A Robust von Neumann Minimax Theorem for Zero-Sum Games under Bounded Payoff Uncertainty*

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Abstract

The celebrated von Neumann minimax theorem is a fundamental theorem in two-person zero-sum games. In this paper, we present a generalization of the von Neumann minimax theorem, called robust von Neumann minimax theorem, in the face of data uncertainty in the payoff matrix via robust optimization approach. We establish that the robust von Neumann minimax theorem is guaranteed for various classes of bounded uncertainties, including the matrix 1-norm uncertainty, the rank-1 uncertainty and the column-wise affine parameter uncertainty.

Key words. Robust von Neumann minimax theorem, minimax theorems under payoff uncertainty, robust optimization, conjugate functions.

1 Introduction

The celebrated von Neumann Minimax Theorem [21] asserts that, for an $(n \times m)$ matrix M,

$$\min_{x \in S^n} \max_{y \in S^m} x^T M y = \max_{y \in S^m} \min_{x \in S^n} x^T M y,$$

where S^n is the n-dimensional simplex. It is a fundamental equality in two-person zero-sum games [19].

Due to its importance in mathematics, decision theory, economics and game theory, numerous generalizations have been given in the literature (see [9, 10, 11, 18] and

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other reference therein). However, these generalizations and their applications have so far been limited mainly to problems without data uncertainty, despite the reality of data uncertainty in many real-world problems due to modeling or prediction errors [2, 3, 5, 4, 6, 13, 14, 15]. For related recent work on incomplete-information games, see [1] and other references therein.

The purpose of this paper is to present a new form of the von Neumann minimax theorem, called *robust von Neumann minimax theorem*, for two-person zero sum games under data uncertainty via robust optimization and to establish that the robust von Neumann minimax theorem always holds under various classes of uncertainties, including the matrix 1-norm uncertainty, the rank-1 uncertainty and the column-wise affine parameter uncertainty.

The minimax value $\gamma_1 := \min_{x\in S^n}\max_{y\in S^m} x^T My$ and the maxmin value $\gamma_2 := \max_{y\in S^m}\min_{x\in S^n} x^T My$ can be calculated by the following two optimization problems: $\gamma_1 = \min_{(x,t)\in S^n\times\mathbb{R}}\{t : \max_{y\in S^m} x^T My \le t\}$ and $\gamma_2 = \max_{(y,t)\in S^m\times\mathbb{R}}\{t : \min_{x\in S^n} x^T My \ge t\}$. Whenever the cost function is affected by data uncertainty, the effect of uncertain data on the cost matrix M can be captured by a new matrix $M(u)$ where u is an uncertain parameter and it belongs to the compact uncertainty set $\mathcal{U} \subseteq \mathbb{R}^q$. For instance, the effect of uncertain data (a_1, a_2, a_3) on the cost matrix $M = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$ can be captured by the new matrix $M(u) = \begin{pmatrix} a_1(u) & a_2(u) \\ a_2(u) & a_3(u) \end{pmatrix}$ where $u \in \mathcal{U} \subseteq \mathbb{R}$. So, the minimax value and the maxmin value in the face of cost matrix data uncertainty can be obtained by the following two uncertain optimization problems

$$(UP_I) \qquad \min_{(x,t)\in S^n\times\mathbb{R}}\{t : \max_{y\in S^m} x^T M(u)y \le t\}$$

and

$$(UP_{II}) \qquad \max_{(y,t)\in S^m\times\mathbb{R}}\{t : \min_{x\in S^n} x^T M(u)y \ge t\}.$$

The *robust counterpart* [3, 15, 16] of the uncertain optimization problem (UP_I) is a deterministic optimization problem defined by

$$(RP_I) \qquad \min_{(x,t)\in S^n\times\mathbb{R}}\{t : \max_{y\in S^m} x^T M(u)y \le t, \quad \text{for all } u \in \mathcal{U}\}, \tag{1.1}$$

and the *optimistic counterpart* [2, 15, 16] of the uncertain optimization problem (UP_{II}) is another deterministic optimization problem defined by

$$(OP_{II}) \qquad \max_{(y,t)\in S^m\times\mathbb{R}}\{t : \min_{x\in S^n} x^T M(u)y \ge t, \quad \text{for some } u \in \mathcal{U}\}. \tag{1.2}$$

The robust minimax theorem states that the optimal values of the robust counterpart problem (RP_I) ("worst possible loss of Player I") and the optimistic counterpart (OP_{II}) ("best possible gain of Player II") are equal. Equivalently, it asserts that

$$\min_{x\in S^n}\max_{y\in S^m}\max_{u\in\mathcal{U}} x^T M(u)y = \max_{u\in\mathcal{U}}\max_{y\in S^m}\min_{x\in S^n} x^T M(u)y. \tag{1.3}$$

Employing conjugate analysis [20] and Ky Fan's minimax theorem [8], we derive the robust minimax equality (1.3) under a concave-like condition. We also show that the concave-like condition is also necessary for the robust minimax theorem in the sense that it holds if and only if

$$\inf_{x\in A}\max_{y\in B}\max_{u\in\mathcal{U}}\{x^TM(u)y+x^Ta\}=\max_{u\in\mathcal{U}}\max_{y\in B}\inf_{x\in A}\{x^TM(u)y+x^Ta\},\ \forall a\in\mathbb{R}^n.$$

Importantly, we establish that the robust minimax theorem always holds for various classes of bounded uncertainty sets, including the matrix 1-norm uncertainty set, the rank-1 matrix uncertainty set, the column-wise affine parameter uncertainty set and isotone matrix-data uncertainty set. Consequently, we also derive a robust theorem of the alternative for uncertain linear inequality systems from the robust minimax theorem.

2 A Robust Minimax Theorem under Uncertainty

In this Section, we present a concave-like condition ensuring (1.3). We also show that the condition is also necessary for (1.3) to hold for every linear perturbation.

We begin this section by fixing notation and preliminaries of convex analysis. Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean space with dimension n. The inner product in $\mathbb{R}^n$ is defined by $\langle x,y\rangle := x^Ty$ for all $x,y\in\mathbb{R}^n$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1,\ldots,x_n)\in\mathbb{R}^n : x_i\geq 0\}$. For a set A in $\mathbb{R}^n$, the convex hull of A is denoted by co$A$. We say A is convex whenever $\mu a_1+(1-\mu)a_2\in A$ for all $\mu\in[0,1]$, $a_1,a_2\in A$. A function $f:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is said to be convex if for all $\mu\in[0,1]$ $f((1-\mu)x+\mu y)\leq(1-\mu)f(x)+\mu f(y)$ for all $x,y\in\mathbb{R}^n$. The function f is said to be concave whenever $-f$ is convex.

As usual, for any proper (i.e., dom$f\neq\emptyset$) convex function f on $\mathbb{R}^n$, its conjugate function $f^*:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$ is defined by $f^*(x^*)=\sup_{x\in\mathbb{R}^n}\{\langle x^*,x\rangle-f(x)\}$ for all $x^*\in\mathbb{R}^n$. Clearly, f^* is a proper lower semicontinuous convex function and for any proper lower semicontinuous convex functions f_1,f_2 (cf. [12, 17]),

$$f_1\leq f_2 \ \Leftrightarrow\ f_1^*\geq f_2^* \ \Leftrightarrow\ \text{epi}f_1^*\subseteq\text{epi}f_2^*. \tag{2.1}$$

The following special case of Ky Fan minimax theorem [8] plays a key role in deriving our robust von Neumann minimax theorem. Recall from Ky Fan [8] that the function $f(.,y)$ is said to be convex-like whenever
$(\forall x_1,x_2\in C)$ $(\forall\lambda\in(0,1))$ $(\exists x_3\in C)$ $(\forall y\in D)$ $f(x_3,y)\leq\lambda f(x_1,y)+(1-\lambda)f(x_2,y)$.
The function $f(x,.)$ is said to be concave-like whenever
$(\forall y_1,y_2\in D)$ $(\forall\lambda\in(0,1))$ $(\exists y_3\in D)$ $(\forall x\in C)$ $f(x,y_3)\geq\lambda f(x,y_1)+(1-\lambda)f(x,y_2)$,
where $f:C\times D\to\mathbb{R}$ and C and D are sets.

Theorem 2.1. *[8, 11] Let C be a compact subset of $\mathbb{R}^n$ and let $D\subset\mathbb{R}^m$. Let $f: C\times D\to\mathbb{R}$. Suppose that $f(.,y)$ is concave-like and $f(x,.)$ is convex-like and that $f(.,y)$ is upper-semi-continuous. Then,*

$$\max_{x\in C}\inf_{y\in D}f(x,y)=\inf_{y\in D}\max_{x\in C}f(x,y).$$

Theorem 2.1. (Robust von Neumann Minimax Theorem) *Let A be a closed convex subset of $\mathbb{R}^n$ and let B be a convex compact subset of $\mathbb{R}^m$. Let $\mathcal{U}$ be a convex compact subset of $\mathbb{R}^q$. Assume that*
$(\forall\lambda \in [0,1])$ $(\forall(y_1,u_1),(y_2,u_2) \in B \times U)$ $(\exists(y,u) \in B \times U)$ $(\forall x \in A)$

$$x^T M(u)y \geqq \lambda x^T M(u_1)y_1 + (1-\lambda)x^T M(u_2)y_2. \tag{2.2}$$

Then

$$\inf_{x\in A}\max_{y\in B}\max_{u\in\mathcal{U}} x^T M(u)y = \max_{u\in\mathcal{U}}\max_{y\in B}\inf_{x\in A} x^T M(u)y.$$

Proof. Let $z = (y,u) \in \mathbb{R}^m \times \mathbb{R}^q$ and define $F : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^q \to \mathbb{R}$ by

$$F(x,z) = x^T M(u)y.$$

Then, we see that $x \mapsto F(x,z)$ is linear for any $z \in \mathbb{R}^n \times \mathbb{R}^q$ and $z \mapsto F(x,z)$ is concavelike. So, by Theorem 2.1 gives us that

$$\inf_{x\in A}\max_{z\in B\times\mathcal{U}} F(x,z) = \max_{z\in B\times\mathcal{U}}\inf_{x\in A} F(x,z).$$

Thus, the conclusion follows. □

Corollary 2.1. *Let $A = \{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid x_i \geqq 0,\ i = 1,\ldots,n,\ \sum_{i=1}^n x_i = 1\}$; let B be a convex compact subset of $\mathbb{R}^m$ and let $\mathcal{U}$ be a convex compact subset of $\mathbb{R}^q$. If $(\bigcup_{u\in\mathcal{U},\,y\in B}\{M(u)y\} - \mathbb{R}^n_+)$ is a convex set then*

$$\inf_{x\in A}\max_{y\in B}\max_{u\in\mathcal{U}} x^T M(u)y = \max_{u\in\mathcal{U}}\max_{y\in B}\inf_{x\in A} x^T M(u)y.$$

Proof. The conclusion will follow from Theorem 2.1 if we show that (2.2) holds. To see this, let $\lambda \in [0,1])$ and let $(y_1,u_1),(y_2,u_2) \in B\times\mathcal{U})$. Then, $M(u_i)y_i \in (\bigcup_{u\in\mathcal{U},\,y\in B}\{M(u)y\} - \mathbb{R}^n_+)$, for $i = 1,2$. By the convexity hypothesis, we can find $(y,u) \in B \times \mathcal{U}$ such that

$$M(u)y - \lambda M(u_1)y_1 - (1-\lambda)M(u_2)y_2 \in \mathbb{R}^n_+.$$

This together with the fact that $x \in A$ gives us the required inequality (2.2). □

It is worth noting that whenever A is simplex, i.e. $A = \{(x_1,\cdots,x_n) \in \mathbb{R}^n \mid x_i \geqq 0,\ i = 1,\cdots,n,\ \sum_{i=1}^n x_i = 1\}$, (2.2) is equivalent to the convexity of the set $(\bigcup_{u\in\mathcal{U},\,y\in B}\{M(u)y\} - \mathbb{R}^n_+)$.

As an illustration, we provide a simple numerical example verifying Corollary 2.1.

Example 2.1. *Let $A = B = \{(x_1,x_2) : x_1,x_2 \geq 0$ and $x_1 + x_2 = 1\}$. Let $\mathcal{U} = [0,1]$ and let $M(u) = M_0 + uM_1$ where*

$$M_0 = \begin{pmatrix} 0 & 5/6 \\ 1 & 1/2 \end{pmatrix} \text{ and } M_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then,

$$\begin{aligned}\bigcup_{u\in\mathcal{U},\,y\in B}\{M(u)y\} &= \bigcup_{\substack{u\in[0,1]\\(y_1,y_2)\in\mathrm{co}\{(0,1),(1,0)\}}}\left\{\begin{pmatrix} -u & \frac{5}{6} \\ 1 & \frac{1}{2}+u \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right\} \\ &= \bigcup_{u\in[0,1]}\mathrm{co}\left\{\begin{pmatrix} \frac{5}{6} \\ \frac{1}{2}+u \end{pmatrix},\begin{pmatrix} -u \\ 1 \end{pmatrix}\right\}.\end{aligned}$$

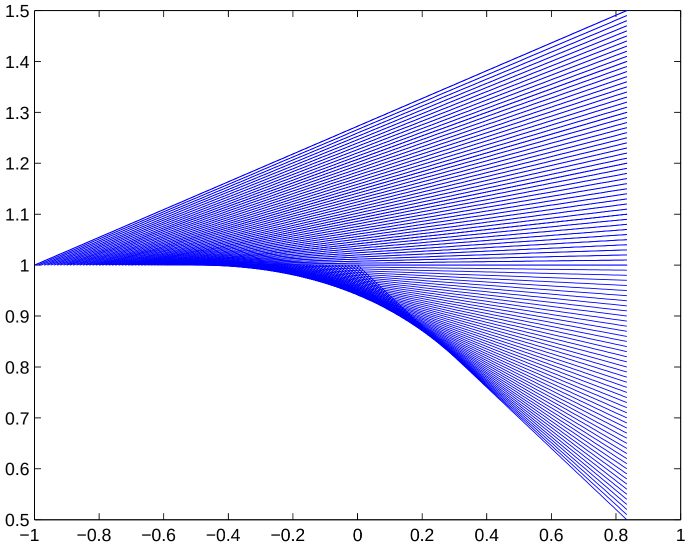

Figure 1

Clearly, the set $\bigcup_{u\in[0,1],\, y\in B}\{(M_0+uM_1)y\}$ *which is shown by the shaded region of figure 1, is not convex; whereas the set*

$$\bigcup_{u\in[0,1],\, y\in B}\{(M_0+uM_1)y\}-\mathbb{R}^2_+=\{(a_1,a_2):a_1\le\frac{5}{6},a_2\le 1.5\}$$

is convex.

To verify the robust minimax equality (1.3), let $x_2=1-x_1$ *and* $y_2=1-y_1$*. Then,*

$$x^T(M_0+uM_1)y=(\frac{1}{3}-u-\frac{4}{3}y_1)x_1+\frac{1}{2}+u+\frac{1}{2}y_1-uy_1.$$

Calculating extreme values with respect to each variable gives us

$$\max_{u\in\mathcal{U}}\max_{y\in B}\min_{x\in A}x^T(M_0+uM_1)y=\max_{u\in[0,1]}\max_{y_1\in[0,1]}\min_{x_1\in[0,1]}f_u(x_1,y_1)=\frac{5}{6}.$$

where $f_u(x_1,y_1)=(\frac{1}{3}-u-\frac{4}{3}y_1)x_1+\frac{1}{2}+u+\frac{1}{2}y_1-uy_1$*. Also,* $\min_{x\in A}\max_{u\in\mathcal{U}}\max_{y\in B}x^T(M_0+uM_1)y=\frac{5}{6}$.

We now show that our jointly concavelike condition of Theorem 2.1 is indeed a characterization for the robust von Neumann minimax theorem in the sense that the condition holds if and only if the robust von Neumann minimax theorem is valid for every linear perturbation, i.e.,

$$\inf_{x\in A}\max_{y\in B}\max_{u\in\mathcal{U}}\{x^TM(u)y+x^Ta\}=\max_{u\in\mathcal{U}}\max_{y\in B}\inf_{x\in A}\{x^TM(u)y+x^Ta\},\ \forall a\in\mathbb{R}^n.$$

Theorem 2.2. (Characterization) *Let* A *be a closed convex subset of* $\mathbb{R}^n$ *and let* B *be a convex compact subset of* $\mathbb{R}^m$*. Let* $\mathcal{U}$ *be a convex compact subset of* $\mathbb{R}^q$*. Then, the following statements are equivalent:*

(1) $(\forall\lambda \in [0,1])$ $(\forall(y_1,u_1),(y_2,u_2) \in B\times U)$ $(\exists(y,u)\in B\times U)$ $(\forall x\in A)$

$$x^T M(u)y \geqq \lambda x^T M(u_1)y_1 + (1-\lambda)x^T M(u_2)y_2. \tag{2.3}$$

(2) $\inf_{x\in A}\max_{y\in B}\max_{u\in\mathcal{U}}\{x^T M(u)y + x^T a\} = \max_{u\in\mathcal{U}}\max_{y\in B}\inf_{x\in A}\{x^T M(u)y + x^T a\}\ \ \forall a\in\mathbb{R}^n.$

Proof. [(1) ⇒ (2)] Let $z=(y,u)\in\mathbb{R}^m\times\mathbb{R}^q$ and define $\tilde{F}:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^q\to\mathbb{R}$ by

$$\tilde{F}(x,z) = x^T M(u)y + a^T x.$$

Then, we see that $x\mapsto\tilde{F}(x,z)$ is linear for any $z\in\mathbb{R}^m\times\mathbb{R}^q$ and $z\mapsto\tilde{F}(x,z)$ is concavelike. So, the Ky Fan's minimax theorem (Theorem 2.1) gives us the statement (2).

[(2) ⇒ (1)] We will establish this implication by the method of contradiction and suppose that (1) fails. Then, there exist $\lambda\in[0,1]$, $y_1,y_2\in B$ and $u_1,u_2\in\mathcal{U}$ such that for all $(y,u)\in B\times\mathcal{U}$, there exists $x\in A$ such that

$$x^T M(u)y < \lambda x^T M(u_1)y_1 + (1-\lambda)x^T M(u_2)y_2. \tag{2.4}$$

Let $a_0 = \lambda M(u_1)y_1 + (1-\lambda)M(u_2)y_2$ and let $a=-a_0$. Then, by (2.4) and statement (2),

$$\inf_{x\in A}\max_{y\in B}\max_{u\in\mathcal{U}} x^T\big(M(u)y-a_0\big) = \max_{u\in\mathcal{U}}\max_{y\in B}\inf_{x\in A} x^T\big(M(u)y-a_0\big) < 0.$$

Let $h(x) := \max_{y\in B}\max_{u\in\mathcal{U}} x^T M(u)y + \delta_A(x)$. Then, h is convex and

$$-h^*(a_0) = \inf_{x\in A}\max_{y\in B}\max_{u\in\mathcal{U}} x^T\big(M(u)y-a_0\big) < 0.$$

Thus, $(a_0,0)\notin \text{epi}h^*$. Let $h_{y,u}(x) = x^T M(u)y$, for $(y,u)\in B\times\mathcal{U}$. As $h\geq h_{y,u}$, for each $(y,u)\in B\times\mathcal{U}$, we see from (2.1) that

$$\text{epi}h^* \supseteq \text{epi}h^*_{y,u} \text{ for each } (y,u)\in B\times\mathcal{U}.$$

This together with the convexity of epih^* gives us that

$$\text{epi}h^* \supseteq \text{co}\bigcup_{y\in B,u\in\mathcal{U}} \text{epi}h^*_{y,u} = \text{co}\bigcup_{y\in B,u\in\mathcal{U}}\{M(u)y\}\times[0,+\infty).$$

Since $(a_0,0)\notin\text{epi}h^*$, it follows that

$$\lambda M(u_1)y_1 + (1-\lambda)M(u_2)y_2 = a_0 \notin \text{co}\bigcup_{y\in B,u\in\mathcal{U}} M(u)y,$$

which is impossible. □

It is easy to see that, the jointly concavelike condition (2.3) holds if the classical condition "$(u,y)\mapsto x^T M(u)y$ is concave" is satisfied, and so, robust von Neumann minimax theorem holds under the classical condition. However, we shall see, in the following simple example, that this classical condition is hard to satisfy even in the case of a linear perturbation:

Example 2.2. *Let* $M_0 = \begin{pmatrix} m_1 & m_2 \\ m_2 & m_3 \end{pmatrix}$*, and consider* $M(\Delta) = M_0 + \Delta$ *where* Δ *is a* (2×2) *symmetric matrix (which can be equivalently regarded as a vector in* $\mathbb{R}^q$ *with* $q = 3$*). Let* $n = m = 2$*. Now, we show that* $(\Delta, y) \mapsto x^T(M_0 + \Delta)y$ *is not concave for any fixed* $x \in \mathbb{R}^2_+ \backslash \{0\}$*. To see this, fix* $x = (x_1, x_2)^T \in \mathbb{R}^2_+ \backslash \{0\}$ *and let*

$$\Delta = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}$$

Then, for each fixed $x = (x_1, x_2)$*, the mapping* $(\Delta, y) \mapsto x^T(M_0 + \Delta)y$ *can be equivalently rewritten (up to an invertible linear transformation) as* $f(a_1, a_2, a_3, y_1, y_2) = (m_1 + a_1)x_1y_1 + (m_2 + a_2)x_1y_2 + (m_2 + a_2)x_2y_1 + (m_3 + a_3)x_2y_2$*. Note that an invertible linear transformation preserves concavity, we only need to show* f *is not concave. To see this, note that, for each* $(a_1, a_2, a_3, y_1, y_2) \in \mathbb{R}^5$*,* $\nabla^2 f(a_1, a_2, a_3, y_1, y_2)$ *is a constant* (5×5) *matrix*

$$C = \begin{pmatrix} 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & x_2 & x_1 \\ 0 & 0 & 0 & 0 & x_2 \\ x_1 & x_2 & 0 & 0 & 0 \\ 0 & x_1 & x_2 & 0 & 0 \end{pmatrix}$$

As $x = (x_1, x_2)^T \in \mathbb{R}^2_+ \backslash \{0\}$*,* $e_5^T C e_5 = 4x_1 + 4x_2 > 0$ *where* $e_5 = (1, 1, 1, 1, 1)^T$*. So,* f *is not concave.*

From the preceding example, we see that the classical sufficient condition "$(\Delta, y) \mapsto x^T(M + \Delta)y$ is concave" is somewhat limited from the application viewpoint. However, we shall see in the next section that our condition (2.2) can be satisfied under various types of simple and commonly used data uncertainty sets, and hence produces various classes of the robust von Neumann minimax theorems in the face of payoff matrix data uncertainty.

3 Classes of Robust Minimax Theorems

In this Section, we establish that robust von Neumann's minimax theorem always holds under various classes of uncertainty sets by verifying the joint concavelike condition in Theorem 2.1.

3.1 Matrix 1-Norm Uncertainty

In the first case, we assume that the matrix data in the bilinear function of the von Neumann's minimax theorem is uncertain and the uncertain data matrix belongs to the matrix 1-norm uncertainty set $\overline{\mathcal{U}}_1 = \{M_0 + \Delta \in \mathbb{R}^{n \times m} \mid \|\Delta\|_1 \leq \rho\}$ where $M_0 \in \mathbb{R}^{n \times m}$, $\|\Delta\|_1$ is the matrix 1-norm defined by $\|\Delta\|_1 = \sup_{x \in \mathbb{R}^m, \|x\|_1 = 1} \|\Delta x\|_1$ and $\|\Delta x\|_1$ is the l_1-norm of the vector $\Delta x \in \mathbb{R}^n$.

Theorem 3.1. (Robust Minimax Theorem I) *Let* $M_0 \in \mathbb{R}^{n\times n}$ *and let* $\overline{\mathcal{U}}_1 = \{M_0 + \Delta \in \mathbb{R}^{n\times m} \mid \|\Delta\|_1 \leq \rho\}$. $\rho > 0$. *Let* $S^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geqq 0,\ i = 1, \dots, n,\ \sum_{i=1}^n x_i = 1\}$ *and let* $S^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_i \geqq 0,\ i = 1, \dots, m,\ \sum_{i=1}^m x_i = 1\}$. *Then, we have*

$$\min_{x\in S^n}\max_{y\in S^m}\max_{M\in\overline{\mathcal{U}_1}} x^T M y = \max_{M\in\overline{\mathcal{U}_1}}\max_{y\in S^m}\min_{x\in S^n} x^T M y. \tag{3.5}$$

Proof. Let $A = S^n$, $B = S^m$. Consider $\mathcal{U}_1 = \{\Delta : \|\Delta\|_1 \leq \rho\} \subseteq \mathbb{R}^{n\times m}$ as a subset of $\mathbb{R}^q$ with $q = mn$ and let $M(\Delta) = M_0 + \Delta$, $\Delta \in \mathcal{U}_1$. Note that (3.5) is equivalent to

$$\min_{x\in S^n}\max_{y\in S^m}\max_{\Delta\in\mathcal{U}_1} x^T M(\Delta) y = \max_{\Delta\in\mathcal{U}_1}\max_{y\in S^m}\min_{x\in S^n} x^T M(\Delta) y$$

Thus, to see the conclusion from Theorem 2.1, it suffices to show that: for any $\lambda \in [0, 1]$, $y_1, y_2 \in B$ and $\Delta_1, \Delta_2 \in \mathcal{U}_1$, there exists $(y, \Delta) \in B \times \mathcal{U}_1$ such that

$$x^T M(\Delta) y \geqq \lambda x^T M(\Delta_1) y_1 + (1-\lambda) x^T M(\Delta_2) y_2 \quad \forall x \in A. \tag{3.6}$$

To see this, fix $\lambda \in [0, 1]$, $y_1, y_2 \in B$, $\Delta_1 \in \mathcal{U}_1$ and $\Delta_2 \in \mathcal{U}_1$. Let $y = \lambda y_1 + (1-\lambda) y_2 \in S^m$ and $a = \lambda\Delta_1 y_1 + (1-\lambda)\Delta_2 y_2$. Now, consider a matrix Δ defined by $\Delta = ae^T$, where $e \in \mathbb{R}^m$ with each coordinate is equal to 1. As $y \in S^m$, we have $\Delta y = a$ and $\|y\|_1 = 1$. Moreover, as $\|y\|_1 = 1$, we have

$$\|\Delta\|_1 = \sup_{\|x\|_1=1} \|\Delta x\|_1 = \|a\|_1 \sup_{\|x\|_1=1} |e^T x| \leq \|a\|_1 \sup_{\|x\|_1=1} \|x\|_1 = \|a\|_1.$$

Note that

$$\|a\|_1 = \|\lambda\Delta_1 y_1 + (1-\lambda)\Delta_2 y_2\|_1 \leq \lambda\|\Delta_1\|_1\|y_1\|_1 + (1-\lambda)\|\Delta_2\|_1\|y_2\|_1 \leq \rho.$$

So, Δ satisfying $\Delta y = a = \lambda\Delta_1 y_1 + (1-\lambda)\Delta_2 y_2$ and $\|\Delta\|_1 \leq \rho$. Now, for any $x \in A$, from $\Delta y = a$, we have

$$\begin{aligned}
\lambda x^T M(\Delta_1) y_1 + (1-\lambda) x^T M(\Delta_2) y_2 &= \lambda x^T (M_0 + \Delta_1) y_1 + (1-\lambda) x^T (M_0 + \Delta_2) y_2 \\
&= x^T M_0 y + \lambda x^T \Delta_1 y_1 + (1-\lambda) x^T \Delta_2 y_2 \\
&= x^T M_0 y + x^T a \\
&= x^T (M_0 + \Delta) y = x^T M(\Delta) y.
\end{aligned}$$

So, the conclusion follows from Theorem 2.1. □

3.2 Rank-1 Matrix Uncertainty

Secondly, we derive the robust minimax theorem in terms of rank-1 uncertainty sets $\overline{\mathcal{U}}_2 = \{M_0 + \rho uv^T \mid u \in \mathbb{R}^n, v \in \mathbb{R}^m, \|u\|_\infty \leq 1 \text{ and } \|v\|_\infty \leq 1\}$ where $\|u\|_\infty$ (resp. $\|v\|_\infty$) is the l_∞-norm of $u = (u_1, \dots, u_n) \in \mathbb{R}^n$ (resp. $v = (v_1, \dots, v_m) \in \mathbb{R}^m$) defined by $\|u\|_\infty = \max_{1\leq i\leq n} |u_i|$ (resp. $\|v\|_\infty = \max_{1\leq i\leq m} |v_i|$).

Theorem 3.2. (Robust Minimax Theorem II) *Let $M_0 \in \mathbb{R}^{n\times n}$. Let $S^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \geqq 0,\ i = 1, \cdots, n,\ \sum_{i=1}^n x_i = 1\}$ and let $S^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geqq 0,\ i = 1, \ldots, m,\ \sum_{i=1}^m x_i = 1\}$. Let $\overline{\mathcal{U}}_2 = \{M_0 + \rho uv^T \mid u \in \mathbb{R}^n, v \in \mathbb{R}^m, \|u\|_\infty \leq 1$ and $\|v\|_\infty \leq 1\}$ where $\rho > 0$. Then,*

$$\min_{x\in S^n} \max_{y\in S^m} \max_{M\in\overline{\mathcal{U}}_2} x^T My = \max_{M\in\overline{\mathcal{U}}_2} \max_{y\in S^m} \min_{x\in S^n} x^T My. \tag{3.7}$$

Proof. Let $A = S^n$, $B = S^m$. Consider $\mathcal{U}_2 = \{\rho uv^T : u \in \mathbb{R}^n, v \in \mathbb{R}^m, \|u\|_\infty \leq 1 \text{ and } \|v\|_\infty \leq 1\} \subseteq \mathbb{R}^{n\times m}$ as a subset of $\mathbb{R}^q$ with $q = mn$ and let $M(\Delta) = M_0 + \Delta$, $\Delta \in \mathcal{U}_2$. Note that (3.7) is equivalent to

$$\min_{x\in S^n} \max_{y\in S^m} \max_{\Delta\in\mathcal{U}_2} x^T M(\Delta)y = \max_{\Delta\in\mathcal{U}_2} \max_{y\in S^m} \min_{x\in S^n} x^T M(\Delta)y.$$

The conclusion will follow from Theorem 2.1, if we show that for any $\lambda \in [0, 1]$, $y_1, y_2 \in B$ and $\Delta_1, \Delta_2 \in \mathcal{U}_2$, there exists $(y, \Delta) \in B \times \mathcal{U}_2$ such that

$$x^T M(\Delta)y \geqq \lambda x^T M(\Delta_1)y_1 + (1 - \lambda)x^T M(\Delta_2)y_2 \quad \forall\, x \in A. \tag{3.8}$$

To see this, fix $\lambda \in [0, 1]$, $y_1, y_2 \in B$ and $\Delta_1, \Delta_2 \in \mathcal{U}_2$. Then, we can find $u_1, u_2 \in \mathbb{R}^n$ and $v_1, v_2 \in \mathbb{R}^m$ such that $\|u_1\|_\infty \leq 1$, $\|u_2\|_\infty \leq 1$, $\|v_1\|_\infty \leq 1$, $\|v_2\|_\infty \leq 1$, $\Delta_1 = \rho u_1 v_1^T$ and $\Delta_2 = \rho u_2 v_2^T$. Now, consider a matrix Δ defined by $\Delta = \rho a e^T$, where $e \in \mathbb{R}^m$ with each coordinate is equal to 1 and $a = \lambda u_1 v_1^T y_1 + (1-\lambda)u_2 v_2^T y_2$. Letting $y = \lambda y_1 + (1-\lambda)y_2$, we see that $y \in S^m$ and $\Delta y = \rho a$. Moreover, as $\|y_1\|_1 = 1$ and $\|y_2\|_1 = 1$, it follows that

$$\begin{aligned}
\|a\|_\infty &= \|\lambda u_1 v_1^T y_1 + (1 - \lambda)u_2 v_2^T y_2\|_\infty \\
&\leq \lambda\|u_1\|_\infty |v_1^T y_1| + (1 - \lambda)\|u_2\|_\infty |v_2^T y_2| \\
&\leq \lambda|v_1^T y_1| + (1 - \lambda)|v_2^T y_2| \\
&\leq \lambda\|v_1\|_\infty \|y_1\|_1 + (1 - \lambda)\|v_2\|_\infty \|y_2\|_1 \leq 1.
\end{aligned}$$

So,

$$\Delta = \rho a e^T \in \{\rho uv^T : u, v \in \mathbb{R}^n, \|u\|_\infty \leq 1 \text{ and } \|v\|_\infty \leq 1\}.$$

Then, we see that $\Delta \in \mathcal{U}_2$ and it satisfies $\Delta y = \rho a = \rho(\lambda u_1 v_1^T y_1 + (1-\lambda)u_2 v_2^T y_2)$. Now, for each $x \in A$, we have

$$\begin{aligned}
x^T M(\Delta_1)y_1 + (1 - \lambda)x^T M(\Delta_2)y_2 &= \lambda x^T (M_0 + \rho u_1 v_1^T)y_1 + (1 - \lambda)x^T (M_0 + \rho u_2 v_2^T)y_2 \\
&= x^T M_0 y + \rho x^T (\lambda u_1 v_1^T y_1 + (1 - \lambda)u_2 v_2^T y_2) \\
&= x^T M_0 y + \rho x^T a \\
&= x^T (M_0 + \Delta)y = x^T M(\Delta)y.
\end{aligned}$$

So, the conclusion follows from Theorem 2.1. $\square$

3.3 Column-wise Affine Parameter Uncertainty

Thirdly, we obtain our robust minimax theorem in the case where the matrix data is uncertain and the uncertain data matrix is columnwise affinely parameterized, i.e., the matrix data M belongs to the uncertainty set

$$\overline{\mathcal{U}}_3 = \{\big(a_0^1 + \sum_{i=1}^{q_1} u_i^1 a_i^1 \,, \ldots, \, a_0^m + \sum_{i=1}^{q_m} u_i^m a_i^m\big) : (u_1^j, \ldots, u_{q_j}^j) \in Z^j, j = 1, \ldots, m\},$$

where Z^j, $j = 1, \ldots, m$ is a compact convex set in $\mathbb{R}^{q_j}$, $a_i^j \in \mathbb{R}^n$, $i = 0, 1, \ldots, q_j$, $j = 1, \ldots, m$. To begin with, we first derive the following proposition as a preparation.

Proposition 3.1. *Let A be a closed convex set in $\mathbb{R}^n$ and let $S^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_j \geqq 0,\ j = 1, \ldots, m,\ \sum_{j=1}^m x_i = 1\}$. Let $a_j : \mathbb{R}^{q_j} \to \mathbb{R}^n$, $j = 1, \cdots, m$, be affine functions and let $\overline{\mathcal{U}} = \{(a_1(u_1), \ldots, a_m(u_m)) : u_j \in \mathcal{U}_j\}$ where $\mathcal{U}_j \subseteq \mathbb{R}^{q_j}$ is a convex compact set, $j = 1, \ldots, m$. Then,*

$$\inf_{x\in A} \max_{y\in S^m} \max_{M\in\overline{\mathcal{U}}} x^T M y = \max_{M\in\overline{\mathcal{U}}} \max_{y\in S^m} \inf_{x\in A} x^T M y. \tag{3.9}$$

Proof. Let $B = S^m$ and consider $\mathcal{U} = \prod_{j=1}^m \mathcal{U}_j$ as a subset of $\mathbb{R}^q$ with $q = \sum_{j=1}^m q_j$ and define

$$M(u) = (a_1(u_1), \ldots, a_m(u_m))\,,\ u = (u_1, \ldots, u_m) \in \mathcal{U}.$$

Note that (3.9) is equivalent to

$$\inf_{x\in A} \max_{y\in S^m} \max_{u\in\mathcal{U}} x^T M(u) y = \max_{u\in\mathcal{U}} \max_{y\in S^m} \inf_{x\in A} x^T M(u) y.$$

As it was seen earlier, the conclusion will follow from Theorem 2.1 if we show that for any $\lambda \in [0, 1]$, $y^1, y^2 \in B$ and $u^1, u^2 \in \mathcal{U}$, there exists $(y, u) \in B \times \mathcal{U}$ such that

$$x^T M(u) y \geqq \lambda x^T M(u^1) y^1 + (1 - \lambda) x^T M(u^2) y^2 \quad \forall\, x \in A. \tag{3.10}$$

To see this, fix $\lambda \in [0, 1]$, $y^1, y^2 \in B$, $u^1 = (u_1^1, \ldots, u_m^1) \in \mathcal{U}$ and $u^2 = (u_1^2, \ldots, u_m^2) \in \mathcal{U}$. So, for any $x \in A$, we have

$$\begin{aligned}
& \lambda x^T M(u^1) y^1 + (1 - \lambda) x^T M(u^2) y^2 \\
= \;& \lambda x^T \left(a_1(u_1^1), \ldots, a_m(u_m^1)\right) y^1 + (1 - \lambda) x^T \left(a_1(u_1^2), \ldots, a_m(u_m^2)\right) y^2 \\
= \;& \lambda \sum_{j=1}^m y_j^1 a_j(u_j^1)^T x + (1 - \lambda) \sum_{j=1}^m y_j^2 a_j(u_j^2)^T x \\
= \;& \sum_{j=1}^m \left(\lambda y_j^1 a_j(u_j^1) + (1 - \lambda) y_j^2 a_j(u_j^2)\right)^T x.
\end{aligned}$$

Let $y = \lambda y^1 + (1 - \lambda) y^2$. Then, $y = (y_1, \ldots, y_m)$ with $y_j = \lambda y_j^1 + (1 - \lambda) y_j^2$. Let $u = (u_1, \ldots, u_m)$ where each u_j, $j = 1, \ldots, m$, is given by

$$u_j = \begin{cases} \frac{\lambda y_j^1 u_j^1 + (1-\lambda) y_j^2 u_j^2}{y_j} & \text{if} \quad y_j \neq 0 \\ u_j^1 & \text{else.} \end{cases}$$

So, $u_j \in \mathcal{U}_j$ and $y_j u_j = \lambda y_j^1 u_j^1 + (1-\lambda) y_j^2 u_j^2$ (this equality is straightforward by the construction of u_j when $y_j \neq 0$. On the other hand, if $y_j = 0$, then $y_j^1 = y_j^2 = 0$ so the equality again follows). This implies that $u \in \mathcal{U}$ and $y_j a(u_j) = \lambda y_j^1 a_j(u_j^1) + (1-\lambda) y_j^2 a_j(u_j^2)$. Thus,

$$\lambda x^T M(u^1) y^1 + (1-\lambda) x^T M(u^2) y^2 = \sum_{j=1}^{m} y_j a_j(u_j)^T x = x^T M(u) y.$$

Thus, the conclusion follows. □

Remark 3.1. *Using a similar method of proof, if we further assume that $A \subseteq \mathbb{R}^n_+$, then the assumption "each a_j is an affine function" can be relaxed to "each a_j is a concave function".*

Now, we establish the robust minimax theorem for columnwise affine parameterization case.

Theorem 3.3. (Robust Minimax Theorem III) *Let $S^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \geqq 0,\ i = 1, \cdots, n,\ \sum_{i=1}^n x_i = 1\}$ and let $S^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geqq 0,\ i = 1, \ldots, m,\ \sum_{i=1}^m x_i = 1\}$. Let $\overline{\mathcal{U}}_3 = \{\big(a_0^1 + \sum_{i=1}^k u_i^1 a_i^1, \ldots, a_0^m + \sum_{i=1}^k u_i^m a_i^m\big) : (u_1^j, \ldots, u_k^j) \in Z^j, j = 1, \ldots, m\}$, where Z^j, $j = 1, \ldots, m$ is a compact convex set in $\mathbb{R}^k$, $a_i^j \in \mathbb{R}^n$, $i = 0, 1, \ldots, k$, $j = 1, \ldots, m$ Then,*

$$\min_{x \in S^n} \max_{y \in S^m} \max_{M \in \overline{\mathcal{U}}_3} x^T M y = \max_{M \in \overline{\mathcal{U}}_3} \max_{y \in S^m} \min_{x \in S^n} x^T M y. \tag{3.11}$$

Proof. The conclusion follows by the preceding proposition by letting $A = S^n$ (which is convex compact and so the infimum is attained on A), $\mathcal{U}_j = Z^j$ and letting a_j, $j = 1, \ldots, m$, be an affine mapping defined by

$$a_j(u_j) = a_0^j + \sum_{i=1}^{q_j} u_i^j a_i^j, \quad u_j = (u_1^j, \ldots, u_{q_j}^j) \in \mathbb{R}^{q_j}.$$

□

As a simple application of Proposition 3.1, we derive a robust theorem of the alternative for a parameterized linear inequality system.

Corollary 3.1. (Robust Gordan Alternative Theorem) *For each $j = 1, \cdots, m$, let $a_j : \mathbb{R}^{q_j} \to \mathbb{R}^n$ be an affine function. Let $\mathcal{U}_j$ be a convex compact subset of $\mathbb{R}^{q_j}$, $j = 1, \cdots, m$. Then exactly one of the following two statements holds:*

(i) $(\exists x \in \mathbb{R}^n)\ (\forall u_j \in \mathcal{U}_j)\ \ a_j(u_j)^T x < 0, \quad j = 1, \cdots, m$

(ii) $(\exists 0 \neq \bar{\lambda} \in \mathbb{R}^m_+)\ (\exists \bar{u}_j \in \mathcal{U}_j,\ j = 1, \cdots, m),\ \ \sum_{j=1}^m \bar{\lambda}_j a_j(\bar{u}_j) = 0.$

Proof. As both (i) and (ii) can not have a solution simultaneously, we only need to show that [Not$(i) \Rightarrow (ii)$]. To see this, let $M(u) = (a_1(u_1), \ldots, a_m(u_m)) \in \mathbb{R}^{n\times m}$ and let $u_j \in \mathbb{R}^{q_j},\ j = 1, 2, \cdots, m$. Then,

$$x^T M(u)y = \sum_{j=1}^{m} y_j a_j(u_j)^T x.$$

Let $A = \mathbb{R}^n$, $B = \{(y_1, \ldots, y_m) : \sum_{j=1}^m y_j = 1, y_j \geq 0\}$ and let $\mathcal{U} = \prod_{j=1}^m \mathcal{U}_j$. Then Not$(i)$ implies that

$$\inf_{x\in A} \max_{y\in B} \max_{u\in\mathcal{U}} x^T M(u)y \;=\; \inf_{x\in\mathbb{R}^n} \max_{\sum_{j=1}^m y_j=1, y_j\geq 0} \max_{u_j\in\mathcal{U}_j} \sum_{j=1}^{m} y_j a_j(u_j)^T x \geq 0.$$

(Otherwise, $\inf_{x\in A} \max_{\sum_{j=1}^m y_j=1,y_j\geq 0} \max_{u_j\in\mathcal{U}_j} \sum_{j=1}^m y_j a_j(u_j)^T x < 0$, and so, there exists $x_0 \in A$ such that $\sum_{j=1}^m y_j a_j(u_j)^T x_0 < 0$ for all $y_j \geq 0$ with $\sum_{j=1}^m y_j = 1$ and for all $u_j \in \mathcal{U}_j$. This means that the statement (i) is true which contradicts our assumption.) Hence, by Proposition 3.1, we have

$$\begin{aligned} \max_{\sum_{j=1}^m y_j=1, y_j\geq 0} \max_{u_j\in\mathcal{U}_j} \inf_{x\in\mathbb{R}^n} \sum_{j=1}^{m} y_j a_j(u_j)^T x &= \max_{y\in B} \max_{u\in\mathcal{U}} \inf_{x\in A} x^T M(u)y \\ &= \inf_{x\in A} \max_{y\in B} \max_{u\in\mathcal{U}} x^T M(u)y \geq 0. \end{aligned}$$

Thus, there exist $\bar{\lambda}_j \geqq 0$, $j = 1, \cdots, m$, not all zero, and $\bar{u}_j \in \mathcal{U}_j,\ j = 1, \cdots, m$, such that, for each $x \in \mathbb{R}^n$, $\sum_{j=1}^m \bar{\lambda}_j a_j(\bar{u}_j)^T x \geq 0$. So, the conclusion follows. ☐

Remark 3.2. *If $\mathcal{U}_j$, $j = 1, \cdots, m$, are singletons, then Corollary 3.1 collapses to the classical Gordan's alternative theorem [7].*

3.4 Isotone Matrix Data Uncertainty

Now, we obtain a form of robust minimax theorem in the case where the matrix data is uncertain and the uncertain matrix is isotone on $\mathcal{U}$ in the sense that the mapping $u \mapsto M(u)$ satisfies the condition that, for any $u_1, u_2 \in \mathcal{U}$, $\max\{u_1, u_2\} \in \mathcal{U}$ and

$$u_1, u_2 \in \mathcal{U},\ u_1 \geq u_2 \;\Rightarrow\; M(u_1) \geq M(u_2).$$

Note that $\max\{u_1, u_2\}$ is the vector whose ith coordinate is the maximum of the ith coordinate of u_1 and u_2, and that $C_1 \geq C_2$ means each entry in the matrix $C_1 - C_2$ is nonnegative. For a simple example of an isotone matrix data uncertainty, let $\mathcal{U}_0 = \{(u_1, \ldots, u_q) \in \mathbb{R}^q : 0 \leq u_i \leq 1, i = 1, \ldots, q\}$, $\widehat{\mathcal{U}_0} = \{M_0 + \sum_{i=1}^q u_i M_i \;:\; M_i \in \mathbb{R}^n \times \mathbb{R}^n, M_i \geq 0, i = 1, \ldots, u_q,\ u = (u_1, \ldots, u_q) \in \mathcal{U}_0\}$.

Theorem 3.4. (Robust Minimax Theorem IV) *Let $S^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \geqq 0,\ i = 1, \cdots, n,\ \sum_{i=1}^n x_i = 1\}$ and let $S^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m \mid x_i \geqq 0,\ i = 1, \ldots, m,\ \sum_{i=1}^m x_i = 1\}$. Suppose that $\mathcal{U}$ is a convex compact set in $\mathbb{R}^q$ and $u \mapsto M(u)$ is an isotone mapping on $\mathcal{U}$. Then,*

$$\min_{x\in S^n} \max_{y\in S^m} \max_{u\in\mathcal{U}} x^T M(u)y = \max_{u\in\mathcal{U}} \max_{y\in S^m} \min_{x\in S^n} x^T M(u)y. \tag{3.12}$$

Proof. Let $A = S^n$, $B = S^m$. Then, the conclusion will follow from Theorem 2.1 if we show that for any $\lambda \in [0, 1]$, $y^1, y^2 \in B$ and $u^1, u^2 \in \mathcal{U}$, there exists $(y_0, u_0) \in B \times \mathcal{U}$ such that

$$x^T M(u_0) y_0 \geqq \lambda x^T M(u^1) y^1 + (1 - \lambda) x^T M(u^2) y^2 \quad \forall\, x \in A. \tag{3.13}$$

To see this, fix $\lambda \in [0, 1]$, $y^1, y^2 \in B$, $u^1 = (u_1^1, \ldots, u_m^1) \in \mathcal{U}$ and $u^2 = (u_1^2, \ldots, u_m^2) \in \mathcal{U}$. Let $u_0 = \max\{u^1, u^2\}$ and $y_0 = \lambda y^1 + (1 - \lambda) y^2$. As $u \mapsto M(u)$ is isotone on $\mathcal{U}$, it follows that $u_0 \in \mathcal{U}$,

$$M(u_0) \geq M(u^1) \text{ and } M(u_0) \geq M(u^2).$$

Now, for each $x \in A$, noting that $x \in \mathbb{R}^n_+$ and $y^1, y^2 \in \mathbb{R}^m_+$, we obtain that

$$x^T M(u^1) y^1 - x^T M(u_0) y^1 = x^T (M(u^1) - M(u_0)) y^1 \leq 0$$

and

$$x^T M(u^2) y^2 - x^T M(u_0) y^2 = x^T (M(u^1) - M(u_0)) y^2 \leq 0.$$

This gives us that

$$\begin{aligned} \lambda x^T M(u^1) y^1 + (1 - \lambda) x^T M(u^2) y^2 &\leq \lambda x^T M(u_0) y^1 + (1 - \lambda) x^T M(u_0) y^2 \\ &= x^T M(u_0) y_0. \end{aligned}$$

□

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