On Polynomial Optimization over Non-compact Semi-algebraic Sets *

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Abstract

The optimal value of a polynomial optimization over a compact semialgebraic set can be approximated as closely as desired by solving a hierarchy of semidefinite programs and the convergence is finite generically under the mild assumption that a quadratic module generated by the constraints is Archimedean. We consider a class of polynomial optimization problems with non-compact semialgebraic feasible sets, for which the associated quadratic module, that is generated in terms of both the objective function and the constraints, is Archimedean. We show that for such problems, the corresponding hierachy converges and the convergence is finite generically. Moreover, we prove that the Archimedean condition (as well as a sufficient coercivity condition) can be checked numerically by solving a similar hierarchy of semidefinite programs. In other words, under reasonable assumptions the now standard hierarchy of semidefinite programming relaxations extends to the non-compact case via a suitable modification.

Key words. Polynomial optimization, non-compact semi-algebraic sets, semidefinite programming relaxations, Positivstellensatzë

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1 Introduction

The last decade has seen several developments in polynomial optimization [1,2,3]. In particular, a systematic procedure has been established to solve Polynomial Optimization Problems (POP) on compact basic semi-algebraic sets. It consists of a hierarchy of (convex) semidefinite relaxations of increasing size whose associated sequence of optimal values is monotone non-decreasing and converges to the global optimum. The proof of this convergence is based on powerful theorems from real algebraic geometry on the representation of polynomials that are positive on a basic semi-algebraic set, the so-called Positivstellensatze of Schmüdgen [4] and Putinar [5].

Under mild assumptions the convergence has been proved to be finite for the class of convex POPs and even at the first step of the hierarchy for the subclass of convex POPs defined with SOS-convex polynomials* [3,6]. In addition, as recently proved by Nie [7] and Marshall [8], finite convergence is generic for POPs on compact basic semi-algebraic sets.

However, all the above results hold in the compact case, i.e., when the feasible set K is a compact basic semi-algebraic set and (for the most practical hierarchy) its defining polynomials satisfy an additional Archimedean assumption. A notable exception is the case of SOS-convex POPs for which convergence is finite even if K is not compact (and of course if f has a minimizer in K).

When the feasible set is a non compact basic semi-algebraic set, Schmüdgen and Putinar's Positivstellensatzë do not hold any more and in fact, as shown in Scheiderer [9], there are fundamental obstructions to such representations in the noncompact case. The non compact case $K = \mathbb{R}^n$ reduces to the compact case if one guesses a ball in which a minimizer exists or one may optimize over the gradient ideal via the specialized hierarchy proposed in Nie et al. [10]. In both cases one assumes that a minimizer exists which can be enforced if instead one minimizes an appropriate perturbation of the initial polynomial f as proposed in Hanzon and Jibetean [11] and Jibetean and Laurent [12]. To avoid assuming existence of a minimizer Schweighofer [13] introduced the notion of gradient tentacle along with an appropriate hierarchy of SDP relaxations, later improved by Hà and Vui [14] who instead use the truncated tangency variety. Remarkably, both hierarchies converge to the global minimum even if there is no minimizer.

On the other hand the so-called Krivine-Stengle Positivstellensatz provides a certificate of positivity even in the non compact case. Namely it states that a polynomial f is positive on K if and only if $pf = 1 + q$ for some polynomials p and q that both must belong to the preordering associated with the polynomials that define K. However, the latter representation is not practical for two reasons: Firstly, requiring that the polynomials p and q belong to the preordering introduces 2^{m+1} unknown SOS polynomials (as opposed to $m+1$ SOS polynomials in Putinar's Positivstellensatz for the compact case). And so, for example, given a polynomial f, checking whether or not $pf = 1 + q$ for some polynomials p, q in the preordering, is very costly from a computational viewpoint. Secondly, as the unknown polynomial p multiplies f, this representation is not practical for optimization purposes when f is replaced with $f - \lambda$ where λ has to be maximized. Indeed one cannot define a hierarchy of semidefinite programs because of the nonlinearities introduced by the product p.

In this paper, we consider a class of polynomial optimization problems with non-compact

*A polynomial f is SOS-convex if its Hessian $\nabla^2 f(x)$ factors as $L(x)L(x)^T$ for some matrix polynomial $L(x)$.

semialgebraic feasible sets, for which the associated quadratic module, that is generated in terms of both the objective function and the constraints, is Archimedean. We show that for such problems, the corresponding hierachy converges and the convergence is finite generically. Moreover, we prove that the Archimedean condition (as well as a sufficient coercivity condition) can be checked numerically by solving a similar hierarchy of semidefinite programs. We present the technical details of our contribution to polynomial optimization in Section 3.

2 Preliminaries

We write $f \in \mathbb{R}[x]$ (resp. $f \in \mathbb{R}_d[x]$) if f is a real polynomial on $\mathbb{R}^n$ (resp. f is a real polynomial with degree at most d). One can associate the l_1-norm on $\mathbb{R}[x]$ defined by $\|f\|_1 = \sum_\alpha |f_\alpha|$ where $f(x) = \sum_\alpha f_\alpha x^\alpha$. We say that a real polynomial f is sum of squares (SOS) [1,2,3] if there exist real polynomials f_j, $j = 1, \ldots, r$, such that $f = \sum_{j=1}^r f_j^2$. The set of all sum of squares real polynomials is denoted by $\Sigma^2[x]$ while $\Sigma_d^2[x] \subset \Sigma^2[x]$ denotes its subset of all sum of squares of degree at most d.

Coerciveness. We say that a polynomial f is coercive if $\lim_{\|x\| \to +\infty} f(x) = +\infty$. Clearly, a coercive polynomial must be of even degree. A typical example of a coercive polynomial is that $f(x) = \sum_{i=1}^n a_i x_i^{2d} + g(x)$ where $a_i > 0$ and $g \in \mathbb{R}[x]_{2d-1}$. This fact also implies that the set of coercive polynomials is dense in $\mathbb{R}[x]$ for l_1-norm because, for any $f \in \mathbb{R}[x]$ with degree p, $f + \epsilon \sum_{i=1}^n x_i^{2p}$ is coercive.

Archimedean property. With a semi-algebraic set K defined as

$$K := \{x : g_j(x) \geq 0, \ j = 1, \ldots, m; \quad h_l(x) = 0, \ l = 1, \ldots, r\}, \tag{1}$$

is associated its quadratic module $M(g; h) = M(g_1, \ldots g_m; h_1, \ldots, h_r) \subset \mathbb{R}[x]$ defined as

$$\begin{aligned} M(g; h) \ := \ & \{\sigma_0 + \sum_{j=1}^m \sigma_j(x) g_j(x) + \sum_{l=1}^r \phi_l(x) h_l(x) : \\ & \sigma_j \in \Sigma^2[x], \ j = 0, 1, \ldots, m; \quad \phi_l \in \mathbb{R}[x], \ l = 1, \ldots, r\}. \end{aligned}$$

For practical computation we also have the useful truncated version $M_k(g; h)$ of $M(g; h)$, i.e., letting $v_j := \lceil \deg(g_j)/2 \rceil$, $j = 1, \ldots, m$, and $w_l := \deg(h_l)$, $l = 1, \ldots, r$,

$$\begin{aligned} M_k(g; h) \ := \ & \{\sigma_0 + \sum_{j=1}^m \sigma_j(x) g_j(x) + \sum_{l=1}^r \phi_l(x) h_l(x) : \\ & \sigma_j \in \Sigma^2[x]_{k-v_j}, \ j = 0, 1, \ldots, m; \quad \phi_l \in \mathbb{R}[x]_{2k-w_l}, \ l = 1, \ldots, r\}. \end{aligned} \tag{2}$$

Of course, membership in $M(g; h)$ provides immediately with a certificate of non negativity on K. The quadratic module $M(g; h)$ is said to be Archimedean if there exists a polynomial $p \in M(g; h)$ such that the superlevel set $\{x \in \mathbb{R}^n : p(x) \geq 0\}$ is compact. An equivalent definition is that there exists $R > 0$ such that $x \mapsto R - \sum_{i=1}^n x_i^2 \in M(g; h)$. Observe that if $M(g; h)$ is Archimedean then the set K is compact.

The following important result on the representation of polynomials that are positive on K is from Putinar [5].

Theorem 2.1. (Putinar Positivstellensatz [5]) *Let $K \subset \mathbb{R}^n$ be as in (1) and assume that $M(g;h)$ is Archimedean. If $f \in \mathbb{R}[x]$ is positive on K then $f \in M(g;h)$.*

An even more powerful result due to Krivine-Stengle is valid on more general semi-algebraic set (not necessarily compact). Denote by $P(g;h) \subset \mathbb{R}[x]$ the preordering associated with K in (1), i.e., the set defined by

$$P(g;h) := \left\{ \sum_{\alpha\in\{0,1\}^m} \sigma_\alpha \, g_1^{\alpha_1} \dots g_m^{\alpha_m} + \sum_{l=1}^{r} \phi_l h_l : \sigma_\alpha \in \Sigma^2[x], \phi_l \in \mathbb{R}[x] \right\}.$$

Theorem 2.2. (Krivine-Stengle Positivstellensatz) *Let $K \subseteq \mathbb{R}^n$ be as in (1).*

(i) If $f \in \mathbb{R}[x]$ is nonnegative on K then $pf = f^{2s} + q$ for some $p, q \in P(g;h)$ and some integer s.

(ii) If $f \in \mathbb{R}[x]$ is positive on K then $pf = 1 + q$ for some $p, q \in P(g;h)$.

Notice the difference between Putinar and Krivine-Stengle certificates. On the one hand, the latter is valid for non-compact sets K but requires knowledge of two elements $p, q \in P(g;h)$, i.e., 2^{m+1} SOS polynomial weights associated with the g_j's and $2r$ polynomials associated with the h_l's, in their representation. On the other hand, the former is valid only for compact sets K with the Archimedean property, but it requires knowledge of only $m+1$ SOS weights and r polynomials.

3 Contribution

In the present paper, we consider the polynomial optimization problem (POP)

$$(\mathbf{P}) \qquad f^* = \inf\{f(x) : x \in K\},$$

for a possibly non-compact basic semi-algebraic set

$$K := \{x \in \mathbb{R}^n : g_j(x) \le 0;\ j = 1, \dots, m; \quad h_l(x) = 0;\ l = 1 \dots, r\},$$

for some polynomials $(f, g_j, h_l) \subset \mathbb{R}[x]$. We assume that $(\mathbf{P})$ is well-posed in the sense that $f^* = f(x^*)$ for at least one minimizer $x^* \in K$. A typical counter example is

$$f(x,y) = \inf\{x + (1 - xy)^2 : x \ge 0\}$$

where $f^* = 0$ is not attained and $(\frac{1}{k}, k)$ is an unbounded minimizing sequence.

An important class of well-posed POPs are those for which $\tilde{K} := \{x \in K : f(x) \le c\}$ is compact, where $c \ge f(x_0)$ for some $x_0 \in K$. In particular notice that this holds true with $c = f^*$ whenever f has finitely many (global) minimizers in K (the generic case).

Our contribution is to show that if the quadratic module,

$$M(g;h;c-f)\ (:= M(g_1,\ldots,g_m;h_1,\ldots,h_r;c-f))$$

generated by the g_j's, the $\pm h_l$'s and the polynomial $c-f$, is Archimedean, then one may approximate as closely as desired the optimal value f^* of (**P**). Indeed it suffices to replace K with $\tilde{K} := K \cap \{x : c - f(x) \geq 0\}$ and apply the the associated standard hierarchy of semidefinite relaxations defined for the compact case. That is, one solves the hierarchy of semidefinite programs:

$$f_k = \sup\{\lambda : f - \lambda \in M_k(g;h;c-f)\}, \qquad k \in \mathbb{N}, \tag{3}$$

where $M_k(g;h;f)$ is the restricted version of the quadratic module $M(g;h;c-f)$ in which the polynomial weights have a degree bound that depend on k. And if the quadratic module $M(g;h;c-f)$ is Archimedean then the monotone convergence $f_k \uparrow f^*$ as $k \to \infty$, is guaranteed† and, is finite generically. Moreover, in such a context our result is to be interpreted as a simplified version of the celebrated Krivine-Stengle Positivstellensatz.

From a mathematical point of view this is a relatively straightforward extension as it reduces the non compact case to the compact case by using $\tilde{K}$ instead of K. However, one main goal of the paper is to show that, under some numerically checkable assumptions, the standard hierarchy of SDP relaxations defined for the compact case indeed can be adapted to the non compact case modulo a slight modification. For instance, when f is coercive, the level set $\{x \in \mathbb{R}^n : c - f(x) \geq 0\}$ is compact (and so the Archimedean condition is satisfied). This coercivity condition of the objective function f to minimize is very natural in many POPs as it simply means that $f(x)$ grows to infinity as $\|x\| \to \infty$ (e.g. when f is a strongly convex polynomial).

Importantly, we also show that the Archimedean and coercivity conditions can be checked numerically by solving a hierarchy of semidefinite programs until some test is passed.

4 Main Results

With K as in (1) and $f \in \mathbb{R}[x]$, let $d := \max[\deg f; \deg g_j; \deg h_l]$ and let $s(d) := \binom{n+d}{n}$. We say that a property holds generically for $f, g_j, h_l \in \mathbb{R}[x]_d$ if the coefficients of the polynomials f, g_j, h_l (as vectors of $\mathbb{R}^{s(d)}$) do not satisfy a system of finitely many polynomial equations. Equivalently, if the coefficients of f, g_j, h_l belong to an open Zariski subset of $\mathbb{R}^{(1+m+r)s(d)}$.

A consequence of Theorem 2.2 is that, computing the global minimum of f on K reduces to solving the optimization problem

$$\begin{aligned} f^* &= \sup_{\lambda}\{\lambda \,:\, f - \lambda > 0 \text{ on } K\} \\ &= \sup_{p,q,\lambda}\{\lambda \,:\, p(f-\lambda) = 1 + q; \quad p,\, q \in P(g;h)\}. \end{aligned} \tag{4}$$

†The hierarchy of semidefinite programs (3) was studied in [15] for convex POPs for which the monotone convergence, $f_k \uparrow f^*$ as $k \to \infty$, has been proved to be true always (i.e. without any regularity condition).

But as already mentioned for Krivine-Stengle's Positivstellensatz, the above formulation (4) is not appropriate because of the product $p\lambda$ which precludes from reducing (4) to a semidefinite program. Moreover there are 2^{m+2} SOS polynomial weights in the definition of p and q in (4). However, inspired by Theorem 2.2 we now provide sufficient conditions on the data (f, g, h) of problem ($\mathbf{P}$) to provide a converging hierarchy of semidefinite programs for solving ($\mathbf{P}$).

Theorem 4.1. *Let $\emptyset \neq K \subset \mathbb{R}^n$ be as in (1) for some polynomials g_j, h_l, $j = 1, \ldots, m$ and $l = 1, \ldots, r$. Let $x_0 \in K$ and let $c \in \mathbb{R}_+$ be such that $c > f(x_0)$. Suppose that the quadratic module $M(g; h; c - f)$ is Archimedean. Then:*

$$\begin{aligned} f^* &= \inf_x \{ f(x) \,:\, x \in K \} \\ &= \sup_\lambda \{ \lambda \,:\, f - \lambda \in M(g; h; c - f) \} \\ &= \lim_{k\to\infty} \sup_\lambda \{ \lambda \,:\, f - \lambda \in M_k(g; h; c - f) \}. \end{aligned} \tag{5}$$

Moreover $f^* = f(x^*)$ for some $x^* \in K$, and generically

$$f^* = \max_\lambda \{ \lambda \,:\, f - \lambda \in M_k(g; h; c - f) \}, \tag{6}$$

for some index k. That is, f^* is obtained after solving finitely many semidefinite programs.

Proof. It suffices to observe that $f^* = \inf_x \{ f(x) : x \in K;\ c - f(x) \geq 0 \}$. And so if the quadratic module $M(g; h; c - f)$ is Archimedean then the set $\tilde{K} := K \cap \{ x : c - f(x) \geq 0 \}$ is compact. Therefore $f^* = f(x^*)$ for some $x^* \in \tilde{K}$. Moreover, $f_k \uparrow f^*$ as $k \to \infty$, where f_k^* is the value of the semidefinite relaxation [1,2,3])

$$f_k^* := \sup \{ \lambda \,:\, f - \lambda \in M_k(g; h; c - f) \}, \qquad k \in \mathbb{N}.$$

Finally, invoking Nie [7], finite convergence takes place generically. □

On the other hand, in the case of a convex polynomial optimization problem ($\mathbf{P}$), exploiting the special structure of convex polynomials, the monotone convergence, $f_k \uparrow f^*$ as $k \to \infty$, has been proved in [15] to be true always without any regularity condition.

Note that in passing that if $f > 0$ on K then $f > 0$ on $\tilde{K} = K \cap \{ x : c - f(x) \geq 0 \}$, and so if $M(g; h; c - f)$ is Archimedean then by Theorem 2.1,

$$f \;=\; \underbrace{\sigma_0 + \sum_{j=1}^m \sigma_j\, g_j + \sum_{l=1}^r \phi_l\, h_l}_{q \in M(g;h)} + \psi (c - f),$$

for some $q \in M(g; h)$ and some SOS polynomial $\psi \in \Sigma^2[x]$. Equivalently, $(1 + \psi) f = q + \underbrace{c\psi}_{SOS}$, i.e., $(1 + \psi) f \in M(g; h)$ for some SOS polynomial $\psi \in \Sigma^2[x]$.

In other words, one has shown:

Corollary 4.1. *Let $\emptyset \neq K \subset \mathbb{R}^n$ be as in (1) for some polynomials g_j, h_l, $j = 1, \ldots, m$ and $l = 1, \ldots, r$. Let $x_0 \in K$ and let $c \in \mathbb{R}_+$ be such that $c > f(x_0)$. Suppose that the quadratic module $M(g; h; c - f)$ is Archimedean.*

If $f > 0$ on K then $(1+\psi)f \in M(g; h)$ for some SOS polynomial $\psi \in \Sigma^2[x]$. And generically, if $f \geq 0$ on K then $(1 + \psi)f \in M(g; h)$ for some SOS polynomial $\psi \in \Sigma^2[x]$.

Corollary 4.1 can be regarded as a simplified form of Krivine-Stengle's Positivstellensatz which holds whenever the quadratic module $M(g; h; c - f)$ is Archimedean.

It is also worth noting that the assumption "quadratic module $M(g; h; c-f)$ is Archimedean" is weaker than the assumption "the quadratic module $M(g; h)$ is Archimedean" used in the Putinar's Positivstellensatz. Indeed, obviously if $M(g; h)$ is Archimedean then $M(g; h; c - f)$ is also Archimedean, whereas the converse is not true in general (see Example 4.1).

4.1 Checking the Archimedean Property

We have seen that the quadratic module $M(g; h; c - f)$ is Archimedean if and only if there exists $N > 0$ such that the quadratic polynomial $x \mapsto N - \|x\|^2$ belongs to $M(g; h; c - f)$. This is equivalent to:

$$\inf \{\lambda \,:\, \lambda - \|x\|^2 \,\in\, M(g; h; c - f)\} \,<\, +\infty,$$

which, in turn, is equivalent to

$$\rho_k \,:=\, \inf \{\lambda \,:\, \lambda - \|x\|^2 \,\in\, M_k(g; h; c - f)\} \,<\, +\infty \tag{7}$$

for some $k \in \mathbb{N}$.

We note that for each fixed $k \in \mathbb{N}$, solving (7) reduces to solving a semi-definite program. And so checking whether the Archimedean property is satisfied reduces to solving the hierarchy of semidefinite programs (7), $k \in \mathbb{N}$, until $\rho_k < +\infty$ for some k.

Example 4.1. Consider the two-dimensional illustrative example where $f(x_1, x_2) = x_1^2 + 1$, $g_1(x_1, x_2) = 1 - x_2^2$ and $g_2(x_1, x_2) = x_2^2 - 1/4$. Let $c = 2$. The corresponding hierarchy (7) reads

$$\begin{aligned} \rho_k \;=\; \inf\{\lambda \,:\, \lambda - \|x\|^2 \;&=\; \sigma_0 + \sigma(2 - f) + \sigma_1 g_1 + \sigma_2 g_2 \\ &\quad \text{for some } \sigma, \sigma_0, \sigma_1, \sigma_2 \in \Sigma_k^2[x]\}. \end{aligned}$$

Using the following simple code

```
sdpvar x1 x2 lower;
f=x1^2+1;
g=[1-x2^2;x2^2-1/4;2-f];
h=x1^2+x2^2
[s1,c1]=polynomial([x1,x2],2)
[s2,c2]=polynomial([x1,x2],2)
[s3,c3]=polynomial([x1,x2],2)
F = [sos(lower-h-[s1 s2 s3]*g), sos(s1), sos(s2), sos(s3)];
solvesos(F,lower,[],[c1;c2;c3;lower]);
```

from the matlab toolbox Yamlip [16,17], we obtain $\rho_2 = 2 > 0$, and an optimal solution

$$\begin{aligned}
\sigma_0 &= 1.325558711 - 1.3103x_1^2 - 1.3408x_2^2 + 0.4609x_1^4 + 0.3885x_1^2x_2^2 + 0.4761x_2^4 \\
\sigma &= 0.3443415642 + 0.4609x_1^2 + 0.1947x_2^2 \\
\sigma_1 &= 0.3300997245 - (1.4964e-15)x_1 + (1.4078e-15)x_2 + 0.1938x_1^2 + 0.4761x_2^2 \\
&\quad +(9.0245e-15)x_1x_2 \\
\sigma_2 &= (2.570263721e-11) + (1.0557e-16)x_1 + (1.0549e-16)x_2 + (2.6157e-11)x_1^2 \\
&\quad +(2.5534e-11)x_2^2 + (1.0244e-15)x_1x_2.
\end{aligned}$$

So we conclude that $M(g;h;c-f)$ is Archimedean. On the other hand, clearly, $f(x) > 0$ for all $x \in K = \{x : g_i(x_1,x_2) \le 0, i = 1,2\} = \{(x_1,x_2) : x_1 \in \mathbb{R}, x_2 \in [-1,-1/2] \cup [1/2,1]\}$. Direct verification gives that f is not coercive and $M(g_1,g_2)$ is not Archimedean (as K is noncompact). Let $\bar{x} = (0,-1)$ and let $c = 2 > 1 = f(\bar{x})$. We have already shown that $M(g,h,c-f)$ is Archimedean. Direct verification shows that

$$x_1^2 + 1 = \frac{1}{2} + 2x_1^2(1-x_2^2) + 2x_1^2(x_2^2 - 1/4) + \frac{1}{2}[2 - (x_1^2+1)].$$

So, letting $\delta = \frac{1}{2}$, $\sigma_1(x) = 2x_1^2$, $\sigma_2(x) = 2x_1^2$ and $\sigma(x) = \frac{1}{2}$, we see that the following positivity certification holds

$$f = \delta + \sigma_1 g_1 + \sigma_2 g_2 + \sigma(c-f).$$

So Example 4.1 illustrates the case where even if K is not compact and f is not coercive, still the quadratic module $M(g;h;c-f)$ is Archimedean.

We next provide an easily verifiable condition guaranteeing that the quadratic module $M(g;h;c-f)$ is Archimedean in terms of coercivity of the functions involved.

Proposition 4.1. *Let K be as in (1), $x_0 \in K$ and let $c \in \mathbb{R}_+$ be such that $c > f(x_0)$. Then, the quadratic module $M(g;h;c-f)$ is Archimedean if there exist $\alpha_0, \lambda_j \ge 0$, $j = 1,\ldots,m$, and $\mu_l \in \mathbb{R}$, $l = 1,\ldots,p$, such that the polynomial $\alpha_0 f - \sum_{j=1}^m \lambda_j g_j - \sum_{l=1}^p \mu_l h_l$ is coercive.*

In particular, $M(g;h;c-f)$ is Archimidean if f is coercive[‡].

Proof. To see that the quadratic module $M(g;h;c-f)$ is Archimedean, note that from its definition,

$$p := \alpha_0(c-f) + \sum_{j=1}^m \lambda_j g_j + \sum_{l=1}^p \mu_l h_l \in M(g;h;c-f).$$

Now, $\{x : p(x) \ge 0\} = \{x : \alpha_0 f(x) - \sum_{j=1}^m \lambda_j g_j(x) - \sum_{l=1}^p \mu_l h_l(x) \le \alpha_0 c\}$ is nonempty (as $x_0 \in \{x : p(x) \ge 0\}$) and compact (by our coercivity assumption). This implies that the quadratic module $M(g;h;c-f)$ is Archimedean.

The particular case when f is coercive follows from the general case with $\alpha_0 = 1$ and $\lambda_j, \mu_l = 0$ for all j, l. □

We now show how the coercivity of a nonconvex polynomial can easily be checked by solving a sequence of semi-definite programming problems.

[‡]As shown in [15], for a convex polynomial f, the positive definiteness of the Hessian of f at a single point guarantees coercivity of f.

4.2 Checking the Coercive Property

For a non convex polynomial $f \in \mathbb{R}[x]_d$ with d even, decompose f as the sum

$$f = f_0 + f_1 + \dots, +f_d,$$

where each f_i, $i = 0, 1, \dots, d$, is an homogeneous polynomial of degree i. Let $\theta \in \mathbb{R}[x]$ be defined by $\theta(x) := \|x\|^2 - 1$ and let $M(\theta)$ be the quadratic module

$$M(\theta) := \{\sigma + \phi\,\theta \,:\, \sigma \in \Sigma^2[x]; \quad \phi \in \mathbb{R}[x]\}.$$

Lemma 4.1. *If there exists $\delta > 0$ such that $f_d(x) \geq \delta\|x\|^d$, then f is coercive and in addition:*

$$f_d(x) \geq \delta\|x\|^d \;\Leftrightarrow\; 0 < \sup_{\|x\|=1} \{\mu : f_d(x) - \mu \geq 0\} \tag{8}$$

$$\Leftrightarrow\; 0 < \sup\{\mu : f_d - \mu \in M(\theta)\}. \tag{9}$$

Proof. Assume that there exists $\delta > 0$ such that $f_d(x) \geq \delta\|x\|^d$. To see that f is coercive, suppose, on the contrary, that there exists $\{x_k\} \subseteq \mathbb{R}^n$ and $M > 0$ such that $\|x_k\| \to \infty$ and $f(x_k) \leq M$ for all $k \in \mathbb{N}$. This implies that

$$\frac{f(x_k)}{\|x_k\|^d} \leq \frac{M}{\|x_k\|^d} \to 0, \text{ as } k \to \infty. \tag{10}$$

On the other hand, as each f_i is a homogeneous function with degree i, for each $i = 0, 1, \dots, d-1$ we have

$$\frac{f_i(x_k)}{\|x_k\|^d} = f_i\left(\frac{x_k}{\|x_k\|}\right)\frac{1}{\|x_k\|^{d-i}} \to 0, \text{ as } k \to \infty.$$

So, this together with the hypothesis gives us that, for sufficiently large k,

$$\frac{f(x_k)}{\|x_k\|^d} = \frac{f_0(x_k)}{\|x_k\|^d} + \frac{f_1(x_k)}{\|x_k\|^d} + \dots + \frac{f_d(x_k)}{\|x_k\|^d} \geq \frac{\delta}{2},$$

which contradicts (10). Hence, f is coercive.

By homogeneity, the condition, $f_d(x) \geq \delta\|x\|^d$ for all $x \in \mathbb{R}^n$, is equivalent to the condition that $f_d(x) \geq \delta$, for all $x \in B := \{x : \|x\| = 1\}$. And so, $0 < \delta \leq \rho := \sup_{x \in B}\{\mu \,:\, f_d(x) - \mu \geq 0\}$. Conversely, if $\rho > 0$ then $f_d(x) \geq \rho\|x\|^d$ for all x and so the equivalence (8) follows. Then the equivalence (9) also follows because B is compact and the quadratic module $M(\theta)$ is Archimedean. □

It easily follows from Lemma 4.1 that the sufficient coercivity condition that $f_d(\cdot) \geq \delta\|\cdot\|^d$ for some $\delta > 0$ can be numerically checked by solving the following hierarchy of semidefinite programs:

$$\rho_k = \sup\{\mu : f_d - \mu \in M_k(\theta)\}, \qquad k \in \mathbb{N},$$

until $\rho_k > 0$ for some k. The following simple example illustrates how to verify the coercivity of a polynomial by solving semidefinite programs.

Example 4.2. With $n = 2$ consider the degree 6 polynomial

$$x \mapsto f(x) := x_1^6 + x_2^6 - x_1^3 x_2^3 + x_1^4 - x_2 + 1.$$

To test whether f is coercive, consider its highest degree term

$$x \mapsto f_6(x) = x_1^6 + x_2^6 - x_1^3 x_2^3$$

and the associated hierarchy of semidefinite programs:

$$\rho_k = \sup_{\mu,\phi,\sigma} \{\mu : f_6 + \phi\,\theta - \mu = \sigma; \quad \phi \in \mathbb{R}_k[x], \quad \sigma \in \Sigma_k^2[x]\}, \quad k \in \mathbb{N}.$$

Running the following simple code

```
p=x^6+y^6-x^3*y^3;
g=[x^2+y^2-1]
[s1,c1]=polynomial([x,y],4)
F = [sos(p-lower-s1*g)];
solvesos(F,-lower,[],[c1;lower]);
```

from the SOS matlab toolbox Yamlip [16,17], one obtains $\rho_4 = 0.125 > 0$, which proves that f is coercive. Indeed, from an optimal solution, one can directly check that the polynomial $x \mapsto f_6(x) - 0.125(x_1^2 + x_2^2)^3$ is an SOS polynomial of degree 6. So, $f_6(x) \geq 0.125(x_1^2 + x_2^2)^3$, and hence f is coercive.

```
u = p-0.125*(x^2+y^2)^3;
F = sos(u);
solvesos(F);
```

5 Conclusions

In this paper we have first provided a simplified version of Krivine-Stengle's Positivstellensatz which holds generically. The resulting positivity certificate is much simpler as it only involves an SOS polynomial and an element of the quadratic module rather than two elements of the preordering. And so it is also easier to check numerically by semidefinite programming.

Inspired by this simplified form we have also shown how to handle POPs on non compact basic semi-algebraic sets provided that some quadratic module is Archimedean. The latter condition (or a sufficient coercive condition) can both be checked by solving a now standard hierarchy of semidefinite programs. The quadratic module is an easy and slight modification of the standard quadratic module when the feasible set is compact, which shows that essentially the non compact case reduces to the compact case when this Archimedean assumption is satisfied. It is worth noting that, in minimizing a polynomial f, the coerciveness of f is natural as it simply means that $f(x)$ grows to infinity as $\|x\| \to \infty$.

It would be of great interest to examine how our scheme can be implemented to solve large scale polynomial optimization problems with possibly non-compact feasible sets by exploiting structures such as sparisty and symmetry. This will be carried out in a forthcoming study.

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