Global Quadratic Minimization over Bivalent Constraints: Necessary and Sufficient Global Optimality Condition *

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Abstract

In this paper, we establish global optimality conditions for quadratic optimization problems with quadratic equality and bivalent constraints. We first present a necessary and sufficient condition for a global minimizer of quadratic optimization problems with quadratic equality and bivalent constraints. Then, we examine situations where this optimality condition is equivalent to checking the positive semidefiniteness of a related matrix, and so, can be verified in polynomial time by using elementary eigenvalues decomposition techniques. As a consequence, we also present simple sufficient global optimality conditions, which can be verified by solving a linear matrix inequality problem, extending several known sufficient optimality conditions in the existing literature.

Key words: Quadratic optimization, Bivalent constraints, Global optimality conditions.

AMS subject classification: 90C26, 90C46, 90C20, 90C30

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1 Introduction

In this paper, we consider the quadratic optimization problem with quadratic equality and bivalent constraints (QP). This optimization problem covers broad model problems, such as max cut problem and p-dispersion problems, and has important applications in financial analysis, biology and signal processing [1,2]. As the problem (QP) covers the max cut problem, which is known as a NP-hard problem [3,4], the problem (QP) is also NP-hard in general, and so is an intrinsic hard optimization problem.

We are interested in the following fundamental question: How to develop a mathematical criteria to identify the global minimizer of the problem (QP). The corresponding mathematical criteria is called the global optimality condition for (QP). In particular, researchers are often interested in finding verifiable global optimality conditions in the sense that the corresponding global optimality condition can be verified in polynomial time either directly or by resorting to some optimization techniques (for example, by solving a linear matrix inequality problem or a semidefinite programming problem).

It is known that, for convex optimization problem, any local minimizer is a global minimizer. However, in general, local optimality condition is not sufficient for identifying the global minimizer for nonconvex optimization problem. Recently, some new global optimality condition have been proposed for a general nonlinear programming without assuming convexity (cf. [5-8]). Note that the quadratic problems with quadratic equality and bivalent constraints can be equivalently rewritten as a concave quadratic optimization problem with quadratic constraints (see [5,6,9]). These conditions have been used to produce global optimality conditions for quadratic problems with bivalent constraints, and some computation results have also been presented to show the significance of the theoretical development.

On the other hand, a different approach is used in [12,13] where the authors explored the hidden convexity structure of the underlying problem and established global optimality conditions for quadratic optimization problems with bivalent constraints. More explicitly, in the special case of (QP) with only bivalent constraints, sufficient global optimality conditions as well as necessary global optimality conditions for problem (QP), were established in [12] in terms of the problem data, by using the elegant convex duality. Recently, the sufficient global optimality condition in [12] was extended to model problem (QP) in [13]. An interesting feature about the sufficient optimality condition in [13] is that the corresponding optimality condition can be verified by solving a linear matrix inequality problem, which can be solved in polynomial time (e.g. by the interior point method). However, the sufficient global optimality conditions and necessary global optimality conditions presented in [12] are treated separately and it is not clear when the sufficient global optimality condition presented in [13] becomes necessary. For other related work on global optimality condition for classes of continuous optimization problem see [14-22].

In this paper, following the research line of [12,13], we complement the study of [12,13] by making the following three contributions: For a feasible point of (QP), we first present a necessary and sufficient condition characterizing the case where this feasible point is indeed a global minimizer of (QP). Secondly, we examine situations where this global optimality condition is equivalent to checking the positive semidefiniteness of a related matrix, and so can be verified in polynomial time by using elementary eigenvalue decomposition techniques. Thirdly, we present a simple sufficient global optimality condition for a class of (QP), which extends the corresponding results in [12,13].

The organization of this paper is as follows. In Section 2, we fix the notation and recall some basic facts on quadratic functions. In Section 3, we present a necessary and sufficient condition for a global minimizer of quadratic optimization problems with quadratic equality and bivalent constraints. In Section 4, we examine situations, where the optimality condition is equivalent to checking the positive semidefiniteness of a related matrix. In Section 5, we obtain a simple sufficient global optimality condition extending the corresponding results in [12,13]. Finally, we conclude our discussion in Section 6 and point out some of the possible future research directions.

2 Preliminaries

In this section, we fix the notation and recall some basic facts on quadratic functions, that will be used throughout this paper. The n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. The dimension of a subspace C of $\mathbb{R}^n$ is denoted by $\dim C$. The set of all non-negative vectors of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$, and the interior of $\mathbb{R}^n_+$ is denoted by $\mathrm{int}\mathbb{R}^n_+$. The space of all $(n \times n)$ symmetric matrices is denoted by S^n. The notation $A \succeq B$ means that the matrix $A - B$ is positive semidefinite. Moreover, the notation $A \succ B$ means the matrix $A - B$ is positive definite. The positive semidefinite cone is defined by $S^n_+ := \{M \in S^n : M \succeq 0\}$. Let $A, B \in S^n$. The (trace) inner product of A and B is defined by $A \cdot B = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ji}$, where a_{ij} is the (i,j) element of A and b_{ji} is the (j,i) element of B. A useful fact about the trace inner product is $A \cdot (xx^T) = x^T A x$ for all $x \in \mathbb{R}^n$ and $A \in S^n$. For a vector $a = (a_1, \ldots, a_n)^T \in \mathbb{R}^n$, we use $\mathrm{diag}\, a$ to denote the diagonal matrix whose diagonal elements are $a_1, \ldots, a_n$.

Recall that the quadratic optimization problem with quadratic equality and bivalent constraints takes the following form:

$$
\begin{aligned}
(QP) \qquad\qquad & \min_{x \in \mathbb{R}^n} \quad x^T A x + 2a^T x + \alpha \\
& s.t. \quad x^T B_i x + 2b_i^T x + \beta_i = 0, \ i = 1, \ldots, m, \\
& \qquad\quad x \in \prod_{j=1}^n \{-1, 1\},
\end{aligned}
$$

where A, B_i, $i = 1, \ldots, m$, are $(n \times n)$ symmetric matrices, a, b_i, $i = 1, \ldots, m$, are vectors in $\mathbb{R}^n$, and α, β_i, $i = 1, \ldots, m$, are real numbers. A closed related problem, which has also been well investigated, is the quadratic optimization problem with quadratic equality and $0-1$ constraints where the constraint $x \in \prod_{j=1}^n \{-1, 1\}$ is replaced by $x \in \prod_{j=1}^n \{0, 1\}$. Let $a = \frac{x+e}{2}$ where $e \in \mathbb{R}^n$ is the vector whose components are all one. It is clear that $x \in \prod_{j=1}^n \{-1, 1\}$, if and only if $a \in \prod_{j=1}^n \{0, 1\}$. So, by doing a one-to-one linear transformation if necessary, one can see that these two problem are essentially equivalent. Therefore, in this paper, we only focus on the problem (QP).

The quadratic programming problem with bivalent constraint problem (QP) can be rewritten as the following nonlinear programming:

$$
\begin{array}{lll}
(QP_1) & \min\limits_{x\in\mathbb{R}^n} & x^TAx + 2a^Tx + \alpha \\
 & s.t. & x^TB_ix + 2b_i^Tx + \beta_i = 0,\ i = 1,\ldots,m,\ x^TE_jx - 1 = 0,\ j = 1,\ldots,n,
\end{array}
$$

where $E_j = \operatorname{diag} e_j$ and $e_j \in \mathbb{R}^n$ is a vector whose jth element is 1 and the other elements are all equal to 0. Let $f(x) = x^TAx + 2a^Tx + \alpha$, $g_i(x) = x^TB_ix + 2b_i^Tx + \beta_i$, $i = 1,\ldots,m$ and $h_j(x) = x^TE_jx - 1$, $j = 1,\ldots,n$. Define the feasible set F by

$$F := \{x \in \mathbb{R}^n : g_i(x) = 0, h_j(x) = 0,\ i = 1,\ldots,m,\ j = 1,\ldots,n\}.$$

Recall that $\overline{x} \in F$ is called a KKT point of (QP) iff there exist $\mu \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^n$ such that

$$\nabla(f + \sum_{i=1}^{m} \mu_i g_i + \gamma_j h_j)(\overline{x}) = 0. \tag{1}$$

The corresponding $\mu \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^n$ satisfying (1) is called the KKT multipliers associated with $\overline{x}$.

3 Global Optimality Characterization

In this section, we derive a global optimality characterization for the problem (QP). We begin by establishing a Lagrange multiplier condition characterizing the case where a KKT point of (QP) is indeed a *global* minimizer. To do this, recall that, for a matrix $M \in S^n$ and $C \subseteq \mathbb{R}^n$, we say M is positive semidefinite over the set C iff $d^TMd \geq 0$ for all $d \in C$.

Lemma 3.1. *For (QP), let $\overline{x}$ be a KKT point, and let $\mu \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^n$ be the associated KKT multipliers. Then $\overline{x}$ is a global minimizer if and only if*

$$A + \sum_{i=1}^{m} \mu_i B_i + \sum_{j=1}^{n} \gamma_j E_j$$

is positive semidefinite over the set $Z(\overline{x})$, where

$$
\begin{aligned}
Z(\overline{x}) := \{v = (v_1,\ldots,v_n)^T \in \mathbb{R}^n : 2(B_i\overline{x} + b_i)^Tv + v^TB_iv = 0,\ i = 1,\ldots,m,& \\
2\overline{x}_jv_j + v_j^2 = 0,\ j = 1,\ldots,n\}.& \qquad (2)
\end{aligned}
$$

Proof. Let $f(x) = x^T Ax + 2a^T x + \alpha$, $g_i(x) = x^T B_i x + 2b_i^T x + \beta_i$, $i = 1, \ldots, m$, and $h_j(x) = x^T E_j x - 1$, $j = 1, \ldots, n$, where $E_j = \mathrm{diag}(e_j)$ and e_j is a vector whose jth element is 1 and the other elements are all equal to 0. Since $\overline{x}$ is a KKT point with the KKT multipliers $\mu \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^n$, we have $g_i(\overline{x}) = 0$, $h_j(\overline{x}) = 0$ and

$$\nabla(f + \sum_{i=1}^{m} \mu_i g_i + \sum_{j=1}^{n} \gamma_j h_j)(\overline{x}) = 0.$$

Then, for each feasible point x_0 of (QP),

$$\begin{aligned} f(x_0) - f(\overline{x}) &= (f + \sum_{i=1}^{m} \mu_i g_i + \sum_{j=1}^{n} \gamma_j h_j)(x_0) - (f + \sum_{i=1}^{m} \mu_i g_i + \sum_{j=1}^{n} \gamma_j h_j)(\overline{x}) \\ &= \left(\nabla(f + \sum_{i=1}^{m} \mu_i g_i + \sum_{j=1}^{n} \gamma_j h_j)(\overline{x})\right)^T (x_0 - \overline{x}) \\ &\quad + (x_0 - \overline{x})^T (A + \sum_{i=1}^{m} \mu_i B_i + \sum_{j=1}^{n} \gamma_j E_j)(x_0 - \overline{x}) \\ &= (x_0 - \overline{x})^T (A + \sum_{i=1}^{m} \mu_i B_i + \sum_{j=1}^{n} \gamma_j E_j)(x_0 - \overline{x}). \end{aligned} \tag{3}$$

Now, suppose that $A + \sum_{i=1}^{m} \mu_i B_i + \sum_{j=1}^{n} \gamma_j E_j$ be positive semidefinite over the set $Z(\overline{x})$. Let $v = x_0 - \overline{x}$ and denote $v = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$. Note that $x_0, \overline{x} \in F$ and so, for all $i = 1, \ldots, m$,

$$0 = g_i(x_0) - g_i(\overline{x}) \;\; = \;\; \nabla g_i(\overline{x})^T v + \frac{1}{2} v^T \nabla^2 g_i(\overline{x}) v = 2(B_i \overline{x} + b_i)^T v + v^T B_i v,$$

and, for all $j = 1, \ldots, n$,

$$0 = h_j(x_0) - h_j(\overline{x}) \;\; = \;\; \nabla h_j(\overline{x})^T v + \frac{1}{2} v^T \nabla^2 h_j(\overline{x}) v = 2\overline{x}_j v_j + v_j^2.$$

This implies that $v \in Z(\overline{x})$. Since $A + \sum_{i=1}^{m} \mu_i B_i$ is positive semidefinite over the set $Z(\overline{x})$, this together with (3) gives that for each feasible x_0 of (QP), $f(x_0) \geq f(\overline{x})$. Hence, $\overline{x}$ is a global minimizer of (QP).

Conversely, let $\overline{x}$ be a global minimizer of (QP). We proceed by the method of contradiction and suppose that there exists $v \in Z(\overline{x})$ such that

$$v^T \left(A + \sum_{i=1}^{m} \mu_i B_i + \sum_{j=1}^{n} \gamma_j E_j \right) v < 0.$$

Let $x_0 = \overline{x} + v$ and $v = (v_1, \ldots, v_n)^T$. Since $\overline{x} \in F$ (so, $g_i(\overline{x}) = 0$ and $h_j(\overline{x}) = 0$) and $v \in Z(\overline{x})$, we have for each $i = 1, \ldots, m$,

$$\begin{aligned} g_i(x_0) = g_i(\overline{x}) + \big(g_i(x_0) - g_i(\overline{x})\big) &= \nabla g_i(\overline{x})^T (x_0 - \overline{x}) + \frac{1}{2}(x_0 - \overline{x})^T \nabla^2 g_i(\overline{x})(x_0 - \overline{x}) \\ &= 2(B_i \overline{x} + b_i)^T v + v^T B_i v = 0, \end{aligned}$$

and for each $j = 1, \ldots, n$,

$$\begin{aligned} h_j(x_0) = h_j(\overline{x}) + \big(h_j(x_0) - h_j(\overline{x})\big) &= \nabla h_j(\overline{x})^T (x_0 - \overline{x}) + \frac{1}{2}(x_0 - \overline{x})^T \nabla^2 h_j(\overline{x})(x_0 - \overline{x}) \\ &= 2\overline{x}_j v_j + v_j^2 = 0. \end{aligned}$$

Thus, x_0 is feasible for (QP). So, it follows from (3) that

$$\begin{aligned} f(x_0) - f(\overline{x}) &= (x_0 - \overline{x})^T \big(A + \sum_{i=1}^m \mu_i B_i + \sum_{j=1}^n \gamma_j E_j\big)(x_0 - \overline{x}) \\ &= v^T \big(A + \sum_{i=1}^m \mu_i B_i + \sum_{j=1}^n \gamma_j E_j\big) v < 0. \end{aligned}$$

This implies that $\overline{x}$ is not a global minimizer which contradicts our assumption. Thus, the conclusion follows. □

Using the preceding lemma, we now derive a characterization for the global minimizer for (QP). To do this, we need the following notation: for $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)^T \in \mathbb{R}^n$, we define

$$\overline{X} := \mathrm{diag}(\overline{x}_1, \ldots, \overline{x}_n). \tag{4}$$

Theorem 3.1. *For (QP), let $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)^T \in \mathbb{R}^n$ be a feasible point, and let $Z(\overline{x})$ be defined as in (2). Then, the following statements are equivalent:*

(i) *$\overline{x}$ is a global minimizer;*

(ii) *the matrix $A - \mathrm{diag}\big(\overline{X} A \overline{x} + \overline{X} a\big)$ is positive semidefinite over the set $Z(\overline{x})$.*

(iii) *there exists $\mu \in \mathbb{R}^m$ with $\mu = (\mu_1, \ldots, \mu_m)^T$ such that the matrix*

$$M(\mu) := A - \mathrm{diag}\big(\overline{X} A \overline{x} + \overline{X} a\big) + \sum_{i=1}^m \mu_i \big(B_i - \mathrm{diag}\big(\overline{X} B_i \overline{x}\big) - \overline{X} \mathrm{diag}(b_i)\big)$$

is positive semidefinite over the set $Z(\overline{x})$.

Proof. Let $f(x) = x^T Ax + 2a^T x + \alpha$, $g_i(x) = x^T B_i x + 2b_i^T x + \beta_i$, $i = 1, \ldots, m$, and $h_j(x) = x^T E_j x - 1$, $j = 1, \ldots, n$, where $E_j = \operatorname{diag} e_j$ and e_j is a vector whose jth element is 1 and the other elements are all equal to 0.

We first claim that, for each $\mu = (\mu_1, \ldots, \mu_m)^T \in \mathbb{R}^m$, the feasible point $\overline{x}$ is a KKT point of (QP) with KKT multiplier $\mu \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^n$, where $B = (b_1, \ldots, b_m) \in \mathbb{R}^{n \times m}$, $\overline{X}$ is defined as in (4) and

$$\gamma := -\Big(\overline{X}(A + \sum_{i=1}^m \mu_i B_i)\overline{x} + \overline{X}a + \overline{X}B\mu\Big). \tag{5}$$

To prove our claim, let $a = (a_1, \ldots, a_n)^T \in \mathbb{R}^n$ and $\gamma = (\gamma_1, \ldots, \gamma_n)^T \in \mathbb{R}^n$. Then, we note that, for each $k = 1, \ldots, n$,

$$\gamma_k = -\overline{x}_k\Big((A + \sum_{i=1}^m \mu_i B_i)\overline{x}\Big)_k - \overline{x}_k a_k - \overline{x}_k \sum_{i=1}^m (\mu_i b_i)_k,$$

where $(u)_k$ is the kth coordinate of a vector $u \in \mathbb{R}^n$. Since $\overline{x}_k^2 = \overline{x}^T E_k \overline{x} = 1$, it follows that, for each $k = 1, \ldots, n$,

$$\begin{aligned}
&\Big(\nabla(f + \sum_{i=1}^m \mu_i g_i + \sum_{j=1}^n \gamma_j h_j)(\overline{x})\Big)_k \\
= &\Big(2(A + \sum_{i=1}^m \mu_i B_i + \sum_{j=1}^n \gamma_j E_j)\overline{x} + 2(a + \sum_{i=1}^m \mu_i b_i)\Big)_k \\
= &\ 2\Big((A + \sum_{i=1}^m \mu_i B_i)\overline{x}\Big)_k + 2(a + \sum_{i=1}^m \mu_i b_i)_k + 2\gamma_k \overline{x}_k = 0,
\end{aligned}$$

and so, $\overline{x}$ is a KKT point with KKT multipliers $\mu \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^n$ for (QP).

[(i) $\Rightarrow$ (ii)] Let $\overline{x}$ be a global minimizer. Consider $\mu = (0, \ldots, 0)^T \in \mathbb{R}^m$ and $\gamma = -\big(\overline{X}A\overline{x} + \overline{X}a\big) \in \mathbb{R}^n$. Then $\overline{x}$ is a KKT point with KKT multipliers $\mu \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^n$. Note that, in this case,

$$A + \sum_{i=1}^m \mu_i B_i + \sum_{j=1}^n \gamma_j E_j = A - \operatorname{diag}\big(\overline{X}A\overline{x} + \overline{X}a\big).$$

Thus, applying the preceding lemma, we see that (2) holds.

[(ii) $\Rightarrow$ (iii)] This implication follows directly as $M(0) = A - \operatorname{diag}\big(\overline{X}A\overline{x} + \overline{X}a\big)$.

[(iii) ⇒ (i)] Suppose that there exists $\mu = (\mu_1, \ldots, \mu_m)^T \in \mathbb{R}^m$ such that the matrix $M(\mu)$ is positive semidefinite over the set $Z(\overline{x})$. Note that $\overline{x}$ is a KKT point with KKT multipliers $\mu \in \mathbb{R}^m$ and $\gamma \in \mathbb{R}^n$ for (QP), and

$$\begin{aligned} & A + \sum_{i=1}^{m} \mu_i B_i + \sum_{j=1}^{n} \gamma_j E_j \\ = \ & A + \sum_{i=1}^{m} \mu_i B_i - \mathrm{diag}\Big(\overline{X}(A + \sum_{i=1}^{m} \mu_i B_i)\overline{x} + \overline{X}a + \overline{X}B\mu\Big) \\ = \ & A - \mathrm{diag}\big(\overline{X}A\overline{x} + \overline{X}a\big) + \sum_{i=1}^{m} \mu_i (B_i - \mathrm{diag}\big(\overline{X}B_i\overline{x}\big) - \overline{X}\mathrm{diag}(b_i)). \end{aligned}$$

Then, we see that $\overline{x}$ is a global minimizer by applying the preceding lemma again. □

As a corollary, we obtain a sufficient global optimality condition presented in [6].

Corollary 3.1. *For (QP), let $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)^T \in \mathbb{R}^n$ be a feasible point and let*

$$\overline{X} = \mathrm{diag}(\overline{x}_1, \ldots, \overline{x}_n).$$

Suppose that there exists $\mu = (\mu_1, \ldots, \mu_m)^T \in \mathbb{R}^m$ such that

$$A - \mathrm{diag}\big(\overline{X}A\overline{x} + \overline{X}a\big) + \sum_{i=1}^{m} \mu_i (B_i - \mathrm{diag}\big(\overline{X}B_i\overline{x}\big) - \overline{X}\mathrm{diag}(b_i)) \succeq 0. \quad (6)$$

Then, $\overline{x}$ is a global minimizer of (QP).

Proof. Suppose that (6) holds. Then, in particular, the matrix

$$A - \mathrm{diag}\big(\overline{X}A\overline{x} + \overline{X}a\big) + \sum_{i=1}^{m} \mu_i (B_i - \mathrm{diag}\big(\overline{X}B_i\overline{x}\big) - \overline{X}\mathrm{diag}(b_i))$$

is positive semidefinite over $Z(\overline{x})$. Thus, the conclusion follows from Theorem 3.1. □

Below, we present an example verifying Theorem 3.1, where the sufficient global optimality condition (6) fails at a global minimizer (thus, in general, the sufficient global optimality condition (6) may not to be necessary).

Example 3.1. *Consider the following two dimensional (QP)*

$$\min_{x=(x_1,x_2)\in\mathbb{R}^2} \quad 2x_1x_2 + x_1 + x_2 \text{ s.t. } x_1 - x_2 = 0, \ x \in \prod_{i=1}^{2}\{-1, 1\}.$$

It is of the form of (QP) with $n = 2$, $m = 1$,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, a = (1,1)^T, B_1 = 0 \textit{ and } b_1 = (1,-1)^T.$$

Direct verification gives that the feasible set $F = \{(1,1)^T, (-1,-1)^T\}$ and $\overline{x} = (-1,-1)^T$ is the global minimizer. On the other hand, the global optimality can also be verified by Theorem 3.1. Indeed, for any $\mu \in \mathbb{R}$,

$$M(\mu) = A - \text{diag}\big(\overline{X}A\overline{x} + \overline{X}a\big) + \mu\big(B_1 - \text{diag}\big(\overline{X}B_1\overline{x}\big) - \overline{X}\text{diag}(b_1)\big) = \begin{pmatrix} \mu & 1 \\ 1 & -\mu \end{pmatrix}.$$

Note that

$$\begin{aligned} Z(\overline{x}) &= \{(v_1,v_2)^T \in \mathbb{R}^2 : 2(B_1\overline{x} + b_1)^T v + v^T B_1 v = 0,\ 2\overline{x}_j v_j + v_j^2 = 0,\ j = 1,2\} \\ &= \{(v_1,v_2)^T \in \mathbb{R}^n : v_1 - v_2 = 0,\ v_1 \in \{0,2\},\ v_2 \in \{0,2\}\} = \{(0,0)^T, (2,2)^T\} \end{aligned}$$

and

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix}^T \begin{pmatrix} \mu & 1 \\ 1 & -\mu \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}^T \begin{pmatrix} 2+2\mu \\ 2-2\mu \end{pmatrix} = 8.$$

Thus, we see that $M(\mu)$ is positive semidefinite over $Z(\overline{x})$.

Finally, we note that, the sufficient global optimality condition (6) *fails at the global minimizer (thus, in general, the sufficient global optimality condition* (6) *may not to be necessary). To see this, note that*

$$\text{Det}\, M(\mu) = \text{Det} \begin{pmatrix} \mu & 1 \\ 1 & -\mu \end{pmatrix} = -\mu^2 - 1 < 0,$$

where Det A *denotes the determinant of a matrix A. Thus, we see that* (6) *fails and so, in this example,* (6) *is not necessary for global optimality.*

As pointed out in [5,6], the discrete optimization problem (QP) is equivalent to the following quadratic minimization problem with box constraint for all large $\mu > 0$,

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & x^T A x + 2a^T x + \alpha + \mu \sum_{i=1}^n (1 - x_i^2) \\ \text{s.t.} \quad & x^T B_i x + 2b_i^T x + \beta_i = 0,\ x \in \prod_{i=1}^n [-1,1]. \end{aligned}$$

By choose a sufficient large $\mu > 0$, one can see that the discrete optimization problem (QP) can be equivalently rewritten as a concave quadratic optimization problem with quadratic equality and box constraints. Thus, one could apply the necessary and sufficient global optimality condition for concave quadratic optimization problem developed in [5-9] to obtain necessary and sufficient global optimality condition for problem (QP). It is worth noting that the corresponding conditions in [5-9] often reduces to checking the copositivity of some related matrix. Moreover, although checking the copositivity of a matrix is, in general, an NP-hard problem, numerous effective approximation schemes have been proposed recently for solving optimization problems with copositivity matrix constraint. For recent excellent surveys on copositive matrix and copositive optimization, see [10,11].

As our condition in Theorem 3.1 and the the condition presented in [5-9] are both necessary and sufficient for a global minimizer for (QP), these two types of optimality conditions are logically equivalent for problem (QP). Moreover, since finding a global minimizer for (QP) is an NP-hard problem. It can be expected that the verification of these two conditions is hard, in general. One interesting and challenge task is to identify some particular structure (QP) problem such that these necessary and sufficient optimality conditions can be verified in polynomial time. In the next section, we show that our condition in Theorem 3.1 can be verified in polynomial time for (QP) problem involving suitable sign structure.

4 Positive Semidefiniteness Characterization

In this section, we examine situations, where the necessary and sufficient global optimality condition is equivalent to checking the positive semidefiniteness of a related matrix. Note that a matrix is positive semidefinite if and only if all its eigenvalues are all nonnegative. So, our condition can be verified in polynomial time by using elementary eigenvalues decomposition techniques.

Consider (QP) with only bivalent constraints:

$$(QP_b) \qquad \min_{x\in\mathbb{R}^n} \; x^T Ax + 2a^T x + \alpha \text{ s.t. } x \in \prod_{j=1}^{n}\{-1,1\}.$$

Since there is no quadratic equality constraints (i.e., $B_i = 0$, $b_i = 0$ and $\beta_i = 0$), in this case, the sufficient global optimality condition (6) reduces to

$$A - \mathrm{diag}\big(\overline{X}A\overline{x} + \overline{X}a\big) \succeq 0. \tag{7}$$

We now show that it is indeed a characterization of global optimality for (QP_b) involving Z-matrix structure.

Recall that a matrix $A = (A_{ij})_{1\le i,j\le n} \in S^n$ is called a Z-matrix iff $A_{ij} \le 0$ for all $i \ne j$ (S^n is the set consisting of all real $n \times n$ symmetric matrix). From the definition, any diagonal matrix is a Z-matrix. The Z-matrix arises naturally in solving Dirichlet problem numerically, and plays an important role in the theory of linear complementary problem (cf. [17,23-25]).

Theorem 4.1. *For (QP_b), let $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)^T \in \mathbb{R}^n$ be a feasible point and let*

$$\overline{X} := \mathrm{diag}(\overline{x}_1, \ldots, \overline{x}_n).$$

Let A be a Z-matrix and $a \in -\mathbb{R}^n_+$. Then, the following three statements are equivalent:

(i) *$\overline{x}$ is a global minimizer of (QP_b);*

(ii) *The matrix $A - \mathrm{diag}\big(\overline{X}A\overline{x} + \overline{X}a\big)$ is positive semidefinite over*

$$Z_b(\overline{x}) = \{(v_1, \ldots, v_n)^T \in \mathbb{R}^n : 2\overline{x}_j v_j + v_j^2 = 0,\ j = 1, \ldots, n\};$$

(iii) $A - \mathrm{diag}\big(\overline{X}A\overline{x} + \overline{X}a\big) \succeq 0.$

Proof. First of all, clearly (iii) $\Rightarrow$ (ii). Applying Corollary 3.1 with $B_i = 0$, $b_i = 0$ and $\beta_i = 0$, we see that (ii) $\Leftrightarrow$ (i). Thus, to finish the proof, we only need to show (i) $\Rightarrow$ (iii). Let $\overline{x}$ be a global minimizer of (QP_b). Let $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 0, 1, \ldots, n$, be defined by

$f_0(x) = x^T A x + 2a^T x + \alpha$ and $f_i(x) = x_i^2 - 1 = x^T E_i x - 1$, $i = 1, \ldots, n$. We define $H_i \in S^{n+1}$, $i = 0, 1, \ldots, n$, by

$$H_0 = \begin{pmatrix} A & a \\ a^T & \alpha - f_0(\overline{x}) \end{pmatrix}, \tag{8}$$

and

$$H_i = \begin{pmatrix} E_i & 0 \\ 0 & -1 \end{pmatrix} \quad i = 1, \ldots, n. \tag{9}$$

It can be verified that H_0 is a Z-matrix (as A is a Z-matrix and $a \in -\mathbb{R}^n_+$) and each H_i is a diagonal matrix, $i = 1, \ldots, n$. Define a set Ω by

$$\Omega := \{(u^T H_0 u, u^T H_1 u, \ldots, u^T H_n u) : u \in \mathbb{R}^{n+1}\} + \mathrm{int}\mathbb{R}_+ \times \{0\}_{\mathbb{R}^n}.$$

Next, we show that Ω is a convex set which does not contain the origin.

We first show that $0 \notin \Omega$. Otherwise, there exists $u = (x^T, t)^T \in \mathbb{R}^{n+1}$ such that $u^T H_0 u < 0$ and $u^T H_i u = 0$ $i = 1, \ldots, n$. If $t \neq 0$, then $f_0(x/t) - f_0(\overline{x}) = t^{-2} u^T H_0 u < 0$ and $f_i(x/t) = t^{-2} u^T H_i u = 0$, $i = 1, \ldots, n$. This contradicts the fact that $\overline{x}$ is a global minimizer of (QP). If $t = 0$, then $x^T A x = u^T H_0 u < 0$ and $x_i^2 = x^T E_i x = u^T H_i u = 0$, $i = 1, \ldots, n$. This is impossible and so, $0 \notin \Omega$.

To prove the convexity of Ω, since $u^T H_i u = H_i \cdot (uu^T)$, $i = 0, 1, \ldots, n$, we observe that

$$\begin{aligned} \Omega : &= \{(u^T H_0 u, u^T H_1 u, \ldots, u^T H_n u) : u \in \mathbb{R}^{n+1}\} + \mathrm{int}\mathbb{R}_+ \times \{0\}_{\mathbb{R}^n} \\ &= \{(H_0 \cdot X, H_1 \cdot X, \ldots, H_n \cdot X) : X = uu^T,\ u \in \mathbb{R}^{n+1}\} + \mathrm{int}\mathbb{R}_+ \times \{0\}_{\mathbb{R}^n} \\ &\subseteq \{(H_0 \cdot X, H_1 \cdot X, \ldots, H_n \cdot X) : X \in S^{n+1}_+\} + \mathrm{int}\mathbb{R}_+ \times \{0\}_{\mathbb{R}^n}. \end{aligned}$$

Note that $\{(H_0 \cdot X, H_1 \cdot X, \ldots, H_n \cdot X) : X \in S^{n+1}_+\}$ is convex, and hence

$$\{(H_0 \cdot X, H_1 \cdot X, \ldots, H_n \cdot X) : X \in S^{n+1}_+\} + \mathrm{int}\mathbb{R}_+ \times \{0\}_{\mathbb{R}^n}$$

is also convex. Thus, to establish the convexity of Ω, it suffices to show that,

$$\begin{aligned} &\{(H_0 \cdot X, H_1 \cdot X, \ldots, H_n \cdot X) : X \in S^{n+1}_+\} + \mathrm{int}\mathbb{R}_+ \times \{0\}_{\mathbb{R}^n} \\ \subseteq\ &\{(H_0 \cdot X, H_1 \cdot X, \ldots, H_n \cdot X) : X = uu^T,\ u \in \mathbb{R}^{n+1}\} + \mathrm{int}\mathbb{R}_+ \times \{0\}_{\mathbb{R}^n}. \end{aligned}$$

To prove this, take

$$(z_0, z_1, \ldots, z_n) \in \{(H_0 \cdot X, H_1 \cdot X, \ldots, H_n \cdot X) : X \in S_+^{n+1}\} + \mathrm{int}\mathbb{R}_+ \times \{0\}_{\mathbb{R}^n}.$$

Then, there exists a matrix $\overline{X} \in S_+^{n+1}$ such that $H_0 \cdot \overline{X} < z_0$ and $H_k \cdot \overline{X} = z_k$, $k = 1, \ldots, n$. We now show that there exists a vector $\overline{u}$ such that

$$H_0 \cdot \overline{X} \geq \overline{u}^T H_0 \overline{u} = H_0 \cdot (\overline{u}\,\overline{u}^T) \tag{10}$$

and

$$H_k \cdot \overline{X} = \overline{u}^T H_k \overline{u} = H_k \cdot (\overline{u}\,\overline{u}^T),\ k = 1, \ldots, n. \tag{11}$$

To establish this, we use x_{ij} to denote the element of $\overline{X}$ which lies at the i^{th} row and j^{th} column. Since $\overline{X} \in S_+^{n+1}$, one has $x_{ii} \geq 0$ $(i = 1, \ldots, n+1)$ and

$$x_{jj}x_{ii} - x_{ji}^2 \geq 0 \quad i, j \in \{1, \ldots, n+1\}. \tag{12}$$

Now, define $\overline{u} = (\overline{u}_1, \ldots, \overline{u}_{n+1})^T$ where $\overline{u}_i = \sqrt{x_{ii}}$ for each $i = 1, \ldots, n+1$. Then, the (j, i) element of $\overline{u}\,\overline{u}^T$ is $\sqrt{x_{jj}x_{ii}}$, and hence

$$\begin{aligned} \overline{u}^T H_0 \overline{u} - H_0 \cdot \overline{X} &= H_0 \cdot (\overline{u}\,\overline{u}^T) - H_0 \cdot \overline{X} = H_0 \cdot (\overline{u}\,\overline{u}^T - \overline{X}) \\ &= \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} a_{ij}(\sqrt{x_{jj}x_{ii}} - x_{ji}) = \sum_{i,j=1, i\neq j}^{n+1} a_{ij}(\sqrt{x_{jj}x_{ii}} - x_{ji}) \leq 0, \end{aligned}$$

where a_{ij} is the (i, j) element of H_0 and the last inequality follows from $a_{ij} \leq 0$ for all $i \neq j$ (since H_0 is a Z-matrix) and (12). Hence, $H_0 \cdot (\overline{u}\,\overline{u}^T) \leq H_0 \cdot \overline{X} < z_0$. Moreover, since each H_k is a diagonal matrix, $k = 1, \ldots, n$, we have

$$\begin{aligned} \overline{u}^T H_k \overline{u} - H_k \cdot \overline{X} = H_k \cdot (\overline{u}\,\overline{u}^T) - H_k \cdot \overline{X} &= H_k \cdot (\overline{u}\,\overline{u}^T - \overline{X}) \\ &= \sum_{i,j=1, i\neq j}^{n+1} h_{ij}^k(\sqrt{x_{jj}x_{ii}} - x_{ji}) = 0, \end{aligned}$$

where h_{ij}^k is the (i, j) element of the matrix H_k and the last equality follows as H_k is diagonal (and so, for each $k = 1, \ldots, n$, $h_{ij}^k = 0$ for all $i \neq j$). Thus, (10) holds and hence, Ω is convex.

Now, since $0 \notin \Omega$ and Ω is convex, the convex separation theorem implies that there exists $(\lambda, \mu) \in (\mathbb{R}_+ \times \mathbb{R}^n)\backslash\{0\}$ such that $\lambda a^T H_0 a + \sum_{i=1}^n \mu_i a^T H_i a \geq 0$ for all $a \in \mathbb{R}^{n+1}$. Letting $a = (x^T, 1)^T$ with $x \in \mathbb{R}^n$, this implies that

$$\lambda(f_0(x) - f_0(\overline{x})) + \sum_{i=1}^n \mu_i f_i(x) \geq 0 \text{ for all } x \in \mathbb{R}^n. \tag{13}$$

In particular, $\lambda > 0$. (Otherwise, we have $\lambda = 0$, and so, for all $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$,

$$\sum_{i=1}^n \mu_i(x_i^2 - 1) = \sum_{i=1}^n \mu_i f_i(x) \geq 0. \tag{14}$$

This forces $\mu_i = 0$, $i = 1, \ldots, n$. Thus $(\lambda, \mu) = 0$ which is impossible.) Therefore, by dividing λ on both sides of (13), we obtain $f_0(x) - f_0(\overline{x}) + \sum_{i=1}^n \lambda_i f_i(x) \geq 0$ for all $x \in \mathbb{R}^n$, where $\lambda_i = \mu_i/\lambda$. Note that $f_i(\overline{x}) = 0$. This means that $\overline{x}$ is a global minimizer of $f_0 + \sum_{i=1}^n \lambda_i f_i$. Thus, we see that

$$2(A + \sum_{i=1}^n \lambda_i E_i)\overline{x} + 2a = \nabla(f_0 + \sum_{i=1}^n \lambda_i f_i)(\overline{x}) = 0 \tag{15}$$

and

$$A + \sum_{i=1}^n \lambda_i E_i = \frac{1}{2}\nabla^2(f_0 + \sum_{i=1}^n \lambda_i f_i)(\overline{x}) \succeq 0. \tag{16}$$

Write $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)^T$. From (15), we see that, for each $i = 1, \ldots, n$,

$$\lambda_i \, \overline{x}_i = -\big(A\overline{x}\big)_i - a_i. \tag{17}$$

Multiplying both sides of (17) by $\overline{x}_i$ and noting that $\overline{x}_i^2 = 1$, we obtain that

$$\lambda_i = -\big(A\overline{x}\big)_i \overline{x}_i - a_i \overline{x}_i.$$

This together with (16) gives that

$$A - \operatorname{diag}\big(\overline{X}A\overline{x} + \overline{X}a\big) = A + \operatorname{diag}\Big(-\overline{X}A\overline{x} - \overline{X}a\Big) = A + \sum_{i=1}^n \lambda_i E_i \succeq 0.$$

So, statement (iii) holds. This completes the proof. □

Corollary 4.1. *For (QP_b), let $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n)^T \in \mathbb{R}^n$ be a feasible point and let*

$$\overline{X} := \operatorname{diag}(\overline{x}_1, \ldots, \overline{x}_n).$$

Suppose that $a \in -\mathbb{R}^n_+$ and A is a diagonal matrix with the form that $A = \operatorname{diag}(\alpha_{11}, \ldots, \alpha_{nn})$. Then, the following statements are equivalent:

(i) $\overline{x}$ *be a global minimizer of* (QP_b)*;*

(ii) *For each* $j = 1, \dots, n$, $\alpha_{jj} \geq \left(\overline{X}A\overline{x} + \overline{X}a\right)_j$.

Proof. Since A is a diagonal matrix, we see that the matrix $A - \operatorname{diag}\left(\overline{X}A\overline{x} + \overline{X}a\right)$ is also a diagonal matrix. Note that a diagonal matrix is positive semidefinite if and only if each of its diagonal element is nonnegative. Thus the conclusion follows from the preceding Theorem. $\square$

It is worth noting that in the special case when $a = 0$, the equivalence of (i) and (ii) in corollary 4.1 was established in [26].

5 Classes with Simple Sufficient Global Optimality

Consider the following quadratic optimization problem with linear equality constraints and bivalent constraints:

$$(QP_l) \quad \min_{x \in \mathbb{R}^n} \quad x^T A x + 2a^T x + \alpha \text{ s.t. } Hx = d, \ x \in \prod_{j=1}^{n} \{-1, 1\},$$

where $H \in \mathbb{R}^{m \times n}$ and $d \in \mathbb{R}^m$. The model (QP_l) is a special case of (QP) with $B_i \equiv 0$, $i = 1, \dots, m$. As a corollary of Theorem 3.1, we now obtain a simple sufficient global optimality condition for (QP_l), which can be verified by solving a linear matrix inequality.

Corollary 5.1. *For* (QP_l)*, let* $\overline{x} = (\overline{x}_1, \dots, \overline{x}_n)^T \in \mathbb{R}^n$ *be a feasible point and let*

$$\overline{X} = \operatorname{diag}(\overline{x}_1, \dots, \overline{x}_n).$$

Suppose that there exists $z \in \mathbb{R}^m$ *such that the matrix* $A - \operatorname{diag}\left(\overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z\right)$ *be positive semidefinite over* $\ker(\mathrm{H}) := \{x \in \mathbb{R}^n : Hx = 0\}$*. Then* $\overline{x}$ *is a global minimizer.*

Proof. Let

$$H = (h_1, \dots, h_m)^T \text{ and } d = (d_1, \dots, d_m)^T,$$

where $h_i \in \mathbb{R}^n$ and $d_i \in \mathbb{R}$, $i = 1, \dots, m$. Then, applying Theorem 3.1 with $B_i \equiv 0$, $b_i = h_i/2$ and $\beta_i = -d_i$, $i = 1, \dots, m$, we see that $\overline{x}$ is a global minimizer if and only if there exists $\mu = (\mu_1, \ldots, \mu_m)^T \in \mathbb{R}^m$ such that the matrix

$$A - \operatorname{diag}\big(\overline{X}A\overline{x} + \overline{X}a\big) - \frac{1}{2}\sum_{i=1}^{m} \mu_i \overline{X}\operatorname{diag}(h_i)$$

is positive semidefinite over

$$Z_l(\overline{x}) := \{v = (v_1, \ldots, v_n)^T \in \ker(H) : 2\overline{x}_j v_j + v_j^2 = 0,\ j = 1, \ldots, n\}.$$

Let $z = \frac{\mu}{2} = (\mu_1/2, \ldots, \mu_m/2)^T$. Note that $\frac{1}{2}\sum_{i=1}^{m} \mu_i \overline{X}\operatorname{diag}(h_i) = \operatorname{diag}(\overline{X}H^T z)$ and $Z_l(\overline{x}) \subseteq \ker(H)$. Thus, the conclusion follows. □

It is worth noting that the condition in Corollary 5.1 "there exists $z \in \mathbb{R}^m$ such that the matrix $A - \operatorname{diag}\big(\overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z\big)$ is positive semidefinite over ker(H)" can be checked by solving a linear matrix inequality. To see this, let $k = \dim\ker(H)$ and let $Q \in \mathbb{R}^{k\times n}$ be the full rank matrix such that $Q(\mathbb{R}^n) = \ker(H)$, where $Q(\mathbb{R}^n) = \{Qx \in \mathbb{R}^k : x \in \mathbb{R}^n\}$. Then, the condition in Corollary 5.1 is equivalent to the linear matrix inequality problem: there exists $z \in \mathbb{R}^m$ such that the matrix $Q^T(A - \operatorname{diag}\big(\overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z\big))Q$ is positive semidefinite. For other related conditions on positive semidefiniteness of a matrix over a subspace, see [27].

Moreover, our Corollary 5.1 extends Corollary 2.1 of Pinar [13] and Theorem 2.3 of Beck and Teboulle [12] where they imposed a slightly stronger condition: there exists $z \in \mathbb{R}^m$ such that $\lambda_{\min}(A)e \geq \overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z$ where $\lambda_{\min}(A)$ is the minimum eigenvalue of A and $e \in \mathbb{R}^n$ is a vector whose coordinates are all equal to one. (To prove the fact that the condition in Corollary 5.1 "there exists $z \in \mathbb{R}^m$ such that the matrix $A - \operatorname{diag}\big(\overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z\big)$ is positive semidefinite over ker(H)" is indeed weaker, we only need to observe that the condition "$\lambda_{\min}(A)e \geq \overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z$" implies that the matrix $A - \operatorname{diag}\big(\overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z\big)$ is positive semidefinite.)

Finally, to conclude this section, we present an example verifying Corollary 5.1, where the condition "$\lambda_{\min}(A)e \geq \overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z$ for some $z \in \mathbb{R}^m$" fails (and so, the result in [12,13] is not applicable).

Example 5.1. *Consider the same example as in Example 3.1:*

$$\min_{x=(x_1,x_2)\in\mathbb{R}^2} \quad 2x_1x_2 + x_1 + x_2 \text{ s.t. } x_1 - x_2 = 0, x \in \prod_{i=1}^{2}\{-1, 1\}.$$

It is of the form of (QP_l) *with* $n = 2$, $m = 1$,

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, a = (1,1)^T \text{ and } H = (1,-1).$$

Direct verification gives that the feasible set $F = \{(1,1)^T, (-1,-1)^T\}$ *and* $\overline{x} = (-1,-1)^T$ *is the global minimizer. On the other hand, the global optimality can also be verified by Corollary 5.1. Indeed, for any* $z \in \mathbb{R}$, *we see that*

$$A - \mathrm{diag}\big(\overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z\big) = \begin{pmatrix} z & 1 \\ 1 & -z \end{pmatrix}.$$

Note that $\ker(H) = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 = x_2\}$. *Letting* $z = 0$, *then for any* $x \in \ker(H)$, $x^T(A - \mathrm{diag}\big(\overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z\big))x \geq 0$. *Thus, for* $z = 0$, *the matrix* $A - \mathrm{diag}\big(\overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z\big)$ *is positive semidefinite over* $\ker(H)$.

Moreover, note that for any $z \in \mathbb{R}$, $\mathrm{Det}(A - \mathrm{diag}\big(\overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z\big)) = -z^2 - 1 < 0$. *We see that* $A - \mathrm{diag}\big(\overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z\big)$ *is not positive semidefinite, and so, the condition "*$\lambda_{\min}(A)e \geq \overline{X}A\overline{x} + \overline{X}a + \overline{X}H^T z$ *for some* $z \in \mathbb{R}^m$*" must fail.*

6 Conclusion

In this paper, we present a necessary and sufficient condition for a global minimizer of quadratic optimization problems with quadratic equality and bivalent constraints (QP). Then, we examine situations when this global optimality condition is equivalent to the positive semidefiniteness of a related matrix, and so can be verified by using elementary eigenvalue decomposition techniques. Finally, comparison of this optimality condition with some existing global optimality conditions in [12,13] are also presented.

It would be interesting to explore more on the necessary and sufficient global optimality condition in Theorem 3.1 to see when it can be verified in polynomial time, other than the Z-matrix structural case presented in this paper. Moreover, as pointed out before, by transforming the problem (QP) into an equivalent concave quadratic optimization problem, necessary and sufficient global optimality condition can also be derived for problem

(QP). Although the corresponding conditions in [5-9] is logically equivalent to the global optimality condition in Theorem 3.1 for problem (QP), as they are both necessary and sufficient for a global minimizer of (QP). It is not clear how one could link these two approach together. Investigations about the link between the approach here and the approach used in [5-9] will be an interesting and useful topic. Another interesting research question is that: how the approach in this paper can be extended to handle the quadratic inequality constraints. These will be possible further research topics and will be examined in a forthcoming paper.

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