# Some Robust Convex Programs without a Duality Gap*

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**Abstract**

In this paper, we examine the duality gap between the robust counterpart of a primal uncertain convex optimization problem and the optimistic counterpart of its uncertain Lagrangian dual and identify the classes of uncertain problems which do not have a duality gap. The absence of a duality gap (or equivalently zero duality gap) means that the primal worst value equals the dual best value. We first present a new constraint qualification characterizing zero duality gap for convex programming problems under uncertainty. We then show that the constraint qualification always holds for several important classes of robust convex programming problems. They include convex programs with separable inequality constraints under scenario uncertainty, convex optimization problems over faithfully convex inequality constraints under scenario uncertainty and convex programs with quadratic inequality constraints under spectral norm uncertainty.

**Key words.** Robust convex programming, zero duality gap, robust optimization, convex optimization under uncertainty.

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# 1 Introduction

A standard form of convex programming problem [7, 22] in the absence of data uncertainty is the problem

$$\begin{array}{lll} (P) & \inf & f(x) \\ & \text{s.t.} & g_i(x) \leq 0,\ i = 1, \ldots, m, \end{array}$$

where $f, g_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \ldots, m$, are convex functions. The convex programming problem $(P)$ in the face of data uncertainty in the constraints can be captured by the problem

$$\begin{array}{lll} (UP) & \inf & f(x) \\ & \text{s.t.} & g_i(x, v_i) \leq 0,\ i = 1, \ldots, m, \end{array}$$

where $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $g_i(\cdot, v_i)$ is convex and $v_i \in \mathbb{R}^q$ is the uncertain parameter which belongs to the uncertainty set $\mathcal{V}_i \subseteq \mathbb{R}^q$, $i = 1, \ldots, m$.

The *robust counterpart* [4] of the uncertain convex programming problem is the problem

$$\begin{array}{llll} (RP) & v(RP) := & \inf & f(x) \\ & & \text{s.t.} & g_i(x, v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m, \end{array}$$

where the constraints are enforced for all uncertain parameters $v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$.

On the other hand, the Lagrangian dual of the uncertain convex program $(UP)$ is the uncertain problem

$$(DUP) \quad \sup_{\lambda_i \geq 0} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i)\}.$$

The optimistic counterpart of this dual problem is the program

$$(OLD) \quad v(OLD) := \sup_{\substack{\lambda_i \geq 0 \\ v_i \in \mathcal{V}_i}} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i)\},$$

where the supremum of the values of $(DUP)$ is taken over all uncertain parameters $v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$. By construction, the primal worst value, $v(RP)$, is greater than or equal to the dual best value, $v(OLD)$. Symbolically, $v(RP) \geq v(OLD)$.

The duality gap between the robust counterpart (RP) and the optimistic counterpart (OLD), is defined as the difference between the primal worst value and the dual best value, i.e.,

$$\text{duality gap } = v(RP) - v(OLD).$$

Thus, zero duality gap in robust convex programming means that the primal worst value equals the dual best value.

Recently, strong duality was established in [2, 11] between $(RP)$ and $(OLD)$ in the sense that $v(RP) = v(OLD)$ with the attainment in $(OLD)$ (i.e. $v(RP) = v(OLD) = \max_{\substack{\lambda_i \geq 0 \\ v_i \in \mathcal{V}_i}} \inf_{x\in\mathbb{R}^n}\{f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i)\}$). Thus, strong duality ensures that the primal worst value equals the dual best value. However, the converse is often not true. For a simple one dimensional example, let $f(x) = x$ and $g_1(x, v_1) = x^2 + v_1$ where $v_1$ is uncertain and $v_1 \in \mathcal{V}_1 = [-1, 0]$. Then, $v(RP) = \inf\{x : x^2 + v_1 \leq 0, \forall\, v_1 \in [-1, 0]\} = 0$ and

$$\inf_{x\in\mathbb{R}}\{x + \lambda_1(x^2 + v_1)\} = \begin{cases} -\infty, & \text{if } \lambda_1 = 0, \\ -\frac{1}{4\lambda_1} + \lambda_1 v_1, & \text{if } \lambda_1 > 0. \end{cases}$$

So, $v(RP) = v(OLD) = 0$ but the value $v(OLD)$ is not attained. Indeed, strong duality may fail frequently for a robust program which is a semi-infinite convex program, while the duality gap is zero.

The purpose of this paper is to identify the classes of robust convex programs which do not have a duality gap. Identifying such programs is significant because it not only shows that for these problems the primal worst value always equals the dual best value but also points the way to the calculation of the primal worst value by way of examining the value of the optimistic dual which may be computationally tractable. In fact, it has been shown in [15] that the optimistic duals of broad classes of infinite dimensional constrained approximation problems under ellipsoidal uncertainty are computationally tractable finite dimensional convex programming problems.

We show that (1) convex programs with separable convex inequality constraints under scenario uncertainty, (2) convex optimization problems with faithfully convex inequality constraints under scenario uncertainty and (3) convex programs with quadratic inequality constraints under spectral norm uncertainty, exhibit no duality gap. We achieve this by first presenting a constraint qualification characterizing zero duality gap between the robust counterpart (RP) and the optimistic counterpart (OLD), and then verifying that these classes of robust convex programs satisfy the constraint qualification. The verification employs known results of standard convex programs without a duality gap [1, 16, 21].

The organization of the paper is as follows. In Section 2, we review some basic concepts of conjugate function and collect several results of conjugate analysis that will be used later in the paper. In Section 3, we develop conditions characterizing zero duality gap between the robust counterpart (RP) and the optimistic counterpart (OLD). In Section 4, we present three classes of robust convex programs without a duality gap.

# 2 Preliminaries on Conjugate Analysis

We begin this section by fixing notation and preliminaries of convex analysis. Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean space with dimension $n$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y \rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$. For a set $A$ in $\mathbb{R}^n$, the interior (resp. closure, convex hull) of $A$ is denoted by $\text{int}A$ (resp. $\text{cl}(A)$, $\text{co}A$). The set $A$ is a cone if $\lambda A \subseteq A$ for all $\lambda \geq 0$. We say $A$ is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0,1]$, $a_1, a_2 \in A$. The indicator function $\delta_A : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\delta_A(x) := \begin{cases} 0, & \text{if} \quad x \in A, \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.1}$$

For an extended real-valued function $f$ on $\mathbb{R}^n$, the effective domain and the epigraph are respectively defined by $\text{dom}f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ and $\text{epi}f := \{(x, r) \in \mathbb{R}^n \times \mathbb{R} : f(x) \leq r\}$. We say that $f$ is proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^n$ and $\text{dom}f \neq \emptyset$. Moreover, if $\liminf_{x' \to x} f(x') \geq f(x)$ for all $x \in \mathbb{R}^n$, we say $f$ is a lower semicontinuous function. A function $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is called a separable function on $\mathbb{R}^m$ if

$$f(x) = \sum_{l=1}^{m} g_l(x_l) \qquad x = (x_1, \ldots, x_m) \in \mathbb{R}^m \tag{2.2}$$

for some proper lower semicontinuous functions $g_l$ on $\mathbb{R}$ $(1 \leq l \leq m)$. Clearly, any affine function is a separable function. A function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $\mu \in [0,1]$ $f((1-\mu)x + \mu y) \leq (1-\mu)f(x) + \mu f(y)$ for all $x, y \in \mathbb{R}^n$. As usual, for any proper convex function $f$ on $\mathbb{R}^n$, its conjugate function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $f^*(x^*) = \sup\limits_{x \in \mathbb{R}^n} \{\langle x^*, x \rangle - f(x)\}$ for all $x^* \in \mathbb{R}^n$. Clearly, $f^*$ is a proper lower semicontinuous convex function and $\lambda \text{epi}f^* = \text{epi}(\lambda f)^*$ for any $\lambda > 0$. If $f_1, f_2$ are two proper lower semicontinuous convex functions, then we have

$$\text{epi}(f_1 + f_2)^* = \text{cl}(\text{epi}f_1^* + \text{epi}f_2^*). \tag{2.3}$$

Furthermore, the closure is superfluous if $(\text{intdom}f_1) \cap \text{dom}f_2 \neq \emptyset$. For details see [20, 22]. For any proper lower semicontinuous convex functions $f_1, f_2$,

$$f_1 \leq f_2 \;\Leftrightarrow\; f_1^* \geq f_2^* \;\Leftrightarrow\; \text{epi}f_1^* \subseteq \text{epi}f_2^*. \tag{2.4}$$

**Lemma 2.1.** *(cf. [10, 17]) Let $I$ be an arbitrary index set and let $f_i$, $i \in I$, be proper lower semicontinuous convex functions on $\mathbb{R}^n$. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $\sup_{i \in I} f_i(x_0) < \infty$. Then*

$$\text{epi}(\sup_{i \in I} f_i)^* = \text{cl}(\text{co} \bigcup_{i \in I} \text{epi}f_i^*),$$

*where $\sup\limits_{i \in I} f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by $(\sup\limits_{i \in I} f_i)(x) = \sup\limits_{i \in I} f_i(x)$.*

## 3 Characterizing Primal Worst Equals Dual Best

In this section, we develop constraint qualifications which characterize the zero duality gap property. Let $U = \Pi_{i=1}^{m} \mathcal{V}_i$. For each $v = (v_1, \cdots, v_m) \in U$, we define the function $g_v : \mathbb{R}^n \to \mathbb{R}^m$ by

$$g_v(x) = (g_1(x, v_1), \cdots, g_m(x, v_m))$$

and the feasible set of the robust counterpart (RP) by

$$F := \{x : g_i(x, v_i) \leq 0, \forall\, v_i \in \mathcal{V}_i,\, i = 1, \ldots, m\}.$$

**Definition 3.1.** *The function $g^\diamond : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by*

$$g^\diamond(x^*) = \inf_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \big(\langle \lambda, g_v \rangle\big)^*(x^*).$$

The function $g^\diamond$ is called the characteristic function whereas epi$g^\diamond$ is called the characteristic cone as they are used to characterize zero duality gap for robust convex programs.

It is worth noting that for any $v \in U$ and any $\lambda \in \mathbb{R}^m_+$, $\delta_F \geq \langle \lambda, g_v \rangle$. So, by taking conjugation, for each $x^* \in \mathbb{R}^n$,

$$\delta_F^*(x^*) \leq (\langle \lambda, g_v \rangle)^*(x^*).$$

Thus, $g^\diamond$ serves as a majorant to the support function of $F$ in the sense that, for each $x^* \in \mathbb{R}^n$,

$$\delta_F^*(x^*) \leq \inf_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} (\langle \lambda, g_v \rangle)^*(x^*) = g^\diamond(x^*). \tag{3.5}$$

Moreover, it easily follows from the Definition 3.1 that

$$\bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \mathrm{epi}(\langle \lambda, g_v \rangle)^* \subseteq \mathrm{epi} g^\diamond \subseteq \mathrm{cl}\big( \bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \mathrm{epi}(\langle \lambda, g_v \rangle)^* \big). \tag{3.6}$$

**Theorem 3.1.** *Let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, be continuous functions such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is a convex function. Let $F := \{x : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i,\, i = 1, \ldots, m\} \neq \emptyset$. Then*

(i) epi$g^\diamond$ *is a cone.*

(ii) epi$\delta_F^* = \mathrm{cl}\,(\mathrm{co}(\mathrm{epi} g^\diamond))$.

*Proof.* Clearly,

$$g^\diamond(0) = \inf_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \big(\langle \lambda, g_v \rangle\big)^*(0) = -\sup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \inf_{x \in \mathbb{R}^n} \langle \lambda, g_v(x) \rangle \leq -\inf_{x \in \mathbb{R}^n} \langle 0, g_v(x) \rangle = 0.$$

That is, $(0,0) \in \text{epi}g^\diamond$. Let $x^* \in \mathbb{R}^n$ and $a > 0$. Then,

$$g^\diamond(ax^*) = \inf_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \big(\langle \lambda, g_v \rangle\big)^*(ax^*) = a \inf_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \big(\langle \frac{1}{a}\lambda, g_v \rangle\big)^*(x^*) = a g^\diamond(x^*).$$

So, $\text{epi}g^\diamond$ is a cone.

To establish (ii), note from (3.5) that $\text{epi}g^\diamond \subseteq \text{epi}\delta_F^*$. Since $\text{epi}\delta_F^*$ is a closed and convex subset of $\mathbb{R}^{n+1}$, it follows that

$$\text{cl}(\text{co}(\text{epi}g^\diamond)) \subseteq \text{epi}\delta_F^*. \tag{3.7}$$

On the other hand, as $\delta_F = \sup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \langle \lambda, g_v \rangle$, it follows from Lemma 2.1 that

$$\text{epi}\delta_F^* = \text{cl}(\text{co} \bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \text{epi}(\langle \lambda, g_v \rangle)^*). \tag{3.8}$$

Moreover, (3.6) gives us that

$$\text{epi}g^\diamond \supseteq \bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \text{epi}(\langle \lambda, g_v \rangle)^*.$$

So,

$$\text{cl}(\text{co}(\text{epi}g^\diamond)) \supseteq \text{cl}(\text{co} \bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \text{epi}(\langle \lambda, g_v \rangle)^*) = \text{epi}\delta_F^*.$$

This together with (3.7) implies that $\text{epi}\delta_F^* = \text{cl}\,(\text{co}(\text{epi}g^\diamond))$. □

We now see that if $g_i(x, \cdot)$ is concave and $\mathcal{V}_i \subseteq \mathbb{R}^q$ is convex then $\text{epi}g^\diamond$ is a convex cone.

**Proposition 3.1.** *For each $i = 1, \dots, m$, let $\mathcal{V}_i \subseteq \mathbb{R}^q$ be convex and let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ be continuous. Suppose that, for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is convex on $\mathbb{R}^n$ and, for each $x \in \mathbb{R}^n$, $g_i(x, \cdot)$ is concave on $\mathcal{V}_i$. Then, the cone, $\text{epi}g^\diamond$, is convex.*

*Proof.* To complete the proof, it is sufficient to show that $\text{co}(\text{epi}g^\diamond) \subseteq \text{epi}g^\diamond$. Now, let $(x^*, r) \in \text{co}(\text{epi}g^\diamond)$. As $\text{co}(\text{epi}g^\diamond)$ is a cone of dimension $n+1$, from the Carathédory theorem [22, Theorem 17.1, p.155], there exist $(x_l^*, r_l) \in \text{epi}g^\diamond$, $l = 1, \dots, n+2$, such that $(x^*, r) = \sum_{l=1}^{n+2}(x_l^*, r_l)$. Then for each $l = 1, \dots, n+2$, $g^\diamond(x_l^*) \le r_l$, that is,

$$g^\diamond(x_l^*) = \inf_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} (\langle \lambda, g_v \rangle)^*(x_l^*) \le r_l.$$

Hence, for each $l = 1, \dots, n$ and $\varepsilon_l > 0$, there exist $\lambda^l = (\lambda_1^l, \cdots, \lambda_m^l) \in \mathbb{R}^m_+$ and $v^l = (v_1^l, \cdots, v_m^l) \in \Pi_{i=1}^m \mathcal{V}_i$ such that $(\langle \lambda^l, g_{v^l} \rangle)^*(x_l^*) \le r_l + \varepsilon_l$, that is,

$$(x_l^*, r_l + \varepsilon_l) \in \text{epi}(\sum_{i=1}^m \lambda_i^l g(\cdot, v_i^l))^*.$$

So,

$$\sum_{l=1}^{n+2}(x_l^*, r_l + \varepsilon_l) \in \sum_{l=1}^{n+2} \operatorname{epi}\big(\sum_{i=1}^{m} \lambda_i^l g_i(\cdot, v_i^l)\big)^* = \sum_{i=1}^{m} \operatorname{epi}\big(\sum_{l=1}^{n+2} \lambda_i^l g_i(\cdot, v_i^l)\big)^*. \tag{3.9}$$

Let $\lambda_i = \sum_{l=1}^{n+2} \lambda_i^l \geq 0$. Now, we show that, for each $i = 1, \ldots, m$, there exists $v_i \in \mathcal{V}_i$ such that, for each $x \in \mathbb{R}^n$,

$$\sum_{l=1}^{n+2} \lambda_i^l g_i(x, v_i^l) \leq \lambda_i g_i(x, v_i). \tag{3.10}$$

If $\lambda_i = 0$, then $\lambda_i^l = 0$ for all $l = 1, \ldots, n+2$, and so, (3.10) holds for any $v_i \in \mathcal{V}_i$. If $\lambda_i > 0$, then let $v_i = \sum_{l=1}^{n+2} \frac{\lambda_i^l v_i^l}{\lambda_i}$. Because $\mathcal{V}_i$ is convex, it follows that $v_i \in \mathcal{V}_i$. Since $g_i(x, \cdot)$ is a concave function, for each $x \in \mathbb{R}^n$ and for each $i = 1, \ldots, m$,

$$\sum_{l=1}^{n+2} \lambda_i^l g_i(x, v_i^l) = \lambda_i \sum_{l=1}^{n+2} \frac{\lambda_i^l}{\lambda_i} g_i(x, v_i^l) \leq \lambda_i g_i(x, \sum_{l=1}^{n+2} \frac{\lambda_i^l v_i^l}{\lambda_i}) = \lambda_i g_i(x, v_i).$$

So, (3.10) holds, and thus, $\operatorname{epi}\big(\sum_{l=1}^{n+2} \lambda_i^l g_i(\cdot, v_i^l)\big)^* \subseteq \operatorname{epi}(\lambda_i g_i(\cdot, v_i))^*$. Now it follows from (3.9) that

$$\sum_{l=1}^{n+2}(x_l^*, r_l + \varepsilon_l) \in \sum_{i=1}^{m} \operatorname{epi}\big(\lambda_i g_i(\cdot, v_i)\big)^* \subseteq \operatorname{epi}\big(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\big)^*. \tag{3.11}$$

Letting $(\varepsilon_1, \cdots, \varepsilon_{n+2}) \to (0, \cdots, 0)$ in (3.11), we obtain that

$$(x^*, r) \in \operatorname{epi}\big(\sum_{i=1}^{m} \lambda_i g_i(\cdot, v_i)\big)^* \subseteq \operatorname{epi} g^\diamond,$$

which completes the proof. □

The following lemma, which provides basic characterizations of the zero duality gap property, will be useful in deriving a constraint qualification characterizing the same property.

**Lemma 3.1.** *Consider the problems* $(RP)$ *and* $(OLD)$. *Suppose that* $v(RP)$ *is finite. Then the following statements are equivalent:*

(i) $v(RP) = v(OLD)$.

(ii) $(\forall \epsilon > 0)$ $(\exists \lambda \in \mathbb{R}_+^m,\ v \in U)$ $(\forall x \in \mathbb{R}^n)$ $f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i) \geq v(RP) - \epsilon$.

(iii) $(\forall \epsilon > 0)$ $(0, \epsilon - v(RP)) \in \bigcup_{\substack{\lambda \in \mathbb{R}_+^m \\ v \in U}} \operatorname{epi}(f + \langle \lambda, g_v \rangle)^*$.

*Proof.* [(i) $\Leftrightarrow$ (ii)]. As $v(RP) \geq v(OLD)$ always holds, $v(RP) = v(OLD)$ is equivalent to $v(RP) \leq v(OLD)$, which means that

$$v(RP) \leq \sup_{\substack{\lambda_i \geq 0 \\ v_i \in \mathcal{V}_i}} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i)\},$$

which is in turn equivalent to (ii).

[(ii) $\Leftrightarrow$ (iii)]. Observe that (ii) is equivalent to the property: for each $\epsilon \geq 0$, there exist $\lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m_+$ and $v = (v_1, \cdots, v_m) \in U$ such that

$$(f + \langle \lambda, g_v \rangle)^*(0) = -\inf_{x \in \mathbb{R}^n} (f + \langle \lambda, g_v \rangle)(x) = -\inf_{x \in \mathbb{R}^n} (f + \langle \lambda, g_v \rangle)(x) \leq \epsilon - v(RP),$$

which is in turn equivalent to (iii). □

**Theorem 3.2. (Primal Worst equals Dual Best: Characterizations)** *Let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, be continuous functions such that for each $v_i \in \mathbb{R}^q$, $g_i(\cdot, v_i)$ is convex. Let $\mathcal{V}_i \subseteq \mathbb{R}^q$, for each $i = 1, \cdots, m$, and let $F := \{x : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \ldots, m\} \neq \emptyset$. Then the following statements are equivalent.*

(i) epi$g^\diamond$*, is convex and closed.*

(ii) *For each continuous convex function $f$ on $\mathbb{R}^n$,*

$$\inf\{f(x) : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \ldots, m\} = \sup_{\substack{\lambda_i \geq 0 \\ v_i \in \mathcal{V}_i}} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i)\}.$$

(iii) *For each affine function $f$ on $\mathbb{R}^n$,*

$$\inf\{f(x) : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \ldots, m\} = \sup_{\substack{\lambda_i \geq 0 \\ v_i \in \mathcal{V}_i}} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i)\}.$$

(i) $\Rightarrow$ (ii). To see this, fix an arbitrary continuous convex function $f$ on $\mathbb{R}^n$. If $v(RP) = -\infty$ then $v(RP) = v(OLD)$ by weak duality which always holds. As $F \neq \emptyset$, we assume without loss of generality that $v(RP) \in \mathbb{R}$. The conclusion (ii) will follow from Lemma 3.1(iii) if we show that for each $\epsilon > 0$,

$$(0, \epsilon - v(RP)) \in \bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \text{epi}(f + \langle \lambda, g_v \rangle)^*. \tag{3.12}$$

Since $v(RP) = \inf_{x \in F} f(x)$, $(0, -v(RP)) \in \text{epi}(f + \delta_F)^*$. Observe that $\text{epi}(f + \delta_F)^* = \text{epi}f^* + \text{epi}\delta_F^*$. So,

$$(0, -v(RP)) \in \text{epi}f^* + \text{epi}\delta_F^*. \tag{3.13}$$

Using Theorem 3.1(iii) and assumption (i), we obtain that

$$\text{epi}\delta_F^* = \text{cl}\,(\text{co}(\text{epi}g^\diamond)) = \text{epi}g^\diamond.$$

Combining this with (3.13) gives us that $(0, -v(RP)) \in \text{epi}f^* + \text{epi}g^\diamond$. Now, we can find $(a^*, \alpha) \in \text{epi}f^*$ and $(b^*, \beta) \in \text{epi}g^\diamond$ such that $a^* + b^* = 0$ and $\alpha + \beta = -v(RP)$. Since $(b^*, \beta) \in \text{epi}g^\diamond$, it follows that

$$g^\diamond(b^*) = \inf_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \big(\langle \lambda, g_v \rangle\big)^*(b^*) \leq \beta.$$

So, for each $\epsilon > 0$. there exist $\lambda_\epsilon \in \mathbb{R}^m_+$ and $v^\varepsilon \in U$ such that $(\langle \lambda_\epsilon, g_{v^\varepsilon} \rangle)^*(b^*) \leq \beta + \epsilon$. Then, $(b^*, \beta + \epsilon) \in \text{epi}(\langle \lambda_\epsilon, g_{v^\varepsilon} \rangle)^*$ and hence

$$(0, \epsilon - v(RP)) = (a^*, \alpha) + (b^*, \beta + \epsilon) \in \text{epi}f^* + \text{epi}(\langle \lambda_\epsilon, g_{v^\varepsilon} \rangle)^* \subseteq \text{epi}(f + \langle \lambda_\epsilon, g_{v^\varepsilon} \rangle)^*.$$

where the last inclusion follows from (2.3). Thus (3.12) follows and so (ii) holds.

[(ii) $\Rightarrow$ (iii)] This implication directly follows.

[(iii) $\Rightarrow$ (i)] Let $(\alpha, r) \in \text{cl}(\text{co}(\text{epi}g^\diamond))$. To complete the proof, it is sufficient to show that

$$(\alpha, r) \in \text{epi}g^\diamond. \tag{3.14}$$

In fact, by Theorem 3.1(iii), $\text{epi}\delta_F^* = \text{cl}\,(\text{co}(\text{epi}g^\diamond))$ and so $(\alpha, r) \in \text{epi}\delta_F^*$. This gives that $\delta_F^*(\alpha) \leq r$, that is,

$$\sup_{x \in \mathbb{R}^n} \{\langle \alpha, x \rangle - \delta_F(x)\} \leq r,$$

which implies that

$$-\alpha^T x + r \geq 0 \quad \text{ for each } x \in F. \tag{3.15}$$

Define the affine function $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) := -\alpha^T x + r$. Apply assumption (iii) to $f$ and using (3.15), we obtain that

$$\sup_{\substack{\lambda_i \geq 0 \\ v_i \in \mathcal{V}_i}} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i)\} = \inf\{f(x) : x \in F\} \geq 0.$$

Hence, for each $\varepsilon > 0$, there exist $\lambda^\varepsilon = (\lambda_1^\varepsilon, \cdots, \lambda_m^\varepsilon) \in \mathbb{R}^m_+$ and $v^\varepsilon = (v_1^\varepsilon, \cdots, v_m^\varepsilon) \in \Pi_{i=1}^m \mathcal{V}_i = U$ such that, for each $x \in \mathbb{R}^n$,

$$-\alpha^T x + r + \sum_{i=1}^m \lambda_i^\varepsilon g_i(x, v_i^\varepsilon) \geq -\varepsilon \quad .$$

That is, for each $x \in \mathbb{R}^n$,

$$\alpha^T x - \sum_{i=1}^m \lambda_i^\varepsilon g_i(x, v_i^\varepsilon) \leq r + \varepsilon \quad ,$$

and so, $(\langle \lambda^\varepsilon, g_{v^\varepsilon} \rangle)^*(\alpha) \leq r + \varepsilon$. As $g^\diamond \leq (\langle \lambda^\varepsilon, g_{v^\varepsilon} \rangle)^*$, we have

$$g^\diamond(\alpha) \leq r + \varepsilon. \tag{3.16}$$

Letting $\varepsilon \to 0$ in (3.16), we get $g^\diamond(\alpha) \leq r$. Thus, $(\alpha, r) \in \text{epi}g^\diamond$ and so (3.14) holds. □

**Corollary 3.1.** *Let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, $i = 1, \ldots, m$, be continuous functions such that $g_i(\cdot, v_i)$ is convex for each $v_i \in \mathbb{R}^q$ and $g_i(x, \cdot)$ is concave for each $x \in \mathbb{R}^n$. Let $\mathcal{V}_i \subseteq \mathbb{R}^q$ be convex, for each $i = 1, \cdots, m$, and let $F := \{x : g_i(x, v_i) \le 0, \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\} \ne \emptyset$. Then the following statements are equivalent.*

(i) $\mathrm{epi}g^\diamond$ *is closed.*

(ii) *For each continuous convex function $f$ on $\mathbb{R}^n$,*

$$\inf\{f(x) : g_i(x, v_i) \le 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\} = \sup_{\substack{\lambda_i \ge 0 \\ v_i \in \mathcal{V}_i}} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i)\}.$$

(iii) *For each affine function $f$ on $\mathbb{R}^n$,*

$$\inf\{f(x) : g_i(x, v_i) \le 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m\} = \sup_{\substack{\lambda_i \ge 0 \\ v_i \in \mathcal{V}_i}} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i)\}.$$

*Proof.* The conclusion follows from Theorem 3.2 and Proposition 3.1. □

From (3.6), we see that $\mathrm{epi}g^\diamond$ is closed whenever $\bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \mathrm{epi}(\langle \lambda, g_v \rangle)^*$ is closed. On the other hand, as shown in [11], the robust Slater condition that there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0, v_i) < 0$, $\forall\ v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$, guarantees that the set $\bigcup_{\substack{\lambda \in \mathbb{R}^m_+ \\ v \in U}} \mathrm{epi}(\langle \lambda, g_v \rangle)^*$ is closed. Thus, if we assume the robust Slater condition and the concavity of $g_i(x, \cdot)$ for each $i = 1, \ldots, m$, then the primal worst value equals the dual best value. This can also be seen from the result of Beck and Bental [2] and Jeyakumar and Li [11] and the fact that robust strong duality implies the zero duality gap property between the robust counterpart (RP) and the optimistic dual (OLD).

Additional sufficient conditions (e.g. compactness of the robust feasible set and concavity of $v_i \mapsto g_i(x, v_i)$) which ensure that $\mathrm{epi}g^\diamond$ is closed and convex can be found in the appendix.

The following example illustrates the case where for a robust convex programming problem the characteristic cone $\mathrm{epi}g^\diamond$ is a closed and convex cone and the zero duality property holds whereas strong duality fails.

**Example 3.1.** *Consider the convex programming problem with data uncertainty*

$$\inf_{x = (x_1, x_2, x_3) \in \mathbb{R}^3} \{x_1 + x_3 : v_1 x_1^2 + x_2 \le 0,\ -v_2 x_2 \le 0,\ -x_3 + v_3 \le 0\},$$

*where $v_1, v_2, v_3$ are the uncertain parameters, and $v_1 \in [0, 1]$, $v_2 \in [0.5, 1]$ and $v_3 \in [0, 1]$. Let $f(x) = x_1 + x_3$, $g_1(x, v_1) = v_1 x_1^2 + x_2$, $g_2(x, v_2) = -v_2 x_2$, $g_3(x, v_3) = -x_3 + v_3$, $\mathcal{V}_1 = [0, 1]$, $\mathcal{V}_2 = [0.5, 1]$ and $\mathcal{V}_3 = [0, 1]$. We now verify that* $\mathrm{epi}g^\diamond$ *is closed and convex. To see this, we first note that, for each $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+$ and $v_i \in \mathcal{V}_i$, $i = 1, 2, 3$,*

$$\Big(\sum_{i=1}^3 \lambda_i g(\cdot, v_i)\Big)^*(x^*) = \begin{cases} \lambda_3 v_3 + \delta_{\{0\} \times \{\lambda_1 - \lambda_2 v_2\} \times \{\lambda_3\}}(x^*), & \textit{if} \quad \lambda_1 v_1 = 0, \\ \lambda_3 v_3 + \frac{1}{4\lambda_1 v_1} + \delta_{\{\frac{1}{2\lambda_1 v_1}\} \times \{\lambda_1 - \lambda_2 v_2\} \times \{\lambda_3\}}(x^*), & \textit{if} \quad \lambda_1 v_1 \ne 0. \end{cases}$$

*This shows that, for each* $x^* \in \mathbb{R}^3$,

$$g^{\diamond}(x^*) = \inf_{\lambda\in\mathbb{R}^3_+,\, v\in\prod_{i=1}^3 \mathcal{V}_i} \big(\sum_{i=1}^{3} \lambda_i g(\cdot, v_i)\big)^*(x^*) = 0,$$

*and so,* $\mathrm{epi}g^{\diamond}$ *is closed and convex. So, by Theorem 3.2, there is no gap between the value of the robust counterpart and the value of the optimistic dual. Indeed, the feasible set of the robust counterpart is*

$$\begin{aligned} F &= \{(x_1, x_2, x_3) : g_i(x, v_i) \leq 0,\ \forall\ v_i \in \mathcal{V}_i\} \\ &= \{(x_1, x_2, x_3) : v_1 x_1^2 + x_2 \leq 0,\ \forall\ v_1 \in [0, 1],\ -v_2 x_2 \leq 0,\ \forall\ v_2 \in [0.5, 1], \\ &\qquad v_3 - x_3 \leq 0,\ \forall\ v_3 \in [0, 1]\} \\ &= \{(0, 0, x_3) : x_3 \geq 1\}. \end{aligned}$$

*So, the value of the robust counterpart of the primal problem is*

$$v(RP_1) := \inf_{x\in\mathbb{R}^3} \{f(x) : g_i(x, v_i) \leq 0,\ \forall\ v_i \in \mathcal{V}_i,\ i = 1, 2, 3\} = 1.$$

*On the other hand, for each* $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3_+$ *and* $v_i \in \mathcal{V}_i$, $i = 1, 2, 3$,

$$\begin{aligned} &\inf_{x\in\mathbb{R}^3} \{f(x) + \sum_{i=1}^{3} \lambda_i g_i(x, v_i)\} \\ =\ &\inf_{x\in\mathbb{R}^3} \{x_1 + x_3 + \lambda_1(v_1 x_1^2 + x_2) + \lambda_2(-v_2 x_2) + \lambda_3(v_3 - x_3)\} \\ =\ &\inf_{x\in\mathbb{R}^3} \{\lambda_1 v_1 x_1^2 + x_1 + (\lambda_1 - \lambda_2 v_2)x_2 + (1 - \lambda_3)x_3 + \lambda_3 v_3\} \\ =\ &\begin{cases} -\infty, & \textit{if} \quad \lambda_3 \neq 1, \\ -\infty, & \textit{if} \quad \lambda_1 \neq \lambda_2 v_2, \\ -\infty, & \textit{if} \quad \lambda_1 v_1 = 0, \\ -\frac{1}{4\lambda_1 v_1} + v_3, & \textit{if} \quad \lambda_1 = \lambda_2 v_2, \lambda_1 v_1 \neq 0 \ \textit{and}\ \lambda_3 = 1. \end{cases} \end{aligned}$$

*So, the value of the optimistic dual is*

$$v(OLD_1) : \ = \sup_{\lambda_i\geq 0,\, v_i\in\mathcal{V}_i} \inf_{x\in\mathbb{R}^3} \{f(x) + \sum_{i=1}^{3} \lambda g_i(x, v_i)\} = 1,$$

*and the supremum in* $v(OLD_1)$ *is not attained. Hence, zero duality gap holds between the robust counterpart and the optimistic dual while strong duality fails.*

# 4 Robust Convex Problems without a Duality Gap

In this section, we provide classes of uncertain convex optimization problems without a duality gap. We begin this Section by examining convex programming problems with convex separable constraints under scenario uncertainty [4].

## 4.1 Separable inequality constraints & scenario uncertainty

Consider the following convex optimization problems with separable inequality constraints under scenario uncertainty

$$
\begin{array}{lll}
(SP) & \min_{x\in\mathbb{R}^n} & f(x) \\
 & \text{s.t.} & g_i(x,v_i)\le 0,\ i=1,\dots,m,
\end{array}
$$

where $g_i(x,v_i) = g_i^0(x) + \sum_{s=1}^q v_i^s g_i^s(x)$ where $g_i^s$, $i = 1,\dots,m$ and $s = 0,\dots,q$, are all continuous convex separable functions and the uncertain parameter $v_i = (v_i^1,\dots,v_i^q)^T \in \mathbb{R}^q$ belongs to the scenario uncertainty set $Z_i := \text{co}\{w_{i1},\dots,w_{ip_i}\}$ where $p_i \in \mathbb{N}$ and $w_{ij} \in \mathbb{R}^q$, $j = 1,\dots,p_i$ and $i = 1,\dots,m$.

We shall apply the results in the preceding section to show that the zero duality gap property holds for the convex optimization problems with separable inequalities constraint under scenario uncertainty. To do this, let us recall the following zero duality gap result of Tseng [21] which shows that separable convex programming problems without data uncertainty have no duality gap.

**Lemma 4.1.** *([21, Theorem 1]) Let $f$ and $g_i$ $(1 \le i \le m)$ be proper lower semi-continuous separable convex functions on $\mathbb{R}^n$ with* $\text{dom}f \subseteq \bigcap_{i=1}^m \text{dom}g_i$ *and* $\text{dom}f \cap (\cap_{i=1}^m\{x \in \mathbb{R}^n : g_i(x) \le 0\}) \neq \emptyset$. *Then,*

$$\inf\{f(x) : g_i(x) \le 0,\ i = 1,\dots,m\} = \sup_{\lambda_i\ge 0}\inf_{x\in\mathbb{R}^m}\{f(x) + \sum_{i=1}^m \lambda_i g_i(x)\}.$$

**Theorem 4.1.** *For the problem (SP), let $F := \{x : g_i(x,v_i) \le 0,\ \forall v_i \in Z_i\ i = 1,\dots,m\} \neq \emptyset$. Then, for each continuous convex function $f$ on $\mathbb{R}^n$,*

$$
\begin{aligned}
&\inf\{f(x) : g_i(x,v_i) \le 0,\ \forall v_i \in Z_i\ i = 1,\dots,m\} \\
=\ & \sup_{\lambda_i\ge 0, v_i\in Z_i}\ \inf_{x\in\mathbb{R}^n}\{f(x) + \sum_{i=1}^m \lambda_i g_i(x,v_i)\}.
\end{aligned}
$$

*Proof.* The conclusion will follow from Theorem 3.2 if we show that $\text{epi}g^\diamond$, is convex and closed. This will follows if we can verify that $\text{epi}g^\diamond = \text{epi}\delta_F^*$. As $g_i(x,\cdot)$ is affine and $w_{i1},\dots,w_{ip_i}$ are the extreme points of $Z_i$, we have

$$\sup_{v_i\in Z_i} g_i(x,v_i) = \max_{1\le j\le p_i} g_i(x,w_{ij}),\ i = 1,\dots,m.$$

So,

$$F = \{x : g_i(x,w_{ij}) \le 0, i = 1,\dots,m,\ j = 1,\dots,p_i\}.$$

Fix an arbitrary affine function $a$ on $\mathbb{R}^n$ and consider the following convex optimization problem with separable inequality constraints

$$
\begin{array}{lll}
(P_0)\quad v(P_0) := & \inf_{x\in\mathbb{R}^n} & a(x) \\
 & \text{s.t.} & g_i(x,w_{ij}) \le 0,\ i = 1,\dots,m,\ j = 1,\dots,p_i,
\end{array}
$$

and its dual

$$(D_0) \quad v(D_0) := \sup_{\lambda_{ij}\geq 0} \inf_{x\in\mathbb{R}^n} \{a(x) + \sum_{1\leq i\leq m, 1\leq j\leq p_i} \lambda_{ij} g_i(x, w_{ij})\}.$$

Note that $g_i(x, w_{ij})$ are all separable functions, Lemma 4.1 gives us that $v(P_0) = v(D_0)$ for each affine objective function $a$. Applying Theorem 3.2 with the singleton uncertainty set, we obtain that epi $\inf_{\lambda_{ij}\geq 0} \big(\sum_{1\leq i\leq m, 1\leq j\leq 2^q} \lambda_{ij} g_i(\cdot, w_{ij})\big)^*$ is closed and so,

$$\text{epi}\delta_F^* = \text{epi} \inf_{\lambda_{ij}\geq 0} \big(\sum_{1\leq i\leq m, 1\leq j\leq p_i} \lambda_{ij} g_i(\cdot, w_{ij})\big)^*.$$

Now, we show that

$$\text{epi} \inf_{\lambda_{ij}\geq 0} \big(\sum_{1\leq i\leq m, 1\leq j\leq p_i} \lambda_{ij} g_i(\cdot, w_{ij})\big)^* \subseteq \text{epi}g^\diamond.$$

To verify this, take $(x^*, r) \in$ epi $\inf_{\lambda_{ij}\geq 0} \big(\sum_{1\leq i\leq m, 1\leq j\leq p_i} \lambda_{ij} g_i(\cdot, w_{ij})\big)^*$. Then

$$\inf_{\lambda_{ij}\geq 0} \big(\sum_{1\leq i\leq m, 1\leq j\leq p_i} \lambda_{ij} g_i(\cdot, w_{ij})\big)^*(x^*) \leq r,$$

and so, for each $k \in \mathbb{N}$, there exists $\lambda_{ij}^k \geq 0$ such that

$$\big(\sum_{1\leq i\leq m, 1\leq j\leq p_i} \lambda_{ij}^k g_i(\cdot, w_{ij})\big)^*(x^*) \leq r + \frac{1}{k}.$$

That is, for each $x \in \mathbb{R}^n$

$$\begin{aligned}\langle x^*, x\rangle - \sum_{1\leq i\leq m, 1\leq j\leq p_i} \lambda_{ij}^k (g_i^0(x) + \sum_{s=1}^q w_{ij}^s g_i^s(x)) &= \langle x^*, x\rangle - \sum_{1\leq i\leq m, 1\leq j\leq p_i} \lambda_{ij}^k g_i(x, w_{ij}) \\ &\leq r + \frac{1}{k},\end{aligned}$$

where $w_{ij} = (w_{ij}^1, \ldots, w_{ij}^q)^T \in \mathbb{R}^q$. Let $\lambda_i^k = \sum_{1\leq j\leq p_i} \lambda_{ij}^k$, $1 \leq i \leq m$ and

$$v_i^k = (v_i^{k1}, \ldots, v_i^{kq})^T := (\frac{\sum_{1\leq j\leq p_i} \lambda_{ij}^k w_{ij}^1}{\lambda_i^k}, \ldots, \frac{\sum_{1\leq j\leq p_i} \lambda_{ij}^k w_{ij}^q}{\lambda_i^k})^T \in Z_i,$$

we obtain that, for each $x \in \mathbb{R}^n$,

$$\begin{aligned}\langle x^*, x\rangle - \sum_{i=1}^m \lambda_i^k g_i(x, v_i^k) &= \langle x^*, x\rangle - \sum_{1\leq i\leq m} \lambda_i^k (g_i^0(x) + \sum_{s=1}^q v_i^{ks} g_i^s(x)) \\ &= \langle x^*, x\rangle - \sum_{1\leq i\leq m, 1\leq j\leq p_i} \lambda_{ij}^k (g_i^0(x) + \sum_{s=1}^q w_{ij}^s g_i^s(x)) \\ &\leq r + \frac{1}{k}.\end{aligned}$$

This implies that

$$\big(\sum_{i=1}^{m}\lambda_i^k g_i(\cdot,v_i^k)\big)^*(x^*)\le r+\frac{1}{k}.$$

As $g^\diamond \le \big(\sum_{i=1}^m \lambda_i^k g_i(\cdot,v_i^k)\big)^*$, we obtain that $g^\diamond(x^*)\le r+\frac{1}{k}$. Letting $k\to\infty$, we see that $(x^*,r)\in \mathrm{epi}g^\diamond$. So,

$$\mathrm{epi}\delta_F^* = \mathrm{epi}\inf_{\lambda_{ij}\ge 0}\big(\sum_{1\le i\le m,1\le j\le p_i}\lambda_{ij}g_i(\cdot,w_{ij})\big)^*\subseteq \mathrm{epi}g^\diamond.$$

Note that $\mathrm{epi}g^\diamond\subseteq \mathrm{epi}\delta_F^*$ always holds. Hence $\mathrm{epi}\delta_F^*=\mathrm{epi}g^\diamond$. Thus, the conclusion follows. □

## 4.2 Faithfully convex constraints & scenario uncertainty

Recall that a real valued function $h$ is said to be faithfully convex (cf. [22]) if

$$h(x)=\phi(Ax+b)+l^Tx+\beta,$$

where $\phi:\mathbb{R}^p\to\mathbb{R}$ is a strictly convex function, $A:\mathbb{R}^n\to\mathbb{R}^p$ is a linear transform and $b\in\mathbb{R}^p, l\in\mathbb{R}^n,\beta\in\mathbb{R}$. Consider the following convex optimization problem with faithfully convex inequality constraints under scenario uncertainty

$$\begin{array}{lll}(FP) & \min_{x\in\mathbb{R}^n} & f(x)\\ & \text{s.t.} & g_i(x,v_i)\le 0,\ i=1,\dots,m,\end{array}$$

where $f$ is a convex function, $g_i(x,v_i)=g_i^0(x)+\sum_{s=1}^q v_i^s g_i^s(x)$ where $g_i^s$, $i=1,\dots,m$ and $s=0,1,\dots,q$, are all faithfully convex functions and the uncertain parameter $v_i=(v_i^1,\dots,v_i^q)^T\in\mathbb{R}^q$ belongs to the scenario uncertainty set $Z_i:=\mathrm{co}\{w_{i1},\dots,w_{ip_i}\}$ where $p_i\in\mathbb{N}$ and $w_{ij}\in\mathbb{R}^q$, $j=1,\dots,p_i$ and $i=1,\dots,m$.

To establish the desired result, we need the zero duality gap result for weakly analytic convex programming problems without uncertainty. Recall that a real-valued convex function $f$ on $\mathbb{R}^n$ is called weakly analytic whenever the function $f$ takes a constant value on a nonempty open interval, it takes the same constant value on the whole line containing this interval. The class of weakly analytic function was first introduced in [16] and has been investigated thoroughly in [1]. Clearly, any analytic function is weakly analytic. Moreover, any faithfully convex function is also weakly analytic (cf [1, 23]).

**Lemma 4.2.** *(cf. [1, Theorem 5.4.2]) Let $f$ and $g_i$ $(1\le i\le m)$ be weakly analytic convex functions. Then*

$$\inf\{f(x):g_i(x)\le 0,\ i=1,\dots,m\}=\sup_{\lambda_i\ge 0}\inf_{x\in\mathbb{R}^m}\{f(x)+\sum_{i=1}^m\lambda_i g_i(x)\}.$$

The following theorem shows that the robust zero duality gap property holds for convex optimization problems with faithfully convex inequality constraints under scenario uncertainty.

**Theorem 4.2.** *For problem (FP), let* $F := \{x : g_i(x, v_i) \leq 0, \ \forall v_i \in Z_i \ i = 1, \ldots, m\} \neq \emptyset$*. Then, for each continuous convex function* $f$ *on* $\mathbb{R}^n$*,*

$$\begin{aligned} & \inf\{f(x) : g_i(x, v_i) \leq 0, \ \forall v_i \in z_I, \ i = 1, \ldots, m\} \\ = & \sup_{\lambda_i \geq 0, v_i \in Z_i} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i)\}. \end{aligned}$$

*Proof.* The proof is similar to Theorem 4.1 and so is omitted. □

It is worth noting that the preceding Theorem is indeed true when $g_i$ are all weakly analytic convex functions.

## 4.3 Quadratic constraints & spectral norm uncertainty

Recall that $S^n$ denotes the space of all the symmetric $(n \times n)$ matrices and that $M \succeq 0$ means that $M$ is positive semidefinite. Consider the following convex optimization problem with quadratic inequality constraints under spectral norm uncertainty [2, 4]

$$\begin{array}{lll} (QP) & \min_{x \in \mathbb{R}^n} & f(x) \\ & \text{s.t.} & \frac{1}{2} x^T B_i x + b_i^T x + \beta_i \leq 0, \ i = 1, \ldots, m, \end{array}$$

where the matrix $B_i \in S^n$ are uncertain and they belong to the spectral norm uncertainty set $\overline{\mathcal{V}_i} = \{\overline{B}_i + \Delta_i : \Delta_i \in S^n, \|\Delta_i\|_{\text{spec}} \leq \rho_i\}$ with $\overline{B}_i \succeq 0$ and $\rho_i \geq 0$. Here the spectral norm $\|A\|_{\text{spec}}$ of a matrix $A \in S^n$ is defined as $\|A\|_{\text{spec}} := (\lambda_{\max}(A^T A))^{1/2}$ where $\lambda_{\max}$ is the maximum eigenvalue. This problem can be equivalently rewritten as a form of our model problem (P) by identifying $S^n$ as $\mathbb{R}^{\frac{n(n+1)}{2}}$ and letting $g_i(x, B_i) = \frac{1}{2} x^T B_i x + b_i^T x + \beta_i$ where the uncertain parameter $B_i \in S^n = \mathbb{R}^{\frac{n(n+1)}{2}}$. We assume that for each $B_i \in \overline{\mathcal{V}}_i$, $g_i(x, \cdot)$ is convex.

**Theorem 4.3.** *For the problem (QP), let* $F := \{x : g_i(x, B_i) \leq 0, \ \forall B_i \in \overline{\mathcal{V}}_i, \ i = 1, \ldots, m\} \neq \emptyset$*. Then, for each continuous convex function* $f$ *on* $\mathbb{R}^n$*,*

$$\begin{aligned} & \inf\{f(x) : g_i(x, B_i) \leq 0, \ \forall B_i \in \overline{\mathcal{V}_i}, \ i = 1, \ldots, m\} \\ = & \sup_{\lambda_i \geq 0, B_i \in \overline{\mathcal{V}_i}} \inf_{x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i (\frac{1}{2} x^T B_i x + b_i^T x + \beta_i)\}. \end{aligned}$$

*Proof.* We divide the proof into the following two cases: Case 1. $\rho_i = 0$ for all $i = 1, \ldots, m$; Case 2. there exists $i_0 \in \{1, \ldots, m\}$ such that $\rho_{i_0} > 0$.

Suppose that Case 1 holds. Then, the problem is free of uncertainty. Note that a convex quadratic function is analytic convex and so, is weakly analytic. So the conclusion follows by Lemma 4.2.

Suppose that Case 2 holds. Then, the robust feasible set $F$ is compact as $\overline{B}_{i_0} + \rho_{i_0} I_n$ is positive definite and

$$\begin{aligned} F &\subseteq \{x : \frac{1}{2}x^T B_{i_0} x + b_{i_0}^T x + \beta_{i_0} \le 0 \ \forall\ B_{i_0} \in \overline{\mathcal{V}_{i_0}}\} \\ &\subseteq \{x : \frac{1}{2}x^T (\overline{B}_{i_0} + \rho_{i_0} I_n) x + b_{i_0}^T x + \beta_{i_0} \le 0\}. \end{aligned}$$

Thus Proposition 4.1 gives us that epi$g^\diamond$ is closed and convex. Therefore, the conclusion follows. □

# Appendix: Additional Constraint Qualifications

In this Section, we provide additional sufficient conditions ensuring that the characteristic cone epi$g^\diamond$ is closed and convex.

**Proposition 4.1.** *Let $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ be continuous. Suppose that $x \mapsto g_i(x, v_i)$ is convex for each $v_i$ and that $v_i \mapsto g_i(x, v_i)$ is concave for each $x \in \mathbb{R}^n$. Assume that the robust feasible set $F := \{x : g_i(x, v_i) \le 0,\ \forall v_i \in Z_i,\ i = 1, \ldots, m\}$ is compact and nonempty. Then,* epi$g^\diamond$ *is closed and convex.*

*Proof.* We show that epi$g^\diamond$ = epi$\delta_F^*$. To see this, define $\tilde{g}_i(x) = \max_{v_i \in Z_i} g_i(x, v_i)$. For any affine functions $w(x)$, consider the following convex optimization problem $\min\{w(x) : \tilde{g}_i(x) \le 0, i = 1, \ldots, m\}$ and its dual $\max_{\lambda \ge 0} \min_{x \in \mathbb{R}^n}\{f(x) + \sum_{i=1}^m \lambda_i \tilde{g}_i(x)\}$. As the feasible set of the primal problem $F$ is compact, the standard zero duality gap holds [22], i.e.,

$$\min\{w(x) : \tilde{g}_i(x) \le 0, i = 1, \ldots, m\} = \max_{\lambda_i \ge 0} \min_{x \in \mathbb{R}^n}\{w(x) + \sum_{i=1}^m \lambda_i \tilde{g}_i(x)\}.$$

Applying Theorem 3.2 with the singleton uncertainty set, we see that

$$\text{epi}\delta_F^* = \text{epi} \inf_{\lambda_i \ge 0} \big( \sum_{1 \le i \le m} \lambda_i \tilde{g}_i \big)^*.$$

Next, we show that

$$\text{epi} \inf_{\lambda_i \ge 0} \big( \sum_{1 \le i \le m} \lambda_i \tilde{g}_i \big)^* \subseteq \text{epi} g^\diamond.$$

To verify this, take $(x^*, r) \in \text{epi} \inf_{\lambda_i \ge 0} \big( \sum_{1 \le i \le m} \lambda_i \tilde{g}_i \big)^*$. Then $\inf_{\lambda_i \ge 0} \big( \sum_{1 \le i \le m} \lambda_i \tilde{g}_i \big)^*(x^*) \le r$, and so, for each $k \in \mathbb{N}$, there exists $\lambda_i^k \ge 0$ such that $\big( \sum_{1 \le i \le m} \lambda_i^k \tilde{g}_i \big)^*(x^*) \le r + \dfrac{1}{k}$.

Thus, for each $x \in \mathbb{R}^n$

$$\begin{aligned}
\sup_{x\in\mathbb{R}^n} \min_{(v_1,\ldots,v_m)\in\prod_{i=1}^m Z_i} \{\langle x^*, x\rangle - \sum_{1\le i\le m} \lambda_i^k g_i(x, v_i)\} &= \sup_{x\in\mathbb{R}^n} \{\langle x^*, x\rangle - \sum_{1\le i\le m} \lambda_i^k \max_{v_i\in Z_i} g_i(x, v_i)\} \\
&= \langle x^*, x\rangle - \sum_{1\le i\le m} \lambda_i^k \tilde{g}_i(x) \\
&\le r + \frac{1}{k}.
\end{aligned}$$

Define $h(x, v) = -\langle x^*, x\rangle + \sum_{1\le i\le m} \lambda_i^k g_i(x, v_i)$. Then, $x \mapsto h(x, v)$ is convex and $v \mapsto h(x, v)$ is concave. By the minimax theorem [22], we have

$$\inf_{x\in\mathbb{R}^n} \max_{(v_1,\ldots,v_m)\in\prod_{i=1}^m Z_i} h(x, v) = \max_{(v_1,\ldots,v_m)\in\prod_{i=1}^m Z_i} \inf_{x\in\mathbb{R}^n} h(x, v).$$

So,

$$\min_{(v_1,\ldots,v_m)\in\prod_{i=1}^m Z_i} \sup_{x\in\mathbb{R}^n} \{\langle x^*, x\rangle - \sum_{1\le i\le m} \lambda_i^k g_i(x, v_i)\} \le r + \frac{1}{k}$$

and hence, there exist $v_i^k \in Z_i$ such that

$$\big(\sum_{1\le i\le m} \lambda_i^k g_i(\cdot, v_i^k)\big)^*(x^*) = \sup_{x\in\mathbb{R}^m} \{\langle x^*, x\rangle - \sum_{1\le i\le m} \lambda_i^k g_i(x, v_i^k)\} \le r + \frac{1}{k}.$$

This shows that $g^\diamond(x^*) = \inf_{\lambda_i\ge 0, v_i\in Z_i} \big(\sum_{i=1}^m \lambda_i g_i(x, v_i)\big)^*(x^*) \le r$. So, $(x^*, r) \in \text{epi}g^\diamond$, and hence

$$\text{epi}\delta_F^* = \text{epi} \inf_{\lambda_{ij}\ge 0} \big(\sum_{1\le i\le m, 1\le j\le 2^q} \lambda_{ij} g_i(\cdot, v_{ij})\big)^* \subseteq \text{epi}g^\diamond.$$

Note that $\text{epi}g^\diamond \subseteq \text{epi}\delta_F^*$ always holds. It follows that $\text{epi}\delta_F^* = \text{epi}g^\diamond$. Thus, the conclusion follows. □

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