ROBUST SOLUTIONS OF MULTI-OBJECTIVE LINEAR SEMI-INFINITE PROGRAMS UNDER CONSTRAINT DATA UNCERTAINTY *

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Abstract. The multi-objective optimization model studied in this paper deals with simultaneous minimization of finitely many linear functions subject to an arbitrary number of uncertain linear constraints. We first provide a radius of robust feasibility guaranteeing the feasibility of the robust counterpart under affine data parametrization. We then establish dual characterizations of robust solutions of our model that are immunized against data uncertainty by way of characterizing corresponding solutions of robust counterpart of the model. Consequently we present robust duality theorems relating the value of the robust model with the corresponding value of its dual problem.

Key words. Linear semi-infinite programming, linear multi-objective optimization, robust optimization, duality

AMS subject classifications. 90C29, 90C31, 90C34

1. Introduction. Consider the deterministic multi-objective linear semi-infinite program of the form

$$(1.1) \qquad (P) \quad \operatorname*{V-min}_{x\in\mathbb{R}^n} \quad \left(c_1^\top x, \ldots, c_m^\top x\right) \quad \text{s.t.} \quad a_t^\top x \geq b_t, \forall t \in T,$$

where V-min stands for *vector minimization*, $c_i \in \mathbb{R}^n$ for all $i \in I := \{1, \ldots, m\}$, the superscript $^\top$ denotes transpose, $(a_t, b_t) \in \mathbb{R}^n \times \mathbb{R}$ for all $t \in T$, and the *index set* T is arbitrary. When T is finite, (P) becomes an ordinary multi-objective linear optimization problem, whereas, when T is infinite, (P) is a multi-objective linear semi-infinite optimization problem. Some potential applications of these models have been discussed in [9]. In particular, whenever $m = 1$, (P) becomes a single-objective linear semi-infinite program which has been extensively studied in the literature (see [8, 13] and other references therein).

When dealing with real-world optimization problems, the input data associated with a multi-objective linear semi-infinite program are often noisy or uncertain due to *prediction or measurement errors*. For example, a multi-objective optimization problem arising in industry or commerce might involve various costs, financial returns, and future demands that might be unknown at the time of the decision. They have to be predicted and are replaced with their forecasts. They often result in prediction errors. Similarly, some of the data, such as the contents associated with raw materials, might be hard to measure exactly. These input data are subject to measurement errors.

In the single-objective optimization case of constraint data uncertainty, Ben-Tal, El Ghaoui and Nemirovski [2] provided a highly successful computationally tractable

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treatment of the robust optimization approach for linear as well as convex optimization problems under data uncertainty. Recently, single-objective linear semi-infinite programming problems under constraint data uncertainty were studied in [10].

In the same vein as in [10], the multi-objective linear problem (P) in the face of *data uncertainty* in the constraints can be captured by a parameterized multi-objective linear problem of the form

$$(1.2)\qquad \begin{array}{lll} (P^u) & \operatorname*{V-min}_{x\in\mathbb{R}^n} & \left(c_1^\top x,\dots,c_m^\top x\right) \\ & \text{s.t.} & v_t^\top x \geq w_t, \forall t\in T, \end{array}$$

where c_i are given deterministic vectors in $\mathbb{R}^n$ for all $i \in I$, and $u = (v, w) : T \to \mathbb{R}^n \times \mathbb{R}$ represents a *selection* of a given *uncertain set-valued mapping* $\mathcal{U} : T \rightrightarrows \mathbb{R}^{n+1}$ (in short, $u \in \mathcal{U}$). Let $\mathcal{U}_t := \mathcal{U}(t) \subset \mathbb{R}^{n+1}$ for all $t \in T$. Hence, in this robust model, the uncertainty set is the *graph* of $\mathcal{U}$, that is, $\operatorname{gph}\mathcal{U} = \{(t, u_t) : u_t \in \mathcal{U}_t, t \in T\}$.

A robust decision maker facing uncertainty in the constraints intends to guarantee the feasibility of her/his decisions, so that the *robust counterpart* of the parametric problem $(P^u)_{u\in\mathcal{U}}$ is the deterministic problem

$$(1.3)\qquad \begin{array}{lll} (RP) & \operatorname*{V-min}_{x\in\mathbb{R}^n} & \left(c_1^\top x,\dots,c_m^\top x\right) \\ & \text{s.t.} & v_t^\top x \geq w_t, \forall (v_t,w_t)\in\mathcal{U}_t, t\in T, \end{array}$$

where the uncertain constraints are enforced for every possible value of the data within the prescribed uncertainty set $\operatorname{gph}\mathcal{U}$. Notice that (RP) is an ordinary multi-objective linear problem whenever $\operatorname{gph}\mathcal{U}$ is finite (unlikely in practice). Otherwise, it is a multi-objective linear semi-infinite optimization problem.

It is worth noting that if the uncertainty also occurs in the objective functions of problem (P), then its corresponding robust counterpart can be rewritten in the form of (RP). For instance, assume that, for each $i \in I$, the vector c_i is an uncertain parameter belonging to the uncertainty set $\mathcal{V}_i$, then, the *robust counterpart* of the associated parametric problem is given by

$$\begin{array}{ll} \operatorname*{V-min}_{x\in\mathbb{R}^n} & \left(\sup_{c_1\in\mathcal{V}_1} c_1^\top x,\dots,\sup_{c_m\in\mathcal{V}_m} c_m^\top x\right) \\ \text{s.t.} & v_t^\top x \geq w_t, \forall (v_t,w_t)\in\mathcal{U}_t, t\in T, \end{array}$$

which is equivalent to

$$(1.4)\qquad \begin{array}{ll} \operatorname*{V-min}_{x\in\mathbb{R}^n, z\in\mathbb{R}^m} & \left(e_1^\top z,\dots,e_m^\top z\right) \\ \text{s.t.} & e_i^\top z - c_i^\top x \geq 0, \forall c_i\in\mathcal{V}_i, i\in I, \\ & v_t^\top x \geq w_t, \forall (v_t,w_t)\in\mathcal{U}_t, t\in T, \end{array}$$

where $\{e_1, \dots, e_m\}$ is the canonical basis of $\mathbb{R}^m$.

Following the work of scalar robust optimization (see [1, 2, 15]), some of the key questions of multi-objective optimization under data uncertainty include:

I. (*Guaranteeing robust feasibility*) How to guarantee feasibility for all realizations of an uncertain scenario?
II. (*Identifying robust solutions*) How to characterize a robust solution that is immunized against data uncertainty?
III. (*Developing robust duality*) How to relate worst-case (robust counterpart) value with the best-case (optimistic dual) value?

In this paper, we provide some answers to the above questions for the multi-objective linear semi-infinite programming with uncertain constraints within the robust optimization framework. In particular, we first establish a radius of robust feasibility guaranteeing the feasibility of the robust counterpart under affine data parametrization. Then, we provide dual characterizations of robust solutions of our uncertain model by way of characterizing corresponding solutions of robust counterpart of the uncertain model. Finally, we present robust duality theorems for our uncertain multi-objective problem. Some useful lemmas for deriving the radius of robust feasiblity are summarized in the appendix.

2. Radius of robust feasibility. In this section, we discuss the feasibility of the robust counterpart of our uncertain multi-objective model under affine data perturbations.

We begin by introducing some notation and preliminary definitions. Given a subset E of a linear space (equipped with a topology not necessarily compatible with the linear structure), $\operatorname{conv} E$, $\operatorname{cone} E$, $\operatorname{int} E$, $\operatorname{cl} E$, and $\operatorname{bd} E$, denote the convex hull, the convex conical hull, the interior, the closure, and the boundary of E, respectively. By 0_n, $\|\cdot\|$, $\mathbb{B}_n$, $\mathbb{R}^n_+$, and $\mathbb{R}^n_{++}$, we denote the zero vector, the Euclidean norm, the Euclidean closed unit ball, the non-negative orthant, and the positive orthant in $\mathbb{R}^n$, respectively. We also denote by d the Euclidean distance. For a convex cone $K \subset \mathbb{R}^n$, its *positive polar cone* is defined as $K^+ := \left\{y \in \mathbb{R}^n : x^\top y \geq 0, \forall x \in K\right\}$. Let $\mathbb{R}^{(T)}$ be the linear space of mappings $\mu \in \mathbb{R}^T$ such that $\{t \in T : \mu_t \neq 0\}$ is finite. The positive cone in $\mathbb{R}^{(T)}$ is denoted by $\mathbb{R}^{(T)}_+$.

Next, we first discuss the feasibility of the robust counterpart of uncertain multi-objective model under affine data perturbations under the norm data uncertainty case, where the uncertainty is described as a ball. In other words, let $\alpha \geq 0$ and study the feasibility of the problem

$$(RP_\alpha) \quad \operatorname*{V-min}_{x\in\mathbb{R}^n} \quad (c_1^\top x, \ldots, c_m^\top x)$$
$$\text{s.t.} \quad v_t^\top x \geq w_t, \forall (v_t, w_t) \in (\bar{v}_t, \bar{w}_t) + \alpha\mathbb{B}_{n+1}, t \in T,$$

The general case where the uncertainty set is not necessarily a ball will be treated later on.

Let $\mathcal{U}_t = (\bar{v}_t, \bar{w}_t) + \alpha\mathbb{B}_{n+1}$ be the norm data uncertainty set with $\{x \in \mathbb{R}^n : \bar{v}_t^\top x \geq \bar{w}_t, \forall t \in T\} \neq \emptyset$. Let $\mathcal{U} : T \rightrightarrows \mathbb{R}^{n+1}$ be defined as $\mathcal{U}(t) = \mathcal{U}_t$ for all $t \in T$. The *radius of feasibility* of the parameterized robust counterpart problem (RP_α) associated with $\mathcal{U}$ is defined to be

$$\rho(\mathcal{U}) := \sup \{\alpha \in \mathbb{R}_+ : \text{ the feasible set of } (RP_\alpha) \text{ is nonempty } \}. \tag{2.1}$$

We observe that $\rho(\mathcal{U})$ is a non-negative real number whenever $\{x \in \mathbb{R}^n : \bar{v}_t^\top x \geq \bar{w}_t, \forall t \in T\} \neq \emptyset$. Indeed, as $\{x \in \mathbb{R}^n : \bar{v}_t^\top x \geq \bar{w}_t, \forall t \in T\} \neq \emptyset$, by the definition, $\rho(\mathcal{U})$ is always nonnegative. To see $\rho(\mathcal{U}) < +\infty$, we note that for a given $t \in T$, $(0_n, 1) \in (\bar{v}_t, \bar{w}_t) + \alpha\mathbb{B}_{n+1}$ for a positive large enough α, in which case the corresponding problem (RP) is not feasible by Lemma 4.6 in the appendix. This shows that $\rho(\mathcal{U}) < +\infty$.

Moreover, the supremum in the definition of $\rho(\mathcal{U})$ (see (2.1)) may not always be attained, as illustrated in the following example.

Example 2.1. *Let* $T = \{1, 2\}$, $(\bar{v}_1, \bar{w}_1) = (1, 1, 0)$, $(\bar{v}_2, \bar{w}_2) = (-1, 1, 0)$, *and* $\alpha = 1$. *Let* $\mathcal{U}_1 := (1, 1, 0) + \mathbb{B}_3$ *and* $\mathcal{U}_2 := (-1, 1, 0) + \mathbb{B}_3$. *Then, we get*

$$\operatorname{cl} \operatorname{cone} T_R = \operatorname{cl} \left[\left\{x \in \mathbb{R}^3 : x_2 > 0\right\} \cup (\mathbb{R} \times \{0_2\})\right] = \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R},$$

so that $(0_2, 1) \in \operatorname{cl}\operatorname{cone} T_R$ *and so Lemma 4.6 implies that* (RP_α) *is infeasible for* $\alpha = 1$*. Moreover, it is easy to show that* (RP_α) *is feasible for any* $\alpha < 1$*. So,* $\rho(\mathcal{U}) = 1$ *and the supremum in the definition of* $\rho(\mathcal{U})$ *is not attained.*

Next, we provide a sufficient condition guaranteeing that the supremum in the radius of robust feasibility (2.1) is attained. To do this, recall that for a closed and convex set A, its recession cone A^∞ is defined by

$$A^\infty := \{d : x + \gamma d \in A \text{ for all } \gamma \geq 0, x \in A\}.$$

Below, we show that, if the recession cone of the feasible set of the unperturbed problem is a subspace, then the supremum in (2.1) is attained. We note that this assumption is satisfied when the corresponding feasible set can be written as the Minkowski sum of a convex compact set and a subspace.

PROPOSITION 2.2. *Let* $A := \{x \in \mathbb{R}^n : \bar{v}_t^\top x \geq \bar{w}_t, t \in T\} \neq \emptyset$*. Suppose that* A^∞ *is a subspace. Then the supremum in* (2.1) *is attained.*

Proof. Let $\rho(\mathcal{U}) := \sup\{\alpha \in \mathbb{R}_+ : (RP) \text{ is feasible for } \alpha\}$. If $\rho(\mathcal{U}) = 0$, then the supremum is automatically attained as $A \neq \emptyset$. So, we assume that $\rho(\mathcal{U}) > 0$ and let $\alpha^k \in (0, \rho(\mathcal{U}))$ be such that $\alpha^k \to \rho(\mathcal{U})$. Then, for each k, there exists $x^k \in \mathbb{R}^n$ such that $v_t^\top x^k - w_t \geq 0$ for all $(v_t, w_t) \in (\bar{v}_t, \bar{w}_t) + \alpha^k \mathbb{B}_{n+1}, t \in T$. This implies that

$$\bar{v}_t^\top x^k - \bar{w}_t + \inf_{(a_t, b_t) \in \mathbb{B}_{n+1}} \alpha^k (a_t^\top x^k - b_t) \geq 0 \text{ for all } t \in T.$$

So, we have

$$\bar{v}_t^\top x^k - \bar{w}_t - \alpha^k \|(x^k, -1)\| \geq 0 \text{ for all } t \in T. \tag{2.2}$$

We now show that $\{x^k\}$ is a bounded sequence. Granting this and by passing to subsequence if necessary, we may assume that $x^k \to \bar{x}$. Passing to the limit in (2.2), we have $\bar{v}_t^\top \bar{x} - \bar{w}_t - \rho(\mathcal{U})\|(\bar{x}, -1)\| \geq 0$ for all $t \in T$. This implies that, for any $(v_t, w_t) \in (\bar{v}_t, \bar{w}_t) + \rho(\mathcal{U})\mathbb{B}_{n+1}, t \in T$, we have

$$\begin{aligned} v_t^\top \bar{x} - w_t &\geq \bar{v}_t^\top \bar{x} - \bar{w}_t + \inf_{(a_t, b_t) \in \mathbb{B}_{n+1}} \rho(\mathcal{U})\{a_t^\top \bar{x} - b_t\} \\ &\geq \bar{v}_t^\top \bar{x} - \bar{w}_t - \rho(\mathcal{U})\|(\bar{x}, -1)\| \geq 0. \end{aligned}$$

So, $\bar{x}$ is feasible for (RP) for $\rho(\mathcal{U})$ and so, the supremum is attained.

We now establish our claim that $\{x^k\}$ is bounded. Suppose on the contrary that $\|x^k\| \to \infty$. We may assume that $\frac{x^k}{\|x^k\|} \to u \in A^\infty$ with $\|u\| = 1$. Dividing $\|x^k\|$ on both sides of (2.2) and passing to the limit, we have

$$\bar{v}_t^\top u - \rho(\mathcal{U}) \geq 0 \text{ for all } t \in T. \tag{2.3}$$

By our assumption A^∞ is a subspace. As $u \in A^\infty$ and $\|u\| = 1$, we see that $-u \in A^\infty$. Take any $x_0 \in A$. Then $x_0 - \gamma u \in A$ for all $\gamma \geq 0$. This implies that $\bar{v}_t^\top u \leq 0$ for all $t \in T$. This contradicts (2.3) and so, the claim follows. ∎

Example 2.1 violates the condition in Proposition 2.2. Indeed, in this case,

$$\{x \in \mathbb{R}^n : \bar{v}_t^\top x \geq \bar{w}_t, \forall t \in T\}^\infty = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq |x_1|\}$$

is not a subspace.

Below, we provide a formula for computing the radius of robust feasiblity. To do this, we first recall some notations. Consider the parameter space $\Theta := (\mathbb{R}^n)^T \times \mathbb{R}^T$. One can endow the parameter space Θ with the extended metric $\tilde{d}$ of the uniform convergence on T, that is,

$$\tilde{d}\left((v,w),(p,q)\right) := \sup_{t\in T} \left\| (v_t, w_t) - (p_t, q_t) \right\|, \text{ for } (v,w),(p,q) \in \Theta.$$

Observe that we may have $\tilde{d}\left((v,w),(p,q)\right) = +\infty$.

Recall the following sets of parameters:

$$\begin{array}{rcl} \Theta_c & := & \left\{ (v,w) \in (\mathbb{R}^n)^T \times \mathbb{R}^T : \exists x \in \mathbb{R}^n, v_t^\top x \geq w_t, \forall t \in T \right\} \\ \Theta_\infty & := & \left\{ (v,w) \in (\mathbb{R}^n)^T \times \mathbb{R}^T : \tilde{d}\left((v,w),\Theta_c\right) = +\infty \right\}, \quad \text{and} \\ \Theta_s & := & \left\{ (v,w) \in (\mathbb{R}^n)^T \times \mathbb{R}^T : \exists \text{ a finite } S \subset T, \{v_t^\top x \geq w_t, \forall t \in S\} \text{ is not feasible} \right\}. \end{array}$$

Recall also the so-called *hypographical set* ([6]) of the system $\{v_t^\top x \geq w_t, t \in T\}$ defined as

$$H(v,w) := \operatorname{conv}\left\{(v_t, w_t), t \in T\right\} + \mathbb{R}_+ \left\{(0_n, -1)\right\}. \tag{2.4}$$

Some useful results relating the hypographical set and the above set of parameters Θ_c, Θ_∞ and Θ_s, are summarized in the appendix for the reader's convenience.

The next result provides a formula for the radius of robust feasibility. The proof of this formula relies heavily on existing results of semi-infinite linear systems. These are summarized in Lemma 4.6 and Lemma 4.7 in the appendix.

THEOREM 2.3. **[Radius of robust feasibility]** *Let* $(\bar{v}_t, \bar{w}_t) \in \mathbb{R}^n \times \mathbb{R}$, $t \in T$ *with* $\{x : \bar{v}_t^\top x \geq \bar{w}_t, t \in T\} \neq \emptyset$. *Let* $\mathcal{U}_t := (\bar{v}_t, \bar{w}_t) + \alpha\mathbb{B}_{n+1}$, $t \in T$, *and let* $\mathcal{U} : T \rightrightarrows \mathbb{R}^{n+1}$ *be defined by* $\mathcal{U}(t) = \mathcal{U}_t$. *Then,* $\rho(\mathcal{U}) = d\left(0_{n+1}, H(\bar{v},\bar{w})\right)$, *where* $\rho(\mathcal{U})$ *is the radius of robust feasibility given as in* (2.1) *and* $H(\bar{v},\bar{w})$ *is given as in* (2.4).

Proof. We first show that $0_{n+1} \notin \operatorname{int} H(\bar{v},\bar{w})$. Otherwise, Lemma 4.7 (i) gives us

$$(0_n, 1) \in \operatorname{int}\operatorname{cone}\left\{(\bar{v}_t, \bar{w}_t), t \in T\right\} \subset \operatorname{cl}\operatorname{cone}\left\{(\bar{v}_t, \bar{w}_t), t \in T\right\}.$$

Then, Lemma 4.6 implies that $\{x : \bar{v}_t^\top x \geq \bar{w}_t,\ t \in T\} = \emptyset$. This contradicts the feasibility assumption $(\bar{v},\bar{w}) \in \Theta_c$. Hence, one has $0_{n+1} \notin \operatorname{int} H(\bar{v},\bar{w})$ and so,

$$d\left(0_{n+1}, \operatorname{bd} H(\bar{v},\bar{w})\right) = d\left(0_{n+1}, \operatorname{cl} H(\bar{v},\bar{w})\right) = d\left(0_{n+1}, H(\bar{v},\bar{w})\right). \tag{2.5}$$

We also observe that $(\bar{v},\bar{w}) \notin \Theta_\infty$. Otherwise, by Lemma 4.7 (v), $\sup_{t\in T}\left(\bar{w}_t - \bar{v}_t^\top x\right) = +\infty$ for all $x \in \mathbb{R}^n$, but this contradicts the fact that $\{x : \bar{v}_t^\top x \geq \bar{w}_t,\ t \in T\} \neq \emptyset$. Now, it follows from (2.5) and Lemma 4.7 that

$$\begin{aligned} \tilde{d}\left((\bar{v},\bar{w}), \Theta\backslash\Theta_c\right) &= \tilde{d}\left((\bar{v},\bar{w}), \operatorname{bd}\Theta_c\right) \\ &= \tilde{d}\left((\bar{v},\bar{w}), \operatorname{bd}\Theta_s\right) \\ &= d\left(0_{n+1}, \operatorname{bd} H(\bar{v},\bar{w})\right) = d\left(0_{n+1}, H(\bar{v},\bar{w})\right), \end{aligned} \tag{2.6}$$

where the first equality is from Lemma 4.7 (iii), the second equality is from Lemma 4.7 (ii) and the third equality follows from Lemma 4.7 (iv).

Let $\alpha \in \mathbb{R}_+$ be such that (RP) is feasible for α. Then, $(v_t, w_t) \in \mathcal{U}_t$, $t \in T$ (with $\mathcal{U}_t = (\bar{v}_t, \bar{w}_t) + \alpha\mathbb{B})$) implies that $(v,w) \in \Theta_c$. Since $(v_t, w_t) \in \mathcal{U}_t$, $t \in T$

if and only if $\tilde{d}((\bar{v},\bar{w}),(v,w)) \leq \alpha$, we can equivalently say that $(v,w) \in \Theta\backslash\Theta_c$ implies that $\tilde{d}((\bar{v},\bar{w}),(v,w)) > \alpha$. Therefore, (2.6) gives us that $d(0_{n+1}, H(\bar{v},\bar{w})) = \tilde{d}((\bar{v},\bar{w}),\Theta\backslash\Theta_c) \geq \alpha$ and, as a consequence of (2.1),

$$\rho(\mathcal{U}) \leq d(0_{n+1}, H(\bar{v},\bar{w})). \tag{2.7}$$

We now show that $\rho(\mathcal{U}) = d(0_{n+1}, \operatorname{cl} H(\bar{v},\bar{w}))$. To see this, we proceed by the method of contradiction and suppose that there exists $\bar{\alpha} \in \mathbb{R}$ such that $\rho(\mathcal{U}) < \bar{\alpha} < d(0_{n+1}, H(\bar{v},\bar{w}))$. Let $\mathcal{U}_t(\bar{\alpha}) := (\bar{v}_t,\bar{w}_t) + \bar{\alpha}\mathbb{B}_{n+1} \quad \forall t \in T$. Then, by the definition of the $\rho(\mathcal{U})$,

$$\{x : v_t^\top x \geq w_t, \forall (v_t,w_t) \in \mathcal{U}_t, t \in T\} = \emptyset. \tag{2.8}$$

Now, recall that

$$c(\bar{v},\bar{w}) := \sup_{x\in\mathbb{R}^n} \inf_{t\in T} \frac{\bar{v}_t^T x - \bar{w}_t}{\|(x,-1)\|}.$$

Then, from (2.6) and Lemma 4.7 (vi), we have

$$d(0_{n+1}, H(\bar{v},\bar{w})) = \tilde{d}\big((\bar{v},\bar{w}),\Theta\backslash\Theta_c\big) = c(\bar{v},\bar{w}).$$

It then follows that $c(\bar{v},\bar{w}) > \bar{\alpha}$, and so, there exists $\bar{x} \in \mathbb{R}^n$ such that $\inf_{t\in T} \frac{\bar{v}_t^T \bar{x} - \bar{w}_t}{\|(\bar{x},-1)\|} > \bar{\alpha}$. This implies that, for each $t \in T$, we have $\bar{v}_t^T \bar{x} - \bar{w}_t > \bar{\alpha}\|(\bar{x},-1)\|$. Then, for each $(v_t,w_t) \in \mathcal{U}_t$ and for each $t \in T$,

$$\begin{aligned} v_t^T \bar{x} - w_t &= \bar{v}_t^T \bar{x} - \bar{w}_t + \langle (v_t,w_t) - (\bar{v}_t,\bar{w}_t), (\bar{x},-1)\rangle \\ &> \bar{\alpha}\|(\bar{x},-1)\| - \tilde{d}\big((v,w),(\bar{v},\bar{w})\big)\,\|(\bar{x},-1)\| \geq 0. \end{aligned}$$

This contradicts (2.8), and so, the conclusion follows. □

Remark 2.4. *From the proof of the radius of robust feasibility, we indeed have*

$$\rho(\mathcal{U}) = d(0_{n+1}, H(\bar{v},\bar{w})) = \sup_{x\in\mathbb{R}^n} \inf_{t\in T} \frac{\bar{v}_t^T x - \bar{w}_t}{\|(x,-1)\|}.$$

We now provide two examples to illustrate how the radius of robust feasibility can be computed.

Example 2.5. *Consider the same example as in Example 2.1, that is,* $T = \{1,2\}$, $(\bar{v}_1,\bar{w}_1) = (1,1,0)$, $(\bar{v}_2,\bar{w}_2) = (-1,1,0)$, *and* $\alpha = 1$. *As calculated in Example 2.1,* $\rho(\mathcal{U}) = 1$. *Now, since* $H(\bar{v},\bar{w}) = \operatorname{conv}\{(1,1,0),(-1,1,0)\} + \mathbb{R}_+(0,0,-1)$, *the point of* $H(\bar{v},\bar{w})$ *closest to* 0_3 *is* $(0,1,0)$ *and so* $d(0_3, H(\bar{v},\bar{w})) = 1$. *This shows that* $\rho(\mathcal{U}) = H(\bar{v},\bar{w}) = 1$.

Example 2.6. *Consider the multi-objective problem*

$$\begin{aligned} (RP)\quad & \operatorname*{V-min}_{x\in\mathbb{R}^2} && (x_1,x_2) \\ & \quad s.t. && v_t^\top x \geq w_t, \forall (v_t,w_t) \in \mathcal{U}_t, t \in [0,1], \end{aligned}$$

where $\mathcal{U}_t = (\bar{v}_t,\bar{w}_t) + \alpha\mathbb{B}_{n+1}$, *with* $(\bar{v}_t,\bar{w}_t) = \big(t, 1-t, t-t^2\big)$ *for all* $t \in [0,1]$. *The feasible set of* $\{\bar{v}_t^\top x \geq \bar{w}_t, \forall t \in [0,1]\}$ *is* $\{x \in \mathbb{R}^2_+ : \sqrt{x_1} + \sqrt{x_2} = 1\} + \mathbb{R}^2_+$ *(recall [13, Example 1.1]). Moreover,*

$$H(\bar{v},\bar{w}) := \operatorname{conv}\left\{\big(t,1-t,t-t^2\big), t \in [0,1]\right\} + \mathbb{R}_+\{(0,0,-1)\}$$

is contained in the hyperplane $\left\{x \in \mathbb{R}^3 : x_1 + x_2 = 1\right\}$*, so that the point of* $\operatorname{bd} H(\bar{v}, \bar{w}) = H(\bar{v}, \bar{w})$ *closest to the origin is* $\operatorname{proj}_{H(\bar{v},\bar{w})}(0_3) = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$*, and so* $d(0_3, H(\bar{v}, \bar{w})) = d(0_3, \operatorname{bd} H(\bar{v}, \bar{w})) = \left\|\left(\frac{1}{2}, \frac{1}{2}, 0\right)\right\| = \frac{\sqrt{2}}{2}$.

On the other hand, direct verification shows that (RP_α) *is feasible for* $\alpha < \frac{\sqrt{2}}{2}$ *and* (RP_α) *is not feasible for all* $\alpha = \frac{\sqrt{2}}{2}$ *(the constraint corresponding to* $t = \frac{1}{2}$ *is* $0_2^\top x \geq \frac{1}{4}$*). So,* $\rho(\mathcal{U}) = d(0_3, H(\bar{v}, \bar{w})) = \frac{\sqrt{2}}{2}$.

Now we consider a more general case where the uncertain set-valued mapping for affine data perturbations takes the form

$$\mathcal{U}_t := (\bar{v}_t, \bar{w}_t) + \alpha_t Z, \forall t \in T, \tag{2.9}$$

with $(\bar{v}, \bar{w}) \in \Theta_c$, $Z \subset \mathbb{R}^{n+1}$ a compact set such that $0_{n+1} \in Z$, and $\alpha \in l_+^\infty(T)$, where $l_+^\infty(T)$ denotes the positive cone of the Banach space $l^\infty(T)$ of all bounded functions of $\mathbb{R}^T$ equipped with the norm $\|\alpha\|_\infty := \sup_{t\in T} |\alpha(t)|$. The next corollary guarantees the feasibility of the robust counterpart (RP) associated with $\mathcal{U}$ as in (2.9) for small enough α.

Corollary 2.7 (Sufficient feasibility condition). *Let* $(\bar{v}_t, \bar{w}_t) \in \mathbb{R}^n \times \mathbb{R}$, $t \in T$ *with* $\{x : \bar{v}_t^\top x \geq \bar{w}_t, t \in T\} \neq \emptyset$*. Let* $\mu > 0$*,* Z *be a compact set with* $0_{n+1} \in Z \subset \mu\mathbb{B}$ *and* $\mathcal{U}_t := (\bar{v}_t, \bar{w}_t) + \alpha_t Z$, $t \in T$*. Let* $\mathcal{U} : T \rightrightarrows \mathbb{R}^{n+1}$ *be defined by* $\mathcal{U}(t) = \mathcal{U}_t$*. Suppose that* $\{(\bar{v}_t, \bar{w}_t), t \in T\}$ *is a compact set. Then, the robust counterpart associated with* $\mathcal{U}$ *as in* (2.9) *is feasible for any* $\alpha \in l_+^\infty(T)$ *such that*

$$\|\alpha\|_\infty < \frac{d(0_{n+1}, \operatorname{cl} H(\bar{v}, \bar{w}))}{\mu}.$$

Proof. It is an immediate consequence of Theorem 2.3 as $\mu \|\alpha\|_\infty < \varepsilon$ entails that $\alpha_t Z \subset \varepsilon\mathbb{B}_{n+1}$ for all $t \in T$. □

The results of this section, including Corollary 2.7, can be easily adapted to multi-objective linear semi-infinite programming with uncertainty in all data by using its reformulation (1.4), where the uncertainty in the objective has been transferred to the constraints.

3. Robust optimality. In this section, we derive conditions characterizing robust solutions of a multi-objective linear semi-infinite programming problem with uncertain constraints.

We first recall different concepts of a solution for a deterministic multi-objective linear semi-infinite program as in (1.1) where the *feasible set* of (P), denoted by X^0, is assumed to be non-empty. A feasible solution $\bar{x} \in X^0$ is said to be *efficient* for (P) if there is no $x \in X^0$ such that $c_i^\top x \leq c_i^\top \bar{x}$ for all $i \in I$ and $c_j^\top x < c_j^\top \bar{x}$ for at least one $j \in I$. Analogously, $\bar{x} \in X^0$ is said to be *weakly efficient* if there is no $x \in X^0$ such that $c_i^\top x < c_i^\top \bar{x}$ for all $i \in I$. Moreover, $\bar{x} \in X^0$ is said to be *properly efficient* (in Geoffrion's sense) if there exists $\rho > 0$ such that, for all $i \in I$ and $x \in X^0$ satisfying $c_i^\top x < c_i^\top \bar{x}$, there exists $j \in I$ such that $c_j^\top x > c_j^\top \bar{x}$ and $\frac{c_i^\top \bar{x} - c_i^\top x}{c_j^\top x - c_j^\top \bar{x}} \leq \rho$. Let $\Delta_+^m := \{\lambda \in \mathbb{R}_+^m : \sum_{i=1}^m \lambda_i = 1\}$ and $\Delta_{++}^m := \{\lambda \in \mathbb{R}_{++}^m : \sum_{i=1}^m \lambda_i = 1\}$ It is known (for example see [4]) that $\bar{x} \in X^0$ is *weakly efficient* if and only if there exists $\lambda \in \Delta_+^m$ such that $\bar{x} \in \operatorname{argmin}_{x\in X}(\sum_{i=1}^m \bar{\lambda}_i c_i^T x)$ and $\bar{x} \in X^0$ is *properly efficient* (in Geoffrion's sense) if there exists $\lambda \in \Delta_{++}^m$ such that $\bar{x} \in \operatorname{argmin}_{x\in X}(\sum_{i=1}^m \lambda_i c_i^T x)$.

3.1. Robust solutions. Recall the robust counterpart introduced in (1.3) of the multi-objective linear semi-infinite program with uncertain constraints. Observe that, by letting $T_R := \bigcup_{t\in T}\mathcal{U}_t$, the program (RP) can be equivalently written as follows,

$$\begin{array}{ll} \text{V-}\min\limits_{x\in\mathbb{R}^n} & \left(c_1^\top x,\dots,c_m^\top x\right) \\ \text{s.t.} & v^\top x \geq w, \forall (v,w)\in T_R. \end{array}$$

The feasible set of (RP), denoted by X, is said to be the set of *robust feasible solutions.*

Definition 3.1. **[Robust efficient solutions]** *A given $\bar{x}\in\mathbb{R}^n$ is said to be a* robust efficient *(*robust weakly efficient*,* robust properly efficient*) solution of* (P^u) *whenever x is an efficient (weakly efficient, properly efficient) solution of the robust counterpart* (RP)*. Denote by X_E, X_{pE}, and X_{wE} the sets of robust efficient points, robust properly efficient points, and robust weakly efficient points, respectively.*

Obviously, $X_{pE}\subset X_E\subset X_{wE}$, with $X=X_{wE}$ whenever $c_i=0_n$ for some $i\in I$, and $X=X_{pE}$ in the trivial case that $c_i=0_n$ for all $i\in I$.

Let us give an example illustrating the different robust solutions for an uncertain multi-objective linear semi-infinite programming problem.

Example 3.2. *Consider the uncertain problem with deterministic objectives*

$$\begin{array}{lll} (P^u) & \text{V-}\min\limits_{x\in\mathbb{R}^2} & (x_1,x_2) \\ & s.t. & v_{(k,t)}^\top x \geq w_{(k,t)}, \forall (k,t)\in\mathbb{N}\times([0,1]\cup\{2\}), \end{array}$$

where $u\in\mathcal{U}$ and $\mathcal{U}:\mathbb{N}\times([0,1]\cup\{2\})\rightrightarrows\mathbb{R}^3$ is the uncertain set-valued mapping defined by $\mathcal{U}_{(k,t)} := \left(\bar{a}_{(k,t)},\bar{b}_{(k,t)}\right)+\alpha\mathbb{B}_3$ for all $(k,t)\in\mathbb{N}\times([0,1]\cup\{2\})$, with $\alpha>0$ and

$$\left(\bar{a}_{(k,t)},\bar{b}_{(k,t)}\right) := \begin{cases} \left(kt, k\,(1-t)\,, k\left(t-t^2\right)-1\right), & \text{if } (k,t)\in\mathbb{N}\times[0,1]\,, \\ (-k,-k,-2k-1)\,, & \text{if } (k,t)\in\mathbb{N}\times\{2\}\,. \end{cases}$$

It can be checked that the systems

$$\left\{\bar{a}_{(k,t)}^\top x \geq \bar{b}_{(k,t)}, \forall (k,t)\in\mathbb{N}\times([0,1]\cup\{2\})\right\}$$

and

$$\left\{tx_1+(1-t)\,x_2 \geq t-t^2, \forall t\in[0,1]\,; -x_1-x_2\geq -2\right\}$$

have the same feasible set $F:=\operatorname{conv}(D\cup\{2e_1,2e_2\})$, where $D:=\{x\in\mathbb{R}^2_+ : \sqrt{x_1}+\sqrt{x_2}=1\}$ (see [12, Example 1] and [13, Example 1.1] for details). Since $F\subset 3\mathbb{B}_3$, there exists $\delta>0$ such that the feasible set of any system of the form

$$\left\{v_{(k,t)}^1 x_1 + v_{(k,t)}^2 x_2 \geq w_{(k,t)}, \forall (k,t)\in\mathbb{N}\times([0,1]\cup\{2\})\right\} \tag{3.1}$$

such that

$$\left\|\left(v_{(k,t)}^1, v_{(k,t)}^2, w_{(k,t)}\right)-\left(kt, k(1-t), k(t-t^2)-1\right)\right\| \leq \delta, \forall (k,t)\in\mathbb{N}\times[0,1]\,, \tag{3.2}$$

and

$$\left\|\left(v_{(k,2)}^1, v_{(k,2)}^2, w_{(k,2)}\right)-(-k,-k,-2k-1)\right\| \leq \delta, \forall k\in\mathbb{N}, \tag{3.3}$$

is also contained in $3\mathbb{B}_3$ *(see [13, Corollary 6.2.1] for details). Moreover, if the inequalities* (3.2) *and* (3.3) *hold with* δ *replaced with* $\frac{1}{\sqrt{3}}\min\left\{\frac{1}{\sqrt{10}},\delta\right\}$ *then the feasible set of* (3.1) *is* F *too (see [12, Example 1]). So, if* $\alpha \leq \frac{1}{\sqrt{3}}\min\left\{\frac{1}{\sqrt{10}},\delta\right\}$*, the set of robust feasible solutions of* (P^u) *is* $X = F$*, whereas it is easy to see that* $X_{pE} = D\setminus\{e_1, e_2\}$*,* $X_E = D$*, and* $X_{wE} = D \cup \operatorname{conv}\{e_1, 2e_1\} \cup \operatorname{conv}\{e_2, 2e_2\}$*.*

3.2. Characterizations of robust efficient solutions. We now establish some simple characterizations for the robust weakly efficient solutions and robust properly efficient solutions. These characterizations involve the so-called *active cone* at $\bar{x} \in X$,

$$A(\bar{x}) := \operatorname{cone}\left\{v : (v, w) \in T_R \text{ and } v^\top \bar{x} = w\right\} \subset \mathbb{R}^n,$$

defined in terms of the data of the problem (RP), which is closely related to the cone of *feasible directions* at $\bar{x} \in X$, given by

$$D(X; \bar{x}) := \{d \in \mathbb{R}^n : \exists\, \mu > 0 \text{ such that } \bar{x} + \mu d \in X\}.$$

On the other hand, the program (RP) (or its constraints system) is said to satisfy the *Farkas-Minkowski* constraint qualification (FMCQ) when $X \neq \emptyset$ and any linear consequence of $\left\{v^\top x \geq w, (v, w) \in T_R\right\}$ is also consequence of some finite subsystem. FMCQ holds if and only if $\operatorname{cone}\{T_R \cup (0_n, -1)\}$ is closed. Moreover, FMCQ holds whenever $\{\mathcal{U}_t, t \in T\}$ is a finite family of finite sets. We will say that (RP) satisfies the *local Farkas-Minkowski* constraint qualification (LFMCQ) at $\bar{x} \in X$ when $D(X; \bar{x})^+ = A(\bar{x})$ or, equivalently, when any consequence of $\left\{v^\top x \geq w, (v, w) \in T_R\right\}$ determining a supporting hyperplane to X at $\bar{x}$ is also consequence of some finite subsystem. Obviously, if (RP) satisfies the FMCQ, then it also satisfies the LFMCQ at any $\bar{x} \in X$. These constraint qualifications allow us to replace $D(X; \bar{x})^+$ by $A(\bar{x})$, with $A(\bar{x})$ being expressed in terms of the data of the problem.

Below, we present characterizations of robust solutions under the constraint qualifications LFMCQ.

THEOREM 3.3. **[Characterization of robust solution w.r.t weak efficiency]** *Let* X *be the feasible set of problem* (RP)*. Suppose that the LFMCQ at* $\bar{x} \in X$ *holds and* $\mathcal{U}_t$ *is convex for all* $t \in T$*. Then,* $\bar{x}$ *be a robust weakly efficient solution of* (P^u) *if and only if there exists* $\bar{\lambda} \in \Delta^m_+, \bar{y}^i \in \mathbb{R}^{(T)}$*,* $(\bar{w}_t, \bar{v}_t) \in \mathcal{U}_t$*,* $t \in T$ *and* $\bar{r}_i \in \mathbb{R}$ *such that*

$$\left\{\begin{array}{l} \sum_{i=1}^m \bar{\lambda}_i\big(c_i - \sum_{t\in T} \bar{y}^i_t \bar{v}_t\big) = 0_n, \\ \sum_{i=1}^m \bar{\lambda}_i \bar{y}^i \in \mathbb{R}^{(T)}_+, \\ \sum_{i=1}^m \bar{\lambda}_i \bar{r}_i = 0, \\ c_i^T \bar{x} = \sum_{t\in T} \bar{w}_t \bar{y}^i_t + \bar{r}_i,\ i = 1, \ldots, m. \end{array}\right. \tag{3.4}$$

Proof. [$\Rightarrow$] Let $\bar{x}$ be a weakly efficient solution. From [9, Prop. 18 (iii)], there exists $\bar{\lambda} \in \mathbb{R}^m_+\setminus\{0_m\}$ such that $\sum_{i=1}^m \bar{\lambda}_i c_i \in D(X; \bar{x})^+$. As the LFMCQ at $\bar{x}$, there exists $\bar{\lambda} \in \mathbb{R}^m_+\setminus\{0_m\}$ such that $\sum_{i=1}^m \bar{\lambda}_i c_i \in A(\bar{x})$. As $A(\bar{x})$ is a cone, by normalization, we may assume that $\sum_{i=1}^m \bar{\lambda}_i = 1$, and so $\bar{\lambda} \in \Delta^m_+$. Thus, we can write

$$\sum_{i=1}^m \bar{\lambda}_i c_i = \sum_{t\in T'} \sum_{l=1}^{m_t} \mu^l_t v^l_t$$

for some $m_t \in \mathbb{N}$, $\mu^l_t > 0$ and $(v^l_t, w^l_t) \in \mathcal{U}_t$ with $(v^l_t)^\top \bar{x} = w^l_t$, for all $l = 1, \ldots, m_t$, $t \in T'$, with $T' \subset T$ being a finite set. By defining $\mu_t := 0$ and $(\bar{v}_t, \bar{w}_t)$ any point in

$\mathcal{U}_t$, for those $t \in T \backslash T'$, and $\mu_t := \sum_{l=1}^{m_t} \mu_t^l > 0$ and

$$(\bar{v}_t, \bar{w}_t) := \sum_{l=1}^{m_t} \frac{\mu_t^l}{\mu_t}(v_t^l, w_t^l) \in \mathcal{U}_t,$$

for those $t \in T'$, then we get that $\mu \in \mathbb{R}_+^{(T)}$ and $(\bar{v}_t, \bar{w}_t) \in \mathcal{U}_t,\ t \in T$. Moreover, $\sum_{i=1}^m \bar{\lambda}_i c_i = \sum_{t \in T} \mu_t \bar{v}_t$ with $\bar{v}_t^\top \bar{x} = \bar{w}_t$ for those $t \in T$ such that $\mu_t \neq 0$. Letting $\bar{y}^i := \mu$ for all $i = 1, \ldots, m$, we have

$$\sum_{i=1}^m \bar{\lambda}_i \left(c_i - \sum_{t \in T} \bar{y}_t^i \bar{v}_t \right) = \sum_{i=1}^m \bar{\lambda}_i c_i - \sum_{t \in T} \mu_t \bar{v}_t = 0_n.$$

Furthermore,

$$\sum_{i=1}^m \bar{\lambda}_i c_i^\top \bar{x} = \sum_{t \in T} \mu_t \bar{v}_t^\top \bar{x} = \sum_{t \in T} \mu_t \bar{w}_t = \sum_{t \in T} \bar{w}_t \bar{y}_t^i \quad \text{for all } i = 1, \ldots, m.$$

Now, for each $i = 1, \ldots, m$, let $\bar{r}_i := c_i^\top \bar{x} - \bar{w}^\top \bar{y}^i$. Then, $c_i^\top \bar{x} = \bar{w}^\top \bar{y}^i + \bar{r}_i$ for all $i = 1, \ldots, m$ and

$$\sum_{i=1}^m \bar{\lambda}_i \bar{r}_i = \sum_{i=1}^m \bar{\lambda}_i (c_i^\top \bar{x} - \bar{w}^\top \bar{y}^i) = \sum_{i=1}^m \bar{\lambda}_i c_i^\top \bar{x} - \bar{w}^\top \mu = 0.$$

$[\Leftarrow]$ Suppose that there exists $\bar{\lambda} \in \Delta_+^m, \bar{y}^i \in \mathbb{R}^{(T)}$ and $\bar{r}_i \in \mathbb{R}$ such that (3.4) holds. Take any feasible point x of (RP). Then, we have $v_t^T x \geq w_t \ \forall\ (v_t, w_t) \in \mathcal{U}_t,\ t \in T$. It then follows that

$$\begin{aligned}
\sum_{i=1}^m \bar{\lambda}_i (c_i^T x - c_i^T \bar{x}) &= \sum_{i=1}^m \bar{\lambda}_i c_i^T x - \sum_{i=1}^m \bar{\lambda}_i c_i^T \bar{x} \\
&= \sum_{i=1}^m \sum_{t \in T} \bar{\lambda}_i \bar{y}_t^i v_t^T x - \sum_{i=1}^m \bar{\lambda}_i c_i^T \bar{x} \\
&\geq \sum_{i=1}^m \sum_{t \in T} \bar{\lambda}_i \bar{y}_t^i w_t - \sum_{i=1}^m \bar{\lambda}_i (\sum_{t \in T} \bar{w}_t \bar{y}_t^i + \bar{r}_i) \\
&= -\sum_{i=1}^m \bar{\lambda}_i \bar{r}_i = 0.
\end{aligned}$$

This shows that for any feasible solution x of (RP), $c_i^T x < c_i^T \bar{x}$, $i = 1, \ldots, m$, cannot happen simultaneously. So, $\bar{x}$ is a robust weakly efficient solution. □

REMARK 3.4. We note that, in the special case *when T is finite*, the above robust weakly efficient solution characterization was obtained in [11]. In fact, [11] established a characterization for robust weakly efficient solution for multiobjective linear programming problems where the data uncertainty occurs in both the objective function and the constraints.

Next, in the case where $\mathcal{U}_t$ is the scenario uncertainty set and T is the unit ball in $\mathbb{R}^q$, $q \in \mathbb{N}$, we show that, whether a robust feasible point $\bar{x}$ is a robust weakly efficient solution or not can be verified by solving a *second-order cone programming problem.* To do this, we recall that the second-order cone SOC_r, $r \in \mathbb{N} \cup \{0\}$, is given by

$$\mathrm{SOC}_r := \{(x_0, x_1, \ldots, x_r) \in \mathbb{R}^{r+1} : x_0 \geq \|(x_1, \ldots, x_r)\|\},$$

(in particular, $\mathrm{SOC}_0 = \mathbb{R}_+$). It is known that a second-order cone programming problem can be efficiently solved (for example by interior point method) see [16].

COROLLARY 3.5. **[Tractable characterization w.r.t weak efficiency: scenario uncertainty]** *Let $p, q \in \mathbb{N}$. For problem (P^u), suppose that*

$$\mathcal{U}_t := \mathrm{co}\{(v_1^0, w_1^0) + \sum_{j=1}^{q} t_j(v_1^j, w_1^j), \ldots, (v_s^0, w_s^0) + \sum_{j=1}^{q} t_j(v_p^j, w_p^j)\},$$

where $(v_k^j, w_k^j) \times \mathbb{R}^n \times \mathbb{R}$, $k = 1, \ldots, p$, $j = 1, \ldots, q$ and $T = \mathbb{B}_{\mathbb{R}^q}$. Let X be the feasible set of its associated robust counterpart problem (RP). Suppose that the LFMCQ at $\bar{x} \in X$ holds. Then, the following statements are equivalent:

(1) *$\bar{x}$ is a robust weakly efficient solution of (P^u);*

(2) *The following second-order cone system has a solution*

$$A(\lambda, \mu) = 0 \text{ and } (\lambda, \mu) \in (\prod_{i=1}^{m} \mathrm{SOC}_0) \times (\prod_{k=1}^{p} \mathrm{SOC}_q), \tag{3.5}$$

where $A : \mathbb{R}^{m+p(q+1)} \to \mathbb{R}^{n+2}$ is an affine mapping given by

$$A(\lambda, \mu) = \begin{pmatrix} \sum_{i=1}^{m} \lambda_i c_i - \sum_{k=1}^{p} (\mu_k^0 v_k^0 + \sum_{j=1}^{q} \mu_k^j v_k^j) \\ \sum_{k=1}^{p} \big(\mu_k^0((v_k^0)^T \bar{x} - w_k^0) + \sum_{j=1}^{q} \mu_k^j((v_k^j)^T \bar{x} - w_k^j)\big) \\ \sum_{i=1}^{m} \lambda_i - 1 \end{pmatrix}.$$

(3) *The following second-order cone programming has a solution*

$$\min_{(\lambda,\mu) \in \mathbb{R}^m \times \mathbb{R}^{p(q+1)}} \{0 : A(\lambda, \mu) = 0 \text{ and } (\lambda, \mu) \in (\prod_{i=1}^{m} \mathrm{SOC}_0) \times (\prod_{k=1}^{p} \mathrm{SOC}_q)\}.$$

Proof. [(1) ⇒ (2)] Let $\bar{x}$ be a robust weakly efficient solution of (P^u). As the LFMCQ at $\bar{x} \in X$ holds, the preceding theorem implies that there exists $\bar{\lambda} \in \Delta_+^m, \bar{y}^i \in \mathbb{R}^{(T)}$, $(\bar{w}_t, \bar{v}_t) \in \mathcal{U}_t$, $t \in T := \mathbb{B}_{\mathbb{R}^q}$ and $\bar{r}_i \in \mathbb{R}$ such that

$$\begin{cases} \sum_{i=1}^{m} \bar{\lambda}_i \big(c_i - \sum_{t \in T} \bar{y}_t^i \bar{v}_t\big) = 0_n, \\ \sum_{i=1}^{m} \bar{\lambda}_i \bar{y}^i \in \mathbb{R}_+^{(T)}, \\ \sum_{i=1}^{m} \bar{\lambda}_i \bar{r}_i = 0, \\ c_i^T \bar{x} = \sum_{t \in T} \bar{w}_t \bar{y}_t^i + \bar{r}_i, \ i = 1, \ldots, m. \end{cases} \tag{3.6}$$

For each $t \in T = \mathbb{B}_{\mathbb{R}^q}$, as $(\bar{w}_t, \bar{v}_t) \in \mathcal{U}_t$, there exist $\gamma_k \geq 0$, $k = 1, \ldots, p$, with $\sum_{k=1}^{p} \gamma_k = 1$ such that

$$(\bar{w}_t, \bar{v}_t) = \sum_{k=1}^{p} \gamma_k \big((v_k^0, w_k^0) + \sum_{j=1}^{q} t_j(v_k^j, w_k^j)\big).$$

For each $k = 1, \ldots, p$, let $\bar{\mu}_k^0 = \sum_{i=1}^{m} \sum_{t \in T} \bar{\lambda}_i \bar{y}_t^i \gamma_k$ and $\mu_k^j = \sum_{i=1}^{m} \sum_{t \in T} \bar{\lambda}_i \bar{y}_t^i \gamma_k t_j$, $j = 1, \ldots, q$. Then, the second relation of (3.6) and $\gamma_k \geq 0$ imply that $\bar{\mu}_k^0 =$

$\sum_{t\in T}(\sum_{i=1}^m \bar{\lambda}_i \bar{y}_t^i)\gamma_k \geq 0$. Moreover, $t \in T = \mathbb{B}_{\mathbb{R}^q}$ give us that $\bar{\mu}_k^0 \geq \|(\bar{\mu}_k^1, \ldots, \bar{\mu}_k^q)\|$. Note that

$$\sum_{i=1}^m \sum_{t\in T} \bar{\lambda}_i \bar{y}_t^i \bar{v}_t = \sum_{i=1}^m \sum_{t\in T} \sum_{k=1}^p \bar{\lambda}_i \bar{y}_t^i \gamma_k (v_k^0 + \sum_{j=1}^q t_j v_k^j) = \sum_{k=1}^p (\bar{\mu}_k^0 v_k^0 + \sum_{j=1}^q \bar{\mu}_k^j v_k^j).$$

Then, the first relation of (3.6) implies that $\sum_{i=1}^m \bar{\lambda}_i c_i - \sum_{k=1}^p (\bar{\mu}_k^0 v_k^0 + \sum_{j=1}^q \bar{\mu}_k^j v_k^j) = 0_n$. To see the last assertion, we first note that

$$\sum_{i=1}^m \sum_{t\in T} \bar{\lambda}_i \bar{y}_t^i \bar{w}_t = \sum_{k=1}^p (\bar{\mu}_k^0 w_k^0 + \sum_{j=1}^q \bar{\mu}_k^j w_k^j). \tag{3.7}$$

From (3.6), it follows that

$$\sum_{t\in T} \sum_{i=1}^m (\bar{\lambda}_i \bar{y}_t^i)(\bar{v}_t^T \bar{x} - \bar{w}_t) = \sum_{i=1}^m \sum_{t\in T} \bar{\lambda}_i \bar{y}_t^i \bar{v}_t^T \bar{x} - \sum_{i=1}^m \bar{\lambda}_i \sum_{t\in T} \bar{y}_t^i \bar{w}_t = \sum_{i=1}^m \bar{\lambda}_i c_i^T \bar{x} - \sum_{i=1}^m \bar{\lambda}_i (c_i^T \bar{x} - \bar{r}_i) = 0,$$

where the second equality follows from the the first and the last relations in (3.6); and the third equality follows from the third relation in (3.6). So, we have

$$\begin{aligned} \sum_{k=1}^p \left(\bar{\mu}_k^0((v_k^0)^T \bar{x} - w_k^0) + \sum_{j=1}^q \bar{\mu}_k^j ((v_k^j)^T \bar{x} - w_k^j) \right) &= \sum_{i=1}^m \sum_{t\in T} \bar{\lambda}_i \bar{y}_t^i (\bar{v}_t^T \bar{x} - \bar{w}_t) \\ &= \sum_{t\in T} \sum_{i=1}^m (\bar{\lambda}_i \bar{y}_t^i)(\bar{v}_t^T \bar{x} - \bar{w}_t) = 0. \end{aligned}$$

Therefore, we see that there exist $\bar{\lambda} \in \Delta_+^m$ and $\bar{\mu}_k^j \in \mathbb{R}$, $k = 1, \ldots, p$, $j = 0, 1, \ldots, q$ such that

$$\begin{cases} \sum_{i=1}^m \bar{\lambda}_i c_i - \sum_{k=1}^p (\bar{\mu}_k^0 v_k^0 + \sum_{j=1}^q \bar{\mu}_k^j v_k^j) = 0_n \\ \mu_k^0 \geq \|(\mu_k^1, \ldots, \mu_k^q)\|, \ k = 1, \ldots, p \\ \sum_{k=1}^p \left(\bar{\mu}_k^0((v_k^0)^T \bar{x} - w_k^0) + \sum_{j=1}^q \bar{\mu}_k^j ((v_k^j)^T \bar{x} - w_k^j) \right) = 0. \end{cases}$$

This implies that the second-order cone system (3.5) has a solution.

[(2) ⇒ (1)] Suppose that the second-order cone system (3.5) has a solution. Then, there exist $\bar{\lambda} \in \Delta_+^m$ and $\bar{\mu}_k^j \in \mathbb{R}$, $k = 1, \ldots, p$, $j = 0, 1, \ldots, q$ such that

$$\begin{cases} \sum_{i=1}^m \bar{\lambda}_i c_i - \sum_{k=1}^p (\bar{\mu}_k^0 v_k^0 + \sum_{j=1}^q \bar{\mu}_k^j v_k^j) = 0_n \\ \mu_k^0 \geq \|(\mu_k^1, \ldots, \mu_k^q)\|, \ k = 1, \ldots, p \\ \sum_{k=1}^p \left(\bar{\mu}_k^0((v_k^0)^T \bar{x} - w_k^0) + \sum_{j=1}^q \bar{\mu}_k^j ((v_k^j)^T \bar{x} - w_k^j) \right) = 0. \end{cases}$$

Let x be a feasible point of (RP). Then, we have $v_t^T x \geq w_t \ \forall \ (v_t, w_t) \in \mathcal{U}_t, \ t \in T$. It then follows that, for each $k = 1, \ldots, p$ and $t \in \mathbb{B}_{\mathbb{R}^q}$,

$$(v_k^0 + \sum_{j=1}^q t_j v_k^j)^T x - (w_k^0 + \sum_{j=1}^q t_j w_k^j) \geq 0.$$

This implies that for each $k = 1, \dots, p$

$$(3.8)\quad (v_k^0)^T x - w_k^0 \geq \sup_{t\in\mathbb{B}_{\mathbb{R}^q}} \sum_{j=1}^q t_j\big((v_k^j)^T x - w_k^j\big) = \|\big((v_k^1)^T x - w_k^1, \dots, (v_k^q)^T x - w_k^q\big)\|.$$

So, from (3.8), we see that

$$\begin{aligned}\sum_{i=1}^m \bar\lambda_i(c_i^T x - c_i^T\bar x) &= \sum_{i=1}^m \bar\lambda_i\big(\sum_{k=1}^p(\bar\mu_k^0 v_k^0 + \sum_{j=1}^q \bar\mu_k^j v_k^j))^T x\big) - \sum_{i=1}^m \bar\lambda_i\big(\sum_{k=1}^p(\bar\mu_k^0 v_k^0 + \sum_{j=1}^q \bar\mu_k^j v_k^j))^T \bar x\big)\\ &= \sum_{i=1}^m \bar\lambda_i\big(\sum_{k=1}^p(\bar\mu_k^0 v_k^0 + \sum_{j=1}^q \bar\mu_k^j v_k^j))^T x\big) - \sum_{i=1}^m \bar\lambda_i \sum_{k=1}^p(\bar\mu_k^0 w_k^0 + \sum_{j=1}^q \bar\mu_k^j w_k^j)\\ &= \sum_{i=1}^m \bar\lambda_i \sum_{k=1}^p\Big(\bar\mu_k^0\big((v_k^0)^T x - w_k^0\big) + \sum_{j=1}^q \bar\mu_k^j\big((v_k^j)^T x - w_k^j\big)\Big) \geq 0,\end{aligned}$$

where the last inequality follows from the fact that $\mu_k^0 \geq \|(\mu_k^1, \dots, \mu_k^q)\|,\ k = 1, \dots, p$ and (3.8). Thus, we see that $\bar x$ is a robust weakly efficient solution for (P^u).

[(2) ⇔ (3)] This equivalence is immediate. □

THEOREM 3.6. **[Characterization of robust solution w.r.t proper efficiency]** *Let X be the feasible set of problem (RP). Suppose that the LFMCQ at $\bar x \in X$ holds and $\mathcal{U}_t$ is convex for all $t \in T$. Then, $\bar x$ be a robust properly efficient solution of (P^u) if and only if there exists $\bar\lambda \in \Delta_{++}^m, \bar y^i \in \mathbb{R}^{(T)}$, $(\bar w_t, \bar v_t) \in \mathcal{U}_t$, $t \in T$ and $\bar r_i \in \mathbb{R}$ such that*

$$(3.9)\qquad \begin{cases} \sum_{i=1}^m \bar\lambda_i\big(c_i - \sum_{t\in T}\bar y_t^i \bar v_t\big) = 0_n,\\ \sum_{i=1}^m \bar\lambda_i \bar y^i \in \mathbb{R}_+^{(T)},\\ \sum_{i=1}^m \bar\lambda_i \bar r_i = 0,\\ c_i^T\bar x = \sum_{t\in T}\bar w_t \bar y_t^i + \bar r_i,\ i = 1, \dots, m.\end{cases}$$

Proof. [⇒] Let $\bar x$ be a properly efficient solution of (RP) and the LFMCQ at $\bar x$ holds, [9, Prop. 18 (ii)] implies the existence of $\bar\lambda \in \mathbb{R}_{++}^m$ such that $\sum_{i=1}^m \bar\lambda_i c_i \in A(\bar x)$. Again, as $A(\bar x)$ is a cone, by normalization, we may assume that $\sum_{i=1}^m \bar\lambda_i = 1$, and so $\bar\lambda \in \Delta_{++}^m$. Then, following similar arguments as in the proof of Theorem 3.3, we see that (3.9) holds.

[⇐] Suppose that there exists $\bar\lambda \in \Delta_{++}^m, \bar y^i \in \mathbb{R}^{(T)}$ and $\bar r_i \in \mathbb{R}$ such that (3.4) holds. Take any feasible point x of (RP). Then, we have $v_t^T x \geq w_t\ \forall\ (v_t, w_t) \in \mathcal{U}_t,\ t \in T$. Following similar arguments as in the proof of Theorem 3.3, we see that $\sum_{i=1}^m \bar\lambda_i(c_i^T x - c_i^T\bar x) \geq 0$. Thus, the conclusion follows. □

Similar to Corollary 3.5, in the case where $\mathcal{U}_t$ is the scenario uncertainty set and T is the unit ball in $\mathbb{R}^q$, $q \in \mathbb{N}$, we obtain the following numerically checkable robust optimality condition for verifying whether a robust feasible point is robust properly efficient or not.

COROLLARY 3.7. **[Tractable sufficient robust optimality condition w.r.t properly efficiency: scenario uncertainty]** *Let $p, q \in \mathbb{N}$. For problem (P^u), suppose that*

$$\mathcal{U}_t := \mathrm{co}\{(v_1^0, w_1^0) + \sum_{j=1}^q t_j(v_1^j, w_1^j), \dots, (v_s^0, w_s^0) + \sum_{j=1}^q t_j(v_p^j, w_p^j)\},$$

where $(v_k^j, w_k^j) \times \mathbb{R}^n \times \mathbb{R}$, $k = 1, \dots, p$, $j = 1, \dots, q$ and $T = \mathbb{B}_{\mathbb{R}^q}$. Let X be the feasible set of its associated robust counterpart problem (RP). Suppose that the LFMCQ at $\bar{x} \in X$ holds. Consider the second-order cone system (3.5) as in Corollary 3.5. If this second-order cone system has a solution $(\bar{\lambda}, \bar{\mu})$ with $\bar{\lambda} \in \mathbb{R}^m_{++}$, then $\bar{x}$ is a robust properly efficient solution of (P^u).

Proof. Let $(\bar{\lambda}, \bar{\mu})$ be a solution of the second-order cone system (3.5) with $\bar{\lambda} \in \mathbb{R}^m_{++}$. This implies that $\bar{\lambda} \in \Delta^m_{++}, \bar{\mu}_k^j \in \mathbb{R}$, $k = 1, \dots, p$, $j = 0, 1, \dots, q$ and

$$\begin{cases} \sum_{i=1}^m \bar{\lambda}_i c_i - \sum_{k=1}^p (\bar{\mu}_k^0 v_k^0 + \sum_{j=1}^q \bar{\mu}_k^j v_k^j) = 0_n \\ \mu_k^0 \geq \|(\mu_k^1, \dots, \mu_k^q)\|, \ k = 1, \dots, p \\ \sum_{k=1}^p \left(\bar{\mu}_k^0 ((v_k^0)^T \bar{x} - w_k^0) + \sum_{j=1}^q \bar{\mu}_k^j ((v_k^j)^T \bar{x} - w_k^j) \right) = 0. \end{cases}$$

Using similar method of proof as in Corollary 3.5, we see that $\bar{x}$ is a robust properly efficient solution of (P^u). □

4. Robust duality. In this section, we now develop a suitable robust duality framework for the multi-objective linear semi-infinite programming problem with uncertain constraints. Related details for scalar optimization problems can be found in [1, 5, 10, 14, 15].

As stated in Section 1, the multi-objective linear problem (P) in the face of data uncertainty in the constraints can be captured by the parameterized problem (P^u), for each fixed selection $u = (v, w) \in \mathcal{U}$, introduced in (1.2). The robust counterpart of problem (P^u) is obtained by finding the "worst" value over all possible scenario $u \in \mathcal{U}$,

$$(RP) \quad \underset{x \in \mathbb{R}^n}{\text{V-min}} \quad \left(c_1^\top x, \dots, c_m^\top x\right) \quad \text{s.t.} \quad v_t^\top x \geq w_t, \forall (v_t, w_t) \in \mathcal{U}_t, t \in T.$$

For each fixed selection $u = (v, w) \in \mathcal{U}$, we associate to (P^u) the dual problem (D^u) as follows

$$\begin{array}{lll} (D^u) & \underset{\substack{\lambda \in \Delta^m_+ \\ y^i \in \mathbb{R}^{(T)}, r_i \in \mathbb{R}}}{\text{V-max}} & \left(w^\top y^1 + r_1, \dots, w^\top y^m + r_m\right) \\ & \text{s.t.} & \sum_{i=1}^m \lambda_i \left(c_i - \sum_{t \in T} y_t^i v_t\right) = 0_n, \\ & & \sum_{i=1}^m \lambda_i y^i \in \mathbb{R}^{(T)}_+, \\ & & \sum_{i=1}^m \lambda_i r_i = 0, \end{array}$$

where $w^\top y^i := \sum_{t \in T} w_t y_t^i$ for all $i \in I$. In short, we will write $y := \left(y^1, \dots, y^m\right)$ and $r := (r_1, \dots, r_m)$. We note that, for each fixed parameter u, the dual problem (D^u) is nothing but the standard Fenchel-Lagrange type dual of the primal problem which was extensively studied in the literature. For a comprehensive survey, see [4].

We now define a deterministic problem by looking at the optimistic counterpart of (D^u),

$$\begin{array}{lll} (OD) & \underset{\substack{\lambda \in \Delta^m_+, (v,w) \in \mathcal{U} \\ y^i \in \mathbb{R}^{(T)}, r_i \in \mathbb{R}}}{\text{V-max}} & \left(w^\top y^1 + r_1, \dots, w^\top y^m + r_m\right) \\ & \text{s.t.} & \sum_{i=1}^m \lambda_i \left(c_i - \sum_{t \in T} y_t^i v_t\right) = 0_n, \\ & & \sum_{i=1}^m \lambda_i y^i \in \mathbb{R}^{(T)}_+, \\ & & \sum_{i=1}^m \lambda_i r_i = 0. \end{array}$$

The optimal value of (OD) is the "best" value over all possible scenario $u = (v, w) \in \mathcal{U}$ of (D^u). In the special case when $m = 1$ (that is, the scalar value case), the problem (OD) reduces to

$$\max_{y \in \mathbb{R}_+^{(T)},\, (v,w) \in \mathcal{U}} \quad w^\top y \qquad \text{s.t.} \quad c - \sum_{t \in T} y_t v_t = 0_n,$$

which is the optimistic counterpart of the usual Lagrangian dual of a semi-infinite programming problem introduced in [10].

Definition 4.1. *A feasible point* $(\bar{\lambda}, \bar{y}, \bar{r}, (\bar{v}, \bar{w}))$ *of the optimistic counterpart* (OD) *is said to be a* weakly efficient solution *of* (OD) *if there is no feasible point* $(\lambda, y, r, (v, w))$ *of* (OD) *such that*

$$\bar{w}^\top \bar{y}^i + \bar{r}_i < w^\top y^i + r_i \quad \textit{for all } i \in I.$$

Theorem 4.2. **[Robust duality with respect to weak efficiency]** *Let* $\bar{x}$ *be a robust weakly efficient solution of* (P^u). *Suppose that the LFMCQ at* $\bar{x} \in X$ *holds and* $\mathcal{U}_t$ *is convex for all* $t \in T$. *Then, there exists a weakly efficient solution* $(\bar{\lambda}, \bar{y}, \bar{r}, (\bar{v}, \bar{w}))$ *for* (OD) *such that* $c_i^\top \bar{x} = \bar{w}^\top \bar{y}^i + \bar{r}_i \quad$ *for all* $i \in I$.

Proof. We first show that weak duality holds between the robust counterpart (RP) and the optimistic counterpart (OD), that is, there is no $x \in X$ and no $(\lambda, y, r, (v, w))$ feasible for (OD) such that $c_i^\top x < w^\top y^i + r_i$ for all $i \in I$. We proceed by the method of contradiction and assume that there exists $x \in X$ and $(\lambda, y, r, (v, w))$ feasible for (OD) such that $c_i^\top x < w^\top y^i + r_i$ for all $i \in I$. As $x \in X$ and $(\lambda, y, r, (v, w))$ is feasible for problem (OD), we have $v_t^\top x \geq w_t \quad \forall t \in T$, and

$$\begin{cases} \sum_{i=1}^m \lambda_i \left(c_i - \sum_{t \in T} y_t^i v_t\right) = 0_n, \\ \sum_{i=1}^m \lambda_i y^i \in \mathbb{R}_+^{(T)}, \\ \sum_{i=1}^m \lambda_i r_i = 0, \\ \lambda \in \Delta_+^m, y^i \in \mathbb{R}^{(T)}, r_i \in \mathbb{R}, (v, w) \in \mathcal{U}. \end{cases}$$

This implies that $\sum_{i=1}^m \lambda_i \left(c_i^\top x - w^\top y^i - r_i\right) = \sum_{i=1}^m \lambda_i \left(\sum_{t \in T} y_t^i (v_t^\top x - w_t)\right) \geq 0$. On the other hand, as $c_i^\top x < w^\top y^i + r_i$ for all $i \in I$ and $\lambda \in \Delta_+^m$, we see that $\sum_{i=1}^m \lambda_i \left(c_i^\top x - w^\top y^i - r^i\right) < 0$. This makes a contradiction, and so, the conclusion follows.

From Theorem 3.3, there exists $\bar{\lambda} \in \Delta_+^m, \bar{y}^i \in \mathbb{R}^{(T)}$, $(\bar{w}_t, \bar{v}_t) \in \mathcal{U}_t$, $t \in T$ and $\bar{r}_i \in \mathbb{R}$ such that (3.4) holds. In particular, $(\bar{\lambda}, \bar{y}, \bar{r}, (\bar{v}, \bar{w}))$ is feasible for (OD). To see the conclusion, it suffices to show that $(\bar{\lambda}, \bar{y}, \bar{r}, (\bar{v}, \bar{w}))$ is a weakly efficient solution for (OD). To see this, we proceed by the method of contradiction and assume that there exists $(\lambda, y, r, (v, w))$ feasible for (OD) such that, for all $i \in I$, $\bar{w}^\top \bar{y}^i + \bar{r}_i < w^\top y^i + r_i$. This together with the last relation in (3.4) implies that $c_i^\top \bar{x} < w^\top y^i + r^i$ for all $i \in I$. Since $\bar{x} \in X$, this contradicts the weak duality statement, and so, the conclusion follows. □

As a corollary, we obtain a version of the robust duality theorem which was given in [10] for a single-objective linear semi-infinite programming problem under data uncertainty using a local constraint qualification.

Corollary 4.3. *Consider the programs* (RP) *and* (OD) *with* $m = 1$, *and let* $\bar{x}$ *be a robust solution of* (P^u). *Suppose that the LFMCQ at* $\bar{x} \in X$ *holds and* $\mathcal{U}_t$

is convex for all $t \in T$. Then, there exists a solution $(\bar{y}, (\bar{v}, \bar{w}))$ for (OD) such that $c^\top \bar{x} = \bar{w}^\top \bar{y}$.

Proof. Note that the programs (RP) and (OD) with $m = 1$ collapse to

$$(RP_1) \quad \min_{x\in\mathbb{R}^n} \quad c^\top x \qquad \text{s.t.} \quad v_t^\top x \geq w_t, \forall (v_t, w_t) \in \mathcal{U}_t, t \in T,$$

and

$$(OD_1) \quad \max_{y\in\mathbb{R}_+^{(T)}, (v,w)\in\mathcal{U}} \quad w^\top y \qquad \text{s.t.} \quad c - \sum_{t\in T} y_t v_t = 0_n.$$

Thus, the conclusion follows from the preceding robust duality theorem. □

Similarly, one can obtain duality theorems with respect to properly efficient solutions. For that purpose, let $\Delta_{++}^m := \{\lambda \in \mathbb{R}_{++}^m : \sum_{i=1}^m \lambda_i = 1\}$ and consider the following modified optimistic dual problem,

$$(MOD) \quad \underset{\substack{\lambda\in\Delta_{++}^m, (v,w)\in\mathcal{U} \\ y^i\in\mathbb{R}^{(T)}, r_i\in\mathbb{R}}}{\text{V-max}} \quad \left(w^\top y^1 + r_1, \dots, w^\top y^m + r_m\right)$$
$$\text{s.t.} \quad \sum_{i=1}^m \lambda_i \left(c_i - \sum_{t\in T} y_t^i v_t\right) = 0_n, \quad \sum_{i=1}^m \lambda_i y^i \in \mathbb{R}_+^{(T)}, \quad \sum_{i=1}^m \lambda_i r_i = 0,$$

where we replace $\lambda \in \Delta_+^m$ by $\lambda \in \Delta_{++}^m$ in (OD).

DEFINITION 4.4. *A feasible point $(\bar{\lambda}, \bar{y}, \bar{r}, (\bar{v}, \bar{w}))$ of the problem (MOD) is said to be an* efficient solution *of (MOD) if there is no feasible point $(\lambda, y, r, (v, w))$ of (MOD) such that*

$$\begin{cases} \bar{w}^\top \bar{y}^i + \bar{r}_i \leq w^\top y^i + r_i & \textit{for all } i \in I, \textit{and} \\ \bar{w}^\top \bar{y}^j + \bar{r}_j < w^\top y^j + r_j & \textit{for at least one } j \in I. \end{cases}$$

Similarly to Theorem 4.2, we prove the following robust duality theorem with respect to properly efficient solutions.

THEOREM 4.5. **[Robust duality with respect to proper efficiency]** *Let $\bar{x}$ be a robust properly efficient solution of (P^u). Suppose that the LFMCQ holds at $\bar{x} \in X$ and $\mathcal{U}_t$ is convex for all $t \in T$. Then, there exists an efficient solution $(\bar{\lambda}, \bar{y}, \bar{r}, (\bar{v}, \bar{w}))$ for (MOD) such that $c_i^\top \bar{x} = \bar{w}^\top \bar{y}^i + \bar{r}_i$ for all $i \in I$.*

Proof. Using a similar method of proof as in Theorem 4.2, one can show that weak duality holds, that is, there is no $x \in X$ and no $(\lambda, y, (v, w))$ feasible for (MOD) such that

$$\begin{cases} c_i^\top x \leq w^\top y^i + r_i & \text{for all } i \in I, \text{and} \\ c_j^\top x < w^\top y^j + r_j & \text{for at least one } j \in I. \end{cases}$$

As $\bar{x}$ is a properly efficient solution of (P^u) and the LFMCQ at $\bar{x}$ holds, From Theorem 3.6, there exists $\bar{\lambda} \in \Delta_{++}^m, \bar{y}^i \in \mathbb{R}^{(T)}$, $(\bar{w}_t, \bar{v}_t) \in \mathcal{U}_t$, $t \in T$ and $\bar{r}_i \in \mathbb{R}$ such that (3.9) holds. In particular, we have $(\bar{\lambda}, \bar{y}, \bar{r}, (\bar{v}, \bar{w}))$ is feasible for (MOD). The conclusion will follow if we show that $(\bar{\lambda}, \bar{y}, \bar{r}, (\bar{v}, \bar{w}))$ is an efficient solution of (MOD).

To see this, we proceed by the method of contradiction and assume that there exists $(\lambda, y, r, (v, w))$ feasible for (MOD) such that

$$\begin{cases} \bar{w}^\top \bar{y}^i + \bar{r}_i \leq w^\top y^i + r_i & \text{for all } i \in I, \text{and} \\ \bar{w}^\top \bar{y}^j + \bar{r}_j < w^\top y^j + r_j & \text{for at least one } j \in I. \end{cases}$$

This together with the last relation in (3.9) implies that $c_i^\top \bar{x} \leq w^\top y^i + r_i$ for all $i \in I$ and $c_j^\top \bar{x} < w^\top y^j + r_j$ for at least one $j \in I$. Since $\bar{x} \in X$, this contradicts the weak duality statement, and so, the conclusion follows. □

Appendix. In this appendix, we collect some known results which was used to establish the formula of radius of robust feasibility. The first lemma is a useful characterization of feasibility of an infinite inequality system which can be found in [13, Theorem 4.4]. Below we provide a simple proof for the self-containment purpose.

LEMMA 4.6. *Let M be an index set. Then, $\{x \in \mathbb{R}^n : v_t^\top x \geq w_t, t \in M\} \neq \emptyset$ if and only if $(0_n, 1) \notin \operatorname{cl}\operatorname{cone}\{(v_t, w_t) : t \in M\}$.*

Proof. ($\Leftarrow$) Suppose that $(0_n, 1) \notin \operatorname{cl}\operatorname{cone}\{(v_t, w_t) : t \in M\}$. Then, the separation theorem implies that there exists $(\xi, r) \in \mathbb{R}^{n+1} \backslash \{0_{n+1}\}$ such that $r < 0 \leq v_t^\top \xi + w_t r$ for all $t \in M$. So, we have $v_t^\top(-\frac{\xi}{r}) - w_t \geq 0$ for all $t \in M$, and hence $-\frac{\xi}{r} \in \{x : v_t^\top x \geq w_t, t \in M\}$. So, the desired implication follows.

($\Rightarrow$) Suppose that $\{x : v_t^\top x \geq w_t, t \in M\} \neq \emptyset$. Then, there exists $x_0 \in \mathbb{R}^n$ such that $v_t^\top x_0 \geq w_t$ for all $t \in M$. We now show that $(0_n, 1) \notin \operatorname{cl}\operatorname{cone}\{(v_t, w_t) : t \in M\}$. Suppose on the contrary that $(0_n, 1) \in \operatorname{cl}\operatorname{cone}\{(v_t, w_t) : t \in M\}$. Then, there exist $M_k \subset M$ with $|M_k| < \infty$ and $\mu_t \geq 0$, $t \in M_k$ such that $(\xi_k, r_k) = \sum_{t \in M_k} \mu_t (v_t, w_t)$ and $(\xi_k, r_k) \to (0_n, 1)$. Then, we have $\xi_k^\top x_0 - r_k = \sum_{t \in T_k} \mu_t (v_t^\top x_0 - w_t) \geq 0$. Passing to limit, we have $-1 \geq 0$ which is impossible. So, the desired implication follows. □

LEMMA 4.7. *Let $(v, w) \in \Theta$. Then, the following statements hold:*

(i) $0_{n+1} \in \operatorname{int} H(v, w)$ *if and only if* $(0_n, 1) \in \operatorname{int}\operatorname{cone}\{(v_t, w_t), t \in T\}$.

(ii) *If* $(v, w) \notin \Theta_\infty$, *then* $(v, w) \in \operatorname{bd}\Theta_c$ *if and only if* $(v, w) \in \operatorname{bd}\Theta_s$.

(iii) *If* $(v, w) \in \Theta_c$, *then* $\tilde{d}((v, w), \Theta \backslash \Theta_c) = \tilde{d}((v, w), \operatorname{bd}\Theta_c)$.

(iv) $\tilde{d}((v, w), \operatorname{bd}\Theta_s) = d(0_{n+1}, \operatorname{bd} H(v, w))$.

(v) $(v, w) \in \Theta_\infty$ *if and only if* $\sup_{t \in T}\{w_t - v_t^\top x\} \equiv +\infty$ *for all* $x \in \mathbb{R}^n$.

(vi) $\tilde{d}\big((v, w), \Theta \backslash \Theta_c\big) = c(v, w)$ *where* $c(v, w)$ *is the called consistency value of* (v, w) *and is given by*

$$c(v, w) := \sup_{x \in \mathbb{R}^n} \inf_{t \in T} \frac{v_t^T x - w_t}{\|(x, -1)\|}.$$

Proof. Statement (i) is from [6, Lemma 5], statement (ii) is from [6, Theorem 5], statement (iii) is from [6, Corollary 1], statement (iv) is from [6, Theorem 6], statement (v) is from [7, Theorem 3], and finally statement (vi) follows from [6, Theorem 7]. □

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