Hölder Metric Subregularity with Applications to Proximal Point Method *

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Abstract

This paper is mainly devoted to the study and applications of Hölder metric subregularity (or metric q-subregularity of order $q \in (0,1]$) for general set-valued mappings between infinite-dimensional spaces. Employing advanced techniques of variational analysis and generalized differentiation, we derive neighborhood and pointbased sufficient conditions as well as necessary conditions for q-metric subregularity with evaluating the exact subregularity bound, which are new even for the conventional (first-order) metric subregularity in both finite and infinite-dimensions. In this way we also obtain new fractional error bound results for composite polynomial systems with explicit calculating fractional exponents. Finally, metric q-subregularity is applied to conduct a quantitative convergence analysis of the classical proximal point method for finding zeros of maximal monotone operators on Hilbert spaces.

1 Introduction

Recall that a set-valued mapping/multifunction $F\colon X \rightrightarrows Y$ between Banach spaces is *metrically regular* with modulus $c > 0$ around the point $(\bar{x},\bar{y}) \in \operatorname{gph} F := \{(x,y) \in X \times Y |\ y \in F(x)\}$ from its graph if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ with

$$d\big(x; F^{-1}(y)\big) \le c\, d\big(y; F(x)\big) \ \text{ for all } \ x \in U \ \text{ and } \ y \in V, \tag{1.1}$$

where $d(\cdot;\Omega)$ stands for the usual distance between a point and a set in the spaces in question. It has been well recognized that this property plays a fundamental role in many aspects of nonlinear analysis and its applications, particularly those related to variational analysis and optimization. We refer the reader to [8, 10, 20, 30, 34] and the bibliographies therein for well-developed theories of metric regularity and numerous applications. Comprehensive characterizations of metric regularity and related well-posedness properties of set-valued mappings are available in the literature via both primal and dual constructions

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of generalized differentiation. Higher-order (Hölder) metric regularity and related properties of multifunctions have also been studied, e.g., in [9, 13, 35]. Note however that, while metric regularity is known to hold for a variety of constraint systems, it fails in typical variational frameworks; see [1, 4, 15, 29] for more details and discussions.

A relaxation of (1.1) when $y = \bar{y}$ fixed therein, i.e., the validity of the estimate

$$d\big(x; F^{-1}(\bar{y})\big) \le c\, d\big(\bar{y}; F(x)\big) \text{ for all } x \in U, \tag{1.2}$$

with some $c > 0$ and neighborhood U of $\bar{x}$, is known as *metric subregularity* of F at $\bar{x}$; see [10] for the history and terminology. This property as well as its equivalent *calmness* (upper Lipschitzian) counterpart for inverse mappings have drawn much attention in recent publications. The reader can find various results in this direction and their applications in, e.g., [1, 2, 10, 11, 17, 18, 19, 21, 22, 23, 36, 37] and the references therein. In what follows we concentrate on the study of metric subregularity and its higher-order extensions, while the results obtained can be reformulated in calmness terms.

The main goal of this paper is to develop the theory and applications of *Hölder metric subregularity*, or *metric q-subregularity* of order $q \in (0, 1]$, which is understood as the validity of estimate (1.2) with $d(\bar{y}; F(x))$ replaced by $d(\bar{y}; F(x))^q$ on the right-hand side. Besides the case of $q = 1$, there are just a few studies of metric q-subregularity and related properties [14, 22] conducted from the prospectives different from those in this paper. It is also worth mentioning that in the case of smooth mappings $F\colon X \to Y$ the notion of metric q-subregularity with $q = \frac{1}{2}$ is closely related to distance estimates to the manifold $M = \{x \in X|\ F(x) = 0\}$ studied, e.g., in [3, 5] under the so-called 2-regularity condition.

In this paper we employ advanced techniques and constructions of variational analysis and generalized differentiation (some of them are firstly introduced in what follows), which allow us to obtain verifiable conditions for metric q-subregularity of set-valued mappings between infinite-dimensional spaces that seems to be new even for the case of $q = 1$ in finite dimensions. Applications of these results are given first to deriving fractional error bounds for inequality systems described by compositions of smooth functions and convex polynomial, with explicit calculations of fractional exponents. Finally, we apply metric q-subregularity to conduct a qualitative convergence rate analysis of the proximal point method to find zeros of maximal monotone operators in Hilbert spaces.

The rest of the paper is organized as follows. Section 2 contains some preliminaries from generalized differentiation used in formulations and proofs of the main results. In Section 3 we establish various *sufficient* conditions for metric q-subregularity of general set-valued mappings between *Asplund* spaces with evaluating the *exact q-subregularity bound* via *coderivatives*. The conditions obtained are of the two major types: *neighborhood* (involving points nearby the reference one) and *pointbased* that use only the reference point. To formulate and justify q-subregularity conditions of the latter type, we introduce new *limiting q-coderivative* notions for general multifunctions in finite and infinite dimensions.

Section 4 presents new pointbased *sufficient* as well as *necessary* coderivative conditions for metric *subregularity* ($q = 1$) of set-valued mappings between Asplund spaces. Sharper results on the general metric q-subregularity as $q \in (0, 1]$ are derived in Section 5 for mappings that take values in spaces that admit *Fréchet smooth renorming*.

Section 6 is devoted to applications of metric q-subregularity to obtaining local *fractional error bounds* of inequality systems while paying the main attention to those given by compositions of smooth functions and convex polynomials. Employing our approach and q-subregularity results from the previous sections, we are able to *explicitly* calculate *fractional exponents* in error bounds for such systems.

The concluding Section 7 presents applications of metric q-subregularity to the classical *proximal point method* for finding zeros of maximal monotone operators on Hilbert spaces. We develop a detailed *quantitative convergence analysis* for various modifications of this method and justify *convergence rates* depending on the order of metric q-subregularity of the maximal monotone operator in question.

Throughout the paper we basically use the standard notation and terminology of variational analysis and generalized differentiation; see, e.g., [30, 34]. Unless otherwise stated, all the spaces in question are Banach equipped with the corresponding norm $\|\cdot\|$. By X^* we denote the topological dual of X with the canonical paring $\langle\cdot,\cdot\rangle$ between X and X^* and w^* signifying the weak* topology on X^*. The symbol $B_X(x,r)$ stands for the open ball in X with center x and radius $r>0$ while $\mathbb{B}_X$ and $\mathbb{S}_X$ denotes, respectively, the closed unit ball and the unit sphere of the space X. Finally, $\mathbb{N}:=\{1,2,\ldots\}$ and $\overline{I\!R}:=(-\infty,\infty]$.

2 Preliminaries from Generalized Differentiation

In this section we recall some generalized differential constructions of variational analysis that are widely used in the paper; see [30] for more details and references.

Let $f\colon X\to\overline{I\!R}$ be an extended-real-valued function, which is always assumed to be proper, i.e., $\mathrm{dom}\,f:=\{x\in X|\ f(x)<\infty\}\neq\emptyset$. Consider its epigraph

$$\mathrm{epi}\,f:=\big\{(x,r)\in X\times\mathbb{R}\big|\ f(x)\le r\big\}$$

and, given any number $r\in\mathbb{R}$, use the notation $[f\le r]$ and $[f>r]$ for the level sets

$$\big\{x\in X\big|\ f(x)\le r\big\}\ \text{ and }\ \big\{x\in X\big|\ f(x)>r\big\},$$

respectively. The *regular subdifferential* (known also as the Fréchet or viscosity subdifferential) of f at $x\in\mathrm{dom}\,f$ is given by

$$\widehat{\partial}f(x):=\Big\{x^*\in X^*\Big|\ \liminf_{h\to 0}\frac{f(x+h)-f(x)-\langle x^*,h\rangle}{\|h\|}\ge 0\Big\}\tag{2.1}$$

with $\widehat{\partial}f(x):=\emptyset$ for $x\notin\mathrm{dom}\,f$. The set (2.1) is always convex and reduces to the classical subdifferential of convex analysis for the case of convex functions. In the general case we obviously have the generalized Fermat rule: $0\in\widehat{\partial}f(x)$ if x is a local minimizer of f.

Given a nonempty set $\Omega\subset X$, the *regular normal cone* Ω at x is defined by

$$\widehat{N}(x;\Omega):=\widehat{\partial}\delta_\Omega(x),\quad x\in\Omega,\tag{2.2}$$

via the regular subdifferential (2.1) of the indicator function δ_Ω of the set Ω equal to 0 on Ω and ∞ otherwise. We put $\widehat{N}(x;\Omega):=\emptyset$ for $x\notin\Omega$ in accordance with (2.1).

Given further a set-valued mapping $F\colon X\rightrightarrows Y$ with $\mathrm{gph}\,F\neq\emptyset$, we define its *regular coderivative* at $(x,y)\in X\times Y$ as a multifunction $\widehat{D}^*F(x,y)\colon Y^*\rightrightarrows X^*$ with the values

$$\widehat{D}^*F(x,y)(y^*):=\big\{x^*\in X^*\big|\ (x^*,-y^*)\in\widehat{N}\big((x,y);\mathrm{gph}\,F\big)\big\}\ \text{ for all }\ y^*\in Y^*\tag{2.3}$$

and omit $y=F(x)$ in (2.3) when F is single-valued. Note that $\widehat{D}^*F(x)(y^*)=\{\nabla F(x)^*y^*\}$ for all $y^*\in Y^*$ if F is Fréchet differentiable at x with the derivative $\nabla F(x)$.

Recall that a mapping $F\colon X \rightrightarrows Y$ is *partially sequentially normally compact* (PSNC) with respect to Y at $(x, y) \in \operatorname{gph} F$ if for any sequences of quadruples $(x_k, y_k, x_k^*, y_k^*) \in X \times Y \times X^* \times Y^*$ as $k \to \infty$ we have the implication:

$$\Big[x_k^* \in \widehat{D}^* F(x_k, y_k)(y_k^*),\ (x_k, y_k) \to (x, y),\ y_k^* \xrightarrow{w^*} 0,\ \|x_k^*\| \to 0\Big] \Longrightarrow \|y_k^*\| \to 0, \tag{2.4}$$

where $\xrightarrow{w^*}$ signifies the convergence in the weak* topology. The PSNC property (2.4) obviously holds when Y is finite-dimensional. It follows from [30, Theorem 1.49 and Proposition 1.68] that (2.4) also holds when F is metrically regular around (x, y); the reader can find more results on this property and its applications in both volumes of the book [30] and the references therein.

Most of the results in this paper require the Asplund structure of the spaces in question. Recall that a Banach space X is *Asplund* if each of its separable subspaces has a separable dual. There are many equivalent descriptions and specifications of these spaces, which can be found, e.g., in [8, 30] and their bibliographies. Note to this end that any space with Fréchet smooth renorming (and hence any reflexive space) is Asplund. Constructions (2.1)–(2.3) and their limiting counterparts enjoy particularly good properties in Asplund spaces; see [30] for the full account and references.

In this paper we are going to use the following two lemmas concerning subdifferential sum and chain rules. The first one can be found in [30, Proposition 1.107 and Theorem 2.33]; see also [20] for more discussions.

Lemma 2.1 (subdifferential sum rules). *Let X be a Banach space, and let $f_2\colon X \to \overline{I\!R}$ be finite at x. The following assertions hold:*

(i) *Assume that $f_1\colon X \to \overline{I\!R}$ is Fréchet differentiable at x. Then*

$$\widehat{\partial}(f_1 + f_2)(x) = \nabla f_1(x) + \widehat{\partial} f_2(x).$$

(ii) *Assume that X is Asplund, that f_1 is locally Lipschitzian around $x \in (\operatorname{dom} f_1) \cap (\operatorname{dom} f_2)$, and that f_2 is lower semicontinuous (l.s.c.) around this point. Then for any $\epsilon > 0$ there are $x_1, x_2 \in B_X(x, \epsilon)$ with $|f_i(x_i) - f_i(x)| < \epsilon$ as $i = 1, 2$ such that*

$$\widehat{\partial}(f_1 + f_2)(x) \subset \widehat{\partial} f_1(x_1) + \widehat{\partial} f_2(x_2) + \epsilon \mathbb{B}_{X^*}.$$

The next lemma is a simple consequence of the chain rule from [30, Theorem 1.66]; see also [31, Lemma 1] for the explicit formulation.

Lemma 2.2 (subdifferential chain rule for power compositions). *Let $f\colon X \to \overline{I\!R}$ be an extended-real-valued lower semicontinuous function on a Banach space, and let $x \in \operatorname{dom} f$ be such that $f(x) > 0$. Consider the power composition $g(x) := f(x)^\gamma$ with some $\gamma > 0$. Then we have*

$$\widehat{\partial} g(x) = \gamma f(x)^{\gamma - 1} \widehat{\partial} f(x).$$

3 Sufficient Coderivative Conditions for q-Subregularity

Let us start this section with the basic definition of positive-order metric subregularity for arbitrary set-valued mappings between Banach spaces.

Definition 3.1 (**metric q-subregularity**). *Let $F\colon X \rightrightarrows Y$, and let $(\bar{x},\bar{y}) \in \operatorname{gph} F$. Given $q>0$, we say that F is* Hölder metrically subregular *at $(\bar{x},\bar{y})$ of order q, or* metrically q-subregular *at this point, if there are constants $c,\delta>0$ such that*

$$d\big(x;F^{-1}(\overline{y})\big) \le c\, d\big(\overline{y};F(x)\big)^q \quad \text{for all} \quad x \in B_X(\overline{x},\delta). \tag{3.1}$$

The exact q-subregularity bound/modulus *of F at $(\bar{x},\bar{y})$ is defined by*

$$\operatorname{subreg}^q F(\overline{x},\overline{y}) := \inf\big\{c>0\,\big|\, \exists\,\delta>0 \;\; s.t. \;\; d\big(x;F^{-1}(\overline{y})\big) \le c\, d\big(\overline{y};F(x)\big)^q \;\; \text{for all} \;\; x\in B_X(\overline{x},\delta)\big\}.$$

When $q=1$ in Definition 3.1, it reduces to *metric subregularity* (1.2) of the mapping F at $(\bar{x},\bar{y})$ with omitting the index q in the notation. Note that, although the metric q-regularity is defined for any positive order q, the main results and applications of this paper hold for the case of $q\in(0,1]$.

Recall that the *duality mapping* $J\colon Y \rightrightarrows Y^*$ for a Banach space Y is defined by

$$J(y) = \big\{y^* \in Y^* \big|\, \|y^*\| = 1 \;\; \text{with} \;\; \langle y^*,y\rangle = \|y\|\big\}.$$

In particular, in Hilbert spaces we have $J(y) = \big\{\frac{y}{\|y\|}\big\}$ if $y \ne 0$. To study q-subregularity, the following extensions of the duality mapping are needed.

Definition 3.2 (**q-duality mapping and its normalized enlargements**). *Given $q>0$, we define the* q-duality mapping *$J^q\colon Y \rightrightarrows Y^*$ by*

$$J^q(y) := \big\{q\|y\|^{q-1}y^* \big|\, y^*\in J(y)\big\} \quad \text{for all} \quad y \ne 0.$$

Given further $\epsilon \ge 0$, the normalized ϵ-enlargement *of the q-duality mapping is*

$$J^q_\epsilon(y) := \Big\{\frac{y^*+\epsilon u^*}{\|y^*+\epsilon u^*\|} \in \mathbb{S}_{Y^*}\,\Big|\, y^*\in J^q(y),\; \|u^*\|\le 1,\; y^*+\epsilon u^* \ne 0\Big\}, \quad y\ne 0. \tag{3.2}$$

We obviously have that $J^1_0(y) = J^1(y) = J(y)$ for all $y\in Y$.

In this section we derive general sufficient conditions for metric q-subregularity of set-valued mappings between Asplund spaces with modulus estimates via coderivatives. The terminology of [30] distinguishes between *neighborhood* and *pointbased* conditions. The former results involve calculating coderivatives not just at the point in question but also at those from its neighborhood, while the latter ones deal only with the reference point.

We begin with neighborhood sufficient conditions for q-subregularity of general mappings and modulus estimating in terms of the regular coderivative (2.3).

Theorem 3.3 (**neighborhood sufficient conditions for q-subregularity with upper modulus estimate in Asplund spaces**). *Let $F: X \rightrightarrows Y$ be a closed-graph multifunction between Asplund spaces with $(\overline{x},\overline{y}) \in \operatorname{gph} F$, and let $q\in(0,1]$. Define the constant*

$$\begin{aligned}\alpha := \sup_{\epsilon>0}\inf\Big\{q\|x^*\|\cdot\|y'-\overline{y}\|^{q-1}\,\Big|\; & x\in B_X(\overline{x},\epsilon)\backslash F^{-1}(\overline{y}),\; y\in F(x)\cap B_Y\big(\overline{y},\min\{\epsilon,\|x-\overline{x}\|^{\frac{1}{2}}\}\big),\\ & y'\in B_Y(y,\|x-\overline{x}\|^{\frac{1}{q}})\setminus\{\overline{y}\},\; x^*\in \widehat{D}^*F(x,y)\big(J^q_\epsilon(y'-\overline{y})\big)\Big\} \ge 0.\end{aligned}$$

Then the condition $\alpha>0$ is sufficient for metric q-subregularity of F at $(\bar{x},\bar{y})$. Furthermore, in this case we have the upper modulus estimate

$$\operatorname{subreg}^q F(\overline{x},\overline{y}) \le \alpha^{-1}, \tag{3.3}$$

and thus there is $\epsilon_0>0$ such that $d\big(x;F^{-1}(\overline{y})\big) \le \frac{2}{\alpha} d\big(\overline{y};F(x)\big)^q$ for all $x\in B_X(\overline{x},\epsilon_0)$.

Proof. It is clear that estimate (3.3) with $\alpha > 0$ ensures all the statements of the theorem. To justify this, assume the contrary and thus find $\tau > 0$ with $\text{subreg}^q F(\overline{x}, \overline{y}) > \tau > \alpha^{-1}$. Let $r = \tau^{-1}$. Then we have $0 < r < \alpha$. Since $\text{subreg}^q F(\overline{x}, \overline{y}) > \tau = r^{-1}$, there are sequences x_k with $\|x_k - \overline{x}\| \to 0$, and $y_k \in F(x_k)$ such that

$$d\big(\overline{y}; F(x_k)\big)^q \leq \|y_k - \overline{y}\|^q < r d\big(x_k; F^{-1}(\overline{y})\big). \tag{3.4}$$

Denote $\epsilon_k := \|x_k - \overline{x}\| \to 0$ and $\mu_k := d(x_k; F^{-1}(\overline{y}))$, which gives $0 < \mu_k \leq \epsilon_k \to 0$, $x_k \notin F^{-1}(\overline{y})$, and $\|y_k - \overline{y}\| < r\epsilon_k^{\frac{1}{q}} \to 0$. Without loss of generality we assume in what follows that $\epsilon_k < 1$ for all $k \in \mathbb{N}$. Consider the function $g : X \times Y \to \overline{I\!R}$ defined by

$$g(u, v) := \|v - \overline{y}\|^q + \delta_{\text{gph}F}(u, v). \tag{3.5}$$

It is easy to see that g is a proper l.s.c. function on $X \times Y$ with $\inf_{(u,v)\in X\times Y} g(u, v) = 0$. Hence we have the equalities

$$\begin{aligned} g(x_k, y_k) &= \inf_{X\times Y} g + \frac{g(x_k, y_k)}{(1 - \sqrt{\epsilon_k})d\big(x_k; F^{-1}(\overline{y})\big)}(1 - \sqrt{\epsilon_k})d\big(x_k; F^{-1}(\overline{y})\big) \\ &= \inf_{X\times Y} g + \rho_k(1 - \sqrt{\epsilon_k})d(x_k; F^{-1}(\overline{y})), \quad \text{where} \quad \rho_k := \frac{g(x_k, y_k)}{(1 - \sqrt{\epsilon_k})d\big(x_k; F^{-1}(\overline{y})\big)}. \end{aligned}$$

It follows from (3.4) that $\limsup_{k\to\infty} \rho_k \leq r$. Having the sequence ϵ_k from above, for each fixed $k \in \mathbb{N}$ equip the product space $X \times Y$ with the norm

$$\|(x, y)\|_{\epsilon_k} := \|x\| + \epsilon_k\|y\|$$

and observe that $(X \times Y, \|(\cdot, \cdot)\|_{\epsilon_k})$ is a Banach space. The classical Ekeland variational principle (see, e.g., [30, Theorem 2.26]) allows us to find $(a_k, b_k) \in X \times Y$ with $\|(a_k, b_k) - (x_k, y_k)\|_{\epsilon_k} \leq (1-\sqrt{\epsilon_k})d(x_k; F^{-1}(\overline{y}))$, $g(a_k, b_k) \leq g(x_k, y_k)$, and therefore $\|b_k - \overline{y}\| \leq \|y_k - \overline{y}\|$ such that the function

$$\phi_k(u, v) := g(u, v) + \frac{g(x_k, y_k)}{(1 - \sqrt{\epsilon_k})d\big(x_k; F^{-1}(\overline{y})\big)}\|(u, v) - (a_k, b_k)\|_{\epsilon_k} \tag{3.6}$$

attains its minimum at (a_k, b_k). Taking into account the definition of the function $g(u, v)$ in (3.5) and then applying the generalized Fermat rule to (3.6), we have that $(a_k, b_k) \in \text{gph}\, F$, that $\|a_k - x_k\| \leq (1 - \sqrt{\epsilon_k})d(x_k; F^{-1}(\overline{y}))$ (hence $a_k \notin F^{-1}(\overline{y})$), and that

$$0 \in \widehat{\partial}\phi_k(a_k, b_k) = \widehat{\partial}\big(\psi_{1k} + \psi_{2k} + \delta_{\text{gph}F}\big)(a_k, b_k),$$

where $\psi_{1k}(u, v) := \|v - \overline{y}\|^q$ and $\psi_{2k}(u, v) = \rho_k\|(u, v) - (a_k, b_k)\|_{\epsilon_k}$. Since $a_k \notin F^{-1}(\overline{y})$ and $b_k \in F(a_k)$, it follows that $b_k \neq \overline{y}$. Define further the sequence

$$\eta_k := \min\left\{\frac{\epsilon_k^2}{2}, \frac{\epsilon_k\|b_k - \overline{y}\|}{4}, 2^{-\frac{q+1}{q}}\epsilon_k^{1+\frac{2}{q}}\right\} \downarrow 0 \ \text{ as } \ k \to \infty. \tag{3.7}$$

Employing the fuzzy sum rule from Lemma 2.1(ii) to the latter inclusion and taking into account that the functions ψ_{1k} and ψ_{2k} are Lipschitz continuous near (a_k, b_k), we find $(x_{ik}, y_{ik}) \in X \times Y$ as $i = 1, 2, 3$ with $\|(x_{ik}, y_{ik}) - (a_k, b_k)\|_{\epsilon_k} < \eta_k$ such that

$$0 \in \widehat{\partial}\psi_{1k}(x_{1k}, y_{1k}) + \widehat{\partial}\psi_{2k}(x_{2k}, y_{2k}) + \widehat{N}\big((x_{3k}, y_{3k}); \text{gph}\, F\big) + \eta_k(\mathbb{B}_{X^*} \times \mathbb{B}_{Y^*}), \tag{3.8}$$

which implies by the choice of η_k the estimates

$$\|x_{ik}-a_k\| < \frac{\epsilon_k^2}{2} \text{ and } \|y_{ik}-b_k\| < 2^{-\frac{q+1}{q}}\epsilon_k^{\frac{2}{q}}, \quad i=1,2,3.$$

It yields therefore the estimates and convergence

$$\|x_{ik}-\overline{x}\| \leq \|x_{ik}-a_k\| + \|a_k-x_k\| + \|x_k-\overline{x}\| < \epsilon_k^2 + (1-\sqrt{\epsilon_k})\mu_k + \epsilon_k \downarrow 0 \tag{3.9}$$

as $k\to\infty$ for $i=1,2,3$, and thus we get

$$\|y_{ik}-\overline{y}\| \leq \|y_{ik}-b_k\| + \|b_k-\overline{y}\| \leq \|y_{ik}-b_k\| + \|y_k-\overline{y}\| < 2r\epsilon_k^{\frac{1}{q}} \to 0 \tag{3.10}$$

by taking into account that $\|b_k-\overline{y}\| \leq \|y_k-\overline{y}\|$ and $\|y_k-\overline{y}\| \leq r\epsilon_k^{\frac{1}{q}}$ for all large k. In turn it follows from (3.10) that, for all large k,

$$\begin{aligned}
\|x_{ik}-\overline{x}\| &\geq \|x_k-\overline{x}\| - (\|a_k-x_{ik}\| + \|x_k-a_k\|) \\
&\geq \epsilon_k - \Big(\frac{\epsilon_k^2}{2} + (1-\sqrt{\epsilon_k})\mu_k\Big) \\
&\geq \epsilon_k - \Big(\frac{\epsilon_k^2}{2} + (1-\sqrt{\epsilon_k})\epsilon_k\Big) \\
&\geq \frac{\epsilon_k^{\frac{3}{2}}}{2} > (2r\epsilon_k^{\frac{1}{q}})^2 \geq \|y_{ik}-\overline{y}\|^2,
\end{aligned}$$

which allows us to arrive at the estimate

$$\|y_{ik}-\overline{y}\| < \|x_{ik}-\overline{x}\|^{\frac{1}{2}}, \quad i=1,2,3. \tag{3.11}$$

Moreover, as $\|y_{ik}-b_k\| < 2^{-\frac{q+1}{q}}\epsilon_k^{\frac{2}{q}}$ and $\|x_{ik}-\overline{x}\| \geq \frac{\epsilon_k^{\frac{3}{2}}}{2} \geq \frac{\epsilon_k^2}{2}$, $i=1,2,3$, we have

$$\|y_{3k}-y_{1k}\| \leq 2\big(2^{-\frac{q+1}{q}}\epsilon_k^{\frac{2}{q}}\big) = 2^{-\frac{1}{q}}\epsilon_k^{\frac{2}{q}} \leq \|x_{3k}-\overline{x}\|^{\frac{1}{q}}. \tag{3.12}$$

Since $\|(x_{ik},y_{ik})-(a_k,b_k)\|_{\epsilon_k} < \eta_k$, we have that $\epsilon_k\|y_{1k}-b_k\| \leq \eta_k \leq \frac{\epsilon_k\|b_k-\overline{y}\|}{4}$ and hence $\|y_{1k}-b_k\| \leq \frac{\|b_k-\overline{y}\|}{4}$. This together with $b_k\neq\overline{y}$ implies that $y_{1k}\neq\overline{y}$ for all $k\in\mathbb{N}$. Thus it follows from (3.8) by applying Lemma 2.2 to the function ψ_{1k} and elementary convex analysis that there are dual elements $y_{1k}^*\in q\|y_{1k}-\overline{y}\|^{q-1}J(y_{1k}-\overline{y})$, $x_{2k}^*\in\mathbb{B}_{X^*}$, $y_{2k}^*\in\mathbb{B}_{Y^*}$, $(x_{3k}^*,-y_{3k}^*)\in\widehat{N}((x_{3k},y_{3k});\operatorname{gph}F)$, and $(x_{4k}^*,y_{4k}^*)\in\mathbb{B}_{X^*}\times\mathbb{B}_{Y^*}$ such that

$$(0,0) = (0,y_{1k}^*) + \rho_k(x_{2k}^*,\epsilon_k y_{2k}^*) + (x_{3k}^*,-y_{3k}^*) + \eta_k(x_{4k}^*,y_{4k}^*).$$

Using now definition (2.3) of the regular coderivative gives us

$$-\rho_k x_{2k}^* - \eta_k x_{4k}^* = x_{3k}^* \in \widehat{D}^*F(x_{3k},y_{3k})(y_{3k}^*) = \widehat{D}^*F(x_{3k},y_{3k})(y_{1k}^*+\rho_k\epsilon_k y_{2k}^*+\eta_k y_{4k}^*).$$

Since $\|y_{1k}^*\| = q\|y_{1k}-\overline{y}\|^{q-1}$ with $\|y_{1k}-\overline{y}\|\to 0$ and $q\in(0,1]$, we may assume with no loss of generality that $\|y_{1k}^*\|\geq 1$ for all $k\in\mathbb{N}$. Furthermore, we get the estimates $\|\rho_k\epsilon_k y_{2k}^*+\eta_k y_{4k}^*\| \leq \rho_k\epsilon_k+\eta_k\downarrow 0$ as $k\to\infty$, and thus

$$\|y_{1k}^*+\rho_k\epsilon_k y_{2k}^*+\eta_k y_{4k}^*\| \geq \frac{1}{2} \text{ for all large } k\in\mathbb{N}. \tag{3.13}$$

This implies, by taking into account definition (3.2) of the normalized ϵ-enlargement, that

$$\begin{aligned} x_k^* := \frac{-\rho_k x_{2k}^* - \eta_k x_{4k}^*}{\|y_{1k}^* + \rho_k \epsilon_k y_{2k}^* + \eta_k y_{4k}^*\|} &\in \widehat{D}^* F(x_{3k}, y_{3k})\Big(\frac{y_{1k}^* + \rho_k \epsilon_k y_{2k}^* + \eta_k y_{4k}^*}{\|y_{1k}^* + \rho_k \epsilon_k y_{2k}^* + \eta_k y_{4k}^*\|}\Big) \\ &\subset \widehat{D}^* F(x_{3k}, y_{3k})\big(J^q_{(r+1)\epsilon_k}(y_{1k} - \overline{y})\big). \end{aligned} \tag{3.14}$$

Note that the sequence of $\|\rho_k x_{2k}^* + \eta_k x_{4k}^*\|$ is bounded, and thus (3.13) implies that the sequence of $\|x_k^*\|$ is also bounded. Hence we have

$$\begin{aligned} \liminf_{k\to\infty} \big(q\,\|x_k^*\| \cdot \|y_{1k} - \overline{y}\|^{q-1}\big) &= \liminf_{k\to\infty} \big(\|x_k^*\| \cdot \|y_{1k}^*\|\big) \\ &= \liminf_{k\to\infty} \big(\|x_k^*\| \cdot \|y_{1k}^* + \rho_k \epsilon_k y_{2k}^* + \eta_k y_{4k}^*\|\big) \\ &= \liminf_{k\to\infty} \|\rho_k x_{2k}^* + \eta_k x_{4k}^*\| \le r, \end{aligned}$$

where the last inequality follows by $\limsup_{k\to\infty} \rho_k \le r$, $x_{2k}^* \in \mathbb{B}_{X^*}$, $x_{4k}^* \in \mathbb{B}_{X^*}$ and $\eta_k \to 0$. Combining now conclusions (3.9)–(3.14) and passing to the limit as $k \to \infty$ give us $\alpha \le r$. This contradicts the assumption of $r < \alpha$ and thus completes the proof of the theorem. $\square$

Remark 3.4. (calculation/estimate of the constant α for differentiable single-valued mappings). If $f : X \to Y$ is a Fréchet differentiable, then $\widehat{D}^* f(x, f(x))(y^*) = \{\nabla f(x)^* y^*\}$ for all $x \in X$ and $y^* \in Y^*$ via the adjoint derivative operator $\nabla f(x)^*$. Thus in this case the constant α from Theorem 3.3 can be calculated and estimated as follows:

$$\begin{aligned} \alpha \;&= \sup_{\epsilon>0} \inf \Big\{ q\|\nabla f(x)^* y^*\| \cdot \|y' - f(\overline{x})\|^{q-1} \Big| x \in B_X(\overline{x}, \epsilon), 0 < \|f(x) - f(\overline{x})\| \le \min\{\epsilon, \|x - \overline{x}\|^{\frac{1}{2}}\}, \\ &\qquad \|y' - f(x)\| \le \|x - \overline{x}\|^{\frac{1}{q}}, y' \ne f(\overline{x}), y^* \in J^q_\epsilon\big(y' - f(\overline{x})\big) \Big\} \\ &\ge \liminf_{x\to\overline{x}} \Big\{ q\|\nabla f(x)^* y^*\| \cdot \|y' - f(\overline{x})\|^{q-1} \Big| \; 0 < \|f(x) - f(\overline{x})\| \le \|x - \overline{x}\|^{\frac{1}{2}}, \\ &\qquad \|y' - f(x)\| \le \|x - \overline{x}\|^{\frac{1}{q}},\; y' \ne f(\overline{x}), \|y^*\| = 1 \Big\} \\ &= \liminf_{x\to\overline{x}} \Big\{ q\|\nabla f(x)^* y^*\| \cdot \|y' - f(\overline{x})\|^{q-1} \Big| \|y' - f(x)\| \le \|x - \overline{x}\|^{\frac{1}{q}},\; y' \ne f(\overline{x}),\; \|y^*\| = 1 \Big\}. \end{aligned}$$

Our next goal is to derive *pointbased* sufficient conditions for metric q-regularity of F at $(\bar{x}, \bar{y})$ by passing to the limit from those in Theorem 3.3. To accomplish this, we introduce new outer limiting coderivatives for set-valued mappings, which agree with each other in finite-dimensional spaces.

Definition 3.5 (outer mixed and reverse outer mixed q-coderivatives). *Let $F\colon X \rightrightarrows Y$, and let $q > 0$. The* OUTER MIXED q-CODERIVATIVE *of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is a mapping $D^{*>}_{M,q} F(\bar{x}, \bar{y})\colon Y^* \rightrightarrows X^*$ defined as follows: $x^* \in D^{*>}_{M,q} F(\bar{x}, \bar{y})(y^*)$ for any given $y^* \in Y^*$ means that there are sequences $(x_k, y_k, v_k, x_k^*, y_k^*) \in X \times Y \times Y \times X^* \times Y^*$ such that*

$$\begin{aligned} &x_k \to \overline{x} \text{ with } x_k \notin F^{-1}(\overline{y}),\; y_k \in F(x_k) \text{ with } y_k \to \overline{y} \text{ and } \|y_k - \overline{y}\| \le \|x_k - \overline{x}\|^{\frac{1}{2}}, \\ &\|v_k - y_k\| \le \|x_k - \bar{x}\|^{\frac{1}{q}} \text{ with } v_k \ne \bar{y}, x_k^* \in \widehat{D}^* F(x_k, y_k)(y_k^*) \text{ with } \|y_k^*\| = 1,\; y_k^* \to y^*, \text{ and} \\ &\lambda_k x_k^* \xrightarrow{w^*} x^* \text{ with } \lambda_k := q\|v_k - \overline{y}\|^{q-1} \text{ as } k \to \infty. \end{aligned}$$

The REVERSED OUTER MIXED q-CODERIVATIVE $\widetilde{D}^{*>}_{M,q} F(\bar{x}, \bar{y})$ *of F at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ is the reversed version of the outer mixed q-coderivative with the strong convergence on X^* and* *weak* convergence on* Y^*, *i.e.,* $x^* \in \widetilde{D}^{*>}_{M,q}F(\overline{x},\overline{y})(y^*)$ *for any given* $y^* \in Y^*$ *means that there are sequences* $(x_k, y_k, v_k, x_k^*, y_k^*) \in X \times Y \times Y \times X^* \times Y^*$ *such that*

$$x_k \to \overline{x} \text{ with } x_k \notin F^{-1}(\overline{y}),\ y_k \in F(x_k) \text{ with } y_k \to \overline{y} \text{ and } \|y_k - \overline{y}\| \le \|x_k - \overline{x}\|^{\frac{1}{2}},$$
$$\|v_k - y_k\| \le \|x_k - \bar{x}\|^{\frac{1}{q}} \text{ with } v_k \ne \bar{y}, x_k^* \in \widehat{D}^*F(x_k, y_k)(y_k^*) \text{ with } \|y_k^*\| = 1,\ y_k^* \xrightarrow{w^*} y^*, \text{ and}$$
$$\lambda_k x_k^* \to x^* \text{ with } \lambda_k := q\|v_k - \bar{y}\|^{q-1} \text{ as } k \to \infty.$$

If $q = 1$, *we use the notation* $D^{*>}_M F(\bar{x}, \bar{y})$ *and* $\widetilde{D}^{*>}_M F(\bar{x}, \bar{y})$ *for the corresponding coderivatives. We drop the symbol "M" and the word "mixed" in the coderivative (resp. reversed coderivative) notation and terminology when* $\dim X < \infty$ *(resp.* $\dim Y < \infty$*).*

Some remarks on limiting coderivatives are in order.

Remark 3.6. (discussions on coderivatives). (i) If the conditions $x_k \notin F^{-1}(\overline{y})$ and

$$\|y_k - \overline{y}\| \le \|x_k - \overline{x}\|^{\frac{1}{2}} \tag{3.15}$$

are dropped in Definition 3.5 and if $q = 1$ (hence $\lambda_k \equiv 1$) therein, the coderivative constructions introduced go back to the mixed coderivative $D^*_M F(\bar{x}, \bar{y})$ and the reversed mixed coderivative $\widetilde{D}^*_M F(\bar{x}, \bar{y})$ of [30]; see there more details and references.* If both spaces X and Y are finite-dimensional, $q = 1$, and condition (3.15) is dropped in Definition 3.5, we get the outer coderivative $D^*_{>} F(\bar{x}, \bar{y})$ of [21] whose values are larger than $D^{*>}F(\bar{x}, \bar{y})$ due to our additional requirement (3.15); see [37] for an infinite-dimensional extension of $D^*_{>} F(\bar{x}, \bar{y})$. Note that requirement (3.15) is essential in what follows. In particular, it is crucial for proving the necessity of the coderivative condition for metric subregularity established below in Theorem 4.4.

(ii) A characteristic feature of outer coderivatives is the additional requirement on $x_k \notin F^{-1}(\overline{y})$. Adding this requirement not only gives sharper sufficient conditions and modulus estimates for metric q-subregularity but also enables us to completely characterize metric q-subregularity for convex multifunctions; see Theorem 5.3 below. However, the required information on the set $F^{-1}(\overline{y})$ makes it more challenging to develop comprehensive calculus rules for outer coderivatives. On the other hand, the imposed additional requirement can be very useful if we have some partial information about the aforementioned set. In particular, consider the case of $F^{-1}(\overline{y}) = C \cup D$, where the set C can be determined rather easily. This often occurs, e.g., for optimization problems with disjunctive constraints and/or complementarity constraints of the type

$$0 \le Mx \perp G(x) \ge 0,$$

where M is an $n \times n$ matrix, and where $G\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 mapping. By setting $C := \{x|\ Mx = 0\}$, the additional requirement implies that $x_k \notin C$. Thus it can be used to eliminate some unnecessary information and to sharpen the corresponding results in terms of outer coderivatives.

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*Note that Definition 3.5 is more appropriate in the Asplund spaces setting used for the main results of the paper. In the case of general Banach spaces we have to involve ε-enlargements $\widehat{D}^*_\varepsilon F(x, y)$ of $\widehat{D}^*F(x, y)$ in the limiting procedure; cf. [30]. Similarly to [30] the "normal" counterpart $D^{*>}_{N,q}F(\bar{x}, \bar{y})$ of $D^{*>}_{M,q}F(\bar{x}, \bar{y})$ can be defined, which is not needed here.

(iii) It follows from the proof of Theorem 3.3 the condition $\alpha > 0$ is still *sufficient* for metric q-subregularity of F at $(\bar{x},\bar{y})$ if the requirement $y \in F(x) \cap B_Y\big(\overline{y}, \min\{\epsilon, \|x-\overline{x}\|^{\frac{1}{2}}\}\big)$ in the definition of this constant is replaced by

$$y \in F(x) \cap B_Y\big(\overline{y}, \epsilon\|x-\overline{x}\|^{\frac{1}{q}}\big). \tag{3.16}$$

However, such a modification does *not* allow us to get the q-subregularity *modulus estimate* (3.3) and the subsequent distance estimate stated in the theorem. Taking modification (3.16) into account, we can further modify Definition 3.5 of the *outer mixed* and *reversed outer mixed $q-$coderivatives* by replacing the condition $\|y_k - \bar{y}\| \le \|x_k - \bar{x}\|^{\frac{1}{2}}$ therein by

$$\frac{\|y_k - \overline{y}\|}{\|x_k - \overline{x}\|^{\frac{1}{q}}} \to 0 \ \text{ as } \ k \to \infty. \tag{3.17}$$

In this case the modified condition (3.15) reads as

$$\|y_k - \overline{y}\| \le \epsilon\|x_k - \overline{x}\|^{\frac{1}{q}}. \tag{3.18}$$

It follows from the proof of Theorem 3.7 below that modification (3.17) of the corresponding limiting q-coderivative allows us to keep the pointwise sufficient condition (3.19) for metric q-subregularity of F at $(\bar{x},\bar{y})$ whenever $q \in (0,1]$. Furthermore, this modification leads to strengthening the “almost necessity” statement for metric subregularity ($q = 1$) in Theorem 4.4; see Remark 4.5 for more discussions.

The next theorem provides a pointbased sufficient condition for metric q-subregularity of multifunctions in terms of the reverse outer mixed q-coderivative from Definition 3.5.

Theorem 3.7 (pointbased sufficient conditions for metric q-subregularity in Asplund spaces). *Let $F\colon X \rightrightarrows Y$ be a closed-graph multifunction between Asplund spaces, let $(\overline{x},\overline{y}) \in \operatorname{gph} F$, and let $q \in (0,1]$. Assume that F is PSNC at $(\bar{x},\bar{y})$ with respect to Y and that the coderivative condition*

$$\ker \widetilde{D}^{*>}_{M,q}F(\overline{x},\overline{y}) = \{0\} \tag{3.19}$$

holds. Then the mapping F is metrically q-subregular at $(\overline{x},\overline{y})$.

Proof. Employing Theorem 3.3, it suffices to show our assumptions ensure that $\alpha > 0$ for the constant α defined therein. To proceed, suppose on the contrary that $\alpha = 0$. Then there are sequences $(x_k, y_k, v_k, x_k^*, y_k^*)$ satisfying $x_k \to \overline{x}$ with $x_k \notin F^{-1}(\overline{y})$, $y_k \to \overline{y}$ with $\|y_k - \overline{y}\| \le \|x_k - \overline{x}\|^{\frac{1}{2}}$, $x_k^* \in \widehat{D}^*F(x_k,y_k)(y_k^*)$, $y_k^* \in \mathbb{S}_{Y^*}$, $v_k \to \overline{y}$ with $v_k \ne \bar{y}$, and

$$q\|x_k^*\| \cdot \|v_k - \overline{y}\|^{q-1} \to 0 \ \text{ as } \ k \to \infty. \tag{3.20}$$

Since the space Y is Asplund, any bounded subset of its dual Y^* is weak* sequentially compact. By passing to a subsequence if needed, we find $y^* \in Y^*$ such that $y_k^* \xrightarrow{w^*} y^*$ as $k \to \infty$. Furthermore, it follows from (3.20) with $q \in (0,1]$ and $v_k \ne \bar{y}$ that $\|x_k^*\| \to 0$. Then we observe by the reversed outer mixed coderivative construction of Definition 3.5 that the coderivative condition (3.19) ensures that $y^* = 0$. By the assumed PSNC property of F we have that $\|y_k^*\| \to 0$, which contradicts the fact that $y_k^* \in \mathbb{S}_{Y^*}$ for all $k \in \mathbb{N}$. This allows us to conclude that F is metrically q-subregular at $(\bar{x},\bar{y})$. $\square$

Note that, in contrast to the conventional setting of [30], we cannot equivalently rewrite the coderivative condition (3.19) in the form $D^{*>}_{M,q}F(\bar{y},\bar{x})^{-1}(0)=\{0\}$, since the set $D^{*>}_{M,q}F(\bar{y},\bar{x})^{-1}(0)$ is generally different from $\ker\widetilde{D}^{*>}_{M,q}F(\overline{x},\overline{y})$ due to the additional requirements $x_k\notin F^{-1}(\overline{y})$ and (3.15) in the outer coderivative constructions of Definition 3.5.

To conclude this section, we present a simple example illustrating the application of Theorem 3.7 to identify metric q-subregularity with $q\in(0,1)$.

Example 3.8. (identifying q-subregularity with $q\neq 1$ via outer coderivatives). Define $F\colon\mathbb{R}\to\mathbb{R}$ by $F(x):=\{\max\{x,0\}^2\}$ as $x\in\mathbb{R}$ and show that F is metrically $\frac{1}{2}$-subregular at $(0,0)$. By Theorem 3.7 we need to verify that $\ker\widetilde{D}^{*>}_{M,\frac{1}{2}}F(0,0)=\{0\}$. To proceed, take $y^*\in\ker\widetilde{D}^{*>}_{M,\frac{1}{2}}F(0,0)$ and, employing Definition 3.5 find sequences $(x_k,y_k,v_k,x^*_k,y^*_k)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ satisfying the conditions

$$\begin{aligned}&x_k\to 0 \text{ with } x_k>0,\ y_k=x_k^2,\ \|v_k-y_k\|\le\|x_k\|^2 \text{ with } v_k\neq 0,\ x^*_k=2x_ky^*_k \text{ with } y^*_k\to y^*,\\ &\text{and}\ \ \lambda_kx^*_k\to 0\ \ \text{with}\ \ \lambda_k:=\tfrac{1}{2}\|v_k\|^{-\frac{1}{2}} \text{ as } k\to\infty.\end{aligned}$$

It follows from the above that

$$\lambda_k\ge\frac{1}{2}\Big(2\|x_k\|^2\Big)^{-\frac{1}{2}}\ge\frac{1}{4}\|x_k\|^{-1},\quad k\in\mathbb{N},$$

which implies together with $\lambda_k(2x_ky^*_k)=\lambda_kx^*_k\to 0$ that $y^*_k\to 0$. Thus $y^*=0$, and we get therefore that $\ker\widetilde{D}^{*>}_{M,\frac{1}{2}}F(0,0)=\{0\}$. This verifies that F is metrically $\frac{1}{2}$-subregular at $(0,0)$ by Theorem 3.7.

4 Pointbased Characterizations of Metric Subregularity

This section is devoted to the study of *metric subregularity*, i.e., the special case of $q=1$ in the constructions above. In this case our results provide not only new sufficient conditions for metric regularity but also "almost necessary" ones in the sense defined below. First we present the following consequence of Theorem 3.7.

Corollary 4.1 (pointbased sufficient conditions for metric subregularity). *Let $F:X\rightrightarrows Y$ be a closed-graph multifunction between Asplund spaces, and let $(\overline{x},\overline{y})\in\operatorname{gph}F$. Assume that F is PSNC at $(\bar{x},\bar{y})$ with respect to Y and that*

$$\ker\widetilde{D}^{*>}_{M}F(\overline{x},\overline{y})=\{0\}.\tag{4.1}$$

Then the mapping F is metrically subregular at $(\overline{x},\overline{y})$.

A finite-dimensional counterpart of Corollary 4.1 with our coderivative $\widetilde{D}^{*>}=D^{*>}$ replaced by $D^*_>$ (see Remark 3.6) is obtained in [21] and then is further extended in [37] to the Asplund space setting by using the mixed coderivative version of $D^*_>$.

Another sufficient condition for metric subregularity has been recently obtained in [17] for set-valued mappings between Asplund spaces in the following *critical set* form:

$$(0,0)\notin\mathrm{Cr}_0F(\bar{x},\bar{y}),\tag{4.2}$$

where the set $\mathrm{Cr}_0(\bar{x},\bar{y})$ is defined as the collection of all pairs $(v,x^*)\in Y\times X^*$ such that there are sequences $t_k\downarrow 0$, $(u_k,y_k^*)\in\mathbb{S}_X\times\mathbb{S}_{Y^*}$, $(-x_k^*,y_k^*)\in\widehat{N}((\overline{x}+t_ku_k,\overline{y}+t_kv_k);\mathrm{gph}\,F)$ with $(v_k,x_k^*)\to(v,x^*)$ as $k\to\infty$. Let us present a simple example with $X=Y=I\!R$, where the critical set condition (4.2) fails while the outer coderivative condition (4.1) certifies the validity of metric subregularity.

Example 4.2 (outer coderivative versus critical set conditions for metric subregularity). Let $F:\mathbb{R}\rightrightarrows\mathbb{R}$ be given by $F(x):=[\max\{x,0\},\infty)$, and let $\bar{x}=\bar{y}=0$. Taking $t_k=\frac{1}{k}$, $u_k=-1$, and $v_k=0$ gives us

$$(0,-1)\in\widehat{N}\big((\overline{x}+t_ku_k,\overline{y}+t_kv_k);\mathrm{gph}\,F\big)\ \text{ and so }\ (0,0)\in\mathrm{Cr}_0(\bar{x},\bar{y}),$$

i.e., the critical set condition (4.2) fails. On the other hand, it is easy to verify that $F^{-1}(0)=(-\infty,0]$ and that for any $x>0$ we have

$$\widehat{N}\big((x,y);\mathrm{gph}\,F\big)=\begin{cases}\{(t,-t)|\ t\ge 0\} & \text{if } y=x,\\ \{(0,0)\} & \text{if } y>x.\end{cases}$$

Thus the outer coderivative condition $\ker\widetilde{D}^{*>}_{M}F(\overline{x},\overline{y})=\{0\}$ holds, which ensures the metric subregularity of F at $(\bar{x},\bar{y})$ by Corollary 4.1.

The next example shows that the coderivative condition (4.1) is not in general necessary for metric subregularity even in one-dimensional settings.

Example 4.3 (outer coderivative condition is not necessary for metric subregularity). Let $X=Y=\mathbb{R}$ and $(\overline{x},\overline{y})=0$. Consider the function

$$\phi(x):=\begin{cases}0 & \text{if } x\in(-\infty,0],\\ 2^{-2n} & \text{if } x\in[2^{-2n},2^{-2n+1}),\\ \frac{3}{2}x-2^{-2n+1} & \text{if } x\in[2^{-2n+1},2^{-2n+2}),\\ x & \text{if } x\in[1,\infty)\end{cases}$$

and define the set-valued mapping $F\colon I\!R\rightrightarrows I\!R$ by $F(x):=[\phi(x),\infty)$. It is clear that the function ϕ is continuous and thus the mapping F is of closed graph. Furthermore, F is metrically subregular at $(\overline{x},\overline{y})$ since

$$d\big(x;F^{-1}(0)\big)=\max\{x,0\},\ d\big((0;F(x)\big)=\phi(x),\ \text{ and }\ \phi(x)\ge\frac{x}{2}\ \text{ for all }\ x\ge 0.$$

On the other hand, it follows from the construction of ϕ that

$$\phi'(x)=0\ \text{ and }\ |\phi(x)|\le|x|^{\frac{1}{2}}\ \text{ whenever }\ x\in(2^{-2n},2^{-2n+1}).$$

This implies the inclusion

$$(0,1)\in\widehat{N}\big((x,\phi(x));\mathrm{gph}\,F\big)\ \text{ for all }\ x\in(2^{-2n},2^{-2n+1}),$$

which ensures that $-1\in\ker\widetilde{D}^{*>}_{M}F(\overline{x},\overline{y})$. Having so $\ker\widetilde{D}^{*>}_{M}F(\overline{x},\overline{y})\neq\{0\}$, we conclude that the outer coderivative condition (4.1) is not necessary for metric subregularity.

Having in hand Corollary 4.1 and Example 4.3, we can see the difference between coderivative characterizations of metric *subregularity* and metric *regularity.* Recall from [30, Theorem 4.18] that the simultaneous fulfilment of the PSNC property of $F\colon X \rightrightarrows Y$ at $(\bar{x}, \bar{y})$ and the (full) mixed coderivative condition

$$D^*_M F^{-1}(\overline{y}, \overline{x})(0) = \{0\} \ \big(\Longleftrightarrow \ker \widetilde{D}^*_M F(\bar{x}, \bar{y}) = \{0\}\big) \tag{4.3}$$

is *necessary and sufficient* for metric regularity of F around $(\bar{x}, \bar{y})$ in Asplund spaces, where $D^*_M F$ and $\widetilde{D}^*_M F$ are defined as in Remark 3.6. When both spaces X and Y are finite-dimensional, the above PSNC property is automatic, the index "M" and the tilde symbol are not needed in (4.3), and this condition reduces to the *coderivative criterion* of [28]: $\ker D^* F(\bar{x}, \bar{y}) = \{0\}$. The situation with characterizing the metric subregularity via the outer coderivative condition (4.1) is significantly different.

Analyzing Example 4.3 from this end, note that the metric subregularity of F therein can be *destroyed* by adding a smooth function with zero value at the point in question. Indeed, consider the function $f(x) = \gamma x^2$ for any positive constant γ with $f(0) = 0$. Forming now the multifunction $G := F + f$, we observe that this mapping is not metrically subregular at $(0, 0)$ since

$$d\big(x; G^{-1}(0)\big) = |x| \ \text{ and } \ d\big(0; G(x)\big) = \gamma x^2 \ \text{ for all } \ x < 0.$$

Thus Example 4.3 illustrates the possible high *instability* of the metric subregularity property, which may fail under small perturbations of the initial data. This is definitely an undesired phenomenon from both viewpoints of the theory and applications.

The next theorem shows that the refined outer coderivative condition (4.1) not only ensures metric subregularity but also makes the aforementioned instability impossible. Furthermore, this result can be treated as a manifestation that the coderivative condition (4.1) is *almost necessary* for metric subregularity meaning that its violation implies the failure of metric subregularity in the above very natural sense.

The proof of this theorem is technically involved and strongly relies on requirement (3.15) in the new coderivative construction of Definition 3.5 that is not imposed in the outer coderivatives of [21, 37]. Note that an "almost necessity" of the critical set condition (4.2) for metric subregularity with the additional condition of $\nabla f(\bar{x}) = 0$ is proved in [17], which largely inspires the proof below. Note also that (see the footnote in Remark 3.6) it is more appropriate for arbitrary Banach spaces that are not Asplund to modify the coderivative construction $\widetilde{D}^{*>}_M F(\bar{x}, \bar{y})$ of Definition 3.5 by involving therein the ε-coderivative enlargements in the limiting process. This in fact does not effect the proof; so we stay for simplicity with the limiting coderivative given in Definition 3.5 in the general Banach space setting of Theorem 4.4.

Theorem 4.4 (almost necessity of the pointbased coderivative condition for metric subregularity). *Let $F : X \rightrightarrows Y$ be a closed-graph multifunction between arbitrary Banach spaces, and let $(\overline{x}, \overline{y}) \in \operatorname{gph} F$. Then the violation of the coderivative condition* (4.1) *implies the existence of a continuous function $f : X \to Y$ such that $f(\overline{x}) = 0$ and the perturbed mapping $F + f$ is not metrically subregular at $(\overline{x}, \overline{y})$.*

Proof. Having $\ker \widetilde{D}^{*>}_M F(\overline{x}, \overline{y}) \neq \{0\}$ and taking into account Definition 3.5 for $q = 1$, we find sequences $(x_k, y_k, x^*_k, y^*_k) \in X \times Y \times X^* \times Y^*$ such that $x_k \to \overline{x}$ with $x_k \notin F^{-1}(\overline{y})$,

$y_k \in F(x_k)$ with $y_k \to \overline{y}$ and $\|y_k - \overline{y}\| \le \|x_k - \overline{x}\|^{\frac{1}{2}}$ such that $(y_k^*, x_k^*) \in \operatorname{gph} \widehat{D}^* F(x_k, y_k)$, $\|y_k^*\| = 1$, $y_k^* \xrightarrow{w^*} y^* \ne 0$, and $x_k^* \to 0$ as $k \to \infty$. Passing to subsequences, assume that

$$\|x_k - \overline{x}\| \le k^{-1},\ \|y_k - \overline{y}\| \le k^{-1},\ \|x_k^*\| \le (4k)^{-1} \text{ and } (x_k^*, -y_k^*) \in \widehat{N}\big((x_k, y_k); \operatorname{gph} F\big)$$

for all $k \in \mathbb{N}$. It follows from $x_k \notin F^{-1}(\overline{y})$ that $x_k \ne \overline{x}$. Letting $u_k := \frac{x_k - \overline{x}}{\|x_k - \overline{x}\|}$ and $t_k := \|x_k - \overline{x}\|$, we get $(x_k, y_k) = (\overline{x}, \overline{y}) + t_k(u_k, v_k)$ with $\|u_k\| = 1$, $0 < t_k \le k^{-1}$, and

$$t_k\|v_k\| = \|y_k - \overline{y}\| \le \|x_k - \overline{x}\|^{\frac{1}{2}} = t_k^{\frac{1}{2}}.$$

Passing to subsequences again ensures that $t_{k+1} \le \frac{t_k}{4}$. Since $(x_k^*, -y_k^*) \in \widehat{N}((x_k, y_k); \operatorname{gph} F)$, there are numbers $\rho_k \in (0, t_k^{\frac{3}{2}}(2k^2)^{-1})$ such that

$$\langle (x_k^*, -y_k^*), (x, y) - (x_k, y_k)\rangle \ge -\frac{1}{4k}\Big(\|x - x_k\| + \|y - y_k\|\Big) \text{ whenever } (x, y) \in B_{X\times Y}\big((x_k, y_k), \rho_k\big),$$

and thus we get the estimates

$$\langle y_k^*, y - y_k\rangle \le \langle x_k^*, x - x_k\rangle + \frac{1}{4k}\Big(\|x - x_k\| + \|y - y_k\|\Big) \le \frac{1}{2k}\Big(\|x - x_k\| + \|y - y_k\|\Big). \quad (4.4)$$

Further, for each $k \in \mathbb{N}$ pick $q_k^* \in \mathbb{S}_{X^*}$ and $p_{ki}^* \in \mathbb{S}_{X^*}$ as $i = 1, \ldots, k-1$ satisfying

$$\langle q_k^*, \overline{x} - x_k\rangle = \|x_k - \overline{x}\| = t_k \text{ and } \langle p_{ki}^*, x_i - x_k\rangle = \|x_i - x_k\|$$

and then define a nonnegative function ξ_k on X by

$$\xi_k(x) := 8t_k^{-2}\Big(\langle q_k^*, x - x_k\rangle^2 + \sum_{i=1}^{k-1} 16^{i-k}\langle p_{ki}^*, x - x_k\rangle^2\Big).$$

Choose $z_k \in \mathbb{S}_Y$ so that $\langle y_k^*, z_k\rangle \ge \frac{1}{2}\|y_k^*\|$ and, following the scheme in the proof of part 2 in [17, Theorem 3.2], construct $f : X \to Y$ by

$$f(x) := \sum_{k=1}^{\infty} -\max\big\{1 - \xi_k(x), 0\big\}^2\Big(t_k v_k + \frac{\rho_k}{\sqrt{k}} z_k\Big). \quad (4.5)$$

Our goal in what follows is to show that the mapping f constructed in (4.5) possesses all the properties claimed in the theorem. Let us proceed with this verification step by step.

(i) Verifying that f is well defined. Represent f in (4.5) as

$$f(x) = \sum_{k=1}^{\infty} f_k(x) \text{ with } f_k(x) := -\max\big\{1 - \xi_k(x), 0\big\}^2\Big(t_k v_k + \frac{\rho_k}{\sqrt{k}} z_k\Big). \quad (4.6)$$

It is easy to get the estimates

$$\|f_k(x)\| \le \Big\|t_k v_k + \frac{\rho_k}{\sqrt{k}} z_k\Big\| \le t_k\|v_k\| + \frac{\rho_k}{\sqrt{k}} \le t_k^{\frac{1}{2}} + \frac{t_k^{\frac{3}{2}}}{2k^{\frac{3}{2}}}.$$

Since $t_{k+1} \le \frac{t_k}{4}$ and $t_1 \le 1$, we have

$$\sum_{k=1}^{\infty} t_k^{\frac{1}{2}} \le \sum_{k=1}^{\infty} \Big(\frac{1}{4}\Big)^{\frac{k-1}{2}} < \infty,$$

and thus $\sum_{k=1}^{\infty} \|f_k(x)\| < \infty$ for all $x \in X$, which justifies that f is well defined.

(ii) Verifying that $F + f$ is not metrically subregular. Now we prove that the mapping $F + f$ with f defined in (4.5) is not metrically subregular at $(\overline{x}, \overline{y})$. To proceed, fix an arbitrary number $n \in \mathbb{N}$ and first show that for all $x \in B(x_n, \rho_n/2)$,

$$\begin{cases} \xi_k(x) \ge 1, & \text{if } k \neq n, \\ 0 \le \xi_k(x) < 1, & \text{if } k = n. \end{cases} \tag{4.7}$$

To see this, we consider the following three cases.

Case 1: If $k > n$ and $x \in B(x_n, \rho_n/2)$, then

$$\begin{aligned} \langle p_{kn}^*, x - x_k\rangle = \langle p_{kn}^*, x_n - x_k\rangle + \langle p_{kn}^*, x - x_n\rangle \ &\ge\ \|x_n - x_k\| - \frac{\rho_n}{2} \\ &\ge\ \|x_n - \overline{x}\| - \|x_k - \overline{x}\| - \frac{t_n^{\frac{3}{2}}}{4n^2} \\ &\ge\ t_n - t_k - \frac{t_n}{4n} \\ &=\ \frac{3t_n}{4} - t_k \ge t_k\Big(\frac{3}{4}4^{k-n} - 1\Big) \ge \frac{t_k}{2}4^{k-n}, \end{aligned}$$

where the second inequality holds due to $\rho_n < \frac{t_n^{\frac{3}{2}}}{2n^2}$ and the third one holds due to $t_n \le 1$. Thus we have $\xi_k(x) \ge 8t_k^{-2}16^{n-k}(\frac{t_k}{2}4^{k-n})^2 \ge 1$, which justifies (4.7) in this case.

Case 2: If $k < n$ and $x \in B(x_n, \rho_n/2)$, then

$$\begin{aligned} \langle q_k^*, x - x_k\rangle = \langle q_k^*, \overline{x} - x_k\rangle + \langle q_k^*, x_n - \overline{x}\rangle + \langle q_k^*, x - x_n\rangle \ &\ge\ \|x_k - \overline{x}\| - \|x_n - \overline{x}\| - \frac{\rho_n}{2} \\ &=\ t_k - t_n - \frac{\rho_n}{2} \\ &\ge\ t_k - t_n - \frac{t_n}{4n} \\ &\ge\ t_k - \frac{5}{4}t_n \ge \frac{11}{16}t_k, \end{aligned}$$

and thus $\xi_k(x) \ge 8t_k^{-2}(\frac{11}{16}t_k)^2 \ge 1$, which justifies (4.7) in this case.

Case 3: If $k = n$ and $x \in B(x_n, \rho_n/2)$, then we have

$$\begin{aligned} 0 \le \xi_k(x) \ &=\ 8t_k^{-2}\Big(\langle q_k^*, x - x_k\rangle^2 + \sum_{i=1}^{k-1} 16^{i-k}\langle p_{ki}^*, x - x_k\rangle^2\Big) \\ &\le\ 8t_k^{-2}\|x - x_k\|^2\Big(1 + \sum_{i=1}^{k-1} 16^{i-k}\Big) \\ &\le\ \frac{32}{15}t_k^{-2}\rho_k^2 \le \frac{32}{15}t_k\big(\rho_k t_k^{-\frac{3}{2}}\big)^2 \le \frac{8}{15k^2} \le \frac{8}{15} < 1, \end{aligned} \tag{4.8}$$

which completes the proof of the estimates in (4.7).

Having (4.7) in hand, we can represent the mapping f in (4.5) as

$$f(x)=-(1-\xi_k(x))^2\Big(t_kv_k+\frac{\rho_k}{\sqrt{k}}z_k\Big)\ \text{ for all }\ x\in B\Big(x_k,\frac{\rho_k}{2}\Big)\ \text{ and }\ k\in\mathbb{N}.$$

Since $\xi_k(x_k)=0$, it implies that $f(x_k)=-\big(t_kv_k+\frac{\rho_k}{\sqrt{k}}z_k\big)$. Recalling that $(x_k,y_k)\in\operatorname{gph}F$, we get the relationships

$$d\big(\overline{y};(F+f)(x_k)\big)\le\|\overline{y}-(y_k+f(x_k))\|=\Big\|(\overline{y}+t_kv_k-y_k)+\frac{\rho_k}{\sqrt{k}}z_k\Big\|=\Big\|\frac{\rho_k}{\sqrt{k}}z_k\Big\|=\frac{\rho_k}{\sqrt{k}}.$$

Our next step is to justify for all large $k\in\mathbb{N}$ the following distance estimate:

$$d\big(x_k;(F+f)^{-1}(\overline{y})\big)\ge\frac{\rho_k}{2},\ \text{ i.e. }\ \overline{y}\notin(F+f)(x)\ \text{ whenever }\ x\in B\Big(x_k,\frac{\rho_k}{2}\Big).\tag{4.9}$$

Arguing by contradiction and passing to a subsequence if necessary, suppose that are $a_k\in X$ with $\|a_k-x_k\|<\frac{\rho_k}{2}$ such that

$$\overline{y}\in(F+f)(a_k)\ \text{ for all }\ k\in\mathbb{N}.$$

This readily implies that

$$\begin{aligned}\|\overline{y}-y_k-f(a_k)\| &= \Big\|-t_kv_k+\big(1-\xi_k(a_k)\big)^2\Big(t_kv_k+\frac{\rho_k}{\sqrt{k}}z_k)\Big)\Big\|\\ &\le t_k\|v_k\|\big(1-(1-\xi_k(a_k))^2\big)+\big(1-\xi_k(a_k)\big)^2\frac{\rho_k}{\sqrt{k}}.\end{aligned}$$

We have from (4.8) and $\|x_k-a_k\|\le\frac{\rho_k}{2}$ that $\xi_k(a_k)\le\frac{32}{15}t_k^{-2}\rho_k^2$ and $0\le\xi_k(a_k)\le\frac{8}{15}$, which in turn reply the relationships

$$t_k\|v_k\|\big(1-(1-\xi_k(a_k))^2\big)\le 2t_k^{\frac{1}{2}}\xi_k(a_k)\le\frac{64\rho_k}{15t_k^{\frac{3}{2}}}\rho_k\le\frac{32}{15}\frac{\rho_k}{k},\quad\text{and}$$

$$\frac{1}{9}\le\big(1-\xi_k(a_k)\big)^2\le 1.$$

It follows therefore that

$$\|\overline{y}-y_k-f(a_k)\|\le\frac{32}{15}\frac{\rho_k}{k}+\frac{\rho_k}{\sqrt{k}}=\rho_k\Big(\frac{32}{15k}+\frac{1}{\sqrt{k}}\Big)<\frac{\rho_k}{2}$$

for all large k, and thus we get from (4.4) that

$$\langle y_k^*,\overline{y}-f(a_k)-y_k\rangle\le\frac{1}{2k}\Big(\|a_k-x_k\|+\|\overline{y}-f(a_k)-y_k\|\Big)\le\frac{\rho_k}{k}.$$

On the other hand, recall that $\|y_k^*\|=1$ and then observe the relationships

$$\begin{aligned}\langle y_k^*,\overline{y}-f(a_k)-y_k\rangle &= \Big\langle y_k^*,-t_kv_k+\big(1-\xi_k(a_k)\big)^2\Big(t_kv_k+\frac{\rho_k}{\sqrt{k}}z_k\Big)\Big\rangle\\ &= -\langle y_k^*,t_kv_k(1-\big(1-\xi_k(a_k)\big)^2)\rangle+\big(1-\xi_k(a_k)\big)^2\frac{\rho_k}{\sqrt{k}}\langle y_k^*,z_k\rangle\\ &\ge -t_k\|v_k\|\big(1-(1-\xi_k(a_k))^2\big)+\big(1-\xi_k(a_k)\big)^2\frac{\rho_k}{\sqrt{k}}\langle y_k^*,z_k\rangle\\ &\ge -\frac{32}{15}\frac{\rho_k}{k}+\frac{1}{18}\frac{\rho_k}{\sqrt{k}},\end{aligned}$$

which is a contradiction, since

$$-\frac{32}{15}\frac{\rho_k}{k}+\frac{1}{18}\frac{\rho_k}{\sqrt{k}}>\frac{\rho_k}{k}$$

for all large k. Thus we get (4.9), which yields

$$d\big(x_k;(F+f)^{-1}(\overline{y})\big)\ge\frac{\sqrt{k}}{2}d\big(\overline{y};(F+f)(x_k)\big)$$

and shows that $F+f$ is not metrically subregular at $(\overline{x},\overline{y})$.

(iii) Verifying that $f(\overline{x})=0$ and f is continuous. To verify these properties for the mapping f in (4.5), observe first that, using further representation (4.6) of the mapping f, we see that each f_k is a continuous function and $\sum_{k=1}^{\infty}\|f_k(x)\|<\infty$ for all $x\in X$. Hence the classical uniform convergence theorem implies that f is continuous. Finally, note that $\xi_k(\overline{x})\ge 8t_k^{-2}\langle q_k^*,\overline{x}-x_k\rangle^2=8$, which gives $f(\overline{x})=0$ and thus completes the proof of the theorem. □

Remark 4.5. (refined version of Theorem 4.4 via the modified outer coderivative). Inspecting the proof of Theorem 4.4, we observe that its statement can be improved by asserting that the perturbation mapping $f\colon X\to Y$ is in addition continuously differentiable around $\bar{x}$ with $\nabla f(\bar{x})=0$. This can be done by using modification (3.17) of the reversed outer coderivative construction from Definition 3.5 with $q=1$. Indeed, using the corresponding estimate (3.18) allows us to check that the perturbation f constructed in (4.5) is continuously differentiable around $\bar{x}$ and satisfies the condition $\nabla f(\bar{x})=0$.

5 Sharper q-Subregularity Results for Mappings with Values in Fréchet Smooth Spaces

In this section we continue studying the metric q-subregularity of set-valued mappings $F\colon X\rightrightarrows Y$ for any $q\in(0,1]$ assuming in addition the range space Y is *Fréchet smooth*, i.e., it admits an equivalent norm Fréchet differentiable at any nonzero point. This class of Banach spaces is sufficiently large including, in particular, every reflexive space while the class of Asplund spaces is broader; see [8, 30] for more details and references. This additional assumption allows us to improve the metric q-subregularity results obtained above.

Let us start with neighborhood conditions and modulus estimates. Recall that generally different elements $y,y'\in B(\overline{y},\epsilon)$ are used in the definition of the q-subregularity constant α in Theorem 3.3. The next theorem shows that we can choose $y'=y$ if the underlying space Y is Fréchet smooth. This leads therefore to a sharper sufficient condition for the metric q-subregularity of F and a better upper upper estimate of its exact bound. The proof is similar to that of Theorem 3.3 with using the exact sum rule (ii) from Lemma 2.1 on Y in addition to the fuzzy one from assertion (i) therein on $X\times Y$.

Theorem 5.1 (neighborhood sufficient conditions for q-subregularity with upper modulus estimate in Fréchet smooth spaces). *Suppose that in the setting of Theorem* 3.3 *the range space Y is Fréchet smooth. Consider the nonnegative constant*

$$\beta:=\sup_{\epsilon>0}\inf\Big\{q\|x^*\|\cdot\|y-\overline{y}\|^{q-1}\Big|\;\begin{array}{l}x\in B_X(\overline{x},\epsilon)\backslash F^{-1}(\overline{y}),\;y\in F(x)\cap B_Y\big(\overline{y},\min\{\epsilon,\|x-\overline{x}\|^{\frac{1}{2}}\}\big)\\ x^*\in\widehat{D}^*F(x,y)\big(J^q_\epsilon(y-\overline{y})\big)\end{array}\Big\},$$

with J^q_ϵ is defined in (3.2). Then the condition $\beta > 0$ is sufficient for metric q-subregularity of F at $(\bar{x},\bar{y})$. Furthermore, in this case we have the upper modulus estimate

$$\text{subreg}^q F(\overline{x},\overline{y}) \le \beta^{-1},$$

and thus there is $\epsilon_0 > 0$ such that $d\big(x; F^{-1}(\overline{y})\big) \le \frac{2}{\beta} d\big(\overline{y}; F(x)\big)^q$ for all $x \in B_X(\overline{x},\epsilon_0)$.

Proof. It is sufficient to show that $\text{subreg}^q F(\overline{x},\overline{y}) \le \beta^{-1}$ whenever $\beta > 0$. Supposing the contrary, we find a number τ such that $\text{subreg}^q F(\overline{x},\overline{y}) > \tau > \beta^{-1}$. Let $r = \tau^{-1}$. Then we have $0 < r < \beta$. Since $\text{subreg}^q F(\overline{x},\overline{y}) > \tau = r^{-1}$, there are sequences $\epsilon_k := \|x_k - \overline{x}\| \downarrow 0$ and $y_k \in F(x_k)$ such that

$$d\big(\overline{y}; F(x_k)\big)^q \le \|y_k - \overline{y}\|^q < r d\big(x_k; F^{-1}(\overline{y})\big) \text{ for all } k \in \mathbb{N}. \tag{5.1}$$

In particular, it follows that $\|y_k - \overline{y}\| < r\epsilon_k^{\frac{1}{q}} \to 0$ as $k \to \infty$ and $x_k \notin F^{-1}(\overline{y})$. Denote $\mu_k := d(x_k; F^{-1}(\overline{y})) > 0$ with $\mu_k \to 0$. Similarly to the proof of Theorem 3.3, define the function $g : X \times Y \to \overline{I\!R}$ by (3.5) using now, without loss of generality, the equivalent norm on Y Fréchet off the origin. Employing then the Ekeland variational principle for each $k \in \mathbb{N}$, we arrive at the function ϕ_k that attains it minimum at the point (a_k, b_k) satisfying the relationships therein. Further, let

$$\phi_{1k}(u,v) := \|v - \overline{y}\|^q + \delta_{\text{gph}F}(u,v) \text{ and } \phi_{2k}(u,v) := \rho_k\|(u,v) - (a_k,b_k)\|_{\epsilon_k} \tag{5.2}$$

and get by the generalized Fermat rule the inclusion

$$0 \in \widehat{\partial}\big(\phi_{1k} + \phi_{2k}\big)(a_k,b_k), \quad k \in \mathbb{N}.$$

Applying the fuzzy sum rule from Lemma 2.1(i) to this inclusion and selecting $\eta_k \downarrow 0$ as in (3.7), with taking into account that $b_k \neq \bar{y}$, give us

$$0 \in \widehat{\partial}\phi_{1k}(x_{1k},y_{1k}) + \widehat{\partial}\phi_{2k}(x_{2k},y_{2k}) + \eta_k(\mathbb{B}_{X^*} \times \mathbb{B}_{Y^*}), \tag{5.3}$$

where $(x_{ik},y_{ik}) \in X \times Y$ are such that $\|(x_{ik},y_{ik}) - (a_k,b_k)\|_{\epsilon_k} < \eta_k$ for $i = 1,2$. As in the proof of Theorem 3.3, we can check that $y_{1k} \neq \overline{y}$ for all $k \in \mathbb{N}$. Thus the function $h(u,v) := \|v - \overline{y}\|^q$ is Fréchet differentiable at y_{1k}, which allows us to apply the exact sum rule from Lemma 2.1(ii) to the function ϕ_{1k} in (5.2). It gives us

$$\widehat{\partial}\phi_{1k}(x_{1k},y_{1k}) = \nabla h(x_{1k},y_{1k}) + \widehat{N}\big((x_{1k},y_{1k}); \text{gph}\,F\big) = \big(0, J^q(y_{1k} - \overline{y})\big) + \widehat{N}\big((x_{1k},y_{1k}); \text{gph}\,F\big).$$

Substituting this into (5.3) and proceeding then as in the proof of Theorem 3.3, we get $\beta \le r$. It contradicts (5.1) and thus completes the proof of the theorem. $\square$

Remark 5.2. (calculation/estimate of the constant β for differentiable single-valued mappings). If $f\colon X \to Y$ is Fréchet differentiable on X, then we have the representation $\widehat{D}^* f(x, f(x))(y^*) = \{\nabla f(x)^* y^*\}$ for all $x \in X$ and $y^* \in Y^*$, and thus the constant β from Theorem 5.1 can be calculated and estimated as follows:

$$\begin{aligned}
\beta &= \sup_{\epsilon>0} \inf \Big\{ q\|\nabla f(x)^* y^*\| \cdot \|f(x) - f(\overline{x})\|^{q-1} \Big| x \in B_X(\overline{x},\epsilon), 0 < \|f(x) - f(\overline{x})\| \le \min\{\epsilon, \|x - \overline{x}\|^{\frac{1}{2}}\}, \\
&\qquad\qquad y^* \in J^q_\epsilon\big(f(x) - f(\overline{x})\big) \Big\} \\
&\ge \liminf_{x \to \overline{x}} \Big\{ q\|\nabla f(x)^* y^*\| \cdot \|f(x) - f(\overline{x})\|^{q-1} \Big| 0 < \|f(x) - f(\overline{x})\| \le \|x - \overline{x}\|^{\frac{1}{2}},\ \|y^*\| = 1 \Big\} \\
&= \liminf_{x \to \overline{x}, f(x) \neq f(\overline{x})} \Big\{ q\|\nabla f(x)^* y^*\| \cdot \|f(x) - f(\overline{x})\|^{q-1} \Big|\ \|y^*\| = 1 \Big\}.
\end{aligned}$$

Next we show that the condition $\beta > 0$ from Theorem 5.1 is *necessary and sufficient* for metric q-subregularity of *convex* multifunctions defined on *general Banach* spaces.

Theorem 5.3 (characterization of metric q-subregularity of convex multifunctions on Banach spaces). *Let X be an arbitrary Banach space while Y is Fréchet smooth, and let $F\colon X \rightrightarrows Y$ be a set-valued mapping with the closed and convex graph. For any $q \in (0,1]$ and $(\bar{x},\bar{y}) \in \operatorname{gph} F$, we calculate $\beta \geq 0$ by the formula of Theorem 5.1 with*

$$\widehat{D}^*F(x,y)(y^*) = \big\{x^* \in X^* \big|\ \langle x^*, x\rangle - \langle y^*, y\rangle = \max_{(u,v)\in \operatorname{gph} F} \big[\langle x^*, u\rangle - \langle y^*, v\rangle\big]\big\}, \quad y^* \in Y^*.$$

Then F is metric q-subregular at $(\bar{x},\bar{y})$ if and only if $\beta > 0$.

Proof. Observe first that the coderivative representation in the theorem holds due to the fact that the regular normal cone to a convex set reduces to the usual normal cone of convex analysis. The sufficiency part of the theorem follows from the proof of Theorem 5.1 by taking into account the Fréchet differentiability of $\|\cdot\|^q$ at any nonzero point and the convexity of the functions $\delta_{\operatorname{gph} F}$ and ϕ_{2k} therein and by replacing then the application of the fuzzy sum rule of Lemma 2.1(ii) with the exact sum rule from Lemma 2.1(i) and the classical Moreau-Rockafellar theorem on the exact subdifferential sum rule of convex analysis in arbitrary Banach spaces.

To prove the necessity part of the theorem, assume that F is metrically regular at $(\bar{x},\bar{y})$ and by Definition 3.1 find constants $c,\delta > 0$ such that estimate (3.1) is satisfied. It follows from the coderivative representation for convex multifunctions that

$$\langle x^*, u-x\rangle \leq \langle y^*, \overline{y}-y\rangle \text{ for all } x\in X,\ y\in F(x),\ u\in F^{-1}(\overline{y}),\ x^*\in \widehat{D}^*F(x,y)(y^*). \quad (5.4)$$

Pick $\epsilon \in (0, \min\{\delta, \frac{q}{4r}, (q/2)^{\frac{1}{2-q}}\})$ and define the number

$$\eta := \min\Big\{q\epsilon^{q-1}, \frac{q}{4r}\Big\}.$$

Taking any $x \in B_X(\overline{x},\epsilon)\backslash F^{-1}(\overline{y})$, $y \in F(x)\cap B_Y(\overline{y}, \min\{\epsilon, \|x-\overline{x}\|^{\frac{1}{2}}\})$, $y^* \in J^q_\epsilon(y-\overline{y})$, and $x^* \in \widehat{D}^*F(x,y)(y^*)$, we want to show that

$$q\|x^*\|\cdot\|y-\overline{y}\|^{q-1} \geq \eta, \quad (5.5)$$

which surely implies that $\beta > 0$ by the definition of this constant.

Observe first that assuming $\|x^*\| \geq 1$ yields that $q\|x^*\|\cdot\|y-\overline{y}\|^{q-1} \geq q\epsilon^{q-1} \geq \eta$, i.e., (5.5) holds. Thus we proceed in what follows with the case of $\|x^*\| < 1$. Since $y^* \in J^q_\epsilon(y-\overline{y})$, there exist $u^* \in \mathbb{B}_{Y^*}$ and $v^* \in J(y-\overline{y})$ with

$$y^* = \frac{q\|y-\overline{y}\|^{q-1}v^* + \epsilon u^*}{\|q\|y-\overline{y}\|^{q-1}v^* + \epsilon u^*\|},$$

and so we have the relationships

$$\begin{aligned} \big\|q\|y-\overline{y}\|^{q-1}v^* + \epsilon u^*\big\|\cdot\langle y^*, \overline{y}-y\rangle &= \langle q\|y-\overline{y}\|^{q-1}v^* + \epsilon u^*, \overline{y}-y\rangle \\ &\leq -q\|y-\overline{y}\|^q + \epsilon\|\overline{y}-y\|. \end{aligned} \quad (5.6)$$

Further, select $u_k \in F^{-1}(\overline{y})$ from the condition

$$\|u_k - x\| < \min\Big\{\Big(1+\frac{1}{k}\Big)\, d\big(x; F^{-1}(\overline{y})\big), \epsilon\Big\}$$

and derive from (5.4) and (5.6) that

$$\begin{aligned}
\big(\|q\|y-\overline{y}\|^{q-1}v^*+\epsilon u^*\|\big)\cdot\langle x^*,u_k-x\rangle &\le \|q\|y-\overline{y}\|^{q-1}v^*+\epsilon u^*\|\cdot\langle y^*,\overline{y}-y\rangle\\
&\le -q\|y-\overline{y}\|^q+\epsilon\|\overline{y}-y\|\\
&= \|y-\overline{y}\|^q(-q+\epsilon\|\overline{y}-y\|^{1-q})\\
&\le -\frac{q}{2}\|y-\overline{y}\|^q\\
&\le -\frac{q}{2}d\big(\overline{y};F(x)\big)^q\\
&\le -\frac{q}{2r}d\big(x;F^{-1}(\overline{y})\big)\\
&\le -\frac{qk}{2r(k+1)}\|u_k-x\|,
\end{aligned}$$

where the third inequality follows due to the conditions

$$\epsilon\le(q/2)^{\frac{1}{2-q}}\ \text{ and }\ \|\overline{y}-y\|<\epsilon,\ \text{ and so }\ \epsilon\|\overline{y}-y\|^{1-q}\le\epsilon^{2-q}\le\frac{q}{2}.$$

This implies that for all $k\in\mathbb{N}$ we have

$$\big(\|q\|y-\overline{y}\|^{q-1}v^*+\epsilon u^*\|\big)\cdot\|x^*\|\ge\frac{qk}{2r(k+1)},$$

which yields in turn as $k\to\infty$ the estimate

$$\big(\|q\|y-\overline{y}\|^{q-1}v^*+\epsilon u^*\|\big)\cdot\|x^*\|\ge\frac{q}{2r}.$$

Taking into account that $\|u^*\|\le 1$, $\|v^*\|=1$, $\|x^*\|<1$, and $\epsilon\le\frac{q}{4r}$, we get that

$$q\|y-\overline{y}\|^{q-1}\|x^*\|\ge\frac{q}{2r}-\epsilon\ge\frac{q}{4r}\ge\eta.$$

This justifies (5.5) and thus completes the proof of the theorem. $\square$

We finish this section with an enhanced version of the pointbased sufficient condition of Theorem 3.7 via the enhanced limiting coderivative construction introduced below.

Definition 5.4 (enhanced outer reversed mixed q-coderivative). *Let $F\colon X\rightrightarrows Y$, and let $q>0$. The* ENHANCED OUTER REVERSED MIXED q-CODERIVATIVE *of F at $(\bar{x},\bar{y})\in\operatorname{gph}F$ is a set-valued mapping $\widetilde{D}^{*+}_{M,q}F(\bar{x},\bar{y})\colon Y^*\rightrightarrows X^*$ defined as follows: the inclusion $x^*\in\widetilde{D}^{*+}_{M,q}F(\overline{x},\overline{y})(y^*)$ for any given $y^*\in Y^*$ means that there are sequences $(x_k,y_k,x^*_k,y^*_k)\in X\times Y\times X^*\times Y^*$ such that*

$$\begin{aligned}
&x_k\to\overline{x}\ \text{with}\ x_k\notin F^{-1}(\overline{y}),\ y_k\in F(x_k)\ \text{with}\ y_k\to\overline{y}\ \text{and}\ \|y_k-\overline{y}\|\le\|x_k-\overline{x}\|^{\frac{1}{2}},\\
&x^*_k\in\widehat{D}^*F(x_k,y_k)(y^*_k)\ \text{with}\ \|y^*_k\|=1,\ y^*_k\overset{w^*}{\to}y^*,\ \text{and}\ \lambda_k x^*_k\to x^*\ \text{with}\ \lambda_k:=q\|y_k-\overline{y}\|^{q-1}\ \text{as}\ k\to\infty.
\end{aligned}$$

We drop the symbol "M" and the word "mixed" in the coderivative notation and terminology when the space Y is finite-dimensional.

Note that for $q=1$ this construction reduces to the coderivative $\widetilde{D}^{*>}_M F(\bar{x},\overline{y})$ introduced in Definition 3.5, and so we do not need another notation. In the general case

$$\widetilde{D}^{*+}_{M,q}F(\bar{x},\overline{y})(y^*)\subset\widetilde{D}^{*>}_{M,q}F(\bar{x},\overline{y})(y^*)\ \text{ whenever }\ y^*\in Y^*\ \text{ and }\ q>0.$$

Theorem 5.5 (pointbased sufficient conditions for metric q-subregularity in Fréchet smooth spaces). *Let $F : X \rightrightarrows Y$ be a closed-graph multifunction between an Asplund space X and a Fréchet smooth space Y, let $(\overline{x}, \overline{y}) \in \operatorname{gph} F$, and let $q \in (0,1]$. Assume that F is PSNC at $(\bar{x}, \bar{y})$ with respect to Y and that*

$$\ker \widetilde{D}^{*+}_{M,q} F(\overline{x}, \overline{y}) = \{0\}. \tag{5.7}$$

Then the mapping F is metrically q-subregular at $(\overline{x}, \overline{y})$.

Proof. We proceed in the same way as in the proof of Theorem 3.7 by passing to the limit from the neighborhood condition of Theorem 5.1. $\square$

Remark 5.6. (refined coderivative conditions for metric q-subregularity). Similarly to the discussions in Remark 3.6(iii) we can get the refined neighborhood sufficient condition for metric q-subregularity in Theorem 5.1 and the pointwise sufficient condition for this property in Theorem 5.5 by modifying the constant β and the q-coderivative construction in Definition 5.4 as in (3.16) and (3.17), respectively. However, in this way we may loose the q-subregularity modulus estimate obtained in Theorem 5.1.

6 Fractional Error Bounds with Explicit Exponents

Error bounds play a significant role in many quantitative and qualitative aspects of optimization and variational analysis; see, e.g., [12, 24, 31, 32] and the references therein. It has been well recognized for a long time that conditions for metric regularity are closely related to deriving efficient error bounds estimates in various optimization problems.

This section is devoted to applications of the obtained results on metric q-subregularity to establishing new *fractional error bounds* for remarkable classes of inequality systems. Our approach and q-subregularity results lead us to the *explicit calculation* of fractional exponents of error bounds. The notation and terminology of this section are standard in the theory of error bounds; see the references above.

We start with a consequence of the q-subregularity modulus estimate of Theorem 3.3 that allows us to provide a subdifferential sufficient condition for fractional local error bounds in general inequality systems. To proceed, recall that $[\alpha]_+ := \max\{\alpha, 0\}$ for any $\alpha \in \mathbb{R}$ and that $[f \leq 0] := \{x \in X : f(x) \leq 0\}$ (resp. $[f < 0] := \{x \in X : f(x) < 0\}$) for each extended real-valued function $f : X \to \overline{I\!R}$.

Lemma 6.1 (subdifferential sufficient condition for fractional error bound). *Let $f : X \to \overline{I\!R}$ be a proper l.s.c. function on an Asplund space X, and let $\overline{x} \in [f \leq 0]$. Fix $q \in (0,1]$ and assume that there exist positive numbers $\epsilon > 0$ and $\delta > 0$ with*

$$\|x^*\| f(x)^{q-1} \geq \delta \ \text{ for all } \ x \in B_X(\overline{x}, \epsilon) \cap [f > 0] \ \text{ and } \ x^* \in \widehat{\partial} f(x).$$

Then the following fractional local error bound holds at $\overline{x}$: there is $\epsilon_0 > 0$ such that

$$d\big(x; [f \leq 0]\big) \leq \frac{2}{\delta} [f(x)]_+^q \ \text{ for all } \ x \in B_X(\overline{x}, \epsilon_0). \tag{6.1}$$

Proof. Define a set-valued mapping $F \colon X \rightrightarrows I\!R$ by

$$F(x) = \begin{cases} [f(x), \infty) & \text{if } f(x) < \infty, \\ \emptyset & \text{otherwise.} \end{cases} \tag{6.2}$$

Note that $\operatorname{gph} F = \operatorname{epi} f$, and this set is closed in $X \times I\!R$ due to the lower semicontinuity of f. Taking $\overline{y} = 0$, we see that $(\overline{x}, \overline{y}) \in \operatorname{gph} F$. It easily follows from the subdifferential assumption made in the lemma that

$$\sup_{\epsilon>0} \inf \Big\{ q\|x^*\| \cdot \|y - \overline{y}\|^{q-1} \Big| \quad x \in B_X(\overline{x}, \epsilon) \backslash F^{-1}(\overline{y}),\ y \in F(x) \cap B_Y(\overline{y}, \epsilon) \backslash \{\overline{y}\}, \\ x^* \in \widehat{D}^* F(x, y)(\mathbb{S}_{\mathbb{R}}) \Big\} \geq \delta.$$

Thus Theorem 5.1 ensures that the mapping F in (6.2) is metrically q-subregular at $(\bar{x}, \bar{y})$ with the upper modulus estimate

$$\operatorname{subreg}^q F(\overline{x}, \overline{y}) \leq \delta^{-1},$$

which readily implies by the structure of (6.2) the claimed error bound (6.1). $\square$

The next lemma recalls an error bound result for convex polynomials established recently in [25, 26]; see also [24, 27] for some previous related developments.

Lemma 6.2 (error bound for convex polynomials). *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a convex polynomial of degree d, and let $\kappa(m, d) := (d-1)^m + 1$. Then f has a Hölder type global error bound with exponent $\kappa(m, d)^{-1}$, i.e., there is a constant $\tau > 0$ such that*

$$d\big(x; [f \leq 0]\big) \leq \tau\big([f(x)]_+ + [f(x)]_+^{\kappa(m,d)^{-1}}\big) \ \text{ for all } \ x \in \mathbb{R}^n.$$

Now we are ready to establish the main result of this section, which provides a new fractional error bound with calculating the *explicit exponent* for *nonconvex* inequality systems given by compositions of smooth functions and convex polynomials.

Theorem 6.3 (fractional error bounds with explicit exponents for nonconvex composite systems). *Let X be an Asplund space, let the mapping $g \colon X \to \mathbb{R}^m$ given by $g(x) := (g_1(x), \ldots, g_m(x))$ with $g_j \colon X \to \mathbb{R}$ be continuously differentiable around the reference points, and let $\psi : \mathbb{R}^m \to \mathbb{R}$ be a convex polynomial of degree d. Form the composition $f(x) := (\psi \circ g)(x)$, take $\overline{x} \in [f \leq 0]$, and assume that the derivative operator $\nabla g(\overline{x}) \colon X \to \mathbb{R}^m$ is surjective. Then there exist positive numbers τ and ϵ such that*

$$d\big(x; [f \leq 0]\big) \leq \tau [f(x)]_+^{\kappa(m,d)^{-1}} \ \text{ for all } \ x \in B_X(\overline{x}, \epsilon),$$

where $\kappa(m, d) = (d-1)^m + 1$.

Proof. We have by the standard chain rule for smooth functions that

$$f'(x) = \nabla g(x)^* \big(\psi'(g(x))\big) \ \text{ for each } \ x,$$

where $f'(x) \in X^*$ and $\psi'(x) \in \mathbb{R}^m$ indicate the classical derivatives/gradients of the real-valued functions. By the surjectivity assumption on $\nabla g(\bar{x})$ and the C^1 property of g around $\bar{x}$, it follows from the classical Lyusternik-Graves theorem (see, e.g., [30, Theorem 1.57]) that there are numbers $l > 0$ and $\epsilon > 0$ such that

$$l\, \mathbb{B}_{\mathbb{R}^m} \subset \nabla g(x)(\mathbb{B}_X) \ \text{ for all } \ x \in B_X(\overline{x}, \epsilon).$$

This implies that for all $z^* \in \mathbb{R}^m$ we have the estimate

$$\|\nabla g(x)^* z^*\| \geq l\|z^*\| \ \text{ as } \ x \in B_X(\overline{x}, \epsilon). \tag{6.3}$$

Further, let us show the existence of $\delta > 0$ such that for each $x \in [f > 0] \cap B_X(\overline{x}, \epsilon)$ we can find $v \in X$ with $\|v\| = 1$ such that

$$\langle f'(x), v\rangle \leq -\delta\big(f(x)\big)^{1-\kappa(m,d)^{-1}}. \tag{6.4}$$

Having this would imply that

$$-\|f'(x)\| \cdot \|v\| \leq -\delta f(x)^{1-\kappa(m,d)^{-1}},$$

which gives in turn that

$$\|f'(x)\|\, f(x)^{\kappa(m,d)^{-1}-1} \geq \delta > 0,$$

and then the conclusion of the theorem would follow from Lemma 6.1.

To justify (6.4), fix $x \in [f > 0] \cap B(\overline{x}, \epsilon)$. Since $f(x) > 0$, the vector $g(x)$ is not a minimizer of the convex function ψ. Thus it follows from the classical sufficient optimality condition that $\psi'(g(x)) \neq 0$. By this we find $v \in X$ with $\|v\| = 1$ such that

$$\langle \nabla g(x)^*\big(\psi'(g(x))\big), v\rangle \leq -\frac{1}{2}\big\|\nabla g(x)^*\big(\psi'(g(x))\big)\big\|$$

Then it follows from (6.3) due to the aforementioned chain rule that

$$\langle f'(x), v\rangle = \big\langle \nabla g(x)^*\big(\psi'(g(x))\big), v\big\rangle \leq -\frac{1}{2}\big\|\nabla g(x)^*\big(\psi'(g(x))\big)\big\| \leq -\frac{l}{2}\,\big\|\psi'(g(x))\big\|.$$

On the other hand, letting $z := g(x)$ and taking $a \in [\psi \leq 0]$ with $d(z; [\psi \leq 0]) = \|z - a\|$, we get from $\psi(z) > 0$ that $\psi(a) = 0$. Thus it follows from the convexity of ψ that

$$\begin{array}{rcl} \psi(g(x)) = \psi(g(x)) - \psi(a) & \leq & \langle \psi'(g(x)), g(x) - a\rangle \\ & \leq & \|\psi'(g(x))\|\, d\big(g(x); [\psi \leq 0]\big) \\ & = & \|\psi'(g(x))\|\, d\big(z; [\psi \leq 0]\big). \end{array}$$

Next denote $M := \sup\{\psi(g(x))|\ x \in B_X(\overline{x}, \epsilon)\} < \infty$ and observe from Lemma 6.2 that there is $\mu > 0$ such that

$$d\big(z; [\psi \leq 0]\big) \leq \mu\big([\psi(z)]_+ + [\psi(z)]_+^{\kappa(m,d)^{-1}}\big) \ \text{ for each } \ z \in \mathbb{R}^m.$$

It follows therefore that

$$\begin{array}{rcl} \psi(g(x)) & \leq & \|\psi'(g(x))\| d\big(z; [\psi \leq 0]\big) \\ & \leq & \mu\|\psi'(g(x))\|\big(\psi(z) + \psi(z)^{\kappa(m,d)^{-1}}\big) \\ & = & \mu\|\psi'(g(x))\|\big(\psi(g(x))^{1-\kappa(m,d)^{-1}} + 1\big)\psi(g(x))^{\kappa(m,d)^{-1}} \\ & \leq & \mu\|\psi'(g(x))\|\big(M^{1-\kappa(m,d)^{-1}} + 1\big)\psi(g(x))^{\kappa(m,d)^{-1}} \\ & = & \gamma\|\psi'(g(x))\|\psi(g(x))^{\kappa(m,d)^{-1}}, \end{array}$$

where $\gamma := \mu(M^{1-\kappa(m,d)^{-1}} + 1) > 0$. This implies that

$$\|\psi'(g(x))\| \geq \gamma^{-1}\psi(g(x))^{1-\kappa(m,d)^{-1}},$$

and hence we arrive at the estimate

$$\langle f'(x), v\rangle \leq -\frac{\gamma^{-1}l}{2}\,\psi(g(x))^{1-\kappa(m,d)^{-1}} = -\frac{\gamma^{-1}l}{2}\,f(x)^{1-\kappa(m,d)^{-1}},$$

which justifies (6.4) with $\delta = \frac{\gamma^{-1}l}{2}$ and thus completes the proof of the theorem. $\square$

7 Applications to Quantitative Convergence Analysis of Proximal Point Method

The concluding section of this paper develops applications of metric q-subregularity to the quantitative convergence analysis of the classical proximal point method for maximal monotone operators in Hilbert spaces. Recall that the operator $T : H \rightrightarrows H$ on the Hilbert space H is *monotone* if

$$\langle u - v, x - y\rangle \geq 0 \text{ for all } u \in T(x) \text{ and } v \in T(y),$$

and it is *maximal monotone* if its graph is not properly contained in the graph of any other monotone operator on the space H.

Let $\{\lambda_k\}$ be a given sequence of positive numbers. The classical *proximal point method* (PPM) to find zeros of the maximal monotone operator T is constructed by

$$x_{k+1} = (I + \lambda_k T)^{-1}(x_k), \quad k = 0, 1, \ldots, \tag{7.1}$$

where I denotes the usual identity operator on H. This algorithm and its modifications have been well recognized among the most powerful methods to solve variational inequalities and other classes of optimization-related problems. In particular, Rockafellar [33] showed that the sequence $\{x_k\}$ generated by PPM converges weakly to a zero of maximal a monotone operator. Furthermore, Güler [16] and Bauschke et al. [6] provided examples showing that the sequence $\{x_k\}$ generated by PPM may not converge in the norm sense. We refer the reader to the recent book [7] for the excellent exposition, new developments, and the extensive bibliographies therein.

To proceed with our detailed convergence analysis of the PPM from the viewpoint of metric q-subregularity, we first establish some recurrent relationships that play a significant role in the subsequent error estimates.

Lemma 7.1 (recurrent relationships). *Let $p > 0$, and let $\{\delta_k\}_{k=0}^{\infty}$ and $\{\beta_k\}_{k=0}^{\infty}$ be two sequences of positive numbers satisfying the conditions*

$$\beta_{k+1}(\delta_k \beta_{k+1}^p + 1) \leq \beta_k \text{ as } k = 0, 1, \ldots.$$

Then there is a number $\gamma > 0$ such that

$$\beta_k \leq \left(\beta_0^{-p} + \gamma \sum_{i=0}^{k-1} \min\left\{\delta_i, \delta_i^{\frac{p}{p+1}}\right\}\right)^{-\frac{1}{p}} \text{ for all } k \in \mathbb{N}. \tag{7.2}$$

In particular, we have $\lim_{k\to\infty} \beta_k = 0$ whenever $\sum_{k=0}^{\infty} \delta_k = \infty$.

Proof. It follows from our assumption that

$$0 \le \beta_{i+1} \le \beta_i \le \ldots \le \beta_0 \ \text{ and } \ \delta_i \beta_{i+1}^{p+1} \le \beta_i - \beta_{i+1} \ \text{ as } \ i \in \mathbb{N},$$

which ensure that the limit $\lim\limits_{i\to\infty} \beta_i$ exists and that

$$\sum_{i=0}^{\infty} \delta_i \beta_{i+1}^{p+1} \le \beta_0 - \lim_{i\to\infty} \beta_i < \infty.$$

Hence we can find a constant $C > 0$ such that $0 < \delta_i \beta_{i+1}^{p+1} \le C$ for all $i \in \mathbb{N} \cup \{0\}$.

Next take $M > 1$ and let $\gamma := \min\{p/M, C^{-\frac{p}{p+1}}(1 - M^{-\frac{p}{p+1}})\} > 0$. We claim that

$$\beta_{i+1}^{-p} - \beta_i^{-p} \ge \gamma \min\left\{\delta_i, \delta_i^{\frac{p}{p+1}}\right\} \ \text{ for all } \ i \in \mathbb{N} \cup \{0\}. \tag{7.3}$$

To verify (7.3), consider the nonincreasing function $h(x) := x^{-(p+1)}$ and split the proof into the following two cases.

Case 1: $h(\beta_{i+1}) \le M h(\beta_i)$. As $\delta_i\, h(\beta_{i+1})^{-1} = \delta_i \beta_{i+1}^{p+1} \le \beta_i - \beta_{i+1}$ in this case, we get

$$\delta_i \le (\beta_i - \beta_{i+1}) h(\beta_{i+1}) \le M(\beta_i - \beta_{i+1}) h(\beta_i) \le M \int_{\beta_{i+1}}^{\beta_i} h(x)dx = M \frac{\beta_{i+1}^{-p} - \beta_i^{-p}}{p}.$$

This implies that $\beta_{i+1}^{-p} - \beta_i^{-p} \ge \frac{p}{M}\delta_i \ge \gamma\delta_i$, and thus (7.3) holds.

Case 2: $h(\beta_{i+1}) > M h(\beta_i)$. Then $\beta_{i+1}^{-(p+1)} > M\beta_i^{-(p+1)}$ and hence $\beta_{i+1}^{-p} > M^{\frac{p}{p+1}} \beta_i^{-p}$. This ensures the estimate

$$\beta_{i+1}^{-p} - \beta_i^{-p} \ge (1 - M^{-\frac{p}{p+1}})\beta_{i+1}^{-p}. \tag{7.4}$$

It follows from $0 < \delta_i \beta_{i+1}^{p+1} \le C$ that $\beta_{i+1}^{-p} = (\beta_{i+1}^{-(p+1)})^{\frac{p}{p+1}} \ge (C^{-1}\delta_i)^{\frac{p}{p+1}}$. Hence (7.4) yields

$$\beta_{i+1}^{-p} - \beta_i^{-p} \ge C^{-\frac{p}{p+1}}(1 - M^{-\frac{p}{p+1}})\delta_i^{\frac{p}{p+1}} \ge \gamma \delta_i^{\frac{p}{p+1}},$$

which verifies that (7.3) holds in this case as well.

Now fix any $k \in \mathbb{N}$ and, summing (7.3) from $i = 0$ to $i = k - 1$, we get

$$\beta_k^{-p} - \beta_0^{-p} \ge \gamma \sum_{i=0}^{k-1} \min\left\{\delta_i, \delta_i^{\frac{p}{p+1}}\right\},$$

which implies the conclusion in (7.2). To justify finally the last assertion of the lemma, observe by (7.2) that we only need to show that

$$\sum_{k=0}^{\infty} \delta_k = \infty \Longrightarrow \sum_{k=0}^{\infty} \min\{\delta_k, \delta_k^{\frac{p}{p+1}}\} = \infty. \tag{7.5}$$

To verify this, consider first the case of $\delta_k \to 0$ as $k \to \infty$. Then we have for large k that $\delta_k \le \delta_k^{\frac{p}{p+1}}$, and thus $\min\{\delta_k, \delta_k^{\frac{p}{p+1}}\} = \delta_k$, which implies (7.5). In the remaining case for $\{\delta_k\}$ there there are $r > 0$ and a subsequence $\{\delta_{k_l}\}_{l=0}^{\infty}$ such that $\delta_{k_l} \ge r$. Hence

$$\sum_{k=0}^{\infty} \min\{\delta_k, \delta_k^{\frac{p}{p+1}}\} \ge \sum_{l=0}^{\infty} \min\{\delta_{k_l}, \delta_{k_l}^{\frac{p}{p+1}}\} = \infty,$$

which also ensures (7.5) and thus completes the proof of the lemma. $\square$

The next result is certainly of its own interest while playing a crucial role in the subsequent convergence analysis of the PPM.

Theorem 7.2 (error estimate for PPM). *Let $T: H \rightrightarrows H$ be a maximal monotone operator on a Hilbert space, let $\bar{x} \in T^{-1}(0)$, and let $q \in (0,1]$. Assume that T is metrically q-subregular at $(\bar{x},0)$, i.e., there are positive constants c and δ such that*

$$d\big(x;T^{-1}(0)\big) \le c\, d\big(0;T(x)\big)^q \ \textit{ for all } \ x \in B_H(\bar{x},\delta). \tag{7.6}$$

Select a starting point $x_0 \in B_H(\bar{x},\delta)$ and consider a sequence of iterates $\{x_k\}$ generated by the proximal point method. Then there is a positive number γ such that for all $k \in \mathbb{N}$ we have the error estimate

$$d\big(x_k;T^{-1}(0)\big) \le \begin{cases} \left(d\big(x_0;T^{-1}(0)\big)^{-\frac{2(1-q)}{q}} + \gamma\, c^{-\frac{2}{q}} \displaystyle\sum_{i=0}^{k-1} \min\left\{\lambda_i^2, \lambda_i^{2(1-q)}\right\}\right)^{-\frac{q}{2(1-q)}}, & \textit{if } q \in (0,1), \\ \sqrt{\prod_{i=0}^{k-1} \frac{1}{c^{-2}\lambda_k^2+1}}\, d\big(x_0;T^{-1}(0)\big), & \textit{if } q = 1. \end{cases}$$

Proof. Define the resolvent $J_{\lambda T}$ by $J_{\lambda T} := (I+\lambda T)^{-1}$ for all $\lambda \ge 0$. Since T is maximal monotone, the resolvent is *firmly nonexpansive* (see, e.g., [7]) meaning that for any $\lambda \ge 0$, $\widehat{x} \in T^{-1}(0)$, and $x \in H$ we have the condition

$$\|J_{\lambda T}(x) - J_{\lambda T}(\widehat{x})\|^2 \le \|x-\widehat{x}\|^2 - \|(I-J_{\lambda T})(x) - (I-J_{\lambda T})(\widehat{x})\|^2.$$

Since $J_{\lambda T}(\widehat{x}) = \widehat{x}$ for all $\lambda \ge 0$, it follows that

$$\|J_{\lambda T}(x) - \widehat{x}\|^2 \le \|x-\widehat{x}\|^2 - \|x - J_{\lambda T}(x)\|^2 \ \text{ for all } \ x \in H \ \text{ and } \ \widehat{x} \in T^{-1}(0). \tag{7.7}$$

In particular, letting $x = x_k$, $\widehat{x} = \bar{x}$ and $\lambda = \lambda_k$ gives us

$$\|x_{k+1} - x^*\|^2 = \|J_{\lambda T}(x_k) - x^*\|^2 \le \|x_k - x^*\|^2,$$

which ensures that $x_k \in B(x_0,\delta)$ as $k = 0,1,\ldots$. On the other hand, note that

$$x \in (I+\lambda T)\big(J_{\lambda T}(x)\big) = J_{\lambda T}(x) + \lambda T\big(J_{\lambda T}(x)\big),$$

and thus $x - J_{\lambda T}(x) \in \lambda T(J_{\lambda T}(x))$ for all $x \in H$. This implies that

$$\|x - J_{\lambda T}(x)\| \ge \lambda d\big(0;T(J_{\lambda T}(x))\big), \quad x \in H. \tag{7.8}$$

Let $P_A(x)$ stand for the projection of x onto the set A. Then for all $k = 0,1,\ldots$ we have

$$\begin{aligned} d\big(x_{k+1};T^{-1}(0)\big)^2 &\le \|x_{k+1} - \mathrm{P}_{T^{-1}(0)}(x_k)\|^2 \\ &= \|J_{\lambda_k T}(x_k) - \mathrm{P}_{T^{-1}(0)}(x_k)\|^2 \\ &\le \|x_k - \mathrm{P}_{T^{-1}(0)}(x_k)\|^2 - \|x_k - J_{\lambda_k T}(x_k)\|^2 \\ &\le \|x_k - \mathrm{P}_{T^{-1}(0)}(x_k)\|^2 - \lambda_k^2\, d\big(0;T(J_{\lambda_k T}(x_k))\big)^2 \\ &\le \|x_k - \mathrm{P}_{T^{-1}(0)}(x_k)\|^2 - c^{-\frac{2}{q}}\lambda_k^2\, d\big(J_{\lambda_k T}(x_k);T^{-1}(0)\big)^{\frac{2}{q}} \\ &= d\big(x_k;T^{-1}(0)\big)^2 - c^{-\frac{2}{q}}\lambda_k^2\, d\big(x_{k+1};T^{-1}(0)\big)^{\frac{2}{q}}, \end{aligned} \tag{7.9}$$

where the second inequality follows from (7.7) with $x = x_k$, $\widehat{x} = \mathrm{P}_{T^{-1}(0)}(x_k)$, and $\lambda = \lambda_k$, the third inequality follows from (7.8) with $x = x_k$ and $\lambda = \lambda_k$, and the last inequality holds by the metric q-subregularity assumption and by $x_k \in B(x_0, \delta)$.

Further, for each k we define $\beta_k := d(x_k; T^{-1}(0))^2$, $p := \frac{1}{q} - 1 \geq 0$, and $\delta_k := c^{-\frac{2}{q}} \lambda_k^2$. It follows from (7.9) that $\beta_{k+1} \leq \beta_k - \delta_k \beta_{k+1}^{1+p}$, which means that

$$\beta_{k+1}(1 + \delta_k \beta_{k+1}^p) \leq \beta_k. \tag{7.10}$$

Now we split the proof into two cases.

Case 1: $q \in (0, 1)$. In this case we have $p > 0$. Thus the preceding Lemma 7.1 implies that there exists $\gamma > 0$ such that

$$\begin{aligned} \beta_k &\leq \Big(\beta_0^{-p} + \gamma \sum_{i=0}^{k-1} \min\big\{\delta_i, \delta_i^{\frac{p}{p+1}}\big\}\Big)^{-\frac{1}{p}} \\ &= \Big(\beta_0^{-\frac{1-q}{q}} + \gamma \sum_{i=0}^{k-1} \min\big\{c^{-\frac{2}{q}}\lambda_i^2, (c^{-\frac{2}{q}}\lambda_i^2)^{1-q}\big\}\Big)^{-\frac{q}{1-q}}, \quad k \in \mathbb{N}, \end{aligned}$$

which yields in turn that

$$d\big(x_k; T^{-1}(0)\big) \leq \Big(d\big(x_0; T^{-1}(0)\big)^{-\frac{2(1-q)}{q}} + \gamma \sum_{i=0}^{k-1} \min\big\{c^{-\frac{2}{q}}\lambda_i^2, (c^{-\frac{2}{q}}\lambda_i^2)^{1-q}\big\}\Big)^{-\frac{q}{2(1-q)}}, \quad k \in \mathbb{N},$$

and thus justifies the error estimate of the theorem in this case.

Case 2: $q = 1$. Then $\delta_k = c^{-\frac{2}{q}}\lambda_k^2 = \delta_k = c^{-2}\lambda_k^2$ and estimate (7.10) can be simplified as

$$\beta_{k+1}(\delta_k + 1) \leq \beta_k.$$

This readily implies that

$$\beta_k \leq \Big(\prod_{i=0}^{k-1} \frac{1}{\delta_i + 1}\Big)\beta_0,$$

which ensures therefore the error estimate

$$d\big(x_k; T^{-1}(0)\big) \leq \sqrt{\prod_{i=0}^{k-1} \frac{1}{\delta_i + 1}}\, d\big(x_0; T^{-1}(0)\big) = \sqrt{\prod_{i=0}^{k-1} \frac{1}{c^{-2}\lambda_i^2 + 1}}\, d\big(x_0; T^{-1}(0)\big)$$

in this case and thus completes the proof of the theorem. $\square$

It follows from the obtained error estimate of the iterates generated by the PPM that

$$\Big[\sum_{i=0}^{\infty} \lambda_i^2 = \infty\Big] \Longrightarrow \Big[\sum_{i=0}^{\infty} \min\big\{\lambda_i^2, \lambda_i^{2(1-q)}\big\} = \infty\Big],$$

and thus we get the convergence $d(x_k; T^{-1}(0)) \to 0$ as $k \to \infty$ whenever the starting point x_0 is chosen sufficiently closely to $T^{-1}(0)$ provided that the metric q-subregularity holds for T. [†] Furthermore, the error estimate of Theorem 7.2 plays a crucial role in the proof of the following main theorem of this section, which presents various results on the convergence rate of the PPM depending on our choice of the sequence $\{\lambda_k\}$ in (7.1) and the q-subregularity requirements on T. To ensure the validity of the latter requirements, we can use the sufficient conditions obtained in Section 4. Note also that part (iv) of this theorem under the metric subregularity assumption has been established in [23] by somewhat different arguments.

Theorem 7.3 (convergence rate analysis for PPM). *Let $T : H \rightrightarrows H$ be a maximal monotone operator on a Hilbert space H, and let $\bar{x} \in T^{-1}(0)$. Assume that T is metrically q-subregular at $\bar{x}$ with some positive constants c, δ in (7.6). Select a starting point $x_0 \in B(\bar{x}, \delta)$ and consider the sequence of iterates $\{x_k\}$ generated by (7.1) with the given sequence $\{\lambda_k\}$. The following assertions hold:*

- **(i)** *If $q \in (0,1)$ and $\lambda_k \equiv \lambda > 0$, then the error sequence $d\big(x_k; T^{-1}(0)\big)$ converges to zero with the convergence rate at least $O\big(k^{-\frac{q}{2(1-q)}}\big)$.*
- **(ii)** *If $q \in (0,1)$ and $\lambda_k = O(k^s)$ with $0 < s \le \frac{1}{2(1-q)}$, then the error sequence $d\big(x_k; T^{-1}(0)\big)$ converges to zero with the convergence rate at least $O(k^{-2sq})$.*
- **(iii)** *If $q \in (0,1)$ and $\lambda_k = O(k^s)$ with $s \ge \frac{1}{2(1-q)}$, then the error sequence $d(x_k; T^{-1}(0))$ converges to zero with the convergence rate at least $O\big(k^{-\frac{q(1+2s(1-q))}{2(1-q)}}\big)$.*
- **(iv)** *If $q = 1$ and $\lambda_k \equiv \lambda > 0$, then the error sequence $d\big(x_k; T^{-1}(0)\big)$ converges R-linearly to zero in the sense that*
$$\lim_{k\to\infty} \sqrt[k]{d(x_k; T^{-1}(0))} < 1.$$
- **(v)** *If $q = 1$ and $\lambda_k \to \infty$, then the error sequence $d\big(x_k; T^{-1}(0)\big)$ converges R-superlinearly to zero the sense that*
$$\lim_{k\to\infty} \sqrt[k]{d\big(x_k; T^{-1}(0)\big)} = 0.$$

Proof. Assertion (i) follows directly from Theorem 7.2. To justify (ii), observe that under the assumptions made there is $C > 0$ such that

$$\sum_{i=0}^{k-1} \min\big\{\lambda_i^2, \lambda_i^{2(1-q)}\big\}\big\} \ge C \sum_{i=0}^{k-1} \min\big\{i^{2s}, i^{2s(1-q)}\big\}.$$

Note that for all $i \ge 1$ we have $i^{2s} \ge i^{2s(1-q)}$, and so there exists $\delta \in (0,1)$ such that

$$\sum_{i=0}^{k-1} \min\big\{\lambda_i^2, \lambda_i^{2(1-q)}\big\}\big\} \ge \delta \sum_{i=0}^{k-1} i^{2s(1-q)} \quad \text{for all} \quad k \in \mathbb{N}.$$

Letting $\alpha := 2s(1-q)$, we see that $\alpha \le 1$, and so

$$\sum_{i=0}^{k-1} i^\alpha \ge \Big(\sum_{i=0}^{k-1} i\Big)^\alpha = \Big(\frac{k(k-1)}{2}\Big)^\alpha = O\big(k^{2\alpha}\big) = O\big(k^{4s(1-q)}\big).$$

[†] It is worth noting that the convergence $d(x_k; T^{-1}(0)) \to 0$ does not necessarily guarantee that the sequence $\{x_k\}$ converges in the norm sense to a solution in $T^{-1}(0)$.

Thus the error estimate of Theorem 7.2 implies that the sequence $\{d(x_k; T^{-1}(0))\}$ converges to zero with the convergence rate at least $O(k^{-2sq})$.

To justify assertion (iii), observe similarly to the proof of (ii) that there exists a number $\delta \in (0,1)$ for which we have

$$\sum_{i=0}^{k-1} \min\big\{\lambda_i^2, \lambda_i^{2(1-q)}\big\} \geq \delta \sum_{i=0}^{k-1} i^{2s(1-q)} \quad \text{whenever} \quad k \in \mathbb{N}.$$

Furthermore, it follows from $\alpha = 2s(1-q) \geq 1$ and the convexity of the function $f(x) := x^\alpha$ when $x \geq 0$ that we get

$$\sum_{i=0}^{k-1} i^\alpha \geq k^{1-\alpha}\Big(\sum_{i=0}^{k-1} i\Big)^\alpha = k^{1-\alpha}\Big(\frac{k(k-1)}{2}\Big)^\alpha = O(k^{1+\alpha}) = O\big(k^{1+2s(1-q)}\big).$$

Using again the error estimate of Theorem 7.2 ensures that the sequence $\{d(x_k; T^{-1}(0))\}$ converges to zero with the convergence rate at least $O\big(k^{-\frac{q(1+2s(1-q))}{2(1-q)}}\big)$.

Assertion (iv) follows from Theorem 7.2 by observing that

$$\lim_{k\to\infty} \sqrt[k]{d\big(x_k; T^{-1}(0)\big)} = \sqrt{\frac{1}{c^{-2}\lambda^2 + 1}} < 1$$

in this case. Assertion (v) is justified similarly to (iv), which completes the proof. $\square$

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