A New Class of Alternative Theorems for SOS-Convex Inequalities and Robust Optimization*

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Dedicated to Boris Mordukhovich on the occasion of his 65th Birthday

Abstract

In this paper we present a new class of theorems of the alternative for SOS-convex inequality systems without any qualifications. This class of theorems provides an alternative equations in terms of sums of squares to the solvability of the given inequality system. A strong separation theorem for convex sets, described by convex polynomial inequalities, plays a key role in establishing the class of alternative theorems. Consequently, we show that the optimal values of various classes of robust convex optimization problems are equal to the optimal values of related semidefinite programming problems (SDPs) and so, the value of the robust problem can be found by solving a single SDP. The class of problems includes programs with SOS-convex polynomials under data uncertainty in the objective function such as uncertain quadratically constrained quadratic programs. The SOS-convexity is a computationally tractable relaxation of convexity for a real polynomial. We also provide an application of our theorem of the alternative to a multi-objective convex optimization under data uncertainty.

Keywords: Theorems of the alternative, convex optimization, multi-objective optimization, optimization under data uncertainty

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1 Introduction

Theorems of the alternative for arbitrary finite systems of linear or convex inequalities [10, 12, 15, 17] have played key roles often in the development of optimality principles for continuous optimization problems and in the convergence analysis of optimization algorithms. Unlike the theorems of the alternative for linear systems such as the Farkas lemma [10] which provides a numerically checkable alternative certificate of the solvability of the given linear system, alternative theorems for convex inequality systems do not, in general, provide such certificates even under a constraint qualification.

The purpose of this paper is to examine theorems of the alternative for inequality systems involving a subclass of convex functions, called SOS-convex polynomials, and to obtain alternative certificates of the solvability of the inequality systems, and to apply them to find optimal values of classes of convex optimization problems under data uncertainty using semidefinite programs.

The SOS-convexity (see Definition 2.1) is a computationally tractable relaxation of convexity for a real polynomial. An interesting feature of SOS-convexity is that whether a polynomial is SOS-convex or not can be checked by solving a related semi-definite programming problem. The class of SOS-convex polynomials includes the class of separable convex polynomials and the class of convex quadratic functions. For related recent work on SOS-convex polynomials and semidefinite optimization, see [13] and other reference therein. We make the following contributions to convex optimization.

Contributions. We first establish a qualification-free theorem of the alternative for SOS-convex inequality system where an alternative certificate in terms of sums of squares holds for the solvability of the inequality system. We derive this by way of first proving a new qualification-free infeasibility certificate of SOS-convex inequality systems. A strong separation theorem for convex sets, described by convex polynomial inequalities, plays a key role in establishing the alternative theorems. The significance of the certificates involving sums of squares polynomials is that they can be easily verified by solving semidefinite programs.

Using the theorem of the alternative, we then show that the robust optimal value of a SOS-convex optimization problem under data uncertainty in the objective function is always equal to the optimal value of a related semidefinite programming problem (SDP). In particular, we obtain that the value of a robust convex quadratic optimization can be found by solving a single simple SDP. We also provide conditions that guarantee that the value of the semidefinite programming problem is attained. For recent work on robust convex optimization, see [6, 14].

Finally, we show how a robust weak efficient solution of a multi-objective SOS-convex optimization problem under data uncertainty in the objectives can be characterized in terms of equations involving sums of squares polynomials.

The outline of the paper is as follows. In Section 2, we present a strong separation theorem for disjoint convex sets, described by convex polynomials, and consequently, obtain theorems of the alternative SOS-convex inequality systems. In Section 3, we show that the optimal value of a robust SOS-convex program is equal to the optimal value of a related semidefinite programming problem. In Section 4, we provide an application of our theorem of the alternative to a multi-objective convex optimization problem under data uncertainty.

2 Alternative Theorems for SOS-convex Inequality Systems

In this Section, we establish new qualification-free theorems of the alternative for SOS-convex inequality systems.

We begin by fixing some notation and preliminaries on polynomials. We denote by S^n the space of symmetric $n \times n$ matrices and $\succeq$ denotes the Löwner partial order of S^n, that is, for $M, N \in S^n$, $M \succeq N$ if and only if $(M - N)$ is positive semidefinite. Note also that $M \succ 0$ means that M is positive definite. Let $S^n_+ := \{M \in S^n : M \succeq 0\}$ be the closed convex cone of positive semi-definite $n \times n$ (symmetric) matrices, equipped with the trace inner product. We recall that a real polynomial f on $\mathbb{R}^m$ is sum of squares if there exist real polynomials f_j, $j = 1, \ldots, r$, such that $f(x) = \sum_{j=1}^r f_j^2(x)$ for all $x \in \mathbb{R}^m$. The set consisting of all sum of squares real polynomials is denoted by $\Sigma^2[x]$. Moreover, the set consisting of all sum of squares real polynomials with degree at most d is denoted by $\Sigma^2_d[x]$. It is known that $\Sigma^2_d[x]$ is a closed convex cone (see [21, Corollary 3.34]). One of the interesting and important features of a sum-of-squares polynomial is that checking a polynomial is sum of squares or not is equivalent to solving a linear matrix inequality problem.

Similarly, we say that a matrix polynomial $F \in \mathbb{R}[x]^{n\times n}$ is a *SOS matrix polynomial* if $F(x) = H(x)H(x)^T$ for some matrix polynomial $H(x) \in \mathbb{R}[x]^{n\times r}$ and some $r \in \mathbb{N}$. We now introduce the definition of SOS-convex polynomial.

Definition 2.1. (SOS-Convexity [1, 8]) *A real polynomial f on $\mathbb{R}^n$ is called SOS-convex if the Hessian matrix function $H : x \mapsto \nabla^2 f(x)$ is a SOS matrix polynomial. Moreover, we say f is SOS-concave if $-f$ is SOS-convex.*

Note that a real polynomial f is SOS-convex if and only if $\sigma(x, y) := f(y) - f(x) - \nabla f(x)^T(y - x)$ is a sum of squares polynomial. For details see [2].

The class of SOS-convex polynomials include the class of separable convex polynomials and the class of convex quadratic functions. Clearly, a SOS-convex polynomial is convex. However, the converse is not true. That is to say, there exists a convex polynomial which is not SOS-convex [1, 2]. An interesting feature of a SOS-convex polynomial is that whether a polynomial is SOS-convex or not can be checked by solving a related semi-definite programming problem while checking the convexity of a polynomial is in general a very hard problem.

We now recall the following useful existence result of a convex polynomial program that will play an important role later in the paper.

Lemma 2.2. *[5] Let $f_0, f_1, \ldots, f_m$ be convex polynomials on $\mathbb{R}^n$. Let $C := \{x \in \mathbb{R}^n : f_i(x) \le 0, i = 1, \ldots, m\}$. Suppose that $\inf\limits_{x\in C} f_0(x) > -\infty$. Then, $\mathrm{argmin}_{x\in C} f_0(x) \neq \emptyset$.*

Remark 2.3. (Connection between SOS-convexity and sum of squares) *It follows easily from the definition of SOS-convexity that any nonnegative SOS-convex polynomial must be a sum of squares polynomial. To see this, let f be a SOS-convex polynomial which is nonnegative over the whole space. Then Lemma 2.2 shows that $\inf_{x\in\mathbb{R}^n} f(x)$ is attained. Let $\bar{x} \in \mathbb{R}^n$ be such that $f(\bar{x}) = \inf_{x\in\mathbb{R}^n} f(x) \ge 0$. Define $g(x) := f(x) - f(\bar{x})$. Then, $\nabla g(\bar{x}) = 0$ and $g(\bar{x}) = 0$. So, by the equivalent definition that g is a sum of squares* *polynomial. Thus, $f = g + f(\bar{x})$ is also a sum of squares polynomial as $f(\bar{x}) \geq 0$. This fact was first established in [8].*

Lemma 2.4. (Strong separation without compactness) *Let $C_1 = \{x \in \mathbb{R}^n : f_i^1(x) \leq 0, i = 1, \ldots, m\}$ and $C_2 = \{x \in \mathbb{R}^n : f_j^2(x) \leq 0, j = 1, \ldots, l\}$ where f_i^1 and f_j^2 are convex polynomials. If $C_1 \cap C_2 = \emptyset$, then there exists $z \in \mathbb{R}^n \backslash \{0\}$ such that*

$$\sup_{c_1 \in C_1} \{z^T c_1\} < \inf_{c_2 \in C_2} \{z^T c_2\}.$$

Proof. We first see that $C_1 - C_2$ is closed. To see this, let $c_k^1 \in C_1$ and $c_k^2 \in C_2$ be such that $c_k^1 - c_k^2 \to c$. We now show that $c \in C_1 - C_2$. Consider the following convex polynomial optimization problem

$$\begin{array}{lll} (P) & \min\limits_{x \in \mathbb{R}^n, y \in \mathbb{R}^n} & \|(x - y) - c\|^2 \\ & \text{s.t.} & f_i^1(x) \leq 0, i = 1, \ldots, m, \\ & & f_j^2(y) \leq 0, j = 1, \ldots, l. \end{array}$$

Note that (c_k^1, c_k^2) are feasible for (P). So, we see that $\inf(P) = 0$. By Lemma 2.2, we see that the optimal solution of (P) is attained. So, there exist $c_1 \in C_1$ and $c_2 \in C_2$ such that $c = c_1 - c_2 \in C_1 - C_2$. So, $C_1 - C_2$ is closed.

Now, as $C_1 \cap C_2 = \emptyset$, we have $0 \notin C_1 - C_2$. From the fact that $C_1 - C_2$ is closed, we obtain that there exists $\delta > 0$ such that

$$\overline{\mathbb{B}}(0, \delta) \cap (C_1 - C_2) = \emptyset.$$

Then the standard separation theorem [23] gives us that there exists $z \in \mathbb{R}^n \backslash \{0\}$ such that

$$\sup_{c \in C_1 - C_2} \{z^T c\} \leq \inf_{a \in \overline{\mathbb{B}}(0,\delta)} \{z^T a\} = -\delta \|z\| < 0.$$

Thus, the conclusion follows by noting that $\sup\limits_{c \in C_1 - C_2} \{z^T c\} = \sup\limits_{c_1 \in C_1} \{z^T c_1\} - \inf\limits_{c_2 \in C_2} \{z^T c_2\}$. □

Using the above strong separation property of semi-algebraic convex sets, we derive a certificate for infeasibility of SOS-convex inequality systems.

Theorem 2.5. (Infeasibility certificate of SOS-convex systems) *Let $M := \{x \in \mathbb{R}^n : f_k(x) \leq 0, k = 1, \ldots, l\}$ where f_k, $k = 1, \ldots, l$, are SOS-convex polynomials. Let $d = \max\{\deg f_1, \ldots, \deg f_l\}$. Then, exactly one of the following statements holds:*

(i) *$M \neq \emptyset$;*

(ii) $(\exists \sigma_0 \in \Sigma_d^2[x])$ $(\exists \lambda_k \geq 0, \sum_{k=1}^l \lambda_k = 1,\ \delta_0 > 0)$ $\displaystyle\sum_{k=1}^l \lambda_k f_k = \delta_0 + \sigma_0.$

Proof. [NOT(i) $\Rightarrow$ (ii)] Suppose that (i) fails, that is, $M = \emptyset$. This implies that $C_1 \cap C_2 = \emptyset$ where

$$C_1 = \{(x, r) \in \mathbb{R}^n \times \mathbb{R}^l : f_k(x) \leq r_k, k = 1, \ldots, l\}$$

and

$$C_2 = \mathbb{R}^n \times (-\mathbb{R}^l_+) = \{(x,r) \in \mathbb{R}^n \times \mathbb{R}^l : r_k \le 0\}.$$

Clearly, C_1 and C_2 are convex semi-algebraic sets. The preceding strong separation theorem, Lemma 2.4, gives us that there exists $z = (z_1, \ldots, z_{n+l}) \in \mathbb{R}^{n+l}\backslash\{0\}$ such that

$$\sup_{c_1 \in C_1} \{z^T c_1\} < \inf_{c_2 \in C_2} \{z^T c_2\}.$$

As $C_2 = \mathbb{R}^n \times (-\mathbb{R}^l_+)$, it follows that $z_1 = \ldots, z_n = 0$ and $z_{n+1} \le 0, \ldots, z_{n+l} \le 0$. As $z \neq 0$, we see that $(z_{n+1}, \ldots, z_{n+l}) \neq 0$ (and so, $\sum_{k=1}^l z_{n+k} < 0$), and there exists $\delta > 0$ such that, for all $x \in \mathbb{R}^n$,

$$\sum_{k=1}^l z_{n+k} f_k(x) < -\delta.$$

Let $\lambda_k = \frac{z_{n+k}}{\sum_{k=1}^l z_{n+k}} \ge 0$ and let $\bar{\delta} = \frac{-\delta}{\sum_{k=1}^l z_{n+k}} > 0$. Then,

$$\sum_{k=1}^l \lambda_k = 1 \text{ and } \sum_{k=1}^l \lambda_k f_k(x) \ge \delta_0.$$

Define $\delta_0 := \inf_{x \in \mathbb{R}^n}\{\sum_{k=1}^l \lambda_k f_k(x)\}$. Then, $\delta_0 \ge \bar{\delta} > 0$. As $\lambda_k \ge 0$ and each f_k is SOS-convex, $\sum_{k=1}^l \lambda_k f_k$ is a SOS-convex polynomial with degree d. From Lemma 2.2, there exists $\bar{x}$ such that $\sum_{k=1}^l \lambda_k f_k(\bar{x}) = \delta_0$. Define, for each $x \in \mathbb{R}^n$, $\sigma_0(x) := \sum_{k=1}^l \lambda_k f_k(x) - \delta_0$. Then σ_0 is a SOS-convex polynomial with degree d and takes nonnegative value. So, σ_0 is a sum of squares polynomial with degree d. It then follows that, for all $x \in \mathbb{R}^n$,

$$\sum_{k=1}^l \lambda_k f_k(x) = \sigma_0(x) + \delta_0,$$

Thus, (ii) follows.

[(ii) $\Rightarrow$ NOT(i)] Suppose that statement (ii) holds. Then there exist $\sigma_0 \in \Sigma^2_d[x]$, $\lambda_k \ge 0$, $\sum_{k=1}^l \lambda_k = 1$ and $\delta_0 > 0$ such that $\sum_{k=1}^l \lambda_k f_k = \delta_0 + \sigma_0$. We now establish that (i) must fail by contradiction. Suppose that (i) holds. Then $M \neq \emptyset$ and so, there exists $x_0 \in \mathbb{R}^n$ such that $f_k(x_0) \le 0$, $k = 1, \ldots, l$. This implies that

$$0 \ge \sum_{k=1}^l \lambda_k f_k(x_0) = \delta_0 + \sigma_0(x_0) \ge \delta_0 > 0.$$

This is impossible and so, the desired conclusion follows. □

Below, we establish a new alternative theorem for SOS-convex polynomial inequality systems.

Theorem 2.6. (Alternative theorem for SOS-convex polynomial systems) *Let $g_i, h_j : \mathbb{R}^n \to \mathbb{R}$ be SOS-convex polynomials with degree at most d for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, p$. Suppose that $A = \{x \in \mathbb{R}^n : h_j(x) \le 0\} \neq \emptyset$. Then, exactly one of the following two statements holds:*

(1) $\exists\, x \in \mathbb{R}^n,\ h_j(x) \le 0, j = 1, 2, \ldots, p,\ g_i(x) < 0, i = 1, \ldots, m$

(2) $(\forall\, \epsilon > 0)\ (\exists\, \mu_i \ge 0, \sum_{i=1}^{m} \mu_i = 1, \lambda \in \mathbb{R}^p_+, \sigma_0 \in \Sigma^2_d[x])\ \sum_{j=1}^{p} \lambda_j h_j + \sum_{i=1}^{m} \mu_i g_i + \epsilon = \sigma_0.$

Proof. [Not (1) ⇒ (2)] Suppose that (1) fails. Fix an arbitrary $\epsilon > 0$. Then, the following convex polynomial inequality system has no solution

$$[h_j(x) \le 0, j = 1, \ldots, p,\ g_i(x) + \epsilon \le 0, i = 1, \ldots, m].$$

For each $i = 1, \ldots, m + p$, let

$$f_i(x) = \begin{cases} h_i(x), & i = 1, \ldots, p, \\ g_{i-p}(x) + \epsilon, & i = p + 1, \ldots, p + m. \end{cases}$$

Let $M = \{x : f_i(x) \le 0, i = 1, \ldots, p+m\} = \emptyset$. The the infeasibility certificate implies that there exist $\sigma_0 \in \Sigma^2_d[x], \bar{\lambda}_i \ge 0, \sum_{i=1}^{p+m} \bar{\lambda}_i = 1$, and $\delta_0 > 0$ such that $\sum_{i=1}^{p+m} \bar{\lambda}_i f_i = \delta_0 + \sigma_0$. From the definition of f_i, we have

$$\sum_{j=1}^{p} \bar{\lambda}_j h_j + \sum_{i=1}^{m} \bar{\lambda}_{p+i}(g_i + \epsilon) = \delta_0 + \sigma_0$$

We now show that $(\bar{\lambda}_{p+1}, \ldots, \bar{\lambda}_{p+m}) \ne 0$. Otherwise, we have $(\bar{\lambda}_{p+1}, \ldots, \bar{\lambda}_{p+m}) = 0$, and so, $\sum_{j=1}^{p} \bar{\lambda}_j h_j = \delta_0 + \sigma_0$. Let $x_0 \in A$. Then, we have

$$0 \ge \sum_{j=1}^{p} \bar{\lambda}_j h_j(x_0) = \delta_0 + \sigma_0(x_0) \ge \delta_0 > 0,$$

which is impossible. Then, $(\bar{\lambda}_{p+1}, \ldots, \bar{\lambda}_{p+m}) \ne 0$ and hence $\sum_{i=1}^{m} \bar{\lambda}_{p+i} > 0$. Letting

$$\mu_i = \frac{\bar{\lambda}_{p+i}}{\sum_{i=1}^{m} \bar{\lambda}_{p+i}}, i = 1, \ldots, m \text{ and } \lambda_j = \frac{\bar{\lambda}_j}{\sum_{i=1}^{m} \bar{\lambda}_{p+i}}, j = 1, \ldots, p,$$

then we have $\sum_{i=1}^{m} \mu_i = 1$ and

$$\sum_{j=1}^{p} \lambda_j h_j(x) + \sum_{i=1}^{m} \mu_i g_i(x) + \epsilon = \sum_{j=1}^{p} \lambda_j h_j(x) + \sum_{i=1}^{m} \mu_i (g_i(x) + \epsilon) \ge \frac{\delta_0}{\sum_{i=1}^{m} \bar{\lambda}_{p+i}} \ge 0.$$

Note that $\sum_{j=1}^{p} \lambda_j h_j(x) + \sum_{i=1}^{m} \mu_i g_i(x) + \epsilon$ is a SOS-convex polynomial and any nonnegative SOS-convex polynomial is sum-of-squares. Thus, (2) holds.

[(2) ⇒ Not (1)] This implication is immediate. □

As a corollary, we now obtain a numerically checkable certificate for strict feasibility for SOS-convex polynomial inequality systems.

Corollary 2.7. (Certificate for Strict Feasibility) *Let $g_i : \mathbb{R}^n \to \mathbb{R}$ be SOS-convex polynomials with degree at most d for $i = 1, 2, \ldots, m$. Then, exactly one of the following two statements holds:*

(1) $\exists\, x \in \mathbb{R}^n,\ g_i(x) < 0, i = 1, \ldots, m$

(2) $(\exists\, \mu_i \geq 0, \sum_{i=1}^{m} \mu_i = 1, \sigma_0 \in \Sigma_d^2[x]) \ \sum_{i=1}^{m} \mu_i g_i = \sigma_0.$

Proof. [Not (1) $\Rightarrow$ (2)] Suppose that (i) fails. Then applying Theorem 2.6 with $h_j = 0$. Then, for all $\epsilon > 0$, there exist $\mu_i^\epsilon \geq 0, \sum_{i=1}^m \mu_i^\epsilon = 1, \sigma_0^\epsilon \in \Sigma_d^2[x]$ such that

$$\sum_{i=1}^{m} \mu_i^\epsilon g_i + \epsilon = \sigma_0^\epsilon.$$

As each $\mu_i^\epsilon \in [0,1]$, by passing to subsequence, we may assume that $\mu_i^\epsilon \to \mu_i \geq 0$ with $\sum_{i=1}^m \mu_i = 1$. Passing to the limit, we have

$$\sigma_0^\epsilon = \sum_{i=1}^{m} \mu_i^\epsilon g_i + \epsilon \to \sum_{i=1}^{m} \mu_i g_i.$$

Note that $\sum_{i=1}^m \mu_i g_i$ is a degree d polynomial and $\Sigma_d^2[x]$ is a closed and convex cone. It follows that

$$\sum_{i=1}^{m} \mu_i g_i \in \Sigma_d^2[x].$$

[(2) $\Rightarrow$ Not (1)] This implication is immediate. □

3 Robust Scalar SOS-convex Programs

In this Section, we show that the value of a SOS-convex programming problem can be found by solving a semidefinite programming problem. A SOS-convex programming problem which is a special case of a convex programming problem, takes the form

$$(P) \quad \min_{x \in \mathbb{R}^m} \ f(x) \quad \text{s.t.} \quad h_j(x) \leq 0, j = 1, \ldots, p,$$

where f and h_j, $j = 1, \ldots, p$, are SOS-convex polynomials. In reality, The data of real-world optimization problems more often than not, are uncertain (that is, they are not known exactly at the time of the decision) due to estimation errors, prediction errors or lack of information.

In the following, we consider the case where the objective function f is subject to data uncertainty. The effect of data uncertainty in the objective function can be captured by the following parameterized problem

$$(P_y) \quad \min_{x \in \mathbb{R}^m} \ f(x, y) \quad \text{s.t.} \quad h_j(x) \leq 0, j = 1, \ldots, p.$$

where $y \in B$ and B is an uncertainty set which is a convex and compact set in $\mathbb{R}^n$. The robust counterpart [6, 18] of the parameterized problem (P_y) takes the form

$$(RP) \quad \min_{x \in \mathbb{R}^m} \max_{y \in B} \ f(x, y) \quad \text{s.t.} \quad h_j(x) \leq 0, j = 1, \ldots, p.$$

We define the class of robust convex programming problem, called robust SOS-convex polynomial programming problem with affinely parameterized data uncertainty as follows.

Definition 3.1 (Robust SOS-convex programs with affinely parameterized data uncertainty). *The data associated with the objective function $f(x,y)$ in the robust optimization problem is affinely parameterized in the sense that*

$$f(x,y) := f_0(x) + \sum_{i=1}^{n} y_i f_i(x),$$

where each f_i, $i = 0, 1, \ldots, n$, is a SOS-convex polynomial with degree at most d_1. The uncertainty set B is given by $B = \{y \in \mathbb{R}^n_+ : M_0 + \sum_{j=1}^n M_j y_j \succeq 0\}$ where B is a convex compact subset of $\mathbb{R}^n$ and $M_j \in S^l$, $l \in \mathbb{N}$ and S^l is the space consists of all symmetric $(l \times l)$ matrices. The feasible set of (RP) is given by $A = \{x \in \mathbb{R}^m : h_j(x) \leq 0, j = 1, \ldots, p\}$ where each h_j is a SOS-convex polynomial with degree at most d_0,

Below, we will show that the optimal value of the robust SOS-convex polynomial program with affinely parameterized data uncertainty is equal to the following conical linear programming problem whose value is equal to the value of a semidefinite programming problem *without any constraint qualifications*:

$$(SDP_0) \qquad \sup_{\mu \in \mathbb{R}, y \in \mathbb{R}^n_+, \lambda \in \mathbb{R}^p_+} \{\mu : f_0 + \sum_{i=1}^{n} y_i f_i + \sum_{j=1}^{p} \lambda_j h_j - \mu \in \Sigma^2_d[x]$$
$$M_0 + \sum_{i=1}^{n} M_i y_i \succeq 0\},$$

where $d = \max\{d_0, d_1\}$.

Recently, some robust convex polynomial programs has been examined in [20, 19] under constraint qualifications. For example, in the special where B is the simplex $\{y \in \mathbb{R}^n_+ : \sum_{i=1}^n y_i = 1\}$, numerical tractability of the above robust SOS-convex polynomial programs was examined in [20] under strict feasibility conditions. Moreover, [19] established numerical tractable classes of robust convex polynomial optimization problems with constraint data uncertainty again under strict feasibility conditions. For more general robust nonconvex polynomial optimization problem with compact feasible sets, see [22].

Note that (SDP_0) is a conical linear programming problem. To see this, let $\mathbb{R}_d[x]$ denote the space consisting of all real polynomials with degree at most d. Then the sum-of-squares cone $\Sigma^2_d[x]$ is a closed and convex cone in the finite dimensional space $\mathbb{R}_d[x]$ (cf. [21, Corollary 3.34]). Define a linear map $L : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^p \to \mathbb{R}_d[x]$ by

$$L(y, \mu, \lambda) = f_0 + \sum_{i=1}^{n} y_i f_i + \sum_{j=1}^{p} \lambda_j h_j - \mu.$$

Then, (SDP_0) can be rewritten as the following conical linear program

$$\sup_{\mu \in \mathbb{R}, y \in \mathbb{R}^n_+, \lambda \in \mathbb{R}^p_+} \{\mu : L(y, \mu, \lambda) \in \Sigma^2_d[x], \; M_0 + \sum_{i=1}^{n} M_i y_i \in S^l_+\}.$$

The following two examples illustrate the semidefinite linear programming reformulation of the conical linear programming problem. The general case of the reformulation is given in the Appendix.

Example 3.2. *Consider the following robust SOS-convex programs with affine data parametrization*

$$\begin{array}{ll}\min_{x\in\mathbb{R}^m}\max_{y\in B} & f_0(x)+\sum_{i=1}^{n} y_i f_i(x)\\ s.t. & h_j(x)\le 0, j=1,\ldots,p,\end{array}$$

where $f_i(x)=\frac{1}{2}x^TG_ix+v_i^Tx+\beta_i$, $i=0,1,\ldots,n$, G_i are positive semidefinite $(m\times m)$ matrices, $v_i\in\mathbb{R}^m$ and $\beta_i\in\mathbb{R}$, the uncertainty set $B=\{y\in\mathbb{R}^n_+ : M_0+\sum_{j=1}^n M_jy_j\succeq 0\}$, and $h_j(x)=\frac{1}{2}x^TH_jx+u_j^Tx+\alpha_j$, H_j is positive semidefinite $(m\times m)$ matrix, $u_j\in\mathbb{R}^m$ and $\alpha_j\in\mathbb{R}$. We show that $f_0+\sum_{i=1}^n y_if_i+\sum_{j=1}^p\lambda_jh_j-\mu\in\Sigma^2_d[x]$ is equivalent to a linear matrix inequality. To see this, observe that

$$\begin{aligned}&f_0+\sum_{i=1}^n y_if_i+\sum_{j=1}^p\lambda_jh_j-\mu\\ =\;&\frac{1}{2}x^T(G_0+\sum_{i=1}^n y_iG_i+\sum_{j=1}^p\lambda_jH_j)x+(v_0+\sum_{i=1}^n y_iv_i+\sum_{j=1}^p\lambda_ju_j)^Tx+(\beta_0+\sum_{i=1}^n y_i\beta_i+\sum_{j=1}^p\lambda_j\alpha_j-\mu),\end{aligned}$$

and that for a quadratic function $q(x)=\frac{1}{2}x^TGx+g^Tx+\gamma$,

$$q\in\Sigma^2_2[x]\;\Leftrightarrow\; q(x)\ge 0 \text{ for all } x\in\mathbb{R}^m\;\Leftrightarrow\;\begin{pmatrix} G & g\\ g^T & 2\gamma\end{pmatrix}\succeq 0.$$

Then, we see that

$$\begin{aligned}&f_0+\sum_{i=1}^n y_if_i+\sum_{j=1}^p\lambda_jh_j-\mu\in\Sigma^2_d[x]\\ \Leftrightarrow\;&\begin{pmatrix} G_0+\sum_{i=1}^n y_iG_i+\sum_{j=1}^p\lambda_jH_j & v_0+\sum_{i=1}^n y_iv_i+\sum_{j=1}^p\lambda_ju_j\\ (v_0+\sum_{i=1}^n y_iv_i+\sum_{j=1}^p\lambda_ju_j)^T & 2(\beta_0+\sum_{i=1}^n y_i\beta_i+\sum_{j=1}^p\lambda_j\alpha_j-\mu)\end{pmatrix}\succeq 0.\end{aligned}$$

Thus, in this case, (SDP_0) is also equivalent to a semidefinite programming problem.

Example 3.3. *Consider the following robust SOS-convex programs with affine data parametrization*

$$\begin{array}{ll}\min_{x\in\mathbb{R}^m}\max_{y\in B} & f_0(x)+\sum_{i=1}^{n} y_i f_i(x):=x+y_1x^4+y_2x\\ s.t. & -x\le 0,\end{array}$$

where the objective function $f_0(x) + \sum_{i=1}^n y_i f_i(x) = x + y_1x^4 + y_2x$ *with* $f_0(x) = x$, $f_1(x) = x^4$ *and* $f_2(x) = x$, *and the uncertainty set* $B = \{y \in \mathbb{R}^n_+ : M_0 + \sum_{j=1}^n M_j y_j \succeq 0\}$. *To see* (SDP_0) *is a semidefinite programming problem, we only need to show that*

$$f_0 + \sum_{i=1}^n y_i f_i + \sum_{j=1}^p \lambda_j h_j - \mu \in \Sigma^2_d[x]$$

is equivalent to a linear matrix inequality. To see this, we observe, by comparing coefficients, that

$$\begin{aligned} & f_0 + \sum_{i=1}^n y_i f_i + \sum_{j=1}^p \lambda_j h_j - \mu \in \Sigma^2_d[x] \\ \Leftrightarrow \quad & x + y_1x^4 + y_2x + \lambda_1(-x) - \mu = \big(1, x, x^2\big) \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{12} & W_{22} & W_{23} \\ W_{13} & W_{23} & W_{33} \end{pmatrix} \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix},\ W = (W_{ij}) \in S^3_+, \\ \Leftrightarrow \quad & W_{11} = -\mu, 2W_{12} = -\lambda_1 + y_2 + 1, 2W_{13} + W_{22} = 0, W_{23} = 0, W_{33} = y_1,\ W = (W_{ij}) \in S^3_+. \end{aligned}$$

Thus, in this case, (SDP_0) *is equivalent to a semidefinite programming problem.*

Theorem 3.4. (Single SDP Values of Robust SOS-convex Programs) *For the robust convex optimization problem*

$$\begin{aligned} (RP) \quad \min_{x\in\mathbb{R}^m} \max_{y\in B} \quad & f_0(x) + \sum_{i=1}^n y_i f_i(x) \\ s.t. \quad & h_j(x) \le 0, j = 1, \ldots, p, \end{aligned}$$

where $f_i, i = 0, 1, \ldots, n$, *are SOS-convex polynomials with degree at most* d_1, $h_j, j = 1, \ldots, p$, *are SOS-convex polynomials with degree at most* d_0, $M_j \in S^l$, $l \in \mathbb{N}$ *and the uncertainty set* B *is a convex and compact set given by* $B = \{y \in \mathbb{R}^n_+ : M_0 + \sum_{i=1}^n M_i y_i \succeq 0\}$. *Let* $d = \max\{d_0, d_1\}$. *Then,*

$$\begin{aligned} \inf(RP) \quad = \quad \sup_{\mu\in\mathbb{R}, y\in\mathbb{R}^n_+, \lambda\in\mathbb{R}^p_+} & \{\mu : f_0 + \sum_{i=1}^n y_i f_i + \sum_{j=1}^p \lambda_j h_j - \mu \in \Sigma^2_d[x] \\ & M_0 + \sum_{i=1}^n M_i y_i \succeq 0\}. \end{aligned}$$

Proof. Let $A = \{x \in \mathbb{R}^m : h_j(x) \le 0, j = 1, \ldots, p\}$ and $B = \{y \in \mathbb{R}^n_+ : M_0 + \sum_{i=1}^n M_i y_i \succeq 0\}$. We first observe that

$$\inf(RP) = \min_{x\in A} \max_{y\in B} \{f_0(x) + \sum_{i=1}^n y_i f_i(x)\}.$$

We next show that

$$\begin{aligned} \max_{y\in B} \inf_{x\in A} \{f_0(x) + \sum_{i=1}^n y_i f_i(x)\} \le \sup_{\mu\in\mathbb{R}, y\in\mathbb{R}^n_+, \lambda\in\mathbb{R}^p_+} & \{\mu : f_0 + \sum_{i=1}^n y_i f_i + \sum_{j=1}^p \lambda_j h_j - \mu \in \Sigma^2_d[x] \\ & M_0 + \sum_{i=1}^n M_i y_i \succeq 0\}. \end{aligned} \tag{3.1}$$

To see this, we may assume that $\max_{y \in B} \inf_{x \in A} \{f_0(x) + \sum_{i=1}^n y_i f_i(x)\} > -\infty$, and so,

$$r := \max_{y \in B} \inf_{x \in A} \{f_0(x) + \sum_{i=1}^n y_i f_i(x)\} \in \mathbb{R}.$$

Then there exists $\bar{y} \in B$ such that

$$\inf_{x \in A} \{f_0(x) + \sum_{i=1}^n \bar{y}_i f_i(x)\} \geq r.$$

This implies that the following SOS-convex inequality system has no solution

$$[h_j(x) \leq 0, j = 1, \ldots, p, \;\; f_0(x) + \sum_{i=1}^n \bar{y}_i f_i(x) - r < 0].$$

So, Theorem 2.6 implies that for each $\epsilon > 0$, there exist $\mu \in \mathbb{R}, \lambda_j \geq 0, \sigma_0 \in \Sigma_d^2[x]$ such that $f_0 + \sum_{i=1}^n \bar{y}_i f_i + \sum_{j=1}^p \lambda_j h_j - r + \epsilon = \sigma_0$. This proves our claim (3.1).

Now, by the usual convex-concave minimax theorem, we have

$$\inf_{x \in A} \max_{y \in B} \{f_0(x) + \sum_{i=1}^n y_i f_i(x)\} = \max_{y \in B} \inf_{x \in A} \{f_0(x) + \sum_{i=1}^n y_i f_i(x)\}.$$

This implies that

$$\begin{aligned}
\inf_{x \in A} \max_{y \in B} \{f_0(x) + \sum_{i=1}^n y_i f_i(x)\} &= \max_{y \in B} \inf_{x \in A} \{f_0(x) + \sum_{i=1}^n y_i f_i(x)\} \\
&\geq \sup_{y \in B, \lambda_j \geq 0} \inf_{x \in \mathbb{R}^m} \{f_0(x) + \sum_{i=1}^n y_i f_i(x) + \sum_{j=1}^p \lambda_j h_j(x)\} \\
&= \sup_{y \in \mathbb{R}^n_+, \lambda_j \geq 0, \mu \in \mathbb{R}} \{\mu : f_0 + \sum_{i=1}^n y_i f_i + \sum_{j=1}^p \lambda_j h_j - \mu \geq 0 \\
&\qquad\qquad M_0 + \sum_{i=1}^n M_i y_i \succeq 0\} \\
&\geq \sup_{\mu \in \mathbb{R}, \sigma_0 \in \Sigma_d^2[x]} \{\mu : f_0 + \sum_{i=1}^n y_i f_i + \sum_{j=1}^p \lambda_j h_j - \mu \in \Sigma_d^2[x] \\
&\qquad\qquad M_0 + \sum_{i=1}^n M_i y_i \succeq 0\},
\end{aligned}$$

where the last inequality follows as any sums-of-squares polynomial takes nonnegative value. Combining this with (3.1), the conclusion follows. □

In the special case of (RP), where $f_0 \equiv 0$ and $B := \{y \in \mathbb{R}^n_+ : \sum_{i=1}^n y_i = 1\}$, the model (RP) collapses to the discrete minimax problem of [20]:

$$\begin{aligned}
(MP) \quad & \min_{x \in \mathbb{R}^m} \max_{1 \leq i \leq m} f_i(x) \\
& \text{s.t.} \quad h_j(x) \leq 0, j = 1, \ldots, p.
\end{aligned}$$

The equality between the value of (MP) and its corresponding SDP value was given in [20, Theorem 4.3]. We now see that in the case of a convex quadratic game, the optimal value of a robust convex quadratic programs can be computed by a simple SDP.

Corollary 3.5. (Single SDP Values of Robust Convex Quadratic Programs) *Consider (RP) with* $f_i(x) = \frac{1}{2}x^T G_i x + v_i^T x + \beta_i$, $i = 0, 1, \ldots, n$ *where* G_i *are positive semidefinite* $(m \times m)$ *matrices,* $v_i \in \mathbb{R}^m$ *and* $\beta_i \in \mathbb{R}$, *and* $h_j(x) = \frac{1}{2}x^T H_j x + u_j^T x + \alpha_j$, $j = 1, \ldots, p$, H_j *are positive semidefinite* $(m \times m)$ *matrices,* $u_j \in \mathbb{R}^m$ *and* $\alpha_j \in \mathbb{R}$, *and the uncertainty set* B *is a convex compact subset of* $\mathbb{R}^n$ *defined by* $B = \{y \in \mathbb{R}^n_+ : \frac{1}{2}y^T Q_i^T Q_i y + b_i^T y + \gamma_i \leq 0, i = 1, \ldots, l\}$ *where* $Q_i \in \mathbb{R}^{s \times n}$, $s \in \mathbb{N}$, $b_i \in \mathbb{R}^n$ *and* $\gamma_i \in \mathbb{R}$. *Then, we have*

$$\inf(RP) = \sup_{\substack{\mu \in \mathbb{R}, \lambda_j \geq 0, \\ y_i \geq 0}} \{\mu : \begin{pmatrix} G_0 + \sum_{i=1}^n y_i G_i + \sum_{j=1}^p \lambda_j H_j & v_0 + \sum_{i=1}^n y_i v_i + \sum_{j=1}^p \lambda_j u_j \\ (v_0 \sum_{i=1}^n y_i v_i + \sum_{j=1}^p \lambda_j u_j)^T & 2(\beta_0 \sum_{i=1}^n y_i \beta_i + \sum_{j=1}^p \lambda_j \alpha_j - \mu) \end{pmatrix} \succeq 0$$

$$\begin{pmatrix} I_{s+1} & \begin{pmatrix} Q_i y \\ \frac{1}{2} + b_i^T y + \gamma_i \end{pmatrix} \\ \begin{pmatrix} (Q_i y)^T, \frac{1}{2} + b_i^T y + \gamma_i \end{pmatrix} & \frac{1}{2} - (b_i^T y + \gamma_i) \end{pmatrix} \succeq 0\},$$

where I_{s+1} *is the* $(s+1) \times (s+1)$ *identity matrix.*

Proof. It is clear that each f_i is a convex quadratic function (and so, is a SOS-convex polynomial with degree 2). Then, it follows from that

$$\begin{aligned} \inf(RP_q) &= \sup_{y \in \mathbb{R}^n, \mu \in \mathbb{R}, \lambda_j \geq 0} \{\mu : f_0 + \sum_{i=1}^n y_i f_i + \sum_{j=1}^p \lambda_j h_j - \mu \in \Sigma_2^2[x], y \in B\} \\ &= \sup_{y \in \mathbb{R}^n, \mu \in \mathbb{R}, \lambda_j \geq 0} \{\mu : \begin{pmatrix} G_0 + \sum_{i=1}^n y_i G_i + \sum_{j=1}^p \lambda_j H_j & v_0 + \sum_{i=1}^n y_i v_i + \sum_{j=1}^p \lambda_j u_j \\ (v_0 + \sum_{i=1}^n y_i v_i + \sum_{j=1}^p \lambda_j u_j)^T & 2(\beta_0 + \sum_{i=1}^n y_i \beta_i + \sum_{j=1}^p \lambda_j \alpha_j - \mu) \end{pmatrix} \succeq 0 \\ &\qquad y \in B\}. \end{aligned} \tag{3.2}$$

The conclusion now follows from the equivalence that

$$\begin{aligned} y \in B &\Leftrightarrow \frac{1}{2}y^T Q_i^T Q_i y + b_i^T y + \gamma_i \leq 0, i = 1, \ldots, l, \\ &\Leftrightarrow \| \begin{pmatrix} Q_i y \\ \frac{1}{2} + b_i^T y + \gamma_i \end{pmatrix} \|_2 \leq \frac{1}{2} - (b_i^T y + \gamma_i), i = 1, \ldots, l, \\ &\Leftrightarrow \begin{pmatrix} I_{s+1} & \begin{pmatrix} Q_i y \\ \frac{1}{2} + b_i^T y + \gamma_i \end{pmatrix} \\ \begin{pmatrix} (Q_i y)^T, \frac{1}{2} + b_i^T y + \gamma_i \end{pmatrix} & \frac{1}{2} - (b_i^T y + \gamma_i) \end{pmatrix} \succeq 0, i = 1, \ldots, l. \end{aligned}$$

□

As a consequence, we obtain the following known result that the optimal value of a robust linear programs with affine data parametrization can be obtained by solving a linear programming problem [4].

Corollary 3.6. (LP Values of Robust Linear Programs) *Consider (RP) with* $f_i(x) = v_i^T x + \beta_i$, $i = 0, 1, \ldots, n$, *where* $v_i \in \mathbb{R}^m$ *and* $\beta_i \in \mathbb{R}$, $h_j(x) = u_j^T x + \alpha_j$, $j = 1, \ldots, p$ *where* $u_j \in \mathbb{R}^m$ *and* $\alpha_j \in \mathbb{R}$ *and the uncertainty set* B *is the simplex in* $\mathbb{R}^n$, *i.e.,* $B = \{y \in \mathbb{R}^n : \sum_{i=1}^n y_i = 1, y_i \geq 0\}$. *Then,*

$$\inf(RP) \quad = \quad \inf_{\substack{\lambda_j \geq 0, \\ y_i \geq 0, \sum_{i=1}^n y_i = 1}} \{\beta_0 + \sum_{i=1}^n y_i \beta_i + \sum_{j=1}^p \lambda_j \alpha_j : v_0 + \sum_{i=1}^n y_i v_i + \sum_{j=1}^p \lambda_j u_j = 0\}.$$

Proof. Let G_i and H_j be the zero matrices in (3.2). Then,

$$\inf(RP) \quad = \quad \sup_{\substack{\mu \in \mathbb{R}, \lambda_j \geq 0, \\ y_i \geq 0, \sum_{i=1}^n y_i = 1}} \{\mu : \begin{pmatrix} 0 & v_0 + \sum_{i=1}^n y_i v_i + \sum_{j=1}^p \lambda_j u_j \\ (v_0 + \sum_{i=1}^n y_i v_i + \sum_{j=1}^p \lambda_j u_j)^T & 2(\beta_0 + \sum_{i=1}^n y_i \beta_i + \sum_{j=1}^p \lambda_j \alpha_j - \mu) \end{pmatrix} \succeq 0\}.$$

Noting that

$$\begin{pmatrix} 0 & g \\ g^T & \gamma \end{pmatrix} \succeq 0 \;\Leftrightarrow\; g = 0 \text{ and } \gamma \geq 0,$$

we obtain

$$\begin{aligned}
& \sup_{\substack{\mu \in \mathbb{R}, \lambda_j \geq 0, \\ y_i \geq 0, \sum_{i=1}^n y_i = 1}} \{\mu : \begin{pmatrix} 0 & v_0 + \sum_{i=1}^n y_i v_i + \sum_{j=1}^p \lambda_j u_j \\ (v_0 + \sum_{i=1}^n y_i v_i + \sum_{j=1}^p \lambda_j u_j)^T & 2(\beta_0 + \sum_{i=1}^n y_i \beta_i + \sum_{j=1}^p \lambda_j \alpha_j - \mu) \end{pmatrix} \succeq 0\} \\
= & \sup_{\substack{\mu \in \mathbb{R}, \lambda_j \geq 0, \\ y_i \geq 0, \sum_{i=1}^n y_i = 1}} \{\mu : v_0 + \sum_{i=1}^n y_i v_i + \sum_{j=1}^p \lambda_j u_j = 0, \beta_0 + \sum_{i=1}^n y_i \beta_i + \sum_{j=1}^p \lambda_j \alpha_j - \mu \geq 0\} \\
= & \inf_{\substack{\lambda_j \geq 0, \\ y_i \geq 0, \sum_{i=1}^n y_i = 1}} \{\beta_0 + \sum_{i=1}^n y_i \beta_i + \sum_{j=1}^p \lambda_j \alpha_j : v_0 + \sum_{i=1}^n y_i v_i + \sum_{j=1}^p \lambda_j u_j = 0\}.
\end{aligned}$$

□

4 Robust Multi-objective SOS-Convex Programs

Consider the following multi-objective polynomial optimization problem

$$\begin{aligned}
(VP) \qquad & \mathrm{V} - \min_{x \in \mathbb{R}^m} \; \left(f_0^1(x), \ldots, f_0^q(x)\right) \\
& \text{s.t.} \qquad h_j(x) \leq 0, \; j = 1, \ldots, p,
\end{aligned}$$

where for each $l = 1, 2, \ldots, q$, f_0^l is a SOS-convex polynomials and h_j's are SOS-convex polynomials on $\mathbb{R}^m$. The data uncertainty of the objective function can be captured by the following affine data parameterized problem

$$\begin{aligned}
& \mathrm{V} - \min_{x \in \mathbb{R}^m} \; \left(f_1(x, y^1), \ldots, f_q(x, y^q)\right) \\
& \text{s.t.} \qquad h_j(x) \leq 0, \; j = 1, \ldots, p,
\end{aligned}$$

where $y^l \in B_l$, B_l is the uncertainty set given by $B_l = \{y \in \mathbb{R}^n_+ : M_0^l + \sum_{i=1}^n M_i^l y_i \succeq 0\}$,

$$f_l(x, y^l) = f_0^l(x) + \sum_{i=1}^n y_i^l f_i^l(x),\ y^l = (y_1^l, \ldots, y_n^l),$$

and each f_l^i, $l = 1, \ldots, q, i = 0, 1, \ldots, n$, is a SOS-convex polynomial. The robust counterpart of the uncertain problem is given by

$$\begin{array}{lll} (RVP) & \mathrm{V} - \min\limits_{x \in \mathbb{R}^m} & \left(\max\limits_{y^1 \in B_1} f_1(x, y^1), \ldots, \max\limits_{y^q \in B_q} f_q(x, y^q) \right) \\ & \text{s.t.} & h_j(x) \leq 0,\ j = 1, \ldots, p, \end{array}$$

where the worst case value of $f_i(x, \cdot)$ is minimized. Let $A = \{x \in \mathbb{R}^m : h_j(x) \leq 0,\ j = 1, \ldots, p\}$.

We say a feasible point $\bar{x} \in A$ is said to be a robust weak minimum point of (VP) (see [11]) whenever $\bar{x}$ is a weak minimum of (RVP); that is, whenever there is no $x \in A$ such that

$$\max_{y^l \in B_l} f_l(x, y^l) < \max_{y^l \in B_l} f_l(\bar{x}, y^l),\ l = 1, 2, \ldots, q.$$

It is worth noting that, in the special case when each f_i^l, $l = 1, \ldots, q$, $i = 0, 1, \ldots, n$ and h_j, $j = 1, \ldots, p$, are all linear functions, the model (RVP) has been examined in [7] and characterizations of weak minimum were established under suitable constraint qualifications.

In the following Theorem, we present a *qualification-free characterization for a robust weak minimum* of the problem (VP) in terms of sums of squares polynomials.

Theorem 4.1. (Characterizing Robust Weak Minimum) *For problem (VP) with f_l^k, $l = 1, \ldots, q$, $k = 0, 1, \ldots, n$ are all SOS-convex polynomials and h_j, $j = 1, \ldots, p$, are SOS-convex polynomials. Let $d = \max\limits_{1 \leq l \leq q, 1 \leq j \leq p} \{\deg f_l, \deg h_j\}$, $A = \{x : h_j(x) \leq 0, j = 1, \ldots, p\} \neq \emptyset$ and $\bar{x} \in A$. Then, $\bar{x}$ is a robust weak minimum of (VP) if and only if, for any $\epsilon > 0$, there exist $\bar{\mu} \in \mathbb{R}^p_+$ with $\sum_{l=1}^q \bar{\mu}_l = 1$, $\lambda \in \mathbb{R}^p_+$, $y^l = (y_1^l, \ldots, y_n^l) \in \mathbb{R}^n_+$ and $\sigma_0 \in \Sigma_d^2$ such that*

$$\left\{ \begin{array}{c} \sum_{l=1}^q \bar{\mu}_l f_l^0 + \sum_{i=1}^n y_i^l (\sum_{l=1}^q \bar{\mu}_l f_l^i) + \sum_{j=1}^p \lambda_j h_j - r + \epsilon = \sigma_0, \\ M_0^l + \sum_{i=1}^n M_i^l y_i^l \succeq 0, \end{array} \right. \tag{4.1}$$

where $r = \sum_{l=1}^q \bar{\mu}_l \max\limits_{y^l \in B_l} \{f_0^l(\bar{x}) + \sum_{i=1}^n y_i^l\, f_i^l(\bar{x})\}$.

Proof. [$\Leftarrow$] We show that $\bar{x}$ is a robust weak minimum of (VP) by the method of contradiction. Suppose that $\bar{x}$ is not a robust weak minimum of (VP). Then, there exists $\hat{x} \in A$ such that

$$\max_{y^l \in B_l} f_l(\hat{x}, y^l) < \max_{y^l \in B_l} f_l(\bar{x}, y^l), l = 1, \ldots, q.$$

Let $\epsilon > 0$ be such that $\epsilon < \min_{1\le l\le q}\{\max_{y^l\in B_l} f_l(\bar{x},y^l) - \max_{y^l\in B_l} f_l(\hat{x},y^l)\}$. Then, for this ϵ, there exist $\bar{\mu}\in\mathbb{R}^p_+$ with $\sum_{l=1}^q \bar{\mu}_l = 1$, $\lambda\in\mathbb{R}^p_+$, $y^l=(y_1^l,\ldots,y_n^l)\in\mathbb{R}^n_+$ and $\sigma_0\in\Sigma^2_d$ such that (4.1) holds. It then follows that

$$\begin{aligned}
0\le\sigma_0(\hat{x}) &= \sum_{l=1}^q \bar{\mu}_l f_l^0(\hat{x}) + \sum_{i=1}^n y_i^l \sum_{l=1}^q \bar{\mu}_l f_l^i(\hat{x}) + \sum_{j=1}^p \lambda_j h_j(\hat{x}) - r + \epsilon \\
&= \sum_{l=1}^q \bar{\mu}_l (f_l^0(\hat{x}) + \sum_{i=1}^n y_i^l f_l^i(\hat{x})) - r + \sum_{j=1}^p \lambda_j h_j(\hat{x}) + \epsilon \\
&\le \sum_{l=1}^q \bar{\mu}_l \{\max_{y^l\in B_l} f_l(\hat{x},y^l) - \max_{y^l\in B_l} f_l(\bar{x},y^l)\} + \epsilon \\
&\le -\min_{1\le l\le q}\{\max_{y^l\in B_l} f_l(\bar{x},y^l) - \max_{y^l\in B_l} f_l(\hat{x},y^l)\} + \epsilon \\
&< 0.
\end{aligned}$$

This contradicts our choice of ϵ and so, the conclusion follows.

[$\Rightarrow$] Suppose that $\bar{x}\in A$ is the weak minimum of (RVP). Define $g_l(x) = \max_{y^l\in B_l} f_l(x,y^l)$, $l=1,\ldots,q$. Then, clearly, each g_l is a convex function. Then, the system

$$[x\in A,\quad g_l(x) - g_l(\bar{x}) < 0,\ l=1,\ldots,q]$$

has no solution. It follows from the classical basic alternative theorem [11] that there exists $\bar{\mu}\in\mathbb{R}^q_+$ with $\sum_{l=1}^q \bar{\mu}_l = 1$, such that

$$\sum_{l=1}^q \bar{\mu}_l g_l(x) - r = \sum_{l=1}^q \bar{\mu}_l \big(g_l(x) - g_l(\bar{x})\big) \ge 0 \text{ for all } x\in A, \tag{4.2}$$

where $r := \sum_{l=1}^q \bar{\mu}_l g_l(\bar{x}) = \sum_{l=1}^q \bar{\mu}_l \max_{y^l\in B_l}\{f_0^l(\bar{x}) + \sum_{i=1}^n y_i^l f_i^l(\bar{x})\}$. We now claim that, for all $x\in\mathbb{R}^m$,

$$\begin{aligned}
\sum_{l=1}^q \bar{\mu}_l \max_{y^l\in B_l} f_l(x,y^l) &= \max_{y^l\in B_l, 1\le l\le q}\{\sum_{l=1}^q \bar{\mu}_l f_l(x,y^l)\} \\
&= \max_{y^l\in B_l, 1\le l\le q}\{\sum_{l=1}^q \bar{\mu}_l f_l^0(x) + \sum_{i=1}^n y_i^l (\sum_{l=1}^q \bar{\mu}_l f_l^i(x))\}.
\end{aligned} \tag{4.3}$$

Granting this and letting

$$B = \prod_{l=1}^q B_l = \{y = (y^1,\ldots,y^q)\in\mathbb{R}^n_+\times\ldots\times\mathbb{R}^n_+ : M_0^l + \sum_{i=1}^n M_i^l y_i^l \succeq 0\},$$

we see from (4.2) and the definition of A that the optimal value of the following optimization problem is greater or equal to r:

$$\begin{aligned}
\min_{x\in\mathbb{R}^m}\max_{y\in B} &\quad \sum_{l=1}^q \bar{\mu}_l f_l^0(x) + \sum_{i=1}^n y_i^l (\sum_{l=1}^q \bar{\mu}_l f_l^i(x)) \\
\text{s.t.} &\quad h_j(x) \le 0.
\end{aligned}$$

It then follows from Theorem 3.4 that the optimal value of the following optimization problem is greater or equal to r:

$$\sup_{\mu\in\mathbb{R},y\in\mathbb{R}^n_+,\lambda\in\mathbb{R}^p_+}\{\mu:\sum_{l=1}^q\bar{\mu}_l f_l^0+\sum_{i=1}^n y_i^l(\sum_{l=1}^q\bar{\mu}_l f_l^i)+\sum_{j=1}^p\lambda_j h_j-\mu\in\Sigma_d^2[x]$$
$$M_0^l+\sum_{i=1}^n M_i^l y_i^l\succeq 0\}.$$

Thus, (4.1) holds.

To see the claim (4.3), we first note that for all fixed $y^l\in B_l, 1\le l\le q$,

$$\sum_{l=1}^q\bar{\mu}_l f_l(x,y^l)\le\sum_{l=1}^q\bar{\mu}_l\max_{y^l\in B_l}f_l(x,y^l)\text{ for all }x\in\mathbb{R}^m.$$

So, for all $x\in\mathbb{R}^m$

$$\max_{y^l\in B_l,1\le l\le q}\{\sum_{l=1}^q\bar{\mu}_l f_l(x,y^l)\}\le\sum_{l=1}^q\bar{\mu}_l\max_{y^l\in B_l}f_l(x,y^l).$$

To see the reverse inequality, fix an arbitrary $x\in\mathbb{R}^m$. As B_l are all compact, there exist $\bar{y}^l\in B_l$, $l=1,\ldots,q$, such that $\max_{y^l\in B_l}f_l(x,y^l)=f_l(x,\bar{y}^l)$. This implies that

$$\sum_{l=1}^q\bar{\mu}_l\max_{y^l\in B_l}f_l(x,y^l)=\sum_{l=1}^q\bar{\mu}_l f_l(x,\bar{y}^l)\le\max_{y^l\in B_l,1\le l\le q}\{\sum_{l=1}^q\bar{\mu}_l f_l(x,y^l)\}.$$

Thus $\sum_{l=1}^q\bar{\mu}_l f_l(x,y^l)=\sum_{l=1}^q\bar{\mu}_l\max_{y^l\in B_l}f_l(x,y^l)$ for all $x\in\mathbb{R}^m$, and so, the claim holds by the definition of f_l. □

5 Appendix: SDP Reformulation of (SDP_0)

Let $\mathbb{R}_d[x_1,\ldots,x_m]$ be the space consisting of all real polynomials on $\mathbb{R}^m$ with degree d and let $I_{d,m}:=\begin{pmatrix}d+m\\m\end{pmatrix}$ be the dimension of $\mathbb{R}_d[x_1,\ldots,x_m]$. Write the canonical basis of $\mathbb{R}_d[x_1,\ldots,x_m]$ by

$$x^{(d)}:=(1,x_1,x_2,\ldots,x_m,x_1^2,x_1x_2,\ldots,x_2^2,\ldots,x_m^2,\ldots,x_1^d,\ldots,x_m^d)^T$$

where $x_\alpha^{(d)}$ is the α-th coordinate of $x^{(d)}$, $1\le\alpha\le I_{d,m}$. Then, we can write $f(x)=\sum_{\alpha=1}^{I_{d,m}}f_\alpha x_\alpha^{(d)}$. As an example, consider $f(x)=x_1^4+x_1x_2+x_1$ which is a polynomial on $\mathbb{R}^2$ with degree 4. Then, $I_{4,2}=15$, $x^{(4)}=(1,x_1,x_2,x_1^2,x_1x_2,x_2^2,x_1^3,x_1^2x_2,x_1x_2^2,x_2^3,x_1^4,x_1^3x_2,x_1^2x_2^2,x_1x_2^3,x_2^4)$,

$$f_\alpha=\begin{cases}1, & \text{if}\quad\alpha\in\{2,5,11\},\\0, & \text{if}\quad\alpha\in\{1,\ldots,15\}\backslash\{2,5,11\},\end{cases}$$

and $f(x) = \sum_{\alpha=1}^{15} f_\alpha x_\alpha^{(4)}$.

Note that f is a sum-of-squares polynomial on $\mathbb{R}^m$ with degree d if and only if there exists a symmetric positive semi-definite matrix $W \in S_+^{C(r,m)}$ such that

$$f(x) = (x^{(r)})^T W x^{(r)}, \tag{5.4}$$

where r is the smallest even number such that $2r \geq d$ and

$$x^{(r)} = (1, x_1, x_2, \ldots, x_m, x_1^2, x_1x_2, \ldots, x_2^2, \ldots, x_m^2, \ldots, x_1^r, \ldots, x_m^r)^T.$$

For each $\alpha = 1, \ldots, I_{d,m}$, we denote $i(\alpha) = (i_1(\alpha), \ldots, i_n(\alpha)) \in (\mathbb{N} \cup \{0\})^m$ to be the multi-index such that

$$x_\alpha^{(d)} = x^{i(\alpha)} := x_1^{i_1(\alpha)} \ldots x_m^{i_m(\alpha)}.$$

Then, by comparing the coefficients in (5.4), we have the following well known characterization of a sum-of-squares polynomial

Lemma 5.1. *(cf. [21, Lemma 3.8]) For a polynomial f on $\mathbb{R}^m$ with degree d, f is a sum-of-squares polynomial if and only if the following linear matrix inequality problem has a solution*

$$\begin{cases} W \in S_+^{I_{r,m}} \\ f_\alpha = \displaystyle\sum_{1 \leq \beta, \gamma \leq I_{r,m}, i(\beta)+i(\gamma)=i(\alpha)} W_{\beta,\gamma}, \ \alpha = 1, \ldots, I_{d,m}. \end{cases}$$

Proposition 5.2. (SDP reformulation of (SDP_0)) *The problem (SDP_0) is equivalent to the following semi-definite programming problem*

$$\sup_{\mu \in \mathbb{R}, y \in \mathbb{R}_+^n, \lambda \in \mathbb{R}_+^p, W \in S_+^l} \{\mu : (f_0)_1 + \sum_{i=1}^n y_i (f_i)_1 + \sum_{j=1}^p \lambda_j (h_j)_1 - \mu = W_{1,1}$$

$$(f_0)_\alpha + \sum_{i=1}^n y_i (f_i)_\alpha + \sum_{j=1}^p \lambda_j (h_j)_\alpha = \sum_{\substack{1 \leq \beta, \gamma \leq I_{r,m} \\ i(\beta)+i(\gamma)=i(\alpha)}} W_{\beta,\gamma}, \ \alpha = 2, \ldots, I_{d,m}$$

$$M_0 + \sum_{i=1}^n M_i y_i \succeq 0\},$$

where $(f_i)_\alpha$ (resp. $(h_j)_\alpha$ are the coefficients of f_i (resp. h_j) in the canonical basis, that is,

$$f_i(x) = \sum_{\alpha=1}^{I_{m,d}} (f_i)_\alpha x_\alpha^{(d)} \text{ and } h_j(x) = \sum_{\alpha=1}^{I_{m,d}} (h_j)_\alpha x_\alpha^{(d)}.$$

Proof. For each fixed $(y, \mu, \lambda) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^p$, $f := \sum_{i=1}^n y_i f_i + \sum_{j=1}^p \lambda_j h_j - \mu$ is a polynomial with degree at most d. Thus, the conclusion follows by applying Lemma 5.1 to the polynomial f. □

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