Global Optimality Principles for Polynomial Optimization Problems over Box or Bivalent Constraints by Separable Polynomial Approximations*

V. Jeyakumar[†], G. Li[‡] and S. Srisatkunarajah[§]

Revised Version II: December 23, 2012

Abstract

In this paper we present necessary conditions for global optimality for polynomial problems over box or bivalent constraints using separable polynomial relaxations. We achieve this by completely characterizing global optimality of separable polynomial problems with box as well as bivalent constraints. Then, by employing separable polynomial under-estimators, we establish sufficient conditions for global optimality for classes of polynomial optimization problems with box or bivalent constraints. The underestimators are constructed using the sum of squares convex polynomials. The significance of our optimality condition is that they can be numerically checked by solving semi-definite programming problems. We illustrate the versatility of our optimality conditions by simple numerical examples.

---

*The authors are grateful to the referees and the editor for their constructive comments and helpful suggestions which have contributed to the final preparation of the paper. Research was partially supported by a grant from the Australian Research Council

[†]Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia. E-mail:v.jeyakumar@unsw.edu.au

[‡]Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia. E-mail: g.li@unsw.edu.au

[§]Department of Mathematics and Statistics, University of Jaffna, Jaffna, Sri Lanka. Email: srisatku@jfn.ac.lk. The work of this author was completed while he was visiting the University of New South Wales, Sydney, Australia.

1 Introduction

This paper deals with the nonconvex polynomial optimization model problem over box or bivalent constraints:

$$\begin{array}{ll} (P_M) & \min\limits_{x\in\mathbb{R}^n} f(x) \\ \text{s.t.} & x_i \in [-1,1],\ i=1,\dots,l, \\ & x_i \in \{-1,1\},\ i=l+1,\dots,n. \end{array}$$

where f is a polynomial on $\mathbb{R}^n$. This model (P_M) covers a wide range of problems, including quadratic problems with box or binary constraint such as the max cut problem [5] and it arises in many areas of applications, including financial modelling and signal processing [19]. Over the years, a great deal of attention has been focussed on establishing global optimality conditions for quadratic optimization problems with box or bivalent constraint [2, 6, 11, 14, 7, 20]. However, the development of optimality conditions for identifying global minimizers of polynomial problems have so far been limited mainly to special classes of polynomial problems (see [8, 15, 22, 23]). For related work on polynomial optimization problems, see [9, 10, 12, 13, 18].

In this paper we make the following contributions to polynomial optimization over box or bivalent constraints:

First, by establishing a complete characterization of global optimality of a separable polynomial over box as well as bivalent constraints and employing a separable polynomial relaxation, we derive necessary conditions for global optimality of general polynomial problems over box or bivalent constraints. The significance of our optimality condition is that they not only extend the corresponding optimality conditions for quadratic optimization problems but also can easily be numerically checked by solving semi-definite programming problems.

Secondly, using separable polynomial under-estimators, we then present sufficient conditions for global optimality for classes of polynomial optimization problems over box or bivalent constraints. We also show how the sum of squares convex (SOS-convex) polynomials can be used to construct separable polynomial under-estimators. Consequently, we obtain sufficient global optimality conditions for the sum of separable polynomial and SOS convex polynomials over box constraints. We see that our sufficient conditions and necessary conditions coincide in the case of separable polynomial problems (P_M). The versatility of our optimality conditions are illustrated by simple numerical examples.

The layout of the paper is as follows. In Section 2, we present a simple complete characterization of global minimizers of (P_M) for separable polynomial problems over box or bivalent constraints. In Section 3, we apply the characterization to obtain necessary optimality conditions for general polynomial problems (P_M). In Section 4, by employing separable under-estimators, we establish sufficient global optimality conditions for classes of polynomial problems (P_M) beyond the quadratic optimization over box or bivalent constraints.

2 Separability and Checkable Characterizations of Optimality

In this Section, we provide a complete characterization of global optimality for polynomial optimization problems over box or bivalent constraints.

We begin by fixing notation and preliminaries that will be used later in the paper. As usual, we denote the Euclidean space with dimension n by $\mathbb{R}^n$. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. For a set C in $\mathbb{R}^n$, the convex hull of C is denoted by co$C$. The set of all $(n \times n)$ symmetric matrices is denoted by S^n. For a symmetric matrix, $Q \in S^n$, $Q \succeq 0$ (resp. $Q \preceq 0$) means that the matrix Q is positive (resp. negative) semi-definite.

We denote the real polynomial ring on $\mathbb{R}^n$ by $\mathbb{R}[x]$ where $x = (x_1, \ldots, x_n)$. The set of all natural numbers is denoted by $\mathbb{N}$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is a separable polynomial if $f(x) = \sum_{i=1}^n f_i(x_i)$, where $x = (x_1, \ldots, x_n)$ and each f_i is a polynomial on $\mathbb{R}$. The set of all the separable polynomial functions with degree at most d on $\mathbb{R}^n$ is denoted by

$$S_d := \{f \in \mathbb{R}[x] : f(x) = \sum_{i=1}^n \sum_{j=0}^d f_{ij} x_i^j, \quad x = (x_1, \ldots, x_n)\}.$$

Recall that a real polynomial f is called a sum of squares polynomial if there exist $r \in \mathbb{N}$ and real polynomials f_j, $j = 1, \ldots, r$, such that $f = \sum_{j=1}^r f_j^2$. The set of all sum of squares real polynomial is denoted by Σ^2. Moreover, the set of all sum of squares real polynomial with degree at most d is denoted by Σ_d^2. An important property of the sum of squares polynomials is that checking a polynomial is sum of squares or not, is equivalent to solving a semidefinite linear programming problem [21].

Consider the polynomial optimization problem over box or bivalent constraints:

$$\begin{array}{ll} (P_M) & \min\limits_{x \in \mathbb{R}^n} f(x) \\ \text{s.t.} & x_i \in [-1, 1],\ i = 1, \ldots, l, \\ & x_i \in \{-1, 1\},\ i = l+1, \ldots, n. \end{array}$$

The feasible set of (P_M) is denoted by F and is given by $F = \{x \in \mathbb{R}^n : x_i \in [-1, 1], i = 1, \ldots, l, x_i \in \{-1, 1\}, i = l+1, \ldots, n\}$.

We first derive a complete characterization of global optimality of (P_M) whenever f is a separable polynomial. This characterization plays an important role in our derivation of sufficient, and necessary global optimality conditions for a general polynomial minimization problem.

Theorem 2.1. (Global optimality characterization) *For (P_M), let f be a separable polynomial with degree d, i.e., $f(x) = \sum_{i=1}^n \sum_{j=0}^d f_{ij}\, x_i^j$, for all $x = (x_1, \ldots, x_n)$. Let F be the feasible set of (P_M) and let $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) \in F$ Then, $\overline{x}$ is a global minimizer of (P_M) if and only if*

$$\begin{cases} \sum_{j=0}^{d} f_{ij}(-1+x_i^2)^j(1+x_i^2)^{d-j} - (1+x_i^2)^d \sum_{j=0}^{d} f_{ij}\overline{x}_i^j \in \Sigma_{2d}^2, & i = 1, \ldots, l, \\ \overline{x}_i \cdot \left(\sum_{j \text{ is odd}} f_{ij}\right) \le 0 & i = l+1, \ldots, n, \end{cases}$$

where Σ_{2d}^2 denotes the set consisting of all the sum of squares polynomials with degree at most $2d$.

Proof. Let $f \in S_d$ be such that $f(x) \ge f(\overline{x})$, $\forall x \in F$. This means that $f(x) = \sum_{i=1}^n \sum_{j=0}^d f_{ij}\, x_i^j$ and

$$f(x) - f(\overline{x}) \;=\; \sum_{i=1}^{n}\sum_{j=0}^{d} f_{ij}(x_i^j - \overline{x}_i^j) = \sum_{i=1}^{n}\Big(\sum_{j=0}^{d} f_{ij}(x_i^j - \overline{x}_i^j)\Big),$$

where $f_{ij} \in \mathbb{R}$ and $x = (x_1, \ldots, x_n)$. Define $F_i \subseteq \mathbb{R}$, $i = 1, \ldots, n$, by $F_i = [-1, 1]$ for all $i = 1, \ldots, l$ and $F_i = \{-1, 1\}$ for all $i = l+1, \ldots, n$. We now show that $\overline{x}$ is a global minimizer of (P_M) if and only if for all $i = 1, \ldots, n$ and for all $x_i \in F_i$,

$$\sum_{j=0}^{d} f_{ij}x_i^j - \sum_{j=0}^{d} f_{ij}\overline{x}_i^j = \sum_{j=0}^{d} f_{ij}(x_i^j - \overline{x}_i^j) \ge 0.$$

To see this, we note that

$$\inf_{x\in F}\{f(x) - f(\overline{x})\} = \inf_{x\in F}\{\sum_{i=1}^{n}\Big(\sum_{j=0}^{d} f_{ij}(x_i^j - \overline{x}_i^j)\Big)\} = \sum_{i=1}^{n}\inf_{x_i\in F_i}\{\sum_{j=0}^{d} f_{ij}(x_i^j - \overline{x}_i^j)\}.$$

As $\overline{x}_i \in F_i$, $\inf_{x_i\in F_i}\{\sum_{j=1}^d f_{ij}(x_i^j - \overline{x}_i^j)\} \le 0$ for each $i = 1, \ldots, n$, and so, $\overline{x}$ is a global minimizer of (P_M) if and only

$$\inf_{x_i\in F_i}\{\sum_{j=0}^{d} f_{ij}(x_i^j - \overline{x}_i^j)\} = 0,\ \ i = 1, \ldots, n,$$

which is, further equivalent to the condition that, for all $i = 1, \ldots, n$ and for all $x_i \in F_i$, $g_i(x_i) \ge 0$, where $g_i : \mathbb{R} \to \mathbb{R}$ is a polynomial in one variable and is defined by

$$g_i(x_i) := \sum_{j=0}^{d} f_{ij}x_i^j - \sum_{j=0}^{d} f_{ij}\overline{x}_i^j = \sum_{j=0}^{d} f_{ij}(x_i^j - \overline{x}_i^j).$$

To finish the proof, it suffices to show that, for each $i = 1, \ldots, l$,

$$g_i(x_i) \geq 0 \text{ for all } x_i \in [-1, 1] \ \Leftrightarrow \ \sum_{j=0}^{d} f_{ij}(-1 + x_i^2)^j (1 + x_i^2)^{d-j} - (1 + x_i^2)^d \sum_{j=0}^{d} f_{ij}\overline{x}_i^j \in \Sigma_{2d}^2, \tag{2.1}$$

and, for each $i = l + 1, \ldots, n$,

$$g_i(x_i) \geq 0 \text{ for all } x_i \in \{-1, 1\} \ \Leftrightarrow \ \overline{x}_i \cdot (\sum_{j \text{ is odd}} f_{ij}) \leq 0. \tag{2.2}$$

To see (2.1), for each $i \in \{1, \ldots, l\}$, define

$$\begin{aligned} h_i(x_i) &:= (1 + x_i^2)^d g_i(\frac{-1 + x_i^2}{1 + x_i^2}) \\ &= \sum_{j=0}^{d} f_{ij}(-1 + x_i^2)^j (1 + x_i^2)^{d-j} - (1 + x_i^2)^d \sum_{j=0}^{d} f_{ij}\overline{x}_i^j. \end{aligned}$$

So, each h_i is a real polynomial on $\mathbb{R}$ with degree at most $2d$. As $y \mapsto \frac{-1+y^2}{1+y^2}$ is a surjection from $\mathbb{R}$ to $[-1, 1)$, we obtain that (e.g. see [3]),

$$\begin{aligned} g_i(x_i) \geq 0 \text{ for all } x_i \in [-1, 1] &\Leftrightarrow g_i(x_i) \geq 0 \text{ for all } x_i \in [-1, 1) \\ &\Leftrightarrow g_i(\frac{-1 + x_i^2}{1 + x_i^2}) \geq 0 \text{ for all } x_i \in \mathbb{R} \\ &\Leftrightarrow h_i(x_i) \geq 0 \text{ for all } x_i \in \mathbb{R}, \end{aligned}$$

where the first equivalence follows by the continuity of g_i. Note that a one variable polynomial is nonnegative if and only if it is a sum of squares polynomial (e.g. see [15]). So,

$$g_i(x_i) \geq 0 \text{ for all } x_i \in [-1, 1] \ \Leftrightarrow \ h_i \in \Sigma_{2d}^2,$$

where Σ_{2d}^2 denotes all the sum of squares polynomials with degree at most $2d$. Thus, (2.1) follows.

To see (2.2), fix $i \in \{l + 1, \ldots, n\}$. Note that, for any $a, b \in \{-1, 1\}$ and $q \in \mathbb{N}$,

$$a^q - b^q = \begin{cases} a - b, & \text{if } \ q \text{ is odd}, \\ 0, & \text{if } \ q \text{ is even}. \end{cases}$$

This means that for all $x_i \in \{-1, 1\}$

$$g_i(x_i) = \sum_{j=1}^{d} f_{ij}(x_i^j - \overline{x}_i^j) = \sum_{j \text{ is odd}} f_{ij}(x_i - \overline{x}_i). \tag{2.3}$$

Now, consider the following two cases:

Case 1. $\overline{x}_i = -1$. Then, $g_i(x_i) \geq 0$ for all $x_i \in \{-1, 1\}$, is equivalent to $\sum_{j \text{ is odd}} f_{ij} \geq 0$ which can be rewritten as

$$\overline{x}_i \cdot (\sum_{j \text{ is odd}} f_{ij}) = -\sum_{j \text{ is odd}} f_{ij} \leq 0.$$

Case 2. $\overline{x}_i = 1$. Then, $g_i(x_i) \geq 0$ for all $x_i \in \{-1, 1\}$, is equivalent to $\sum_{j \text{ is odd}} f_{ij} \leq 0$ which can be rewritten as

$$\overline{x}_i \cdot (\sum_{j \text{ is odd}} f_{ij}) = \sum_{j \text{ is odd}} f_{ij} \leq 0.$$

Thus (2.2) holds, and so, the conclusion follows. $\square$

Below, we show that the global optimality condition in Theorem 2.1 can be verified by solving l many semidefinite programming problems and verifying $n - l$ many simple inequalities.

Remark 2.1. (Semidefinite Programming Reformulation of the Global Optimality Condition) *The above optimality condition can be equivalently rewritten as l many semidefinite programming problems and $n - l$ many simple inequalities, and so, can be efficiently verified in polynomial time (for example by interior point method). To see this, note that for any one variable polynomial h with degree at most $2d$,*

$$h \in \Sigma^2_{2d} \;\Leftrightarrow\; \exists A \in S^{d+1}_+, \; s.t. \; h(x) = (x^{[d]})^T A x^{[d]}, x^{[d]} = (1, x, x^2 \ldots, x^d)^T,$$

where S^{d+1}_+ denotes the set consisting of all positive semidefinite $(d+1) \times (d+1)$ matrices. So, for each fixed $i = 1, \ldots, l$, $\sum_{j=0}^d f_{ij}(-1 + x_i^2)^j (1 + x_i^2)^{d-j} - (1 + x_i^2)^d \sum_{j=0}^d f_{ij} \overline{x}_i^j \in \Sigma^2_{2d}$ is equivalent to the existence of $A^i = (A^i_{\alpha\beta})_{\alpha,\beta=0,1,\ldots,d} \in S^{d+1}_+$ such that

$$\sum_{j=0}^d f_{ij}(-1 + x_i^2)^j (1 + x_i^2)^{d-j} - (1 + x_i^2)^d \sum_{j=0}^d f_{ij} \overline{x}_i^j = (x_i^{[d]})^T A x_i^{[d]}, x_i^{[d]} = (1, x_i, x_i^2 \ldots, x_i^d)^T ..$$

By comparing the coefficients and noting that

$$\begin{aligned} \sum_{j=0}^d f_{ij}(-1 + x_i^2)^j (1 + x_i^2)^{d-j} - (1 + x_i^2)^d \sum_{j=0}^d f_{ij} \overline{x}_i^j &= \sum_{j=0}^d f_{ij} \sum_{m=0}^j C_j^m (-1)^{j-m} x_i^{2m} \sum_{l=0}^{d-j} C_{d-j}^l x_i^{2l} \\ &= \sum_{p=0}^d x_i^{2p} \Bigg(\sum_{m=0}^p \sum_{j=m}^{d+m-p} f_{ij} C_{d-j}^{p-m} C_j^m (-1)^{j-m} \Bigg), \end{aligned}$$

where the binomial coefficient C_α^β is the number of ways of picking β-unordered outcomes from α-possibilities, and

$$(1 + x_i^2)^d \sum_{j=0}^d f_{ij} \overline{x}_i^j = \sum_{p=0}^d x_i^{2p} (C_d^p \sum_{j=0}^d f_{ij} \overline{x}_i^j),$$

we see that the above optimality condition is equivalent to checking simple inequalities $\overline{x}_i \cdot (\sum_{j \text{ is odd}} f_{ij}) \leq 0,\ i = l+1, \ldots, n$, *and solving the following semidefinite programming problems: find* $A^i \in S_+^{d+1}$, $i = 1, \ldots, l$, *with* $A^i = (A^i_{\alpha\beta})_{\alpha,\beta=0,1,\ldots,d}$ *such that*

$$\begin{cases} A^i_{00} - \sum_{j=0}^{d} f_{ij}(-1)^j + \sum_{j=0}^{d} f_{ij}\overline{x}_i^j = 0, \\ \sum_{\alpha+\beta=2p-1} A^i_{\alpha\beta} = 0,\ p = 1, \ldots, d, \\ \sum_{\alpha+\beta=2p} A^i_{\alpha\beta} = \big(\sum_{m=0}^{p} \sum_{j=m}^{d+m-p} f_{ij} C_{d-j}^{p-m} C_j^m (-1)^{j-m}\big) - C_d^p \sum_{j=0}^{d} f_{ij}\overline{x}_i^j,\ p = 1, \ldots, d. \end{cases}$$

As a corollary, we obtain the first and second-order optimality condition characterizing global optimality of a separable quadratic function. This was first given in [11].

Corollary 2.1. (Separable Quadratic Optimization) *Let f be a separable quadratic function, i.e.,* $f(x) = \frac{1}{2}\sum_{i=1}^n a_i x_i^2 + \sum_{i=1}^n b_i x_i + \gamma$, *for all* $x = (x_1, \ldots, x_n)$. *Let* $F = \{x \in \mathbb{R}^n : x_i \in [-1,1], i = 1, \ldots, l, x_i \in \{-1,1\}, i = l+1, \ldots, n\}$ *and let* $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_n) \in F$. *Then, $\overline{x}$ is a global optimizer of (P_M) if and only if, for each $i = 1, \ldots, l$,*

$$\widetilde{\overline{x}}_i(b_i + a_i\overline{x}_i) - \min\{0, a_i\} \leq 0,$$

where $\widetilde{\overline{x}}_i$ is defined as

$$\widetilde{\overline{x}}_i = \begin{cases} -1 & \text{if} \quad \overline{x}_i = -1, \\ 1 & \text{if} \quad \overline{x}_i = 1, \\ b_i + a_i\overline{x}_i & \text{if} \quad \overline{x}_i \in (-1,1), \end{cases}$$

and, for each $i = l+1, \ldots, n$, $b_i\overline{x}_i \leq 0$.

Proof. Letting $d = 2$ in Theorem 2.1, we see that the global optimality condition reads: for each $i = 1, \ldots, l$,

$$\left(\frac{1}{2}a_i(-1+x_i^2)^2 + b_i(-1+x_i^2)(1+x_i^2)\right) - (1+x_i^2)^2\left(\frac{1}{2}a_i\overline{x}_i^2 + b_i\overline{x}_i\right) \in \Sigma_4^2, \tag{2.4}$$

and for each $i = l+1, \ldots, n$, $b_i\overline{x}_i \leq 0$. Thus, to finish the proof, it suffices to show that (2.4) is equivalent to $\widetilde{\overline{x}}_i(b_i + a_i\overline{x}_i) - \min\{0, a_i\} \leq 0$ for all $i = 1, \ldots, l$. To see this, we now split the discussion into three cases: Case 1, $\overline{x}_i = 1$; Case 2, $\overline{x}_i = -1$; Case 3, $\overline{x}_i \in (-1,1)$.

Suppose that Case 1 holds. Then, (2.4) collapses to $-2a_i x_i^2 - 2b_i(1+x_i^2) \in \Sigma_4^2$. This means $b_i \leq 0$ and $a_i + b_i \leq 0$ which is, in turn equivalent to

$$b_i \leq \min\{0, -a_i\} = -\max\{0, a_i\} = -a_i + \min\{0, a_i\}.$$

So, $(b_i + a_i) - \min\{0, a_i\} \leq 0$ and hence, the conclusion follows in this case.

Suppose that Case 2 holds. Then, (2.4) collapses to

$$2x_i^2\big(-a_i+b_i(1+x_i^2)\big)=-2a_ix_i^2+2b_ix_i^2(1+x_i^2)\in\Sigma_4^2.$$

This is equivalent to $-a_i+b_i(1+x_i^2)\geq 0$, which is, in turn equivalent to $b_i\geq 0$ and $a_i-b_i\leq 0$. Similarly, this can be equivalently rewritten as $-(b_i-a_i)-\min\{0,a_i\}\leq 0$ and hence, the conclusion follows in this case.

Suppose that Case 3 holds. Note that (2.4) is equivalent to

$$\frac{1}{2}a_i(\frac{-1+x_i^2}{1+x_i^2})^2+b_i\frac{-1+x_i^2}{1+x_i^2}-(\frac{1}{2}a_i\overline{x}_i^2+b_i\overline{x}_i)\geq 0 \text{ for all } x_i\in\mathbb{R}.$$

As $x_i\mapsto\frac{-1+x_i^2}{1+x_i^2}$ maps $\mathbb{R}$ onto $[-1,1)$, by the continuity, we see that

$$\frac{1}{2}a_iz^2+b_iz-(\frac{1}{2}a_i\overline{x}_i^2+b_i\overline{x}_i)\geq 0 \text{ for all } z\in[-1,1].$$

Since $\overline{x}_i\in(-1,1)$, we see that $a_i\geq 0$ and $a_i\overline{x}_i+b_i=0$. This can be equivalently rewritten as $(b_i+a_i\overline{x}_i)^2-\min\{0,a_i\}\leq 0$, and hence, the conclusion also follows in this case. □

3 Necessary Global Optimality Conditions

In this section, using a separable polynomial relaxation, we obtain a necessary global optimality condition for general polynomial optimization problems over box or bivalent constraints.

Theorem 3.1. (Necessary Global Optimality) *For problem (P_M), let f be a polynomial with degree $d\in\mathbb{N}$, defined by*

$$f(x)=\sum_{0\leq\sum_{k=1}^n\alpha_k\leq d}f_{\alpha_1\,\alpha_2\ldots\alpha_n}x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n},\quad x=(x_1,\ldots,x_n).$$

If $\overline{x}$ is a global minimizer of (P_M), then,

$$[NC]\quad\begin{cases}\displaystyle\sum_{j=0}^d\tilde{f}_{ij}(-1+x_i^2)^j(1+x_i^2)^{d-j}-(1+x_i^2)^d\sum_{j=0}^d\tilde{f}_{ij}\overline{x}_i^j\in\Sigma_{2d}^2, & i=1,\ldots,l,\\ \overline{x}_i\cdot\big(\sum_{j\text{ is odd}}\tilde{f}_{ij}\big)\leq 0 & i=l+1,\ldots,n,\end{cases}$$

where $\tilde{f}_{ij}$, $j=0,1,\ldots,d$ are defined by $\tilde{f}_{ij}=\displaystyle\sum_{0\leq\sum_{k=1}^n\alpha_k\leq d,\,\alpha_i=j}f_{\alpha_1,\ldots,\alpha_n}\prod_{l\neq i}\overline{x}_l^{\alpha_l}$.

Proof. Let $\overline{x}$ be a global minimizer of (P_M). For each $i=1,\ldots,n$, define $\tilde{f}_i:\mathbb{R}\to\mathbb{R}$ and $\tilde{f}:\mathbb{R}^n\to\mathbb{R}$ by

$$\tilde{f}_i(x_i)=f(\overline{x}_1,\ldots,\overline{x}_{i-1},x_i,\overline{x}_{i+1},\ldots,\overline{x}_n)=\sum_{0\leq\sum_{k=1}^n\alpha_k\leq d}f_{\alpha_1\,\alpha_2\ldots\alpha_n}(x_i^{\alpha_i}\prod_{l\neq i}\overline{x}_l^{\alpha_l})$$

and $\tilde{f}(x) = \sum_{i=1}^{n} \tilde{f}_i(x_i) = \sum_{i=1}^{n}\sum_{j=0}^{d} \tilde{f}_{ij} x_i^j$, where, for each $i = 1, \ldots, n$ and $j = 0, \ldots, d$,

$$\tilde{f}_{ij} = \sum_{0 \le \sum_{k=1}^n \alpha_k \le d,\, \alpha_i = j} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \neq i} \overline{x}_l^{\alpha_l}.$$

Then, $\overline{x}$ is a global minimizer of the following separable polynomial optimization problem:

$$\begin{array}{lll} (\overline{P_M}) & \min & \tilde{f}(x) \\ & s.t. & x \in F, \end{array}$$

where $F = \{x \in \mathbb{R}^n : x_i \in [-1, 1], i = 1, \ldots, l, x_i \in \{-1, 1\}, i = l + 1, \ldots, n\}$. (Otherwise, there exists $a \in F$ such that $\tilde{f}(a) < \tilde{f}(\overline{x})$. Then, there exists $i_0 \in \{1, \ldots, n\}$ such that $\tilde{f}_{i_0}(a_{i_0}) < \tilde{f}_{i_0}(\overline{x}_{i_0}) = f(\overline{x})$. Thus,

$$f(\overline{x}_1, \ldots, \overline{x}_{i_0-1}, a_{i_0}, \overline{x}_{i_0+1}, \ldots, \overline{x}_n) < f(\overline{x}),$$

which is impossible.) Hence, the conclusion follows by applying Theorem 2.1. □

Corollary 3.1. (Polynomial Optimization with Box Constraints) *For problem (P_M) with $l = n$, let f be a polynomial with degree $d \in \mathbb{N}$, defined by*

$$f(x) = \sum_{0 \le \sum_{k=1}^n \alpha_k \le d} f_{\alpha_1\, \alpha_2 \ldots \alpha_n} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}, \quad x = (x_1, \ldots, x_n).$$

Suppose that $\overline{x}$ is a global minimizer of (P_M). Then, for each $i = 1, \ldots, n$,

$$\sum_{j=0}^{d} \tilde{f}_{ij}(-1 + x_i^2)^j (1 + x_i^2)^{d-j} - (1 + x_i^2)^d \sum_{j=0}^{d} \tilde{f}_{ij} \overline{x}_i^j \in \Sigma^2_{2d}$$

where $\tilde{f}_{ij}$, $j = 0, 1, \ldots, d$ are defined by $\tilde{f}_{ij} = \sum_{0 \le \sum_{k=1}^n \alpha_k \le d,\, \alpha_i = j} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \neq i} \overline{x}_l^{\alpha_l}$ and Σ^2_{2d} denotes the set consisting of all the sum of squares polynomials with degree at most $2d$.

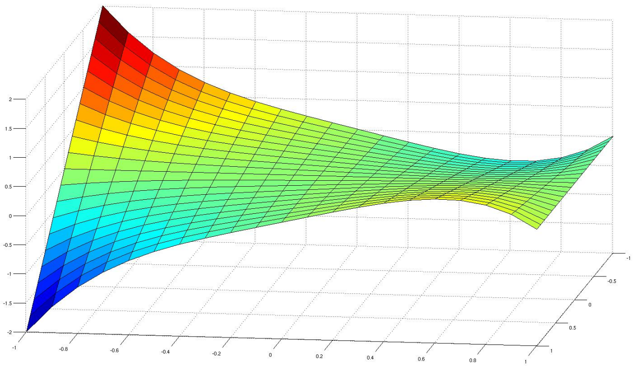
Proof. The conclusion follows from Theorem 3.1 with $l = n$. □

We now provide a simple nonconvex polynomial problem with a box constraint where the necessary optimality condition can be verified numerically, and it may be used to eliminate local minimizers that are not global.

Example 3.1. *Consider the following 2-dimensional nonconvex optimization problem*

$$(P_E) \quad \min f(x_1, x_2) = x_1 x_2 - x_1^4 x_2 \ s.t.\ (x_1, x_2) \in [-1, 1] \times [-1, 1].$$

Clearly, f is a polynomial with degree 5 and $f(x_1, x_2) = \sum f_{i,j} x_1^i x_2^j$ with $f_{4,1} = -1$, $f_{1,1} = 1$ and $f_{ij} = 0$ for all $(i, j) \in (\{0, 1, \ldots, 5\} \times \{0, 1, \ldots, 5\}) \backslash \{(4, 1), (1, 1)\}$. It is not hard to verify that $\overline{w} := (\overline{w}_1, \overline{w}_2) = (1, 1)$ and $\overline{x} := (\overline{x}_1, \overline{x}_2) = (-1, 1)$ are local minimizers, and $\overline{x}$ is the unique global minimizer. (This can also be seen from the plot of the function values of f over $[-1, 1]^2$).



[Verifying $\overline{w}$ is not a global minimizer] For $i = 1$,

$$\tilde{f}_{1j} = \sum_{0 \leq \sum_{k=1}^{2} \alpha_k \leq 5,\, \alpha_1 = j} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \neq 1} \overline{w}_l^{\alpha_l} = \begin{cases} 0 & j = 0, \\ 1 & j = 1, \\ 0 & j = 2, \\ 0 & j = 3, \\ -1 & j = 4, \\ 0 & j = 5. \end{cases}$$

So, the necessary condition at $\overline{w}$ fails as

$$(-1+x_1^2)(1+x_1^2)^4 + (-1)(-1+x_1^2)^4(1+x_1^2) - (1+x_1^2)^5(1-1) = (2+6x_1^4)(x_1^4-1) \notin \Sigma_{10}^2,$$

Indeed, this can be seen by noting that $(2+6x_1^4)(x_1^4-1) = -2.2266 < 0$ at $x_1 = 1/2$. Alternatively, this can also be verified by running the following simple code from the sum of squares matlab toolbox Yamlip [16, 17] to see that the corresponding equivalent semi-definite programming problem is infeasible

```
sdpvar x;
p=(-1+x^2)*(1+x^2)^4+(-1)*(-1+x^2)^4*(1+x^2);
v = monolist([x],degree(p)/2);
Q = sdpvar(length(v));
p_sos = v'*Q*v;
F = [coefficients(p-p_sos,[x]) == 0, Q >= 0];
solvesdp(F)
```

[Verifying $\overline{x}$ satisfies the necessary optimality condition] For $i = 1$,

$$\tilde{f}_{1j} = \sum_{0 \leq \sum_{k=1}^{2} \alpha_k \leq 5,\, \alpha_1 = j} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \neq 1} \overline{x}_l^{\alpha_l} = \begin{cases} 0 & j = 0, \\ 1 & j = 1, \\ 0 & j = 2, \\ 0 & j = 3, \\ -1 & j = 4, \\ 0 & j = 5, \end{cases}$$

and for $i = 2$,

$$\tilde{f}_{2j} = \sum_{0 \le \sum_{k=1}^{2} \alpha_k \le 5,\, \alpha_2 = j} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \ne 2} \overline{x}_l^{\alpha_l} = \begin{cases} 0 & j = 0, \\ -2 & j = 1, \\ 0 & j = 2, \\ 0 & j = 3, \\ 0 & j = 4, \\ 0 & j = 5. \end{cases}$$

So, the necessary condition at $\overline{x}$ reads

$$(-1 + x_1^2)(1 + x_1^2)^4 + (-1)(-1 + x_1^2)^4(1 + x_1^2) - (1 + x_1^2)^5(-1 - 1) \in \Sigma_{10}^2,$$

and

$$(-2)(-1 + x_2^2)(1 + x_2^2)^4 - (1 + x_2^2)^5(-2) \in \Sigma_{10}^2.$$

These two necessary condition can be verified numerically (for example, running the following simple code from the sum of squares matlab toolbox Yamlip, one sees that these two conditions are satisfied).

```
sdpvar x z;
p=(-1+x^2)*(1+x^2)^4+(-1)*(-1+x^2)^4*(1+x^2)-(1+x^2)^5*(-1-1) ;
q=(-2)*(-1+z^2)*(1+z^2)^4-(1+z^2)^5*(-2);
F = sos(p);
G = sos(q);
[sol,v,A] = solvesos(F);
[sol1,u,B] = solvesos(G);
ouput1=clean(p-v{1}'*A{1}*v{1},1e-6)
output2=clean(q-u{1}'*B{1}*u{1},1e-6)
```

It is worth noting that, in Example 3.1, the global minimizer is at the boundary of the feasible set. We now give another simple nonconvex polynomial problem with a box constraint where the global minimizer is at an interior point of the feasible set. We see that, in this case also, the necessary optimality condition can easily be verified, and it may be used to eliminate local minimizers that are not global.

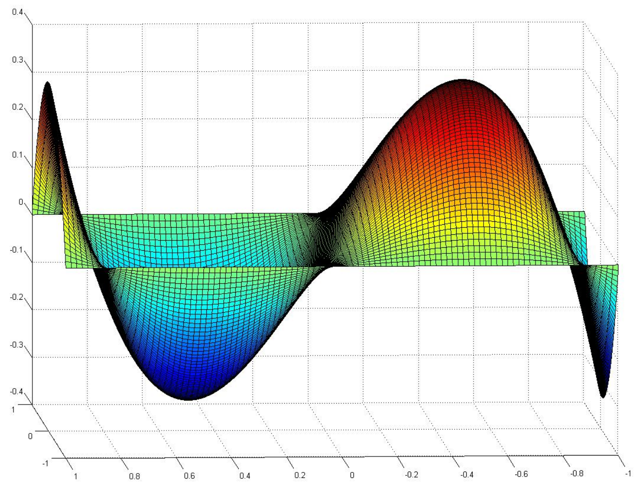
Example 3.2. *Consider the following 2-dimensional nonconvex optimization problem*

$$(P_E) \quad \min f(x_1, x_2) = \frac{4}{3}x_1^3 - x_1 - \frac{4}{3}x_1^3 x_2^2 + x_1 x_2^2 = (\frac{4}{3}x_1^3 - x_1)(1 - x_2^2) \ s.t.\ (x_1, x_2) \in [-1, 1] \times [-1, 1].$$

Clearly, f is a polynomial with degree 5 and $f(x_1, x_2) = \sum f_{i,j} x_1^i x_2^j$ with $f_{3,0} = \frac{4}{3}$, $f_{1,0} = -1$, $f_{3,2} = -\frac{4}{3}$, $f_{1,2} = 1$ and

$$f_{ij} = 0, \ \textit{for all } (i, j) \in (\{0, 1, \ldots, 5\} \times \{0, 1, \ldots, 5\}) \backslash \{(3, 0), (1, 0), (3, 2), (1, 2)\}.$$

It is not hard to verify that $\overline{w} := (\overline{w}_1, \overline{w}_2) = (-1, 1)$, $\overline{v} := (\overline{v}_1, \overline{v}_2) = (-1, -1)$, $\overline{x} := (\overline{x}_1, \overline{x}_2) = (\frac{1}{2}, 0)$ and $\overline{z} := (\overline{z}_1, \overline{z}_2) = (-1, 0)$ are local minimizers, and $\overline{x}$ and $\overline{z}$ are the global minimizers. (This can also be seen from the plot of the function values of f over $[-1, 1]^2$).



[Verifying $\overline{w}$ is not a global minimizer] For $i = 2$,

$$\tilde{f}_{2j} = \sum_{0 \le \sum_{k=1}^{2} \alpha_k \le 5,\, \alpha_2 = j} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \ne 2} \overline{w}_l^{\alpha_l} = \begin{cases} -\frac{1}{3} & j = 0, \\ 0 & j = 1, \\ \frac{1}{3} & j = 2, \\ 0 & j = 3, \\ 0 & j = 4, \\ 0 & j = 5. \end{cases}$$

So, the necessary condition at $\overline{w}$ fails as

$$-\frac{1}{3}(1 + x_2^2)^5 + \frac{1}{3}(-1 + x_2^2)^2(1 + x_2^2)^3 \notin \Sigma_{10}^2,$$

Indeed, this can be seen by noting that $-\frac{1}{3}(1 + x_2^2)^5 + \frac{1}{3}(-1 + x_2^2)^2(1 + x_2^2)^3 < 0$ at $x_2 = 1$. [Verifying $\overline{v}$ is not a global minimizer] For $i = 2$,

$$\tilde{f}_{2j} = \sum_{0 \le \sum_{k=1}^{2} \alpha_k \le 5,\, \alpha_2 = j} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \ne 2} \overline{v}_l^{\alpha_l} = \begin{cases} -\frac{1}{3} & j = 0, \\ 0 & j = 1, \\ \frac{1}{3} & j = 2, \\ 0 & j = 3, \\ 0 & j = 4, \\ 0 & j = 5. \end{cases}$$

So, similarly, the necessary condition at $\overline{v}$ fails. as

$$-\frac{1}{3}(1 + x_2^2)^5 + \frac{1}{3}(-1 + x_2^2)^2(1 + x_2^2)^3 \notin \Sigma_{10}^2.$$

[Verifying $\overline{x}$ satisfies the necessary optimality condition] For $i = 1$,

$$\tilde{f}_{1j} = \sum_{0 \le \sum_{k=1}^{2} \alpha_k \le 5,\, \alpha_1 = j} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \ne 1} \overline{x}_l^{\alpha_l} = 0, \quad j = 0,1,2,3,4,5,$$

and for $i = 2$,

$$\tilde{f}_{2j} = \sum_{0 \le \sum_{k=1}^{2} \alpha_k \le 5,\, \alpha_2 = j} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \ne 2} \overline{x}_l^{\alpha_l} = \begin{cases} -\frac{1}{3} & j = 0, \\ 0 & j = 1, \\ \frac{1}{3} & j = 2, \\ 0 & j = 3, \\ 0 & j = 4, \\ 0 & j = 5. \end{cases}$$

So, the necessary condition at $\overline{x}$ reads $0 \in \Sigma_{10}^2$ and

$$\frac{1}{3}(-1 + x_2^2)^2(1 + x_2^2)^3 = (-\frac{1}{3})(1 + x_2^2)^5 + \frac{1}{3}(-1 + x_2^2)^2(1 + x_2^2)^3 - (1 + x_2^2)^5(-\frac{1}{3}) \in \Sigma_{10}^2.$$

Thus, we see that these two necessary conditions are satisfied at $\overline{x}$.
[Verifying $\overline{z}$ satisfies the necessary optimality condition] For $i = 1$,

$$\tilde{f}_{1j} = \sum_{0 \le \sum_{k=1}^{2} \alpha_k \le 5,\, \alpha_1 = j} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \ne 1} \overline{z}_l^{\alpha_l} = 0, \quad j = 0,1,2,3,4,5,$$

and for $i = 2$,

$$\tilde{f}_{2j} = \sum_{0 \le \sum_{k=1}^{2} \alpha_k \le 5,\, \alpha_2 = j} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \ne 2} \overline{z}_l^{\alpha_l} = \begin{cases} -\frac{1}{3} & j = 0, \\ 0 & j = 1, \\ \frac{1}{3} & j = 2, \\ 0 & j = 3, \\ 0 & j = 4, \\ 0 & j = 5. \end{cases}$$

So, similarly, the necessary conditions at $\overline{z}$ are also satisfied.

Corollary 3.2. (Polynomial Optimization with Bivalent Constraints) *For problem (P_M) with $l = 0$, let f be a polynomial with degree $d \in \mathbb{N}$, defined by*

$$f(x) = \sum_{0 \le \sum_{k=1}^{n} \alpha_k \le d} f_{\alpha_1 \alpha_2 \ldots \alpha_n} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}, \quad x = (x_1, \ldots, x_n).$$

Suppose that $\overline{x}$ is a global minimizer of (P_M). Then, for each $i = 1, \ldots, n$,

$$[NC1] \quad \overline{x}_i \cdot \Big(\sum_{0 \le \sum_{k=1}^{n} \alpha_k \le d,\, \alpha_i \text{ is odd}} f_{\alpha_1,\ldots,\alpha_n} \prod_{j \ne i} \overline{x}_j^{\alpha_j} \Big) \le 0.$$

Proof. The conclusion follows by letting $l = 0$ in Theorem 3.1 and noting that

$$\overline{x}_i \cdot \Big(\sum_{\substack{0 \le \sum_{k=1}^n \alpha_k \le d \\ \alpha_i \text{ is odd}}} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \neq i} \overline{x}_l^{\alpha_l} \Big) = \overline{x}_i \cdot \Big(\sum_{j \text{ is odd}} \sum_{\substack{0 \le \sum_{k=1}^n \alpha_k \le d \\ \alpha_i = j}} f_{\alpha_1,\ldots,\alpha_n} \prod_{l \neq i} \overline{x}_l^{\alpha_l} \Big) = \overline{x}_i \cdot \Big(\sum_{j \text{ is odd}} \tilde{f}_{ij} \Big).$$

□

If f is a quadratic function, then our preceding necessary condition over bivalent constraints collapses to the necessary condition derived in [2].

Corollary 3.3. (Quadratic optimization over bivalent constraints) *For problem (P_M) with $l = 0$, let f be defined by $f(x) = \frac{1}{2}x^T Q x + b^T x$ where $Q \in S^n$ and $b \in \mathbb{R}^n$. Assume that $\overline{x}$ is a global minimizer. Then,*

$$[\text{NC2}] \quad \operatorname{diag}(Q)e \geq XQXe + Xb,$$

where $\operatorname{diag}(Q) = \operatorname{diag}(Q_{11}, \ldots, Q_{nn})$*,* $X = \operatorname{diag}(\overline{x}_1, \ldots, \overline{x}_n) \in S^n$*, and* $e = (1, \ldots, 1)^T \in \mathbb{R}^n$*,* $i = 1, \ldots, n$*.*

Proof. Since f is a quadratic function, our necessary condition [NC1] collapse to the condition that, for each $i = 1, \ldots, n$,

$$\overline{x}_i \cdot \big(\sum_{j \neq i} Q_{ij} \overline{x}_j + b_i \big) \leq 0. \tag{3.5}$$

Using the fact that $\overline{x}_i^2 = 1$, we see that (3.5) is equivalent to the condition that, for each $i = 1, \ldots, n$,

$$\begin{aligned} (\operatorname{diag}(Q)e - XQXe - Xb)_i &= Q_{ii} - \overline{x}_i \sum_{l=1}^n Q_{il} \overline{x}_l - \overline{x}_i b_i \\ &= \overline{x}_i (Q_{ii} \overline{x}_i - \sum_{l=1}^n Q_{il} \overline{x}_l - b_i) \\ &= -\overline{x}_i \cdot \big(\sum_{l \neq i} Q_{il} \overline{x}_l + b_i \big) \geq 0. \end{aligned}$$

Hence, the conclusion follows. □

We now present an example illustrating that our necessary optimality condition can be used to identify a global minimizer. Indeed, in this example, the global minimizer satisfies our necessary condition whereas the other feasible points that are not global minimizers fail to satisfy this necessary condition.

Example 3.3. *Consider the following 2-dimensional nonconvex optimization problem*

$$(P_E) \quad \min f(x_1,x_2) = -3x_1^5 + x_2^4 + 2x_2 - x_1^2 - x_2^2 + x_1x_2 \ s.t.\ x_1,x_2 \in \{-1,1\}.$$

Clearly, f is a polynomial with degree 5 on $\mathbb{R}^2$. Consider $\overline{x} = (1,-1)$. Note that $f(x_1,x_2) = \sum f_{i,j} x_1^i x_2^j$ with $f_{5,0} = -3$, $f_{0,4} = 1$, $f_{0,1} = 2$, $f_{2,0} = -1$, $f_{0,2} = -1$, $f_{1,1} = 1$ and $f_{ij} = 0$ for all $(i,j) \in (\{0,1,\ldots,5\} \times \{0,1,\ldots,5\}) \backslash \{(5,0),(0,4),(2,0),(0,2),(1,1)\}$. So,

$$\overline{x}_1 \cdot \Big(\sum_{0 \le \sum_{k=1}^n \alpha_k \le 5,\, \alpha_1 \text{ is odd}} f_{\alpha_1,\ldots,\alpha_n} \prod_{j \ne 1} \overline{x}_j^{\alpha_j} \Big) = \overline{x}_1 \cdot \Big(\sum_{\alpha_1 = 1,3 \text{ or } 5} f_{\alpha_1,\alpha_2} \overline{x}_2^{\alpha_2} \Big) = 1 \cdot (-3 + (-1)) \le 0$$

and

$$\overline{x}_2 \cdot \Big(\sum_{0 \le \sum_{k=1}^n \alpha_k \le 5,\, \alpha_2 \text{ is odd}} f_{\alpha_1,\ldots,\alpha_n} \prod_{j \ne 2} \overline{x}_j^{\alpha_j} \Big) = \overline{x}_2 \cdot \Big(\sum_{\alpha_2 = 1,3 \text{ or } 5} f_{\alpha_1,\alpha_2} \overline{x}_1^{\alpha_1} \Big) = (-1) \cdot \big(2 + 1\big) \le 0.$$

Therefore, $\overline{x}$ satisfies necessary global optimality condition [NC1]. On the other hand, by considering the other feasible points $\overline{u} = (1,1)$, $\overline{v} = (-1,-1)$ and $\overline{w} = (-1,1)$, we note that

$$\overline{u}_2 \cdot \Big(\sum_{\alpha_2 = 1,3 \text{ or } 5} f_{\alpha_1,\alpha_2} \overline{u}_1^{\alpha_1} \Big) = 1 \cdot (2+1) > 0,$$

$$\overline{v}_1 \cdot \Big(\sum_{\alpha_1 = 1,3 \text{ or } 5} f_{\alpha_1,\alpha_2} \overline{v}_2^{\alpha_2} \Big) = (-1) \cdot (-3 + (-1)) > 0,$$

and

$$\overline{w}_2 \cdot \Big(\sum_{\alpha_2 = 1,3 \text{ or } 5} f_{\alpha_1,\alpha_2} \overline{w}_1^{\alpha_1} \Big) = 1 \cdot (2 + (-1)) > 0.$$

So, $\overline{u} = (1,1)$, $\overline{v} = (-1,-1)$ and $\overline{w} = (-1,1)$ do not satisfy necessary condition [NC1] and hence none of them can be a global minimizer. Thus, $\overline{x}$ is the unique global minimizer.

On the other hand, it can be directly verified that $\overline{x}$ is the unique global minimizer by noting that $f(\overline{x}) = -7$, $f(\overline{u}) = f(\overline{v}) = 1$ and $f(\overline{w}) = -1$.

We now provide an example to illustrate the fact that while necessary condition [NC1] helps to identifying global minimizers, it may not exclude all non-global minimizers.

Example 3.4. *Consider the following 2-dimensional nonconvex optimization problem*

$$(P_E) \quad \min f(x_1,x_2) = x_1x_2 - x_1^5 \ s.t.\ x_1,x_2 \in \{-1,1\}.$$

Clearly, f is a polynomial with degree 5. Consider $\overline{x} = (\overline{x}_1,\overline{x}_2) = (1,-1)$. Note that Note that $f(x_1,x_2) = \sum f_{i,j} x_1^i x_2^j$ with $f_{5,0} = -1$, $f_{1,1} = 1$ and $f_{ij} = 0$ for all $(i,j) \in (\{0,1,\ldots,5\} \times \{0,1,\ldots,5\}) \backslash \{(5,0),(1,1)\}$. So,

$$\overline{x}_1 \cdot \Big(\sum_{0 \le \sum_{k=1}^n \alpha_k \le 5,\, \alpha_1 \text{ is odd}} f_{\alpha_1,\ldots,\alpha_n} \prod_{j \ne 1} \overline{x}_j^{\alpha_j} \Big) = \overline{x}_1 \cdot \Big(\sum_{\alpha_1 \text{ is odd}} f_{\alpha_1,\alpha_2} \overline{x}_2^{\alpha_2} \Big) = 1 \cdot (-1 + (-1)) \le 0$$

and

$$\overline{x}_2 \cdot \Big(\sum_{0 \leq \sum_{k=1}^{n} \alpha_k \leq 5,\, \alpha_2 \text{ is odd}} f_{\alpha_1,\ldots,\alpha_n} \prod_{j \neq 2} \overline{x}_j^{\alpha_j} \Big) = \overline{x}_2 \cdot \Big(\sum_{\alpha_2 \text{ is odd}} f_{\alpha_1,\alpha_2} \overline{x}_1^{\alpha_1} \Big) = (-1) \cdot (1) \leq 0.$$

Therefore, $\overline{x}$ satisfies necessary global optimality condition [NC1].

Consider the another feasible point $\overline{u} = (\overline{u}_1, \overline{u}_2) = (-1, 1)$. Then,

$$\overline{u}_1 \cdot \Big(\sum_{0 \leq \sum_{k=1}^{n} \alpha_k \leq 5,\, \alpha_1 \text{ is odd}} f_{\alpha_1,\ldots,\alpha_n} \prod_{j \neq 1} \overline{u}_j^{\alpha_j} \Big) = \overline{u}_1 \cdot \Big(\sum_{\alpha_1 \text{ is odd}} f_{\alpha_1,\alpha_2} \overline{u}_2^{\alpha_2} \Big) = (-1) \cdot (1 + (-1)) = 0$$

and

$$\overline{u}_2 \cdot \Big(\sum_{0 \leq \sum_{k=1}^{n} \alpha_k \leq 5,\, \alpha_2 \text{ is odd}} f_{\alpha_1,\ldots,\alpha_n} \prod_{j \neq 2} \overline{u}_j^{\alpha_j} \Big) = \overline{u}_2 \cdot \Big(\sum_{\alpha_2 \text{ is odd}} f_{\alpha_1,\alpha_2} \overline{u}_1^{\alpha_1} \Big) = 1 \cdot (-1) \leq 0.$$

Therefore, $\overline{u}$ also satisfies necessary global optimality condition [NC1].

Now, we consider the other feasible points $\overline{v} = (1, 1)$ and $\overline{w} = (-1, -1)$ and note that

$$\overline{v}_2 \cdot \Big(\sum_{0 \leq \sum_{k=1}^{n} \alpha_k \leq 5,\, \alpha_2 \text{ is odd}} f_{\alpha_1,\ldots,\alpha_n} \prod_{j \neq 2} \overline{v}_j^{\alpha_j} \Big) = \overline{v}_2 \cdot \Big(\sum_{\alpha_2 \text{ is odd}} f_{\alpha_1,\alpha_2} \overline{v}_1^{\alpha_1} \Big) = 1 \cdot (1) > 0.$$

and

$$\overline{w}_1 \cdot \Big(\sum_{0 \leq \sum_{k=1}^{n} \alpha_k \leq 5,\, \alpha_2 \text{ is odd}} f_{\alpha_1,\ldots,\alpha_n} \prod_{j \neq 1} \overline{w}_j^{\alpha_j} \Big) = \overline{w}_1 \cdot \Big(\sum_{\alpha_1 \text{ is odd}} f_{\alpha_1,\alpha_2} \overline{w}_2^{\alpha_2} \Big) = (-1) \cdot (-1 + (-1)) > 0.$$

Thus $\overline{v} = (1, 1)$ and $\overline{w} = (-1, -1)$ do not satisfy necessary condition [NC1] and hence they cannot be global minimizers.

On the other hand, it can be directly verified that $f(\overline{v}) = f(\overline{u}) = 0$, $f(\overline{x}) = -2$ and $f(\overline{w}) = 2$. So, $\overline{v} = (1, 1)$ and $\overline{w} = (-1, -1)$ are not global minimizers.

We will later employ sufficient optimality condition [SC] of Section 4 to confirm that $\overline{x}$ is in fact the global minimizer.

4 Sufficient Conditions via Separable Under-Estimators

In this section, we establish sufficient global optimality conditions for general polynomial optimization problems over box or bivalent constraints. We achieve this by employing separable polynomial under-estimators and using the characterization of global optimality of separable polynomials over box or bivalent constraints.

To do this, we first present ways of constructing separable polynomial under-estimators using the sum of squares polynomials. These constructions play key roles in deriving optimality conditions for various classes of bivalent polynomial optimization problems.

SOS-Convexity. Recall that a real-valued function f on $\mathbb{R}^n$ is a sum of squares polynomial (SOS-polynomial) in $\mathbb{R}[x]$ if there exist $k \in \mathbb{N}$, $f_j \in \mathbb{R}[x]$, $j = 1, \ldots, k$ such that for each $x \in \mathbb{R}^n$, $f(x) = \sum_{j=1}^k f_j^2(x)$. The function f is a SOS-convex polynomial on a convex set $\Delta \subseteq \mathbb{R}^n$ if $\nabla^2 f(x) = H(x)^T H(x)$, for all $x \in \Delta$, for some function $H : \mathbb{R}^n \to \mathbb{R}^{s\times n}$ with $H(x) = \big(H_{ij}(x)\big)_{1\le i\le s, 1\le j\le n}$ and each H_{ij} is a polynomial on $\mathbb{R}^n$. Clearly, any SOS-convex polynomial is, in particular, a convex polynomial. Conversely, any convex quadratic function and any convex one dimensional polynomial is a SOS-convex polynomial. It is worth noting that whether a polynomial is SOS-convex or not can efficiently be checked by solving a convex semidefinite programming problem. For details, see ([1, 4]).
Under-Estimators. Let Δ be a subset of $\mathbb{R}^n$, $\overline{x} \in \Delta$ and $d \in \mathbb{N}$. A function $h : \mathbb{R}^n \to \mathbb{R}$ is an *under-estimator* of the function $f : \mathbb{R}^n \to \mathbb{R}$ at $\bar{x}$ over a set $\Delta \subset \mathbb{R}^n$ if, for each $x \in \Delta$, $f(x) \ge h(x)$, and $f(\bar{x}) = h(\bar{x})$.

For a given polynomial f, the following lemma provides a sufficient condition in terms of SOS-convexity for a function h to be a separable polynomial underestimator.

Lemma 4.1. (Separable polynomial estimators) *Let f be a real polynomial on $\mathbb{R}^n$ and $d \in \mathbb{N}$. Let Δ be a subset of $\mathbb{R}^n$ and $\overline{x} \in \Delta$. If $h \in S_d$, $\nabla h(\overline{x}) = \nabla f(\overline{x})$ and $f - h$ is a SOS-convex polynomial on* coΔ*, then, $\tilde{h}(x) := h(x) + f(\overline{x}) - h(\overline{x})$ is a separable polynomial underestimator of f at $\overline{x}$ over Δ.*

Proof. Let $h \in S_d$, $\nabla f(\overline{x}) = \nabla h(\overline{x})$ and $f - h$ is a SOS-convex polynomial on co$D$. Define $\phi = f - h$. Then ϕ is a SOS-convex polynomial on coΔ with $\nabla \phi(\overline{x}) = 0$, and so,

$$\nabla^2 \phi(x) = H(x)^T H(x), \text{ for all } x \in \mathrm{co}\Delta, \tag{4.6}$$

where $H : \mathbb{R}^n \to \mathbb{R}^{s\times n}$ is given by $H(x) = \big(H_{ij}(x)\big)_{1\le i\le s, 1\le j\le n}$ and each H_{ij} is a polynomial on $\mathbb{R}^n$. From the Taylor expansion we see that for each $x \in \Delta$, there exists $t \in [0, 1]$ such that

$$\begin{aligned} \phi(x) - \phi(\overline{x}) &= \nabla \phi(\overline{x}) + (x - \overline{x})^T \nabla^2 \phi(tx + (1-t)\overline{x})(x - \overline{x}) \\ &= (x - \overline{x})^T \nabla^2 \phi(tx + (1-t)\overline{x})(x - \overline{x}). \end{aligned}$$

Since $tx + (1 - t)\overline{x} \in \mathrm{co}\Delta$, it follows from (4.6) that

$$\phi(x) - \phi(\overline{x}) = \|H(\tilde{x})(x - \overline{x})\|^2 \ge 0, \ \forall\, x \in \Delta,$$

where $\tilde{x} = tx + (1-t)\overline{x}$. So, $f(x) \ge h(x) + f(\overline{x}) - h(\overline{x}) = \tilde{h}(x)$. Hence, $f(x) \ge \tilde{h}(x)$ and $f(\overline{x}) = \tilde{h}(\overline{x})$, and hence, $\tilde{h}$ is a separable polynomial underestimator of f. □

In passing, note that our Lemma 4.1 may lead to useful numerical procedures for constructing separable polynomial under-estimators as SOS-convexity can be checked efficiently using convex semidefinite programming methods.

We now establish a sufficient global optimality condition for general polynomial optimization problems over box or bivalent constraints. To do this, we introduce the following definition of canonical decomposition:

Canonical decomposition. Let f be a real polynomial on $\mathbb{R}^n$ and let C be a subset of $\mathbb{R}^n$. We say that f admits a canonical decomposition on C if there exist two polynomials g, h such that g is separable, h is a SOS-convex polynomial on co$C$ and $f(x) = g(x) + h(x)$ for each $x \in \mathbb{R}^n$.

We note that, if f is a quadratic function and C is an arbitrary subset of $\mathbb{R}^n$, then a canonical decomposition for f always exists on C. Indeed, the function $f(x) = \frac{1}{2}x^T Q x + b^T x + c$ admits a canonical decomposition $f = g + h$ where $g(x) = \frac{1}{2}\lambda_{\min}(Q)\|x\|^2 + b^T x + c$, $h(x) = \frac{1}{2}x^T(Q - \lambda_{\min}(Q)I_n)x$ and $\lambda_{\min}(Q)$ is the minimum eigenvalue of Q. Note that g is separable and h is a convex quadratic function (and hence is a SOS-convex polynomial). More generally, a canonical decomposition of f on C can be explicitly given under each of the following cases:

(1) f is the sum of a quadratic function f_1 and a separable polynomial f_2;

(2) f is the sum of a quadratic function f_1 and a SOS-convex polynomial f_2.

We now establish the promised sufficient global optimality condition for general polynomial optimization problems with bivalent constraints, in terms of the canonical decomposition.

Theorem 4.1. (Sufficient Global Optimality) *For problem (P_M), let the feasible set be F and let $\overline{x} \in F$. Let g, h be a canonical decomposition of f on F such that $g(x) = \sum_{i=1}^n \sum_{j=0}^d g_{ij}\, x_i^j$ for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and h is a SOS-convex polynomial on* co$F := \prod_{i=1}^n [-1,1]$. *Suppose that the following condition holds:*

$$[SC] \quad \begin{cases} \displaystyle\sum_{j=0}^d w_{ij}(-1+x_i^2)^j(1+x_i^2)^{d-j} - (1+x_i^2)^d \sum_{j=0}^d w_{ij}\overline{x}_i^j \in \Sigma_{2d}^2 & i = 1, \ldots, l \\ \overline{x}_i\big(\displaystyle\sum_{j \text{ is odd}} w_{ij}\big) \le 0 & i = l+1, \ldots, n. \end{cases}$$

where w_{ij}, $j = 0, \ldots, d$ are defined by

$$w_{ij} = \begin{cases} g_{i1} + \frac{\partial h(\overline{x})}{\partial x_i} & \text{if} \quad j = 1, \\ g_{ij} & \text{if} \quad j \neq 1. \end{cases}$$

Then $\overline{x}$ is a global minimizer of (P_M).

Proof. Define a polynomial w on $\mathbb{R}^n$ by $w(x) = g(x) + \nabla h(\overline{x})^T x$. Then, $w \in S_d$, $\nabla w(\overline{x}) = \nabla f(\overline{x})$ and $\nabla^2(f - w)(x) = \nabla^2 h(x)$, $\forall\, x \in$ co$F$. Thus, $f - w$ is a SOS-convex polynomial on co$F$, and by Lemma 4.1, $w + f(\overline{x}) - w(\overline{x})$ is a separable polynomial under-estimator of f at $\overline{x}$. So, for all $x \in F$

$$w(x) - w(\overline{x}) \le f(x) - f(\overline{x}). \tag{4.7}$$

On the other hand, note that $w(x) = \sum_{i=1}^{n} \sum_{j=0}^{d} w_{ij} x_i^j$ with

$$w_{ij} = \begin{cases} g_{i1} + \frac{\partial h(\overline{x})}{\partial x_i} & \text{if} \quad j = 1 \\ g_{ij} & \text{if} \quad j > 1 \end{cases}, \text{ for each } i = 1, \ldots, n.$$

Then, it follows from Theorem 2.1 and [SC] that $w(x) \geq w(\overline{x})$ for each $x \in F$. So, (4.7) implies that, for each $x \in F$, $f(x) - f(\overline{x}) \geq w(x) - w(\overline{x}) \geq 0$, and hence, $\overline{x}$ is a global minimizer of (P_M). □

It is worth noting that our sufficient optimality condition [SC] and the necessary optimality condition [NC] coincide when the objective function of (P_M) is a separable polynomial.

Corollary 4.1. (Polynomial Optimization with Box Constraints) *For problem (P_M) with $l = n$, let the feasible set be $K = \prod_{i=1}^{n}[-1, 1]$ and let $\overline{x} \in K$. Let g, h be a canonical decomposition of f on K such that $g(x) = \sum_{i=1}^{n} \sum_{j=0}^{d} g_{ij}\, x_i^j$ for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and h is a SOS-convex polynomial on K. Suppose that the following condition holds:*

$$\sum_{j=0}^{d} w_{ij}(-1 + x_i^2)^j (1 + x_i^2)^{d-j} - (1 + x_i^2)^d \sum_{j=0}^{d} w_{ij} \overline{x}_i^j \in \Sigma_{2d}^2 \quad i = 1, \ldots, l,$$

where w_{ij}, $j = 0, \ldots, d$, are defined by

$$w_{ij} = \begin{cases} g_{i1} + \frac{\partial h(\overline{x})}{\partial x_i} & \textit{if} \quad j = 1, \\ g_{ij} & \textit{if} \quad j \neq 1, \end{cases}$$

Then $\overline{x}$ is a global minimizer of (P_M).

Proof. Thus, the conclusion follows from Theorem 4.1 with $l = n$. □

Corollary 4.2. (Polynomial Optimization with Bivalent Constraints) *For problem (P_M) with $l = 0$, let the feasible set be $K_0 = \prod_{i=1}^{n}\{-1, 1\}$ and let $\overline{x} \in K_0$. Let g, h be a canonical decomposition of f on K_0 such that $g(x) = \sum_{i=1}^{n} \sum_{j=0}^{d} g_{ij}\, x_i^j$ for each $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and h is a SOS-convex polynomial on $\mathrm{co}K_0 := \prod_{i=1}^{n}[-1, 1]$. Suppose that the following condition holds*

$$[SC1] \qquad \overline{x}_i\Big(\sum_{j \text{ is odd}} g_{ij} + \frac{\partial h(\overline{x})}{\partial x_i}\Big) \leq 0, \quad \textit{for each } i = 1, \ldots, n.$$

Then $\overline{x}$ is a global minimizer of (P_M).

Proof. Thus, the conclusion follows from Theorem 4.1 with $l = 0$. □

In the special case of quadratic optimization problem with $f(x) = \frac{1}{2}x^T Qx + b^T x$ and bivalent constraints, we can choose the canonical decomposition $f = g + h$ where $g(x) = \frac{1}{2}\lambda_{\min}(Q)\|x\|^2 + b^T x$ and $h(x) = \frac{1}{2}x^T(Q - \lambda_{\min}(Q)I_n)x$. Then, similar arguments as in the proof in Corollary 3.3 shows that [SC1] collapses to "$\lambda_{\min}(Q)e \geq XQXe + Xb$", and hence Corollary 4.2 reduces to the sufficient global optimality condition presented in [2].

We now employ the sufficient global optimality condition [SC1] for identifying the global minimizer of the problem, examined earlier in Example 3.3.

Example 4.1. *Consider again the following 2-dimensional nonconvex optimization problem*

$$(P_E) \quad \min f(x_1, x_2) = x_1 x_2 - x_1^5 \text{ s.t. } x_1, x_2 \in \{-1, 1\}.$$

We write $f(x_1, x_2) = g(x_1, x_2) + h(x_1, x_2)$, *where* $g(x_1, x_2) = -x_1^5 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2$ *and* $h(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + x_1 x_2$. *Note that* $g(x_1, x_2) = \sum_{i=1}^2 \sum_{j=0}^5 g_{ij}\, x_i^j$ *with* $g_{15} = -1$, $g_{12} = -\frac{1}{2}$, $g_{22} = -\frac{1}{2}$ *and* $g_{ij} = 0$ *for all* $(i, j) \in (\{1, 2\} \times \{0, 1, \ldots, 5\}) \backslash \{(1, 5), (1, 2), (2, 2)\}$. *Then, for each* $i = 1, \ldots, n$, *the inequality*

$$\overline{x}_i\Big(\sum_{j \text{ is odd}} g_{ij} + \frac{\partial h(\overline{x})}{\partial x_i}\Big) \leq 0$$

becomes

$$\begin{cases} \overline{x}_1(-1 + \overline{x}_1 + \overline{x}_2) \leq 0 \\ \overline{x}_2(\overline{x}_2 + \overline{x}_1) \leq 0. \end{cases}$$

Let $\overline{x} = (1, -1)$. *Then* $\overline{x}_1(-1 + \overline{x}_1 + \overline{x}_2) = -1$ *and* $\overline{x}_2(\overline{x}_2 + \overline{x}_1) = 0$. *Thus, [SC1] holds at* $\overline{x} = (1, -1)$ *which is indeed the global minimizer. We also note that* $\overline{u} = (-1, 1)$ *does not satisfy [SC1] with the above decomposition* $f = g + h$ *as* $\overline{u}_1(-1 + \overline{u}_1 + \overline{u}_2) = 1 > 0$.

References

[1] A.A. Ahmadi and P. Parrilo, A convex polynomial that is not SOS-convex, *Math. Program.*, **135** (2012), 275-292.

[2] A. Beck and M. Teboulle, Global optimality conditions for quadratic optimization problems with binary constraints, *SIAM. J. Optim*, **11** (2000), 179-188.

[3] D. Bertsimas and I. Popescu, Optimal inequalities in probability theory: a convex optimization approach. *SIAM J. Optim.*, **15** (2005), no. 3, 780-804.

[4] J.W. Helton and J.W. Nie, Semidefinite representation of convex sets, *Math. Program*, **122** (2010), no. 1, Ser. A, 21-64.

[5] R. Horst and M. Pardalos, *Handbook of Global Optimization*, Edited by Reiner Horst and Panos M. Pardalos Nonconvex Optimization and its Applications, 3 Kluwer Academic Publishers, Dordrechet, 1995.

[6] J.B. Hiriart-Urruty, Global optimality conditions in maximizing a convex quadratic function under convex quadratic constraints. *J. Global Optim.*, **21**, 445-455, (2001).

[7] V. Jeyakumar, N.Q. Huy and G. Li, Necessary and sufficient conditions for S-lemma and nonconvex quadratic optimization. *Optim. Eng.* **10** (2009), no. 4, 491-503.

[8] V. Jeyakumar, G. M. Lee and G. Y. Li, Alternative theorems for quadratic inequality systems and global quadratic optimization, *SIAM J. Optim.* **20**, (2009), no. 2, 983-1001.

[9] V. Jeyakumar and G. Y. Li, Necessary global optimality conditions for nonlinear programming problems with polynomial constraints, *Math. Program.*, **126** (2011), 393-399.

[10] V. Jeyakumar, S. Srisatkunarajah and N. Q. Huy, Kuhn-Tucker sufficiency for global minimum of multi-extremal mathematical programming problems, *J. Math. Anal. Appl.*, **335**, (2007), 779-788.

[11] V. Jeyakumar, A. M. Rubinov and Z. Y. Wu, Non-convex quadratic minimization problems with quadratic constraints: Global optimality conditions, *Math. Program.*, Ser. A, **110** (2007), 521-541.

[12] J.B. Lasserre, *Moments, Positive Polynomials and Their Applications*, Imperial College Press, 2009.

[13] J.B. Lasserre, Polynomials nonnegative on a grid and discrete optimization. *Trans. Amer. Math. Soc.*, **354** (2002), no. 2, 631–649.

[14] G. Li, Global quadratic minimization over bivalent constraints: Necessary and sufficient global optimality condition, *J. Optim. Theory Appl.*, **152** (2012), no. 3, 710-726.

[15] M. Laurent, *Sums of squares, moment matrices and optimization over polynomials.* Emerging Applications of Algebraic Geometry, Vol. 149 of IMA Volumes in Mathematics and its Applications, M. Putinar and S. Sullivant (eds.), Springer, pages 157-270, 2009.

[16] J. Löfberg, Pre- and post-processing sum-of-squares programs in practice, *IEEE Transactions on Automatic Control*, **54** (2009), 1007-1011.

[17] J. Löfberg, YALMIP : A Toolbox for Modeling and Optimization in MATLAB. In Proceedings of the CACSD Conference, Taipei, Taiwan, 2004.

[18] A. Neumaier, Second-order sufficient optimality conditions for local and global nonlinear programming. *J. Global Optim.* **9**, 141-151, (1996).

[19] A. T. Phillips and J.B. Rosen, A quadratic assignment formulation of the molecular confirmation problem, *J. Global Optim.*, **4**,229-241 (1994).

[20] M. C. Pinar, Sufficient global optimality conditions for bivalent quadratic optimization, *J. Optim. Theory Appl.* **122**, 433-440, (2004).

[21] P.A. Parrilo, Semidefinite programming relaxations for semialgebraic problems. *Mathematical Programming* Ser. B, **96** (2003), 293-320.

[22] J. Quan, Z. Y. Wu and G.Q. Li, Global optimality conditions for some classes of polynomial integer programming problems, *J. Indust. Manag. Optim.*, **7**, 67-78 (2011).

[23] Y. Wang and Z. Liang, Global optimality conditions for cubic minimization problem with box or binary constraits, *J. Global Optim*, **47**, 583-595, (2010).