of a Product) Let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence of analytic functions defined on Ω such that $\sum |\alpha_n|$ converges uniformly on each compact subset of Ω. Then, at every point z where $P(z) = \prod_{n=1}^{\infty}(1+\alpha(z))$ is nonzero,*

$$\frac{P'(z)}{P(z)} = \sum_{n=1}^{\infty} \frac{\alpha_n'(z)}{1+\alpha_n(z)}.$$

Proof. Since $P(z) \neq 0$, none of the factors $1+\alpha_n(z)$ vanishes.

The product rule lets us differentiate the partial product $P_N(z)$:

$$P_N'(z) = \sum_{n=1}^{N} \alpha_n'(z) \prod_{k \neq n}^{N} (1+\alpha_k(z)).$$

Thus,

$$\frac{P_N'(z)}{P_N(z)} = \sum_{n=1}^{N} \alpha_n'(z) \frac{\prod_{k\neq n}^{N}(1+\alpha_k(z))}{\prod_{k=1}^{N}(1+\alpha_k(z))} = \sum_{n=1}^{N} \frac{\alpha_n'(z)}{1+\alpha_n(z)}$$

since lots of terms cancel.

Now $P_n \to P$ uniformly on each compact set and hence (as the functions are analytic, see Theorem 12) $P_n' \to P'$ uniformly on each compact set.

Thus, if $P(z) \neq 0$, then (noting perhaps that no $P_n(z)$ is zero either),

$$\frac{P_n'(z)}{P_n(z)} \to \frac{P'(z)}{P(z)}$$

as $n \to \infty$. That is

$$\frac{P'(z)}{P(z)} = \sum_{n=1}^{\infty} \frac{\alpha_n'(z)}{1+\alpha_n(z)}.$$

■

Exercise: Investigate the convergence of the series for P'/P given in the corollary.

Recall that we showed that for $|z| < 1$

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

We want to extend this identity to all of $\mathbb{C}$. Now the function $\frac{\sin \pi z}{\pi z}$ is analytic everywhere. To see that the product is also analytic everywhere we consider $\sum |\alpha_n|$ where $\alpha_n(z) = \frac{-z^2}{n^2}$. We need to show that this converges uniformly on any compact subset of $\mathbb{C}$, or equivalently, that it converges uniformly on any closed disk. Suppose then that $|z| \leq R$. Then $|\alpha_n(z)| \leq \frac{R^2}{n^2}$ and $\sum_{n=1}^{\infty} \frac{R^2}{n^2} < \infty$ so that $\sum |\alpha_n|$ converges uniformly on the disk by the Weierstrass M-test. Finally the product is analytic everywhere by Theorem 21 of this section. Therefore

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \quad \text{for all } z \in \mathbb{C}.$$

by the principle of analytic continuation.

Note that the zeros of $\frac{\sin \pi z}{\pi z}$ occur at the nonzero integers. If you were trying to create a function with zeros at those points you might first write down

$$(z-1)(z+1)(z-2)(z+2)\cdots$$

which of course doesn't converge! Your second attempt might be

$$\left(1 - \frac{z}{1}\right)\left(1 + \frac{z}{1}\right)\left(1 - \frac{z}{2}\right)\left(1 + \frac{z}{2}\right)\cdots$$

which is of course what we ended up with.

Question: Is an analytic function determined uniquely, up to a scalar multiple, by its zeros (in the same way that a polynomial is)?

The answer is no, essentially since there are entire functions which are nowhere zero. In particular, if g is any entire function, then $f(z)$ and $e^{g(z)}f(z)$ have exactly the same zeros with the same multiplicities.

Suppose then that we try to construct an entire function f with simple zeros at points $a_1, a_2, a_3, \ldots$ and no other zeros. Obviously the points can't have a limit point or else f would be identically zero by analytic continuation. Since any bounded sequence must have a limit point, this implies that we only have a hope here if $|a_n| \to \infty$ as $n \to \infty$. It turns out that one needs to worry about how fast this happens.

Our first attempt would be

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right).$$

Unfortunately, this doesn't work!

Exercise: (Assignment 2) Examine the convergence of

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{n}\right)$$

which has 'zeros' at the positive integers.

Recall that for this product to converge we need $\sum \operatorname{Log}\left(1-\frac{z}{n}\right)$ to converge and this is essentially $\sum \frac{-z}{n}$. On the other hand, any expression of the form

$$P(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{g_n(z)}$$

will also have zeros at $a_1,\ a_2,\ a_3,\ldots$. A judicious choice of scaling factors fixes the convergence problem. For example, one could let

$$P(z)=\prod_{n=1}^{\infty}\left(1-\frac{z}{n}\right)e^{z/n}.$$

Why the exponential term. Recall that we want to be able to sum the logarithms of the terms. Here

$$\operatorname{Log}\left(\left(1-\frac{z}{n}\right)e^{z/n}\right)=\operatorname{Log}\left(1-\frac{z}{n}\right)+\frac{z}{n}=\left(-\frac{z}{n}-\frac{z^2}{2n^2}-\frac{z^3}{3n^3}-\ldots\right)+\frac{z}{n}=-\frac{z^2}{2n^2}-\ldots$$

Indeed, recalling our earlier estimate:

$$|\operatorname{Log}((1-w)e^{w})|\leq|w|^2,\qquad |w|\leq\frac{1}{2},$$

it follows that if $|z|\leq R$ and $n>2R$ then

$$|\operatorname{Log}\left(\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}\right)|\leq\left|\frac{z}{n}\right|^2\leq\frac{R^2}{n^2}.$$

Thus, by the Weierstrass M-test,

$$\sum_{n=1}^{\infty}|\operatorname{Log}\left(\left(1-\frac{z}{n}\right)e^{\frac{z}{n}}\right)|$$

converges uniformly on the compact disk of radius R, and hence on any compact subset of the plane. It follows from Theorem 18 that the product for $P(z)$ converges

Exercise: Show that the series converges uniformly on compact sets and hence that the function P defined this way is analytic on $\mathbb{C}$.

This gives a clue to a way forward.

Definition: (Weierstrass elementary factors) Let $E_0(z)=1-z$ and, for $k=1,2,\ldots$, let

$$E_k(z)=(1-z)\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^k}{k}\right).$$

The terms in our example then are of the form $E_1(z/n)$.

Lemma 23. *For $|z| \le 1$,*

$$|1 - E_k(z)| \le |z|^{k+1}.$$

Proof. Let $q_k(z) = z + \dfrac{z^2}{2} + \cdots + \dfrac{z^k}{k}$ so $E_k(z) = (1-z)e^{q_k(z)}$. The bound is clearly true if $k = 0$, so suppose $k \ge 1$. Then

$$E_k'(z) = -e^{q_k(z)} + (1-z)q_k'(z)e^{q_k(z)} = e^{q_k(z)}\left(-1 + (1-z)(1 + z + \cdots + z^{k-1}\right) = -z^k e^{q_k(z)}.$$

This could be expanded as a power series, valid for all z,

$$E_k'(z) = -z^k \left(1 + q(z) + \frac{q(z)^2}{2} + \dots\right) = -z^k - z^{k+1} - \text{HOT}.$$

On integrating,

$$E_k(z) = E_k(0) - \frac{z^{k+1}}{k+1} - \frac{z^{k+2}}{k+2} - \text{HOT},$$

or, noting that $E_k(0) = 1$,

$$1 - E_k(z) = \frac{z^{k+1}}{k+1} + \frac{z^{k+2}}{k+2} + \text{HOT} = z^{k+1}\sum_{i=0}^{\infty} b_i z^i.$$

Note that every $b_i \ge 0$, and that

$$1 - E_k(1) = 1 = \sum_{i=0}^{\infty} b_i.$$

Thus, if $|z| \le 1$,

$$|1 - E_k(z)| \le |z|^{k+1}\sum_{i=0}^{\infty}|b_i z^i| \le |z|^{k+1}\sum_{i=0}^{\infty} b_i = |z|^{k+1}$$

as required. ∎

The Weierstrass elementary functions are just the things needed to produce analytic functions with the appropriate zeros.

Theorem 24. *(Weierstrass Version 1) Let $\{a_n\}$ be a sequence of complex numbers such that $a_n \neq 0$ and $|a_n| \to \infty$ as $n \to \infty$. Then the infinite product*

$$P(z) = \prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{a_n}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) e^{q_n(z/a_n)}$$

defines an entire function P which has a zero at each a_n (according to the multiplicity of occurrence in $\{a_n\}$) and no other zeros in $\mathbb{C}$.

In fact we will prove something a little more general. In Version 2, choosing $k_n = n-1$ always works, and this gives Version 1.

Theorem 25. *(Weierstrass Version 2) Let $\{a_n\}$ be a sequence of complex numbers such that $a_n \neq 0$ and $|a_n| \to \infty$ as $n \to \infty$. Let $\{k_n\}$ be a sequence of non-negative integers such that*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{k_n+1} < \infty$$

for every positive r. Then the infinite product

$$P(z) = \prod_{n=1}^{\infty} E_{k_n}\left(\frac{z}{a_n}\right)$$

defines an entire function P which has a zero at each a_n (according to the multiplicity of occurrence in $\{a_n\}$) and no other zeros in $\mathbb{C}$.

Proof. Fix r and consider $|z| \leq r$. Then

$$\sum^{\infty} \left|1 - E_{k_n}\left(\frac{z}{a_n}\right)\right| \leq \sum^{\infty} \left|\frac{z}{a_n}\right|^{k_n+1} \leq \sum_{\infty} \left(\frac{r}{|a_n|}\right)^{k_n+1} < \infty.$$

Hence, by the Weierstrass M-test, the series is uniformly and absolutely convergent on $|z| \leq r$ (and hence on arbitrary compact subsets) so that, according to Theorem21, $\prod_{n=1}^{\infty} E_{k_n}\left(\frac{z}{a_n}\right)$ is uniformly convergent on compact sets and is analytic on $\mathbb{C}$, that is, it is entire. Moreover the zeros of the product are those of the individual factors, which occur at the a_n with appropriate multiplicities. ■

In most examples one can pick k_n to be a fixed integer. If λ is the *smallest integer* that makes the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda+1}}$$

converge then the product

$$z^N \prod_{n=1}^{\infty} E_{\lambda}\left(\frac{z}{a_n}\right)$$

associated with the sequence $0, \ldots, 0, a_1, a_2, \ldots$ with the 0 occurring N times is known as the *canonical product* and λ is called the *rank* of the canonical product.

Example: (a) Let $a_n = \sqrt{n}$. Then

$$\sum_{n=1}^{\infty} \frac{1}{a_n^{\lambda+1}} = \sum_{n=1}^{\infty} \frac{1}{n^{(\lambda+1)/2}}$$

converges if $\lambda > 1$, and so the rank of the canonical product here will be 2. The function

$$P(z) = \prod_{n=1}^{\infty} E_2\left(\frac{z}{\sqrt{n}}\right) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{\sqrt{n}}\right)\exp\left(\frac{z}{\sqrt{n}} + \frac{z^2}{2n}\right)$$

is the canonical product of an analytic function with zeros at $1, \sqrt{2}, \sqrt{3}, 2, \ldots$.

(b) There is no canonical product associated with the sequence $a_n = \log n$ since no fixed choice of λ can be made.

We are now in a position to say exactly what a function with zeros at a certain unbounded sequence has to look like. We first need a small lemma which is often useful.

Lemma 26. *Suppose that h is an entire function which is never zero. The there exists an entire function g such that $h(z) = e^{g(z)}$ for all z.*

Proof. As h never vanishes, the function $\dfrac{h'(z)}{h(z)}$ is also entire, and therefore has an antiderivative, say g_0 (see, for example, Corollary 1 of Section 38 of Brown and Churchill). Now

$$\frac{d}{dz}h(z)e^{-g_0(z)} = h'(z)e^{-g_0(z)} - h(z)\frac{h'(z)}{h(z)}e^{-g_0(z)} = 0$$

and so $h(z) = ce^{g_0(z)}$ for some nonzero constant c. The result follows be letting $g(z) = g_0(z) + \operatorname{Log} c$. ∎

Theorem 27. *(Weierstrass Factorization Theorem) Let f be a non-zero entire function which has*

- *a zero of order N at $z = 0$ and*
- *zeros at the non-zero numbers $a_1, a_2, a_3, \ldots$ (with numbers listed according to the multiplicity of the zero).*

Then there exists an entire function g and a sequence $\{k_n\}$ of non-negative integers such that

$$f(z) = e^{g(z)} z^N \prod_{n=1}^{\infty} E_{k_n}\left(\frac{z}{a_n}\right).$$

Remark: (i) The factorization is clearly not unique.

(ii) You can take $N = 0$ if f doesn't have a zero at $z = 0$.

Proof. Remember that $|a_n| \to \infty$ as $n \to \infty$. Therefore there exists a sequence of integers k_n such that the product $z^N \prod_{n=1}^{\infty} E_{k_n}\left(\frac{z}{a_n}\right)$ exists and is an entire function with the same zeros as f (with the correct multiplicities). Thus

$$h(z) = \frac{f(z)}{z^N \prod_{n=1}^{\infty} E_{k_n}(z/a_n)}$$

has only removable singularities and has no zeros.

By the lemma then there is an entire function g such that $h(z) = \exp(g(z))$, which completes the proof. ■

In the special case where $g(z)$ is a polynomial and the product $\prod_{k=1}^{\infty} E_{k_n}\left(\frac{z}{z_k}\right)$ is a canonical product (so that k_n is some fixed non-negative integer λ), Laguerre introduced the number

$$p = \max(\deg g, \lambda)$$

called the *genus* of the entire function.

Example: (a) Let $f(z) = e^z$. Here we don't have an infinite product, but $\deg g = 1$, so f has genus 1. Similarly, e^{z^2} is of genus 2.

(b) Let $f(z)$ be a polynomial. This can be written as

$$f(z) = c \prod_{n=1}^{N} \left(1 - \frac{z}{a_n}\right)$$

so the exponential term is trivial, and the Weierstrass elementary functions are of order 0. Thus every polynomial if of genus 0.

(c) We saw earlier that

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\pi^2 n^2}\right).$$

This is not the canonical product representation. The rank of the canonical product here is clearly 1 and so the canonical product is

$$\sin z = z \prod_{n=1}^{\infty} \left(1 - \frac{z}{\pi n}\right) e^{\frac{z}{\pi n}}.$$

Again the leading exponential term is trivial, so here the genus is 1.

Remark: We won't look further at the genus in this course. It mainly reappears in the Hadamard Factorization Theorem which is a fancier version of Weierstrass' theorem above. It roughly says that the genus of f is always close to the order of f, which is roughly the smallest μ so that $f(z) = O(\exp(|z|^{\mu})$. Hadamard used his theorem to give an asymptotic law for the distribution of prime numbers.

Recall that a *meromorphic* function is one whose singularities are all poles. Meromorphic functions can be thought of as a natural extension of rational functions.

Theorem 28. *Every function which is meromorphic (in the whole plane) is the quotient of two entire functions.*

Proof. If $F(z)$ is meromorphic we can find an entire function $g(z)$ with the poles of $F(z)$ as zeros. The product $F(z)g(z)$ is then an entire function $f(z)$ so that $F(z) = f(z)/g(z)$. ■