Robust Duality for Generalized Convex Programming Problems under Data Uncertainty*

V. Jeyakumar[†], G. Li[‡] and G. M. Lee[§]

Revised Version: April 2, 2011

Abstract

In this paper we present a robust duality theory for generalized convex programming problems in the face of data uncertainty within the framework of robust optimization. We establish robust strong duality for an uncertain nonlinear programming primal problem and its uncertain Lagrangian dual by showing strong duality between the deterministic counterparts: robust counterpart of the primal model and the optimistic counterpart of its dual problem. A robust strong duality theorem is given whenever the Lagrangian function is convex. We provide classes of uncertain non-convex programming problems for which robust strong duality holds under a constraint qualification. In particular, we show that robust strong duality is guaranteed for non-convex quadratic programming problems with a single quadratic constraint with the spectral norm uncertainty under a generalized Slater condition. Numerical examples are given to illustrate the nature of robust duality for uncertain nonlinear programming problems. We further show that robust duality continues to hold under weakened convexity condition.

Key words. Robust optimization, generalized convexity, duality under uncertainty, robust quadratic optimization.

AMS subject classification. 90C22,90C25,90C46

*he authors are grateful to the referee and the Editor for their helpful comments and valuable suggestions which have contributed to the final preparation of the paper. The first and second authors were partially supported by a grant from the Australian Research Council. The third author was supported by the Korea Science and Engineering Foundation (KOSEF) NRL program grant funded by the Korea government (MEST)(No. ROA-2008-000-20010-0).

[†]Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia

[‡]Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia

[§]Department of Applied Mathematics, Pukong National University, Busan 608-737, Korea

1 Introduction

Consider the standard nonlinear programming problem with inequality constraints

$$\text{(P)} \quad \inf_{x \in X_0} \{f(x) : g_i(x) \le 0,\ i = 1, \cdots, m\},$$

where X_0 is an open set in $\mathbb{R}^n$, $f : X_0 \to \mathbb{R}$ and $g_i : X_0 \to \mathbb{R}$ are continuously differentiable functions. This problem in the face of data uncertainty in the constraints can be captured by the following nonlinear programming problem:

$$\text{(UP)} \quad \inf_{x \in X_0} \{f(x) : g_i(x, v_i) \le 0,\ i = 1, \cdots, m\},$$

where v_i is an uncertain parameter and $v_i \in \mathcal{V}_i$ for some convex compact set $\mathcal{V}_i$ in $\mathbb{R}^q$ and $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ is continuously differentiable. Robust optimization, which has emerged as a powerful deterministic approach for studying mathematical programming under uncertainty [2, 4, 5, 6, 14], associates with the uncertain program (UP) its *robust counterpart* [3],

$$\text{(RP)} \quad \inf_{x \in X_0} \{f(x) : g_i(x, v_i) \le 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \cdots, m\},$$

where the uncertain constraints are enforced for every possible value of the parameters within their prescribed uncertainty sets $\mathcal{V}_i,\ i = 1, \cdots, m$. For other approaches to treating mathematical programming under uncertainty, see [17, 18].

On the other hand, for each fixed $v_i \in \mathcal{V}_i$, the uncertain Lagrangian dual of (UP) [7, 16] is given by

$$\text{(DP)} \quad \max_{\lambda_i \ge 0} \inf_{x \in X_0} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i)\}.$$

The optimistic counterpart of the uncertain dual (DP) is the deterministic problem

$$\text{(ODP)} \quad \max_{v_i \in \mathcal{V}_i, \lambda_i \ge 0} \inf_{x \in X_0} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i)\},$$

where the inner problem is maximized over all $v_i \in \mathcal{V}_i$ and all $\lambda_i \ge 0$. Robust strong duality holds between (UP) and (DP) when

$$\begin{aligned} & \inf_{x \in X_0} \{f(x) : g_i(x, v_i) \le 0, \quad \forall v_i \in \mathcal{V}_i,\ i = 1, \cdots, m\} \\ = \ & \max_{v_i \in \mathcal{V}_i, \lambda_i \ge 0} \inf_{x \in X_0} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i)\}. \end{aligned}$$

Following the work of Beck and Ben-Tal [1], recently robust strong duality theorems have been given for convex programming problems [12].

The purpose of this paper is to establish robust strong duality for classes of uncertain non-convex programming problems (UP). A robust strong duality theorem is first given whenever the Lagrangian function is convex. We show that robust strong duality is guaranteed for non-convex quadratic programming problems with a single quadratic constraint with the spectral norm uncertainty under a generalized Slater condition and for separable homogeneous polynomial problems with the interval uncertainty. Numerical examples are given to illustrate the significance of robust strong duality for uncertain nonlinear programming problems.

The outline of the paper is as follows. Section 2 presents preliminary results that will be used later in the paper. Section 3 develops robust necessary as well as sufficient optimality conditions for the uncertain problem (UP) by examining its robust counterpart. Section 4 presents robust duality theorems for (UP). Section 5 provides classes of non-convex programming problems under different uncertainties for which robust duality holds. Section 6 shows that robust duality continues to hold under weakened convexity condition.

2 Preliminaries

We begin this section by fixing notation and definitions. Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean space with dimension n. The inner product in $\mathbb{R}^n$ is defined by $\langle x, y\rangle := x^T y$ for all $x, y \in \mathbb{R}^n$. The norm of $x \in \mathbb{R}^n$ is defined by $\|x\| = \sqrt{x^T x}$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$. For a set A in $\mathbb{R}^n$, the interior (resp. closure, convex hull) of A is denoted by int$A$ (resp. $\overline{A}$, co$A$). We say A is convex whenever $\mu a_1 + (1-\mu)a_2 \in A$ for all $\mu \in [0,1]$, $a_1, a_2 \in A$. Let X_0 be an open set in $\mathbb{R}^n$ and let C be a subset of X_0. A function $f : X_0 \to \mathbb{R} \cup \{+\infty\}$ is said to be convex on C if $f((1-\mu)x + \mu y) \leq (1-\mu)f(x) + \mu f(y)$ for all $\mu \in [0,1]$ and for all $x, y \in C$. The function f is said to be concave on C whenever $-f$ is convex on C. Moreover, a function $f : X_0 \to \mathbb{R}$ is said to be pseudo-linear if for any $x, x^* \in X_0$ there exists $\alpha : X_0 \times X_0 \to (0, +\infty)$ such that

$$f(x) - f(x^*) = \alpha(x, x^*)\nabla f(x^*)^T(x - x^*).$$

Note that any fractional linear function $f(x) = \frac{a^T x + \alpha}{b^T x + \beta}$ is a pseudo linear function on $X_0 = \{x : b^T x + \beta > 0\}$. We use S^n to denote the space of $(n \times n)$ symmetric matrices. For $A \in S^n$, $A \succeq 0$ (resp., $A \succ 0$) means that A is positive semi-definite (resp., definite). Let $C \subseteq \mathbb{R}^q$. For a continuously differentiable function $g : \mathbb{R}^n \times C \to \mathbb{R}$, we use $\nabla_1 g$ to denote the derivative of g with respect to the first variable.

The following Danskin' theorem on the directional derivative for maximum function will be useful in our later analysis.

Theorem 2.1. *[18, page 352] Let Θ be a nonempty, compact topological space and let $g : \mathbb{R}^n \times \Theta \to \mathbb{R}$ be such that $g(\cdot, \theta)$ is differentiable for every $\theta \in \Theta$ and $\nabla_1 g(x, \theta)$ is continuous on $\mathbb{R}^n \times \Theta$. Let $\phi(x) := \sup_{\theta\in\Theta} g(x, \theta)$. Denote $\bar{\Theta}(x) := \arg\max_{\theta\in\Theta} g(x, \theta)$. Then the function $\phi(x)$ is locally Lipschitz continuous, directionally differentiable and*

$$\phi'(x; h) = \sup_{\theta\in\bar{\Theta}(x)} \nabla_1 g(x, \theta)^T h,$$

where $\phi'(x; h) = \lim_{t\to 0+} \frac{\phi(x+th)-\phi(x)}{t}$.

Next, we establish a robust Gordan type alternative theorem for a parametric inequality system.

Theorem 2.2. *Let $a_i : \mathbb{R}^q \to \mathbb{R}^n$ be a continuous mapping and $\mathcal{V}_i$ is a convex compact set in $\mathbb{R}^q$, $i = 1, \ldots, m$. Suppose that $v_i \mapsto a_i(v_i)^T d$ is concave on $\mathcal{V}_i$ for each fixed $d \in \mathbb{R}^n$, $i = 1, \ldots, m$. Then, exactly one of the following statements holds:*

(i) $(\exists d \in \mathbb{R}^n)$ $(\forall\, v_i \in \mathcal{V}_i)$ $a^T d < 0,\ a_i(v_i)^T d < 0,\ i = 1, \ldots, m,$

(ii) $(\exists 0 \neq \mu = (\mu_0, \mu_1, \ldots, \mu_m) \in \mathbb{R}^{m+1})(\exists v_i \in \mathcal{V}_i),\ \mu_0 a + \sum_{i=1}^m \mu_i a_i(v_i) = 0.$

Proof. [(i) ⇒ Not (ii)] We establish this implication by the method of contradiction. Suppose that (ii) holds. Then, there exist $\mu_0, \mu_1, \ldots, \mu_m \geq 0$ with $\mu_0 + \sum_{i=1}^m \mu_i = 1$ and $v_i \in \mathcal{V}_i$ such that $\mu_0 a + \sum_{i=1}^m \mu_i a_i(v_i) = 0$. So, by (i),

$$0 = \big(\mu_0 a + \sum_{i=1}^m \mu_i a_i(v_i)\big)^T d = \mu_0 a^T d + \sum_{i=1}^m \mu_i a_i(v_i)^T d < 0.$$

This is a contradiction.

[Not (i) ⇒ (ii)] Suppose (i) fails. Then, $0 \notin \Omega$, where

$$\Omega = \{(a^T d, \sup_{v_1\in\mathcal{V}_1} a_1(v_1)^T d, \sup_{v_2\in\mathcal{V}_2} a_2(v_2)^T d, \ldots, \sup_{v_m\in\mathcal{V}_m} a_m(v_m)^T d) : d \in \mathbb{R}^n\} + \mathrm{int}\mathbb{R}^{m+1}_+.$$

Clearly, Ω is convex, and so, by the convex separation theorem [20, Theorem 1.1.5], there exist $\mu_0, \ldots, \mu_m \geq 0$ with $\mu_0 + \sum_{i=1}^m \mu_i = 1$ such that, for each $d \in \mathbb{R}^n$,

$$\mu_0 a^T d + \sum_{i=1}^m \sup_{v_i\in\mathcal{V}_i} \big(\mu_i a_i(v_i)\big)^T d = \mu_0 a^T d + \sum_{i=1}^m \mu_i \sup_{v_i\in\mathcal{V}_i} a_i(v_i)^T d \geq 0.$$

This implies that, for each $d \in \mathbb{R}^n$,

$$\mu_0 a^T d + \sup_{v_i\in\mathcal{V}_i, 1\leq i\leq m} \sum_{i=1}^m \big(\mu_i a_i(v_i)\big)^T d = \mu_0 a^T d + \sum_{i=1}^m \sup_{v_i\in\mathcal{V}_i} \big(\mu_i a_i(v_i)\big)^T d \geq 0.$$

Now, let $h(d) := \mu_0 a^T d + \sup_{v_i\in\mathcal{V}_i, 1\leq i\leq m} \sum_{i=1}^m \big(\mu_i a_i(v_i)\big)^T d$. Clearly, h is a convex function and 0 is a global minimizer of h, and hence, by [20, Theorem 2.18],

$$0 \in \partial h(0) \subseteq \mu_0 a + \overline{\mathrm{co} \bigcup_{v_i\in\mathcal{V}_i, 1\leq i\leq m} \{\sum_{i=1}^m \mu_i a_i(v_i)\}}.$$

The conclusion will follow if we show that $\bigcup_{v_i\in\mathcal{V}_i,1\le i\le m}\{\sum_{i=1}^m \mu_i a_i(v_i)\}$ is convex and closed. To see the convexity, let $x^1,x^2\in\bigcup_{v_i\in\mathcal{V}_i,1\le i\le m}\{\sum_{i=1}^m \mu_i a_i(v_i)\}$ and $\lambda\in[0,1]$. Then, there exist $v_i^j\in\mathcal{V}_i$, $j=1,2$ and $i=1,\ldots,m$ such that $x^j=\sum_{i=1}^m \mu_i a_i(v_i^j)$. So, for all $d\in\mathbb{R}^n$

$$(x^j)^Td=\sum_{i=1}^m \mu_i\big(a_i(v_i^j)\big)^Td.$$

As $v_i^j\in\mathcal{V}_i$, this together with the concavity of $v_i\mapsto a_i(v_i)^Td$ on $\mathcal{V}_i$ for all $d\in\mathbb{R}^n$ implies that for each $d\in\mathbb{R}^n$

$$\lambda(x^1)^Td+(1-\lambda)(x^2)^Td=\sum_{i=1}^m \mu_i\big(\lambda a_i(v_i^1)+(1-\lambda)a_i(v_i^2)\big)^Td\ge\sum_{i=1}^m \mu_i\big(a_i(\lambda v_i^1+(1-\lambda)v_i^2)\big)^Td.$$

Letting $v_i=\lambda v_i^1+(1-\lambda)v_i^2\in\mathcal{V}_i$, we obtain that

$$\lambda x^1+(1-\lambda)x^2=\sum_{i=1}^m \mu_i a_i(v_i)\in\bigcup_{v_i\in\mathcal{V}_i,1\le i\le m}\{\sum_{i=1}^m \mu_i a_i(v_i)\}.$$

To show that $\bigcup_{v_i\in\mathcal{V}_i,1\le i\le m}\{\sum_{i=1}^m \mu_i a_i(v_i)\}$ is closed, let $x^k\to x$ with

$$x^k\in\bigcup_{v_i\in\mathcal{V}_i,1\le i\le m}\{\sum_{i=1}^m \mu_i a_i(v_i)\}.$$

Then, there exist $v_i^k\in\mathcal{V}_i$, $i=1,2,\ldots,m$, such that $x^k=\sum_{i=1}^m \mu_i a_i(v_i^k)$. As $v_i^k\in\mathcal{V}_i$, by passing to subsequence, we may assume that $v_i^k\to v_i\in\mathcal{V}_i$. Thus, passing to limit and noting that a_i is continuous, we have

$$x=\sum_{i=1}^m \mu_i a_i(v_i)\in\bigcup_{v_i\in\mathcal{V}_i,1\le i\le m}\{\sum_{i=1}^m \mu_i a_i(v_i)\}.$$

So, $\displaystyle\bigcup_{v_i\in\mathcal{V}_i,1\le i\le m}\{\sum_{i=1}^m \mu_i a_i(v_i)\}$ is convex and closed, and hence the conclusion follows.

□

3 Robust Optimality Conditions

In this section, we provide necessary, and sufficient optimality conditions for the uncertain optimization problem (UP). To begin with, we recall that the robust feasible set F is defined by

$$F:=\{x\in X_0\mid g_i(x,v_i)\le 0,\ \forall v_i\in\mathcal{V}_i,\ i=1,\cdots,m\}.$$

We say that x^* is a robust global minimizer of (UP) if x^* is a global minimizer of (RP), that is, $x^*\in F$ and $f(x)\ge f(x^*)\ \forall x\in F$. Moreover, we say that x^* is a robust local minimizer of (UP) if $x^*\in F$ and $\exists\epsilon>0$ s.t. $\forall x\in F\cap B_\epsilon(x^*)$, $f(x)\geqq f(x^*)$, where $B_\epsilon(x^*)=\{x\in\mathbb{R}^n\mid ||x-x^*||<\delta\}$.

Robust necessary optimality conditions

Let $x^* \in F$. Let us decompose $I := \{1, \cdots, m\}$ into two index sets $I = I_1(x^*) \cup I_2(x^*)$, where $I_1(x^*) = \{i \in I \ : \ \exists v_i \in \mathcal{V}_i \text{ s.t. } g_i(x^*, v_i) = 0\}$ and $I_2(x^*) = I \setminus I_1(x^*)$. Let $\mathcal{V}_i^0 = \{v_i \in \mathcal{V}_i \mid g_i(x^*, v_i) = 0\}$ for $i \in I_1(x^*)$. Now, we define an Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) as follows:

$$(\exists d \in \mathbb{R}^n)(\forall v_i \in \mathcal{V}_i^0) \ \nabla_1 g_i(x^*, v_i)^T d < 0, \ i \in I_1(x^*).$$

The proof of robust strong duality relies on the robust Karush-Kuhn-Tucker (KKT) necessary optimality condition for (UP) which is given below. As in the classical approach to necessary optimality conditions, the proof of the robust necessary condition employs the robust Gordan's theorem and linearization.

Theorem 3.1. (Robust KKT necessary optimality condition) *Let x^* be a robust local minimizer of (UP). Suppose that $g_i(x, \cdot)$ is concave on $\mathcal{V}_i$, for each $x \in \mathbb{R}^n$ and for each $i = 1, \ldots, m$. Then, there exist $\lambda_i \geq 0$ with $\sum_{i=0}^m \lambda_i = 1$ and $v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$ such that*

$$\lambda_0 \nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla_1 g_i(x^*, v_i) = 0 \ \text{and} \ \lambda_i g_i(x^*, v_i) = 0, \ i = 1, \ldots, m. \tag{3.1}$$

Moreover, if we further assume that the Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds, then

$$\nabla f(x^*) + \sum_{i=1}^m \lambda_i \nabla_1 g_i(x^*, v_i) = 0 \ \text{and} \ \lambda_i g_i(x^*, v_i) = 0, \ i = 1, \ldots, m. \tag{3.2}$$

Proof. As x^* is a robust local minimizer of (UP), there exists $\epsilon > 0$ such that $\mathbb{B}(x^*; \epsilon) \subseteq X_0$ and

$$g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, \ x \in \mathbb{B}(x^*; \epsilon) \ \Rightarrow \ f(x) \geq f(x^*).$$

Define $\psi_i(x) = \max_{v_i \in \mathcal{V}_i} g_i(x, v_i)$, $i = 1, \ldots, m$. Then, we have

$$\psi_i(x) \leq 0, \ x \in \mathbb{B}(x^*; \epsilon) \ \Rightarrow \ f(x) \geq f(x^*). \tag{3.3}$$

[**Linearization**] Let $I_1(x^*) = \{i \in I : \exists\, v_i \in \mathcal{V}_i \text{ s.t. } g_i(x^*, v_i) = 0\}$ and $I_2(x^*) = \{1, \ldots, m\} \backslash I_1(x^*)$. Denote $\mathcal{V}_i^0 = \{v_i \in \mathcal{V}_i : g_i(x^*, v_i) = 0\}$ for all $i \in I_1(x^*)$. Clearly, $\mathcal{V}_i^0$ is a closed subset of the compact set $\mathcal{V}_i$, and hence, is also compact. Note that if $I_1(x^*) = \emptyset$, then, $g_i(x, v_i) < 0$ for all $v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$, and so, (3.1) holds with $\lambda_0 = 1$ and $\lambda_i = 0$, $i = 1, \ldots, m$. Without loss of generality, we may assume that $I_1(x^*)$ is not empty. We first show that the following system has no solution:

$$\nabla_1 g_i(x^*, v_i)^T d < 0, \forall v_i \in \mathcal{V}_i^0, \ i \in I_1(x^*), \ \text{and} \ \nabla f(x^*)^T d < 0. \tag{3.4}$$

Suppose not, then there exists $d_0 \in \mathbb{R}^n$ such that

$$\nabla_1 g_i(x^*, v_i)^T d_0 < 0, \forall v_i \in \mathcal{V}_i^0,\ i \in I_1(x^*), \text{ and } \nabla f(x^*)^T d_0 < 0.$$

This implies that

$$\max_{v_i \in \mathcal{V}_i^0} \nabla_1 g_i(x^*, v_i)^T d_0 < 0,\ i \in I_1(x^*) \text{ and } \nabla f(x^*)^T d_0 < 0.$$

Note from the Danskin's Theorem (Lemma 2.1) that, for each $i \in I_1(x^*)$

$$\psi_i'(x^*; d_0) = \max_{v_i \in \bar{\Theta}_i(x^*)} \nabla_1 g_i(x^*, v_i)^T d_0$$

where $\psi_i(x) = \max_{v_i \in \mathcal{V}_i} g_i(x, v_i)$ and $\psi_i'(x; d_0) = \lim_{t\to 0+} \frac{\psi_i(x+td_0)-\psi_i(x)}{t}$ and for each $i \in I_1(x^*)$,

$$\bar{\Theta}_i(x^*) = \{v_i \in \mathcal{V}_i : g_i(x^*, v_i) = \psi_i(x^*)\} = \{v_i \in \mathcal{V}_i : g_i(x^*, v_i) = 0\} = \mathcal{V}_i^0.$$

So, $\psi_i'(x^*; d_0) = \sup_{v_i \in \mathcal{V}_i^0} \nabla_1 g_i(x^*, v_i)^T d_0$ for each $i \in I_1(x^*)$, and hence,

$$\psi_i'(x^*, d_0) < 0,\ i \in I_1(x^*) \text{ and } \nabla f(x^*)^T d_0 < 0.$$

Now, consider $x(t) = x^* + td_0$, $t > 0$. Since for each $i \in I_1(x^*)$, $\psi_i'(x^*; d_0) < 0$ and $\psi_i(x^*) = 0$, there exists $\delta_1 > 0$ such that for each $t \in (0, \delta_1]$ and for all $i \in I_1(x^*)$

$$\psi_i(x(t)) \quad = \quad \psi_i(x^*) + t\psi_i'(x^*; d_0) + o(t) < 0.$$

As $\psi_i(x^*) < 0$ for all $i \in I_2(x^*)$, we see that there exists $\delta_2 > 0$ such that for each $t \in (0, \delta_2]$ and for all $i \in I_2(x^*)$, $\psi_i(x(t)) < 0$. Moreover, as $\nabla f(x^*)^T d_0 < 0$, there exists $\delta_3 > 0$ such that for all $t \in (0, \delta_3]$

$$f(x(t)) = f(x^*) + t\nabla f(x^*)^T d_0 + o(t) < f(x^*).$$

Thus, whenever $t \in (0, \min\{\delta_1, \delta_2, \delta_3\}]$, we have $\psi_i(x(t)) \leq 0, \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m$ and $f(x(t)) < f(x^*)$. This contradicts (3.3).

[**Verifying conditions in the robust alternative theorem**] Now, we show that $\mathcal{V}_i^0$ is convex for each $i \in I_1(x^*)$. To see the convexity, let $v_i^1, v_i^2 \in \mathcal{V}_i^0$ and $\lambda \in [0, 1]$. Then $g_i(x^*, v_i^j) = 0$, $j = 1, 2$. As $g_i(x^*, \cdot)$ is concave, we see that

$$g_i(x^*, \lambda v_i^1 + (1-\lambda)v_i^2) \geq \lambda g_i(x^*, v_i^1) + (1-\lambda) g_i(x^*, v_i^2) = 0.$$

As x^* is robust feasible, we have $g_i(x^*, \lambda v_i^1 + (1-\lambda)v_i^2) = 0$, and so, $\lambda v_i^1 + (1-\lambda)v_i^2 \in \mathcal{V}_i^0$. Moreover, we also notice that, for each $d \in \mathbb{R}^n$, $v_i \mapsto u_i(v_i) := \nabla_1 g_i(x^*, v_i)^T d$ is concave on $\mathcal{V}_i^0$, $i \in I_1(x^*)$. To see this, fix any $d \in \mathbb{R}^n$. For any $v_i \in \mathcal{V}_i^0$ with $i \in I_1(x^*)$, as $g_i(x^*, v_i) = 0$, we have

$$u_i(v_i) = \nabla_1 g_i(x^*, v_i)^T d = \lim_{t\to 0} \frac{g_i(x^* + td, v_i) - g_i(x^*, v_i)}{t} = \lim_{t\to 0} \frac{g_i(x^* + td, v_i)}{t}.$$

Then, for all $\mu \in [0,1]$ and $v_i^1, v_i^2 \in \mathcal{V}_i^0$

$$\begin{aligned} u_i(\mu v_i^1 + (1-\mu)v_i^2) &= \nabla_1 g_i(x^*, \mu v_i^1 + (1-\mu)v_i^2)^T d \\ &= \lim_{t\to 0^+} \frac{g_i(x^*+td, \mu v_i^1 + (1-\mu)v_i^2)}{t} \\ &\geq \lim_{t\to 0^+} \frac{\mu g_i(x^*+td, v_i^1) + (1-\mu) g_i(x^*+td, v_i^2)}{t} \\ &= \mu\, \nabla_1 g_i(x^*, v_i^1)^T d + (1-\mu)\nabla_1 g_i(x^*, v_i^2)^T d \\ &= \mu\, u_i(v_i^1) + (1-\mu) u_i(v_i^2), \end{aligned}$$

where the inequality follows by the concavity of $g_i(x^*+td,\cdot)$. Thus, $v_i \mapsto \nabla_1 g_i(x^*, v_i)^T d$ is concave on $\mathcal{V}_i^0$, $i \in I_1(x^*)$, for each fixed $d \in \mathbb{R}^n$.

[**Simplification**] So, by applying our robust alternative theorem (Theorem 2.2) with $a_i(\cdot) = \nabla_1 g_i(x^*,\cdot)$, $\mathcal{V}_i = \mathcal{V}_i^0$, $k = |I_1(x^*)|$ and $a = \nabla f(x^*)$, we see that there exist $\mu_0 \geq 0$ and $\mu_i \geq 0$ with $\mu_0 + \sum_{i\in I_1(x^*)} \mu_i = 1$ and $v_i \in \mathcal{V}_i^0$, $i \in I_1(x^*)$ such that

$$\mu_0 \nabla f(x^*) + \sum_{i\in I_1(x^*)} \mu_i \nabla_1 g_i(x^*, v_i) = 0.$$

Note that $v_i \in \mathcal{V}_i^0$, $i \in I_1(x^*)$. We see that $g_i(x^*, v_i) = 0$ for all $i \in I_1(x^*)$. Thus, by letting $\mu_i = 0$ for all $i \in I_2(x^*)$, we see that

$$\mu_0 \nabla f(x^*) + \sum_{i=1}^{m} \mu_i \nabla_1 g_i(x^*, v_i) = 0 \text{ and } \mu_i g_i(x^*, v_i) = 0,$$

and so (3.1) holds. Furthermore, if (EMFCQ) holds, there exists $d \in \mathbb{R}^n$ such that $\nabla_1 g_i(x^*, v_i)^T d < 0$ for all $i \in I_1(x^*)$. Then $\mu_0 > 0$ (otherwise, $\mu_0 = 0$ and so, $\sum_{i\in I_1(x^*)} \mu_i \nabla_1 g_i(x^*, v_i) = 0$ which is impossible as $\sum_{i\in I_1(x^*)} \mu_i = 1$). Thus, the robust KKT condition follows with $\lambda_i = \mu_i/\mu_0$. □

Robust sufficient optimality conditions

In this subsection, we provide robust sufficient global optimality condition under the following generalized convexity condition at $(x^*, v_i) \in X_0 \times \mathcal{V}_i$: for each $x \in X_0$ there exists $\alpha_i : X_0 \times X_0 \to (0, +\infty), i = 0, 1, \ldots, m$ such that for each $i = 1, 2, \ldots, m$,

$$\left.\begin{aligned} f(x) - f(x^*) &\geq \alpha_0(x, x^*) \nabla f(x^*)^T (x - x^*) \\ g_i(x, v_i) - g_i(x^*, v_i) &\geq \alpha_i(x, x^*) \nabla_1 g_i(x^*, v_i)^T (x - x^*). \end{aligned}\right\} \tag{3.5}$$

This condition is satisfied by a broad classes of nonlinear programming problems. Clearly, convex programming problems as well as pseudo-linear programming problems satisfy the condition. More generally, the following class of convex composite programming problems satisfies the generalized convexity condition:

$$\begin{aligned} \inf_{x\in X_0} \quad & (h_0 \circ p_0)(x) \\ \text{s.t.} \quad & (h_i \circ p_i)(x, v_i) \leq 0,\ i = 1, \ldots, m, \end{aligned}$$

where each $h_i : \mathbb{R}^k \to \mathbb{R}$ is a convex function, $p_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^k$ is a pseudo-linear function, for $i = 1, \ldots, m$ and v_i is a uncertain parameter and v_i belongs to some uncertainty set $\mathcal{V}_i \subseteq \mathbb{R}^q$. To see this, note that for a convex function $h : \mathbb{R}^k \to \mathbb{R}$ and a pseudo-linear function $p : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^k$,

$$\begin{aligned} & (h \circ p)(x, v) - (h \circ p)(x^*, v) \\ \geq \ & \nabla h(p(x^*, v))^T (p(x, v) - p(x^*, v)) \\ = \ & \nabla h(p(x^*, v))^T \big(\alpha_0(x, x^*) \nabla_1 p(x^*, v)^T (x - x^*)\big) \\ = \ & \alpha_0(x, x^*) \nabla (h \circ p)(x^*, v)^T (x - x^*). \end{aligned}$$

We now present robust sufficient global optimality condition for the uncertain programming problem.

Theorem 3.2. (Robust KKT sufficient optimality condition) *Assume that the robust Kuhn-Tucker condition,*

$$\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla_1 g_i(x^*, v_i) = 0 \ \textit{and} \ \lambda_i g_i(x^*, v_i) = 0,$$

holds at $(x^*, \lambda_i, v_i) \in X_0 \times \mathbb{R}_+ \times \mathbb{R}^q$. *Suppose that for each* $x \in X_0$ *there exists* $\alpha_i : X_0 \times X_0 \to (0, +\infty), i = 0, 1, \ldots, m$ *such that (3.5) holds for each* $i = 1, 2, \ldots, m$, *Then,* x^* *is a robust global minimizer of (UP).*

Proof. Let x be feasible for (RP). Then,

$$\begin{aligned} & f(x) - f(x^*) \\ \geq \ & \alpha_0(x, x^*) \nabla f(x^*)^T (x - x^*) \\ = \ & -\alpha_0(x, x^*) \sum_{i=1}^{m} \lambda_i \nabla_1 g_i(x^*, v_i)^T (x - x^*). \end{aligned}$$

Since $\lambda_i g_i(x^*, v_i) = 0$ and x is feasible for (RP), it follows that

$$\begin{aligned} f(x) - f(x^*) & \geq -\alpha_0(x, x^*) \sum_{i=1}^{m} \lambda_i \nabla_1 g_i(x^*, v_i)^T (x - x^*) \\ & \geq -\alpha_0(x, x^*) \sum_{i=1}^{m} \frac{\lambda_i}{\alpha_i(x, x^*)} \big(g_i(x, v_i) - g_i(x^*, v_i)\big) \\ & = -\sum_{i=1}^{m} \frac{\lambda_i \alpha_0(x, x^*)}{\alpha_i(x, x^*)} (g_i(x, v_i) \geq 0. \end{aligned}$$

So, x^* is a robust global minimizer. □

In the following example we verify Theorem 3.2 for a non-convex uncertain problem.

Example 3.1. *Consider the following uncertain optimization problem:*

$$\begin{array}{lll}(\text{UP}) & \inf_{x\in X_0} & x_1 \\ & \text{subject to} & \dfrac{v_1^2 x_2}{x_1} - 1 \le 0 \\ & & 1 - x_1 - x_2 \le 0,\end{array}$$

where $X_0 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 > 0,\ x_2 > 0\}$*,* v_1 *is uncertain and it belongs to the uncertainty set* $[\frac{1}{2}, 1]$*. Its robust counterpart is given by*

$$\begin{array}{lll}(\text{UP}) & \inf_{x\in X_0} & x_1 \\ & \text{subject to} & \dfrac{v_1^2 x_2}{x_1} - 1 \le 0 \quad \forall v_1 \in [\dfrac{1}{2}, 1] \\ & & 1 - x_1 - x_2 \le 0.\end{array}$$

Let $f(x_1, x_2) = x_1$*,* $g_1(x_1, x_2, v_1) = \frac{v_1^2 x_2}{x_1} - 1$*,* $\mathcal{V}_1 = [\frac{1}{2}, 1]$*,* $g_2(x_1, x_2, v_2) = 1 - x_1 - x_2$ *and* $\mathcal{V}_2 = \{0\}$*. It is easy to see that* f *and* g_2 *are both convex and* g_1 *is a fractional linear function which is pseudo-linear. Let* $x^* = (x_1^*, x_2^*) = (\frac{1}{2}, \frac{1}{2})$*,* $\lambda_1 = \frac{1}{4}$ *and* $\lambda_2 = \frac{1}{2}$*. Then*

$$\begin{aligned}&\nabla f(x_1^*, x_2^*) + \lambda_1 \nabla g_1(x_1^*, x_2^*, 1) + \lambda_2 \nabla g_2(x_1^*, x_2^*, 0) \\ &= (1, 0) + \frac{1}{4}(-2, 2) + \frac{1}{2}(-1, -1) = (0, 0) \\ \textit{and } &\lambda_1 g_1(x_1^*, x_2^*, 1) = 0 \textit{ and } \lambda_2 g_2(x_1^*, x_2^*, 0) = 0.\end{aligned}$$

So, the generalized Kuhn-Tucker condition holds for (UP) *at* (x_1^*, x_2^*)*.*

On the other hand, we can also verify that $x^* = (x_1^*, x_2^*)$ *is, in fact, a robust global minimizer of (UP). To see this, note that for any* $x = (x_1, x_2)$ *with* $x_1 > 0, x_2 > 0$*,* $x_1 + x_2 \le 1$ *and* $\frac{v_1^2 x_2}{x_1} - 1 \le 0$ *for all* $v_1 \in [\frac{1}{2}, 1]$*, we have* $x_1 + x_2 \le 1$ *and* $x_2 \le x_1$ *which in turn gives us that* $f(x) = x_1 \le 1/2$*. Note that* $f(x^*) = 1/2$*. So,* x^* *is a robust global minimizer.*

Remark 3.1. *A close inspection of the proof of Theorem 3.2 reveals that the generalized convexity condition of Theorem 3.2 can be weakened by replacing* $(x - x^*)$ *by some function* $\eta(x, x^*)$*.*

4 Robust Duality

In this section, we establish robust duality between (UP) and (DP) by establishing strong duality between their robust counterpart (RP) and the optimistic counterpart (ODP).

Theorem 4.1. (Robust Duality) *Let x^* be a robust feasible point of (UP) at which the KKT condition (3.2) is satisfied by $\bar{\lambda} \in \mathbb{R}^m_+$ and $\bar{v}_i \in \mathcal{V}_i$. If $f(\cdot) + \sum_{i=1}^m \bar{\lambda}_i g_i(\cdot, \bar{v}_i)$ is convex, then x^* is a robust global minimizer of (UP) and*

$$\inf_{x \in X_0} \{f(x) : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \cdots, m\} = \max_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \inf_{x \in X_0} \{f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i)\}.$$

Proof. First of all, we note that the weak duality always holds, i.e.,

$$\inf_{x \in X_0} \{f(x) : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \cdots, m\} \geq \max_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \inf_{x \in X_0} \{f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i)\}.$$

To see the reverse inequality, let $h(x) = f(x) + \sum_{i=1}^m \bar{\lambda}_i g_i(x, \bar{v}_i)$ for all $x \in X_0$. Then, by the assumption, h is a convex function with $\nabla h(x^*) = 0$. This implies that, for each $x \in X_0$,

$$f(x) + \sum_{i=1}^m \bar{\lambda}_i g_i(x, \bar{v}_i) = h(x) \geq h(x^*) = f(x^*) + \sum_{i=1}^m \bar{\lambda}_i g_i(x^*, \bar{v}_i) = f(x^*).$$

In particular, for each $x \in X_0$ with $g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \cdots, m, f(x) \geq f(x^*)$. So, x^* is a robust global minimizer. Moreover,

$$\begin{aligned} f(x^*) &\leq \inf_{x \in X_0} \{f(x) + \sum_{i=1}^m \bar{\lambda}_i g_i(x, \bar{v}_i)\} \\ &\leq \max_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \inf_{x \in X_0} \{f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i)\}. \end{aligned}$$

So,

$$\inf_{x \in X_0} \{f(x) : g_i(x, v_i) \leq 0, \forall v_i \in \mathcal{V}_i, i = 1, \cdots, m\} = f(x^*) \leq \max_{v_i \in \mathcal{V}_i, \lambda_i \geq 0} \inf_{x \in X_0} \{f(x) + \sum_{i=1}^m \lambda_i g_i(x, v_i)\}.$$

□

Remark 4.1. *The proof of Theorem 4.1 shows that the convexity condition of Lagrangian function $f(\cdot) + \sum_{i=1}^m \bar{\lambda}_i g_i(\cdot, \bar{v}_i)$ in Theorem 4.1 can be weakened to some generalized convexity condition such as to pseudo-convexity of $f(\cdot) + \sum_{i=1}^m \bar{\lambda}_i g_i(\cdot, \bar{v}_i)$. For related conditions, see [9].*

The following example verifies our robust duality theorem for a non-convex programming problem with interval uncertainty where the Lagrangian function is convex.

Example 4.1. *Consider the following uncertain optimization problem*

$$\begin{array}{ll}\inf_{x\in\mathbb{R}^2} & 2x_1^2 - 2x_1 - x_2\\ \text{s.t.} & -x_1^2 + v_1x_2 - 1 \le 0,\end{array}$$

where v_1 *is uncertain and it belongs to* $[-1,1]$. *Its robust counterpart is given by*

$$\begin{array}{ll}\inf_{x\in\mathbb{R}^2} & 2x_1^2 - 2x_1 - x_2\\ \text{s.t.} & -x_1^2 + v_1x_2 - 1 \le 0 \quad \forall v_1 \in [-1,1].\end{array}$$

Let $f(x) = 2x_1^2 - 2x_1 - x_2$, $g_1(x,v_1) = -x_1^2 + v_1x_2 - 1$ *and* $\mathcal{V}_1 = [-1,1]$. *Let* $x^* = (1,2)$. *It can be verified that* x^* *is a robust global minimizer. Indeed, first of all,* $g_1(x^*,v_1) \le 0$ *for all* $v_1 \in [-1,1]$. *Moreover, the constraint,* $g_1(x,v_1) \le 0$ *for all* $v_1 \in [-1,1]$, *gives us that* $|x_2| \le x_1^2 + 1$, *and so,*

$$\begin{aligned} f(x) = 2x_1^2 - 2x_1 - x_2 &\ge 2x_1^2 - 2x_1 - |x_2|\\ &\ge 2x_1^2 - 2x_1 - (x_1^2+1)\\ &= (x_1-1)^2 - 2 \ge -2 = f(x^*).\end{aligned}$$

Thus, x^* *is a robust global minimizer. Moreover,* $\mathcal{V}_1^0 := \{v_1 \in \mathcal{V}_1 \mid g_1(x^*,v_1) = 0\} = \{1\}$. *For* $d = (1,0)$, $\nabla_1 g_1(x^*,1)^T d = -2$, *and hence* (EMFCQ) *holds at* x^*. *Clearly,* $g_1(x,\cdot)$ *is affine for each* x. *It can be verified by letting* $\lambda_1 = 1$ *and* $\bar{v}_1 = 1$ *that*

$$\nabla f(x^*) + \lambda_1 \nabla_1 g_1(x^*,\bar{v}_1) = (2,-1) + (-2,1) = (0,0), \text{ and } \lambda_1 g_1(x^*,\bar{v}_1) = 0.$$

So, $(x^*,\overline{\lambda}_1,\overline{v}_1)$ *satisfy the robust KKT condition and*

$$f(x) + \overline{\lambda}_1 g_1(x,\overline{v}_1) = x_1^2 - 2x_1 - 1$$

which is a convex function.

We now verify the robust duality. First of all,

$$\inf_{x\in\mathbb{R}^2}\{2x_1^2 - 2x_1 - x_2 : -x_1^2 + v_1x_2 - 1 \le 0 \quad \forall v_1 \in [-1,1]\} = -2.$$

Moreover, the optimistic counterpart of the uncertain dual is

$$\begin{aligned} &\max_{\lambda_1\ge 0, v_1\in[-1,1]} \inf_{x\in\mathbb{R}^2}\{f(x) + \lambda_1 g_1(x,v_1)\}\\ =\; &\max_{\lambda_1\ge 0, v_1\in[-1,1]} \inf_{(x_1,x_2)\in\mathbb{R}^2}\{(2-\lambda_1)x_1^2 - 2x_1 - (1-\lambda_1v_1)x_2 - \lambda_1\}.\end{aligned}$$

Note that

$$\inf_{(x_1,x_2)\in\mathbb{R}^2}\{(2-\lambda_1)x_1^2 - 2x_1 - (1-\lambda_1v_1)x_2 - \lambda_1\} = \begin{cases} -\infty & \text{if } \quad \lambda_1v_1 \ne 1\\ -\frac{1}{2-\lambda_1} - \lambda_1 & \text{if } \quad 0 \le \lambda_1 < 2 \text{ and } \lambda_1v_1 = 1\\ -\infty & \text{if } \quad \lambda_1 \ge 2 \text{ and } \lambda_1v_1 = 1.\end{cases}$$

So,

$$\max_{\lambda_1\geq 0, v_1\in[-1,1]} \inf_{x\in\mathbb{R}^2}\{f(x)+\lambda_1 g_1(x,v_1)\} = \max_{0\leq\lambda_1<2, v_1\in[-1,1]}\{-\frac{1}{2-\lambda_1}-\lambda_1 : \lambda_1 v_1 = 1\}$$
$$= \max_{1\leq\lambda_1<2}\{-\frac{1}{2-\lambda_1}-\lambda_1\} = -2.$$

So, robust duality holds.

Corollary 4.1. (Robust Convex Programming Duality) *For* (UP)*, assume that* f *is convex,* $g_i(\cdot, v_i)$ *is convex for each* $v_i \in \mathcal{V}_i$ *and* $g_i(x,\cdot)$ *is concave for each* $x \in \mathbb{R}^n$*,* $i = 1, 2, \ldots, m$*. If* x^* *is a robust local minimizer of* (UP) *at which the extended Mangasarian-Fromovitz constraint qualification (EMFCQ) is satisfied, then* x^* *is a robust global minimizer of* (UP) *and*

$$\inf_{x\in X_0}\{f(x) : g_i(x,v_i)\leq 0, \quad \forall v_i\in\mathcal{V}_i, \ i=1,\cdots,m\}$$
$$= \max_{v_i\in\mathcal{V}_i,\lambda_i\geq 0}\{f(x)+\sum_{i=1}^m \lambda_i g_i(x,v_i) \ : \ \nabla f(x)+\sum_{i=1}^m \lambda_i \nabla_1 g_i(x,v_i) = 0\}. \quad (4.6)$$

Proof. As x^* is a robust local minimizer of (UP) satisfying the extended Mangasarian-Fromovitz constraint qualification (EMFCQ), Theorem 3.1 shows that there exist $\bar{\lambda}_i \geq 0$ and $\bar{v}_i \in \mathcal{V}_i$ satisfying

$$\nabla f(x^*) + \sum_{i=1}^m \bar{\lambda}_i \nabla g_i(x^*, \bar{v}_i) = 0 \text{ and } \bar{\lambda}_i g_i(x^*, \bar{v}_i) = 0, \, i = 1, \ldots, m.$$

As f and $g_i(\cdot, \bar{v}_i)$ are all convex functions, we see that $f + \sum_{i=1}^m \bar{\lambda}_i g_i(\cdot, \bar{v}_i)$ is also convex. In this case the Lagrangian dual problem collapses to the problem,

$$\max_{v_i\in\mathcal{V}_i,\lambda_i\geq 0}\{f(x)+\sum_{i=1}^m \lambda_i g_i(x,v_i) \ : \ \nabla f(x)+\sum_{i=1}^m \lambda_i \nabla_1 g_i(x,v_i) = 0\}.$$

Hence, the conclusion follows from the preceding theorem. □

We shall see later in Section 6 that strong duality between (UP) and its Wolfe type dual (4.6) [19] continues to hold under weekend convexity conditions.

5 Robust Duality for Classes of Uncertain Non-convex Problems

In this Section, we examine classes of smooth non-convex problems for which robust duality can easily be guaranteed. We shall first look at an uncertain quadratic programming model problem with a single quadratic constraint under spectral norm uncertainty [2].

Single Constraint Quadratic Problem with Spectral Norm Uncertainty

Recall that for a $(m \times n)$ matrix Δ, $\|\Delta\|_{\text{spec}}$ is the spectral norm of Δ and is defined by $\|\Delta\|_{\text{spec}} = \sqrt{\lambda_{\max}(\Delta^T\Delta)}$. Consider the single constraint quadratic problem with the spectral norm uncertainty

$$\begin{aligned} (UQP) \quad \inf_{x\in X_0} \quad & \frac{1}{2}x^T Ax + a^T x \\ \text{s.t.} \quad & \frac{1}{2}x^T Bx + b^T x + \beta \le 0, \end{aligned}$$

where X_0 is an open set in $\mathbb{R}^n$, $A \in S^n$, $a, b \in \mathbb{R}^n$, $\beta \in \mathbb{R}$ and the matrix data $B \in S^n$ is uncertain and belongs to the spectral norm uncertainty set

$$\mathcal{V} = \{B_0 + \Delta : \Delta \in S^n,\ \|\Delta\|_{\text{spec}} \le \rho\}, \quad \rho > 0.$$

The robust counterpart of (UQP) is

$$\begin{aligned} (RQP) \quad \inf_{x\in X_0} \quad & \frac{1}{2}x^T Ax + a^T x \\ \text{s.t.} \quad & \frac{1}{2}x^T Bx + b^T x + \beta \le 0,\ \forall B \in \{B_0 + \Delta : \Delta \in S^n,\ \|\Delta\|_{\text{spec}} \le \rho\} \end{aligned}$$

and the optimistic counterpart of the uncertain dual of (UQP) is

$$(ODQP) \quad \max_{\substack{\|\Delta\|_{\text{spec}}\le\rho \\ \lambda\ge 0}} \inf_{x\in X_0} \{\frac{1}{2}x^T Ax + a^T x + \lambda(\frac{1}{2}x^T(B_0+\Delta)x + b^T x + \beta)\}.$$

Corollary 5.1. *Let x^* be a robust global minimizer of (UQP). Suppose that there exists $x_0 \in X_0$ such that $\frac{1}{2}x_0^T(B+\rho I)x_0 + b^T x_0 + \beta < 0$. Then,*

$$\begin{aligned} & \inf_{x\in X_0} \{\frac{1}{2}x^T Ax + a^T x : \frac{1}{2}x^T Bx + b^T x + \beta \le 0,\ \forall B \in \{B_0 + \Delta : \|\Delta\|_{\text{spec}} \le \rho\}\} \\ = & \max_{\|\Delta\|_{\text{spec}}\le\rho, \lambda\ge 0} \inf_{x\in X_0} \{\frac{1}{2}x^T Ax + a^T x + \lambda(\frac{1}{2}x^T(B_0+\Delta)x + b^T x + \beta)\}. \end{aligned}$$

Proof. The robust global minimizer at x^* ensures that

$$\begin{aligned} & \frac{1}{2}x^T Bx + b^T x + \beta \le 0,\ \forall B \in \{B_0 + \Delta : \|\Delta\|_{\text{spec}} \le \rho\} \\ \Rightarrow\ & \frac{1}{2}x^T Ax + a^T x \ge \frac{1}{2}(x^*)^T Ax^* + a^T x^*. \end{aligned}$$

So, we see that

$$\sup_{\|\Delta\|_{\text{spec}}\le\rho} \frac{1}{2}x^T(B_0+\Delta)x + b^T x + \beta \le 0 \Rightarrow \frac{1}{2}x^T Ax + a^T x - \big(\frac{1}{2}(x^*)^T Ax^* + a^T x^*\big) \ge 0.$$

As $\frac{1}{2}x^T(B_0+\rho I)x = \sup_{\|\Delta\|_{\text{spec}}\le\rho} \frac{1}{2}x^T(B_0+\Delta)x$, it follows that

$$\frac{1}{2}x^T(B_0+\rho I)x + b^T x + \beta \le 0\ \Rightarrow\ \frac{1}{2}x^T Ax + a^T x - \big(\frac{1}{2}(x^*)^T Ax^* + a^T x^*\big) \ge 0.$$

Now, from the S-lemma (see [15, 13, 11]), there exists $\lambda \geq 0$ such that for each $x \in \mathbb{R}^n$

$$\frac{1}{2}x^T Ax + a^T x + \lambda\big(\frac{1}{2}x^T(B_0+\rho I)x + b^T x + \beta\big) \geq \frac{1}{2}(x^*)^T Ax^* + a^T x^*.$$

Letting $x = x^*$, as x^* is robust feasible and $\|\rho I\|_{\text{spec}} \leq \rho$, we have

$$\lambda(\frac{1}{2}(x^*)^T(B_0+\rho I)x^* + b^T x^* + \beta) = 0,$$

and so, for each $x \in \mathbb{R}^n$,

$$\begin{aligned} & \frac{1}{2}x^T Ax + a^T x + \lambda(\frac{1}{2}x^T(B_0+\rho I)x + b^T x + \beta) \\ \geq \;& \frac{1}{2}(x^*)^T Ax^* + a^T x^* + \lambda\big(\frac{1}{2}(x^*)^T(B_0+\rho I)x^* + b^T x^* + \beta\big). \end{aligned}$$

Define $h(x) = \frac{1}{2}x^T Ax + a^T x + \lambda\big(\frac{1}{2}x^T(B_0+\rho I)x + b^T x + \beta\big)$. This implies that

$$\nabla h(\overline{x}) = \big(A + \lambda(B_0+\rho I)\big)\overline{x} + (a+\lambda b) = 0 \text{ and } \nabla^2 h(\overline{x}) = A + \lambda(B_0+\rho I) \succeq 0.$$

Let $f(x) = \frac{1}{2}x^T Ax + a^T x$ and define $g : \mathbb{R}^n \times S^n \to \mathbb{R}$ by

$$g(x,B) = \frac{1}{2}x^T Bx + b^T x + \beta.$$

By identifying S^n as $\mathbb{R}^{\frac{n(n+1)}{2}}$, (RQP) can be rewritten as

$$\begin{array}{ll} \inf_{x\in X_0} & f(x) \\ \quad\text{s.t.} & g(x,B) \leq 0,\ \forall\, B \in \mathcal{V} = \{B_0 + \Delta : \|\Delta\|_{\text{spec}} \leq \rho\}. \end{array}$$

Then, $(\overline{x}, \lambda, \rho I)$ satisfies the robust KKT condition and $f(x) + \lambda g(x,\rho I) = \frac{1}{2}x^T Ax + a^T x + \lambda\big(\frac{1}{2}x^T(B_0+\rho I)x + b^T x + \beta\big)$ is convex as $A + \lambda(B_0+\rho I) \succeq 0$. So, Theorem 4.1 gives us that

$$\begin{aligned} & \inf_{x\in X_0}\{f(x) : g(x,B) \leq 0,\ \forall\, B \in \mathcal{V} := \{B_0+\Delta : \|\Delta\|_{\text{spec}} \leq \rho\}\} \\ = \;& \max_{B\in\mathcal{V},\lambda\geq 0}\ \inf_{x\in X_0}\{f(x) + \lambda g(x,B)\}. \end{aligned}$$

Hence, the conclusion follows. □

Separable Homogeneous Polynomial Problems with Interval Uncertainty

Consider the separable homogeneous polynomial problem with the interval Uncertainty

$$\begin{array}{lll} (USP) & \inf_{x\in X_0} & \frac{1}{2}\sum_{j=1}^n a_j x_j^k \\ & \text{s.t.} & \frac{1}{2}\sum_{j=1}^n b_j^i x_j^k \leq \beta_i,\ i = 1,\ldots,m, \end{array}$$

where X_0 is an open set in $\mathbb{R}^n$, $k \in \mathbb{N}$, the coefficient b_j^i in the constraint is uncertain and $b_j^i \in \mathcal{V}_j^i := [\underline{b}_j^i, \overline{b}_j^i]$. In the special case when $k = 2$ and $\underline{b}_j^i = \overline{b}_j^i$, then (USP) reduces to the usual weighted least square problem.

The robust counterpart of (USP) is

$$(RSP) \quad \inf_{x \in X_0} \quad \frac{1}{2}\sum_{j=1}^{n} a_j x_j^k$$
$$\text{s.t.} \quad \frac{1}{2}\sum_{j=1}^{n} b_j^i x_j^k \leq \beta_i, \ \forall b_j^i \in \mathcal{V}_j^i := [\underline{b}_j^i, \overline{b}_j^i]$$

and the optimistic counterpart of the uncertain dual of (USP) is

$$(ODSP) \quad \max_{b_j^i \in [\underline{b}_j^i, \overline{b}_j^i], \lambda_i \geq 0} \inf_{x \in X_0} \{\frac{1}{2}\sum_{j=1}^{n} a_j x_j^k + \sum_{i=1}^{m} \lambda_i (\frac{1}{2}\sum_{j=1}^{n} b_j^i x_j^k - \beta_i)\}.$$

Corollary 5.2. *Let x^* be a robust global minimizer of (USP). Then*

$$\inf_{x \in X_0} \{\frac{1}{2}\sum_{j=1}^{n} a_j x_j^k : \frac{1}{2}\sum_{j=1}^{n} b_j^i x_j^k \leq \beta_i, \ \forall b_j^i \in \mathcal{V}_j^i := [\underline{b}_j^i, \overline{b}_j^i]\}$$
$$= \max_{b_j^i \in [\underline{b}_j^i, \overline{b}_j^i], \lambda_i \geq 0} \inf_{x \in X_0} \{\frac{1}{2}\sum_{j=1}^{n} a_j x_j^k + \sum_{i=1}^{m} \lambda_i (\frac{1}{2}\sum_{j=1}^{n} b_j^i x_j^k - \beta_i)\}.$$

Proof. The robust global minimizer at x^* means that

$$\frac{1}{2}\sum_{j=1}^{n} b_j^i x_j^k \leq \beta_i, \ \forall b_j^i \in \mathcal{V}_j^i \ \Rightarrow \ \frac{1}{2}\sum_{j=1}^{n} a_j x_j^k \geq \frac{1}{2}\sum_{j=1}^{n} a_j (x_j^*)^k.$$

We now split the proof into two cases:

[Case 1: k is an even number] Suppose that Case 1 holds. Then, by letting $y_j = x_j^k$ and $y_j^* = (x_j^*)^k$, we have

$$y_j \geq 0, \ \frac{1}{2}\sum_{j=1}^{n} b_j^i y_j \leq \beta_i, \ \forall b_j^i \in \mathcal{V}_j^i \ \Rightarrow \ \frac{1}{2}\sum_{j=1}^{n} a_j y_j \geq \frac{1}{2}\sum_{j=1}^{n} a_j y_j^*.$$

Then, by the robust Farkas lemma (see [8, 10]), we see that there exist $\lambda_i \geq 0$, $i = 1, \ldots, m$, $\mu_j \geq 0$, $j = 1, \ldots, n$ and $b_j^i \in \mathcal{V}_j^i$ such that for all $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$

$$(\frac{1}{2}\sum_{j=1}^{n} a_j y_j - \frac{1}{2}\sum_{j=1}^{n} a_j y_j^*) + \sum_{i=1}^{m} \lambda_i (\frac{1}{2}\sum_{j=1}^{n} b_j^i y_j - \beta_i) + \sum_{j=1}^{n} \mu_j (-y_j) \geq 0.$$

Then, $a_j + \sum_{i=1}^{m} \lambda_i b_j^i \geq 0$. Moreover, letting $y = y^* \geq 0$ where $y^* = (y_1^*, \ldots, y_n^*)$ and noting that x^* is robust feasible, we see that

$$\lambda_i (\frac{1}{2}\sum_{j=1}^{n} b_j^i y_j^* - \beta_i) = 0, \ i = 1, \ldots, m.$$

Now, define $f(x) = \frac{1}{2}\sum_{j=1}^{n} a_j x_j^k$ and $g(x, v_i) = \frac{1}{2}\sum_{j=1}^{n} b_j^i x_j^k - \beta_i$ where $v_i = (b_1^i, \dots, b_n^i)$, $i = 1, \dots, m$. It follows from $a_j + \sum_{i=1}^{m} \lambda_i b_j^i \geq 0$ and k is an even number that $f + \sum_{i=1}^{m} \lambda_i g_i(x, v_i) = \frac{1}{2}(a_j + \sum_{j=1}^{n} b_j^i) x_j^k$ is a convex function and (x^*, λ_i, v_i) satisfies the robust KKT condition. So, robust strong duality follows by Theorem 4.1.

[Case 2: k is an odd number] Suppose that Case 2 holds. Then, by letting $y_j = x_j^k$ and $y_j^* = (x_j^*)^k$, we have

$$\frac{1}{2}\sum_{j=1}^{n} b_j^i y_j \leq \beta_i,\ \forall\, b_j^i \in \mathcal{V}_j^i \ \Rightarrow\ \frac{1}{2}\sum_{j=1}^{n} a_j y_j \geq \frac{1}{2}\sum_{j=1}^{n} a_j y_j^*.$$

The, using the similar method of proof as in Case 1, we see that the robust duality also follows. □

6 Robust Duality under Weakened Convexity

In this Section we see that the robust convex programming duality (Corollary 4.1) continues to hold under the following weakened convexity condition (WCC) at x^*: for each $x, y \in X_0$ there exists $\alpha : X_0 \times X_0 \to \mathbb{R}^n$ such that, for each $v_i \in \mathcal{V}_i$, $i = 1, 2, \dots, m$,

$$\begin{aligned} f(x) - f(y) &\geq \nabla f(x^*)^T \alpha(x, y) \\ g_i(x, v_i) - g_i(x^*, v_i) &\geq \nabla_1 g_i(x^*, v_i)^T \alpha(x, y). \end{aligned}$$

Related conditions and classes of functions that satisfy (WCC), when $\mathcal{V}_i$ is singleton, can be found in [9].

Theorem 6.1. (Robust Duality & Weakened Convexity) *Let x^* be a robust local minimizer of (UP). Assume that each constraint function $g_i(x, \cdot)$ is concave for each $x \in \mathbb{R}^n$ and that the Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds. Suppose that the weakened convexity condition (WCC) holds at x^*. Then, x^* is a robust global minimizer of (UP) and*

$$\begin{aligned} &\inf_{x \in X_0} \{f(x) : g_i(x, v_i) \leq 0, \quad \forall v_i \in \mathcal{V}_i,\ i = 1, \cdots, m\} \\ = &\max_{v_i \in \mathcal{V}_i, \lambda_i \geq 0, x \in X_0} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i) : \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla_1 g_i(x, v_i) = 0\}. \end{aligned}$$

Proof. [**Weak Duality**] Let x be feasible for (RP) and let $(y, \lambda, v_1, \cdots, v_m)$ be feasible for (ODP). Then we have,

$$\begin{aligned} f(x) - (f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i)) &\geqq (f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i)) - (f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i)) \\ &\geqq (\nabla f(y) + \sum_{i=1}^{m} \lambda_i \nabla_1 g_i(y, v_i))^T \alpha(x, y)) \\ &= 0. \end{aligned}$$

Thus, $f(x) \geqq f(y) + \sum_{i=1}^{m} \lambda_i g_i(y, v_i)$. So, we see that

$$\begin{aligned} & \inf_{x \in X_0} \{f(x) : g_i(x, v_i) \leq 0, \quad \forall v_i \in \mathcal{V}_i, \ i = 1, \cdots, m\} \\ \geq & \max_{v_i \in \mathcal{V}_i, \lambda_i \geq 0, x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i) : \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla_1 g_i(x, v_i) = 0\}. \end{aligned}$$

[**Strong Duality**] From the weak duality and the robust Kuhn-Tucker necessary optimality conditions, we see that

$$\begin{aligned} f(x^*) & \geq \inf_{x \in X_0} \{f(x) : g_i(x, v_i) \leq 0, \quad \forall v_i \in \mathcal{V}_i, \ i = 1, \cdots, m\} \\ & \geq \max_{v_i \in \mathcal{V}_i, \lambda_i \geq 0, x \in \mathbb{R}^n} \{f(x) + \sum_{i=1}^{m} \lambda_i g_i(x, v_i) : \nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla_1 g_i(x, v_i) = 0\} \\ & \geq f(x^*) + \sum_{i=1}^{m} \lambda_i g_i(x^*, v_i) \\ & = f(x^*). \end{aligned}$$

So, the conclusion follows. □

References

[1] A. Beck and A. Ben-Tal, Duality in robust optimization: primal worst equals dual best, *Oper. Res. Lett.*, **37** (2009), 1-6.

[2] A. Ben-Tal and A. Nemirovski, Robust optimization—methodology and applications, *Math. Program.*, Ser B, 92 (2002), 453-480.

[3] A. Ben-Tal, L.E. Ghaoui and A. Nemirovski, *Robust Optimization*, Princeton Series in Applied Mathematics, 2009.

[4] A. Ben-Tal and A. Nemirovski, A selected topics in robust convex optimization, *Math. Progr. B*, **112** (2008), 125-158.

[5] D. Bertsimas and D. Brown, Constructing uncertainty sets for robust linear optimization, *Oper. Res.*, 57 (2009), 1483-1495.

[6] D. Bertsimas, D. Pachamanova and M. Sim, Robust linear optimization under general norms, *Oper. Res. Lett.*, 32 (2004), 510-516.

[7] V. Jeyakumar, Constraint qualifications characterizing Lagrangian duality in convex optimization, *J. Optim. Theory Appl.*, 136(2008), 31-41.

[8] V. Jeyakumar and G. Li, Characterizing robust set containments and solutions of uncertain linear programs without qualifications, *Oper. Res. Lett.*, 38(2010), 188-194.

[9] V. Jeyakumar and B. Mond, On generalized convex mathematical programming, *J. Austral. Math. Soc.*, Ser. B, 34(1992), 43-53.

[10] V. Jeyakumar and G. Li, Robust Farkas' lemma for uncertain linear systems with applications, *Positivity*, DOI 10. 1007/s11117-010-0078-4.

[11] V. Jeyakumar, N. Q. Huy and G. Li, Necessary and sufficient conditions for S-lemma and non-convex quadratic optimization, *Optim. Eng.*, 10 (2009), 409-503,

[12] V. Jeyakumar and G. Y. Li, Strong duality in robust convex programming: complete characterizations, *SIAM J. Optim.* 20 (2010), 3384-3407.

[13] V. Jeyakumar, G.M. Lee, and G.Y. Li, Alternative theorems for quadratic inequality systems and global quadratic optimization. *SIAM J. Optim.*, 20 (2009), 983-1001.

[14] V. Jeyakumar, G.Y. Li and G.M. Lee, A robust von Neumann minimax theorem for zero-sum games under bounded payoff uncertainty. *Oper. Res. Lett.*, 39 (2011), 109-114

[15] I. Pólik and T. Terlaky, A survey of the S-Lemma, *SIAM Rev.*, 49(2007), 371-418.

[16] R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, N. J, 1970.

[17] A. Shapiro, Stochastic programming approach to optimization under uncertainty, *Math. Program.*, Ser B, 112(2008), 183-220.

[18] A. Shapiro, D. Dentcheva and A. Ruszczynski, *Lectures on Stochastic Programming: Modeling and Theory*, SIAM, Philadelphia, 2009.

[19] P. Wolfe, A duality theorem for nonlinear programming. *Quat. Appl. Math.*, 19 (1961), 239-244.

[20] C. Zalinescu, *Convex Analysis in General Vector Spaces*, World Scientific, 2002.