Strong Duality in Robust Semi-Definite Linear Programming under Data Uncertainty*

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Abstract

This paper develops the deterministic approach to duality for semi-definite linear programming problems in the face of data uncertainty. We establish strong duality between the robust counterpart of an uncertain semi-definite linear programming model problem and the optimistic counterpart of its uncertain dual. We prove that strong duality between the deterministic counterparts holds under a characteristic cone condition. We also show that the characteristic cone condition is also necessary for the validity of strong duality for every linear objective function of the original model problem. In addition, we derive that a robust Slater condition alone ensures strong duality for uncertain semi-definite linear programs with affinely parameterized data. Finally, we present strong duality with computationally easily tractable dual programs for two classes of uncertainties: the scenario uncertainty and the spectral norm uncertainty.

Key words. Robust optimization, semi-definite programming under uncertainty, strong duality, linear matrix inequality problems.

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1 Introduction

A semi-definite linear programming model problem (SDP) in the absence of data uncertainty consists of minimizing a linear objective function over a linear matrix inequality constraint [21]. It is often expressed in the form

$$\inf_{x\in\mathbb{R}^n}\{c^Tx\ :\ A_0+\sum_{i=1}^n x_iA_i\succeq 0\},$$

where the "data" consist of the objective vector $c\in\mathbb{R}^n$ and the matrices $A_i\in S^m$, the space of $(m\times m)$ symmetric matrices. For $A\in S^m$, $A\succeq 0$ (resp., $A\succ 0$) means that A is positive semi-definite (resp., definite). Model problems of this form arise in a wide range of engineering applications [7, 21]. In practice, however, these data are subject to uncertainty due to modeling or forecasting errors [4, 5, 8, 18].

The robust optimization approach to examining semi-definite linear programming problems under data uncertainty, developed in this paper, is to treat uncertainty as deterministic via uncertainty sets which are closed and convex. On the other hand, stochastic approaches to optimization under uncertainty start by assuming the uncertainty has a probabilistic description. The best known technique is based on stochastic programming [20].

We consider the semi-definite linear programming model problem with data uncertainty in the constraint

$$(UP)\qquad \inf_{x\in\mathbb{R}^n}\{c^Tx\ :\ A_0+\sum_{i=1}^n x_iA_i\succeq 0\}$$

and its Lagrangian dual program with the same uncertain data

$$(DP)\qquad \max_{F\in S^m}\{-\mathrm{Tr}(A_0F):\mathrm{Tr}(A_iF)=c_i,\ i=1,\ldots,n,\ F\succeq 0\},$$

where, for each $i=0,1,\ldots,n$, $A_i\in S^m$, is uncertain and belongs to the uncertainty set $\mathcal{V}_i$ which is a closed and convex set in S^m, and $\mathrm{Tr}(A_iF)$ is the trace of the matrix A_iF. We study duality between the uncertain dual pair by examining their deterministic counterparts: the *robust counterpart* of (UP)

$$(RP)\qquad \inf_{x\in\mathbb{R}^n}\{c^Tx\ :\ A_0+\sum_{i=1}^n x_iA_i\succeq 0,\ \ \forall A_i\in\mathcal{V}_i,\ i\in I\},$$

where the index set $I:=\{0,1,\ldots,n\}$, and the optimistic counterpart of the uncertain dual program (DP)

$$(ODP)\ \sup_{A_i\in\mathcal{V}_i,i\in I}\ \max_{F\in S^m}\{-\mathrm{Tr}(A_0F):\mathrm{Tr}(A_iF)=c_i,\ i\in I\backslash\{0\},\ F\succeq 0\}.$$

The strong duality in robust semi-definite programming states that

$$\begin{aligned}&\inf_{x\in\mathbb{R}^n}\{c^Tx:A_0+\sum_{i=1}^n x_iA_i\succeq 0,\ \forall\,A_i\in\mathcal{V}_i,i\in I\}\\ =\ &\max_{A_i\in\mathcal{V}_i,i\in I}\ \max_{F\in S^m}\{-\mathrm{Tr}(A_0F):\mathrm{Tr}(A_iF)=c_i,\ i\in I\backslash\{0\},\ F\succeq 0\}\end{aligned}$$

and it describes, in particular, that the value of the robust counterpart (RP) ("primal worst value") is equal to the value of the optimistic counterpart (ODP) ("dual best value") and the optimal value of optimistic counterpart is attained.

Due to the importance of solving semi-definite programs under data uncertainty, a great deal of attention has recently been focussed on identifying and obtaining (uncertainty-immunized) robust solutions of uncertain semi-definite programs [3, 6, 8, 9, 15, 19]. More recently, strong duality has been established for robust convex programming problems with inequality constraints under uncertainty in [1]. For such inequality constrained problems, various characterizations of strong duality have also been given by the authors in [16].

In this paper, we aim to establish strong duality for linear optimization problems with uncertain linear matrix inequality constraints. We first prove strong duality between the deterministic dual pair (RP) and (ODP) under the condition that the *robust characteristic cone*

$$D = \bigcup_{A_i \in \mathcal{V}_i, i \in I} \{(-\mathrm{Tr}(A_1 F), \dots, -\mathrm{Tr}(A_n F), \mathrm{Tr}(A_0 F) + r) : F \succeq 0, r \geq 0\}$$

is *closed and convex*. We further prove that this robust characteristic cone condition is also necessary for the validity of robust strong duality for every linear objective function $c^T x$. Related results for convex programming problems without uncertainty can be found in [12, 13, 14]. In particular, we show that a robust Slater strict feasibility condition [17] together with the convexity of the characteristic cone ensures strong duality. However, we illustrate by an example that the robust Slater condition is, in general, not necessary for strong duality.

Second, in the case of uncertain semi-definite linear programs with affinely parameterized data [4], we show that the robust characteristic cone is a convex cone and that a robust Slater condition alone ensures strong duality.

Third, we derive two classes of uncertain semi-definite linear programs for which strong duality holds under a robust Slater condition and the optimistic counterparts are computationally tractable semi-definite linear programs: the cases of scenario uncertainty [4] and spectral norm uncertainty [1, 4].

The organization of the paper is as follows. Section 2 presents strong duality results in terms of the robust characteristic cone. Section 3 establishes strong duality for uncertain semi-definite linear programs with affinely parameterized data under a robust Slater condition. Section 4 provides classes of uncertain semi-definite linear programs with computationally tractable optimistic counterparts. Section 5 concludes with a discussion on further research on the tractability of optimistic counterparts.

2 Strong Duality for Robust SDPs

In this Section, we present strong duality results in terms of the *robust characteristic cone*, $D \subset \mathbb{R}^{n+1}$, which is defined by

$$D = \bigcup_{A_i \in \mathcal{V}_i, i \in I} \{(-\mathrm{Tr}(A_1 F), \ldots, -\mathrm{Tr}(A_n F), \mathrm{Tr}(A_0 F) + r) : F \succeq 0, r \geq 0\}.$$

We begin by showing that D is indeed a cone in $\mathbb{R}^{n+1}$.

Lemma 2.1. *For each $i \in I$, let $\mathcal{V}_i \subseteq S^m$ be closed and convex. Then, D is a cone.*

Proof. Clearly, $0 \in D$. Let $x \in D \subseteq \mathbb{R}^{n+1}$ and $\mu > 0$. Then, there exist $A_i \in \mathcal{V}_i, M \succeq 0$ and $r \geq 0$ such that $x_i = -\mathrm{Tr}(A_i M)$, $1 \leq i \leq n$ and $x_{n+1} = \mathrm{Tr}(A_0 M) + r$. So, letting $\overline{M} = \mu M \succeq 0$ and $\overline{r} = \mu r \geq 0$, we obtain that $\mu x_i = -\mathrm{Tr}(A_i \overline{M})$, $1 \leq i \leq n$, and $\mu x_{n+1} = \mathrm{Tr}(A_0 \overline{M}) + \overline{r}$. So, $\mu x \in D$, and hence, D is a cone. □

We now show that strong duality holds whenever the robust characteristic cone is closed and convex.

Theorem 2.1. (Strong Duality) *Let $\mathcal{V}_i \subseteq S^m$ be closed and convex, for $i \in I$, and let the robust feasible set $\mathcal{F} := \{x \in \mathbb{R}^n : A_0 + \sum_{i=1}^n x_i A_i \succeq 0, \forall\, A_i \in \mathcal{V}_i\} \neq \emptyset$. Suppose that the robust characteristic cone D is closed and convex. Then,*

$$\begin{aligned} & \inf_{x \in \mathbb{R}^n} \{c^T x : A_0 + \sum_{i=1}^n x_i A_i \succeq 0, \forall\, A_i \in \mathcal{V}_i, i \in I\} \\ = & \max_{A_i \in \mathcal{V}_i, i \in I} \max_{F \in S^m} \{-\mathrm{Tr}(A_0 F) : \mathrm{Tr}(A_i F) = c_i,\ i \in I \backslash \{0\},\ F \succeq 0\}. \end{aligned}$$

Proof. [**Weak Duality**]. We first observe that, for each $x \in \mathbb{R}^n$ with $A_0 + \sum_{i=1}^n x_i A_i \succeq 0, \forall\, A_i \in \mathcal{V}_i,\ i \in I$, and for each $F \succeq 0$, $A_i' \in \mathcal{V}_i$, $i \in I \backslash \{0\}$, with $\mathrm{Tr}(A_i' F) = c_i$ and $A_0' \in \mathcal{V}_0$, we have

$$c^T x = \sum_{i=1}^n x_i \mathrm{Tr}(A_i' F) = \mathrm{Tr}\Big((A_0' + \sum_{i=1}^n x_i A_i')F\Big) - \mathrm{Tr}(A_0' F) \geq -\mathrm{Tr}(A_0' F),$$

and so, the following weak duality holds:

$$\begin{aligned} & \inf_{x \in \mathbb{R}^n} \{c^T x : A_0 + \sum_{i=1}^n x_i A_i \succeq 0, \forall\, A_i \in \mathcal{V}_i\} \\ \geq & \max_{A_i \in \mathcal{V}_i, i \in I} \max_{F \in S^m} \{-\mathrm{Tr}(A_0 F) : \mathrm{Tr}(A_i F) = c_i,\ i \in I \backslash \{0\},\ F \succeq 0\}. \end{aligned}$$

[**Reverse Inequality**] To see the reverse inequality, we may assume that $\inf_{x \in \mathbb{R}^n} \{c^T x : A_0 + \sum_{i=1}^n x_i A_i \succeq 0, \forall\, A_i \in \mathcal{V}_i,\ i \in I\} > -\infty$. As the robust feasible set $\mathcal{F} \neq \emptyset$,

$$\gamma := \inf_{x \in \mathbb{R}^n} \{c^T x : A_0 + \sum_{i=1}^n x_i A_i \succeq 0, \forall\, A_i \in \mathcal{V}_i,\ i \in I\} \in \mathbb{R}.$$

Then, we see that the following implication holds:

$$A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0, \, \forall \, A_i \in \mathcal{V}_i, \, i \in I \;\Rightarrow\; c^T x - \gamma \geq 0. \tag{2.1}$$

[**Homogenization**]. Using (2.1), we now show, by the method of contradiction, that

$$t \geq 0 \text{ and } tA_0 + \sum_{i=1}^{n} x_i A_i \succeq 0, \, \forall \, A_i \in \mathcal{V}_i, \, i \in I \;\Rightarrow\; c^T x - t\gamma \geq 0. \tag{2.2}$$

Suppose, to the contrary, that there exists $(x^0, t^0) \in \mathbb{R}^n \times \mathbb{R}_+$ such that $t^0 A_0 + \sum_{i=1}^n x_i^0 A_i \succeq 0, \forall A_i \in \mathcal{V}_i$ and $c^T x^0 - t^0\gamma < 0$. If $t^0 > 0$, then we have $A_0 + \sum_{i=1}^n \frac{x_i^0}{t_0} A_i \succeq 0, \forall A_i \in \mathcal{V}_i$ and $c^T(\frac{x_i^0}{t_0}) - \gamma < 0$. This contradicts (2.1). If $t_0 = 0$, then $\sum_{i=1}^n x_i^0 A_i \succeq 0, \forall A_i \in \mathcal{V}_i, \, i \in I \backslash \{0\}$ and $c^T x^0 < 0$. Now, fix $\overline{x} \in \mathcal{F}$ such that $\forall A_i \in \mathcal{V}_i$, $A_0 + \sum_{i=1}^n \overline{x}_i A_i \succeq 0$ and consider $x^\alpha = \overline{x} + \alpha x^0$, $\alpha \geq 0$. Then,

$$A_0 + \sum_{i=1}^{n} x_i^\alpha A_i = A_0 + \sum_{i=1}^{n} (\overline{x}_i + \alpha x_i^0) A_i = A_0 + \sum_{i=1}^{n} \overline{x}_i A_i + \alpha \sum_{i=1}^{n} x_i^0 A_i \succeq 0, \; \forall \; A_i \in \mathcal{V}_i, \, i \in I$$

and

$$c^T x^\alpha - \gamma = (c^T \overline{x} - \gamma) + \alpha c^T x^0 \to -\infty \text{ as } \alpha \to +\infty.$$

This also contradicts (2.1), and so, (2.2) holds. Now, fix $A_i \in \mathcal{V}_i$. Define $\tilde{c} \in \mathbb{R}^{n+1}$ by $\tilde{c} = (c, -\gamma)$ and define $\tilde{A} : \mathbb{R}^{n+1} \to S^m \times \mathbb{R}$ by

$$\tilde{A}z = (-tA_0 - \sum_{i=1}^{n} x_i A_i, -t), \;\; \text{for } z = (x, t) \in \mathbb{R}^n \times \mathbb{R}.$$

Then (2.2) can be equivalently rewritten as

$$\forall A_i \in \mathcal{V}_i, \; \tilde{A}z \in -(S^m_+ \times \mathbb{R}_+) \;\Rightarrow\; \tilde{c}^T z \geq 0. \tag{2.3}$$

Now, we show that

$$(-c, -\gamma) \in \overline{\text{co}D}, \tag{2.4}$$

where $\overline{\text{co}D}$ denotes the closure of the convex hull of D.
[**Hyperplane separation**]. Suppose that $(-c, -\gamma) \notin \overline{\text{co}D}$. Then, by the convex separation theorem, there exists $z = (x, t) \in (\mathbb{R}^n \times \mathbb{R}) \backslash \{0\}$ such that

$$\langle u, (x, t) \rangle < \langle (-c, -\gamma), (x, t) \rangle, \; \forall u \in D.$$

As

$$D = \bigcup_{A_i \in \mathcal{V}_i, i \in I} \{(-\text{Tr}(A_1 F), \ldots, -\text{Tr}(A_n F), \text{Tr}(A_0 F) + r) : F \succeq 0, r \geq 0\}$$

is a cone, we obtain that

$$\langle u, (x, t) \rangle \leq 0 < \langle (-c, -\gamma), (x, t) \rangle, \; \forall u \in D.$$

This gives us that

$$\sup_{u\in D}\langle u,(x,t)\rangle \le 0 \text{ and } \langle (c,\gamma),(x,t)\rangle < 0.$$

On the other hand, it can be verified that, for each $(F,r)\in S^m\times\mathbb{R}$, $\langle (F,r),\tilde{A}(x,t)\rangle = \langle (F,r),(-tA_0-\sum_{i=1}^n x_iA_i,-t)\rangle = -\sum_{i=1}^n x_i\mathrm{Tr}(A_iF)+(-\mathrm{Tr}(A_0F)-r)t$. Recall that for a linear mapping $L:\mathbb{R}^p\to\mathbb{R}^q$, $p,q\in\mathbb{N}$, its conjugate mapping $L^*:\mathbb{R}^q\to\mathbb{R}^p$ is defined by

$$\langle L^*(y),x\rangle = \langle y,L(x)\rangle \text{ for all } x\in\mathbb{R}^p \text{ and } y\in\mathbb{R}^q.$$

Then, the conjugate mapping of $\tilde{A}$, denoted as $\tilde{A}^*$, is given by

$$\tilde{A}^*(F,r) = (-\mathrm{Tr}(A_1F),\ldots,-\mathrm{Tr}(A_nF),-\mathrm{Tr}(A_0F)-r).$$

So, letting $w=(x,-t)\in(\mathbb{R}^n\times\mathbb{R})\backslash\{0\}$, we have

$$\sup_{A_i\in\mathcal{V}_i,F\succeq 0,r\ge 0}\langle \tilde{A}w,(F,r)\rangle = \sup_{A_i\in\mathcal{V}_i,F\succeq 0,r\ge 0}\langle \tilde{A}^*(F,r),w\rangle = \sup_{u\in D}\langle u,(x,t)\rangle \le 0$$

and

$$\langle \tilde{c},w\rangle = \langle (c,-\gamma),(x,-t)\rangle = \langle (c,\gamma),(x,t)\rangle < 0.$$

Hence, from the bipolar theorem, $\tilde{A}w\in -(S^m_+\times\mathbb{R}_+)$ and $\langle \tilde{c},w\rangle<0$. This contradicts (2.3), and so (2.4) holds.
[**Dual Attainment**]. By assumption, $\overline{\mathrm{co}D}=D$, and so, $(-c,-\gamma)\in D$. Then, there exist $A_i\in\mathcal{V}_i$, $F\succeq 0$ and $r\ge 0$ such that

$$\mathrm{Tr}(A_iF)=c_i, 1\le i\le n \text{ and } \gamma=-\mathrm{Tr}(A_0F)-r\le -\mathrm{Tr}(A_0F).$$

So,

$$\gamma \le \max_{A_i\in\mathcal{V}_i,i\in I}\max_{F\in S^m}\{-\mathrm{Tr}(A_0F):\mathrm{Tr}(A_iF)=c_i,\ i\in I\backslash\{0\},\ F\succeq 0\},$$

and hence the conclusion follows. □

We now show that the requirement for strong duality, that D is convex and closed, is in fact necessary and sufficient for the validity of strong duality for every linear objective function c^Tx.

Theorem 2.2. (Characterization of Strong Duality) *Let $\mathcal{V}_i\subseteq S^m$ be closed and convex and let the robust feasible set $\mathcal{F}:=\{x\in\mathbb{R}^n: A_0+\sum_{i=1}^n x_iA_i\succeq 0,\ \forall A_i\in\mathcal{V}_i\}\ne\emptyset$. Then, the following statements are equivalent:*

(i) *The robust characteristic cone D is closed and convex .*

(ii) *For each $c\in\mathbb{R}^n$,*

$$\begin{aligned}&\inf_{x\in\mathbb{R}^n}\{c^Tx: A_0+\sum_{i=1}^n x_iA_i\succeq 0,\ \forall A_i\in\mathcal{V}_i, i\in I\}\\ =\ &\max_{A_i\in\mathcal{V}_i,i\in I}\max_{F\in S^m}\{-\mathrm{Tr}(A_0F):\mathrm{Tr}(A_iF)=c_i,\ i\in I\backslash\{0\},\ F\succeq 0\}.\end{aligned}$$

Proof. The conclusion will follow from Theorem 2.1 if we show that [(ii) $\Rightarrow$ (i)]. Assume that (ii) holds. Let $z := (z_1, \dots, z_n, z_{n+1}) \in \mathbb{R}^{n+1}$ such that $z \in \overline{\mathrm{co} D}$. Let $w = (z_1, \dots, z_n)$. Then, we claim that

$$\inf\{-w^T x : A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0, \forall A_i \in \mathcal{V}_i\} \geq -z_{n+1}. \tag{2.5}$$

To see this, as $z \in \overline{\mathrm{co} D}$ and D is a cone of dimension $n+1$, there exist $z^{jk} \in D$ and $\lambda^{jk} \in [0, 1]$, $j = 1, \dots, n+1$ with $\sum_{j=1}^{n+1} \lambda^{jk} = 1$ such that, as $k \to \infty$, $\sum_{j=1}^{n+1} \lambda^{jk} z^{jk} \to z$. Since $z^{jk} = (z_1^{jk}, \dots, z_n^{jk}, z_{n+1}^{jk}) \in D = \bigcup_{A_i \in \mathcal{V}_i, i \in I} \{(-\mathrm{Tr}(A_1 F), \dots, -\mathrm{Tr}(A_n F), \mathrm{Tr}(A_0 F) + r) : F \succeq 0, r \geq 0\}$, there exist $A_i^{jk} \in \mathcal{V}_i$, $i = 1, \dots, n$, $F^{jk} \succeq 0$ and $r^{jk} \geq 0$ such that

$$z_i^{jk} = -\mathrm{Tr}(A_i^{jk} F^{jk}),\ 1 \leq i \leq n, \text{ and } z_{n+1}^{jk} = \mathrm{Tr}(A_0^{jk} F^{jk}) + r^{jk}.$$

Let $w^{jk} = (z_1^{jk}, \dots, z_n^{jk})$. Then, $\sum_{j=1}^{n+1} \lambda^{jk} w^{jk} \to w$ and

$$\begin{aligned} -(\sum_{j=1}^{n+1} \lambda^{jk} w^{jk})^T x &= -\sum_{j=1}^{n+1} \lambda^{jk} \sum_{i=1}^{n} z_i^{jk} x_i \\ &= \sum_{j=1}^{n+1} \lambda^{jk} \sum_{i=1}^{n} \mathrm{Tr}(x_i A_i^{jk} F^{jk}). \end{aligned}$$

So, for each $x = (x_1, \dots, x_n)$ with $A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0$, $\forall A_i \in \mathcal{V}_i$, we have

$$\begin{aligned} -(\sum_{j=1}^{n+1} \lambda^{jk} w^{jk})^T x &= \sum_{j=1}^{n+1} \lambda^{jk} \sum_{i=1}^{n} \mathrm{Tr}(x_i A_i^{jk} F^{jk}) \\ &= \sum_{j=1}^{n+1} \lambda^{jk} \mathrm{Tr}\Big((A_0^{jk} + \sum_{i=1}^{n} x_i A_i^{jk}) F^{jk} \Big) - \sum_{j=1}^{n+1} \lambda^{jk} \mathrm{Tr}(A_0^{jk} F^{jk}) \\ &\geq \sum_{j=1}^{n+1} \lambda^{jk} (-z_{n+1}^{jk} + r^{jk}) \geq -\sum_{j=1}^{n+1} \lambda^{jk} z_{n+1}^{jk}. \end{aligned}$$

Passing to limit and noting that $\sum_{j=1}^{n+1} \lambda^{jk} w^{jk} \to w = (z_1, \dots, z_n)$ and $\sum_{j=1}^{n+1} \lambda^{jk} z_{n+1}^{jk} \to z_{n+1}$, we obtain that for each $x = (x_1, \dots, x_n)$ with $A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0$, $\forall A_i \in \mathcal{V}_i$

$$-w^T x \geq -z_{n+1}.$$

So, (2.5) follows.

Now, applying (ii) with $c = -w$, where $w = (z_1, \dots, z_n)$, we obtain that

$$\max_{A_i \in \mathcal{V}_i, i \in I} \max_{F \in S^m} \{-\mathrm{Tr}(A_0 F) : \mathrm{Tr}(A_i F) = -z_i,\ i \in I \backslash \{0\},\ F \succeq 0\} \geq -z_{n+1}.$$

So, there exist $F \succeq 0$ and $A_i \in \mathcal{V}_i$, $i \in I$ with $\mathrm{Tr}(A_i F) = -z_i$, such that

$$z_{n+1} \geq \mathrm{Tr}(A_0 F).$$

This implies that

$$z = (z_1, \dots, z_n, z_{n+1}) \in D := \bigcup_{A_i \in \mathcal{V}_i, i \in I} \{(-\mathrm{Tr}(A_1 F), \dots, -\mathrm{Tr}(A_n F), \mathrm{Tr}(A_0 F) + r) : F \succeq 0, r \geq 0\}.$$

This shows that $\overline{\mathrm{co} D} \subseteq D$ and so, $\overline{\mathrm{co} D} = D$. So, D is closed and convex. □

Remark 2.1. *Now, we show that the more general case, i.e., linear semi-definite problems where uncertainty occurs in both objective function and in the constraints can also be handled by our approach. Indeed, assume that the data c in the objective function $c^T x$ is also uncertain and c belongs to the uncertainty set $\mathcal{C}$, then the corresponding uncertain semi-definite linear programs is given by*

$$\begin{array}{ll} \inf\limits_{(x_1,\dots,x_n)^T \in \mathbb{R}^n} & c^T x \\ s.t. & A_0 + \sum\limits_{i=1}^{n} x_i A_i \succeq 0 \end{array}$$

where the datum $c \in \mathbb{R}^n$ and $A_i \in S^m$ are uncertain, and the data (c, A_i) belongs to the uncertainty set $\mathcal{C} \times \mathcal{V}_i$, $i = 0, 1, \dots, n$. This can be equivalently rewritten as

$$\begin{array}{ll} \inf\limits_{(x_1,\dots,x_{n+1})^T \in \mathbb{R}^{n+1}} & x_{n+1} \\ s.t. & A_0 + \sum\limits_{i=1}^{n} x_i A_i \succeq 0 \\ & x_{n+1} - c^T x \geq 0, \end{array}$$

where the datum $c \in \mathbb{R}^n$ and $A_i \in S^m$ are uncertain, and the data (c, A_i) belongs to the uncertainty set $\mathcal{C} \times \mathcal{V}_i$, $i = 0, 1, \dots, n$. Denote $c = (c_1, \dots, c_n)^T \in \mathbb{R}^n$ and the $(n \times n)$ identity matrix by I_n. Then, this situation can be modeled as the following uncertain linear semi-definite problem where uncertainty only occurs in the constraint

$$\begin{array}{ll} \inf\limits_{(x_1,\dots,x_{n+1})^T \in \mathbb{R}^{n+1}} & x_{n+1} \\ s.t. & A_0' + \sum\limits_{i=1}^{n+1} x_i A_i' \succeq 0 \end{array}$$

where the matrix data A_i' is uncertain, $i = 0, 1, \dots, n+1$, and the matrix A_i belongs to the uncertainty set $\mathcal{V}_i'$ where

$$\mathcal{V}_0' = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A_0 \end{pmatrix} : A_0 \in \mathcal{V}_0 \right\}, \mathcal{V}_i' = \left\{ \begin{pmatrix} -c_i I_n & 0 \\ 0 & A_i \end{pmatrix} : A_0 \in \mathcal{V}_0 \right\}, \ i = 1, \dots, n$$

and

$$\mathcal{V}_{n+1}' = \left\{ \begin{pmatrix} I_n & 0 \\ 0 & A_i \end{pmatrix} \right\}.$$

In the special case of (UP) without data uncertainty, where each $\mathcal{V}_i$, $i \in I$, is a singleton set $\{A_i\}$, we see that Theorem 2.2 yields the recent characterization of strong duality for semi-definite linear programs, given in [13].

Corollary 2.1. *Let the feasible set* $\mathcal{F} := \{x \in \mathbb{R}^n : A_0 + \sum_{i=1}^n x_i A_i \succeq 0\} \neq \emptyset$. *Then, the following statements are equivalent:*

(i) *The cone* $D_0 := \{(-\mathrm{Tr}(A_1 F), \dots, -\mathrm{Tr}(A_n F), \mathrm{Tr}(A_0 F) + r) : F \succeq 0, r \geq 0\}$ *is closed.*

(ii) *For each* $c \in \mathbb{R}^n$,

$$\inf_{x \in \mathbb{R}^n} \{c^T x : A_0 + \sum_{i=1}^n x_i A_i \succeq 0\} = \max_{F \in S^m} \{-\mathrm{Tr}(A_0 F) : \mathrm{Tr}(A_i F) = c_i,\, i = 1, \dots, n,\, F \succeq 0\}.$$

Proof. For each $i \in I$, let $\mathcal{V}_i$, be a singleton set $\{A_i\}$. Then the cone D collapses to the cone $D_0 := \{(-\mathrm{Tr}(A_1 F), \dots, -\mathrm{Tr}(A_n F), \mathrm{Tr}(A_0 F) + r) : F \succeq 0, r \geq 0\}$. Note that D_0 is always convex. So, the conclusion follows Theorem 2.2. □

We also show that under the *Robust Slater Condition* that $\mathcal{F}_{\succ} := \{x \in \mathbb{R}^n : A_0 + \sum_{i=1}^n x_i A_i \succ 0,\, \forall\, A_i \in \mathcal{V}_i\} \neq \emptyset$, the characteristic cone D is closed and that strong duality is characterized by the convexity of D.

Theorem 2.3. *Let* $\mathcal{V}_i \subseteq S^m$ *be compact and convex. Suppose that* $\mathcal{F}_{\succ} = \{x \in \mathbb{R}^n : A_0 + \sum_{i=1}^n x_i A_i \succ 0,\, \forall\, A_i \in \mathcal{V}_i\} \neq \emptyset$. *Then the following statements hold*

(i) *The characteristic cone* D *is closed.*

(ii) *The cone* D *is convex if and only if, for each* $c \in \mathbb{R}^n$,

$$\begin{aligned} &\inf_{x \in \mathbb{R}^n} \{c^T x : A_0 + \sum_{i=1}^n x_i A_i \succeq 0,\, \forall\, A_i \in \mathcal{V}_i, i \in I\} \\ = &\max_{A_i \in \mathcal{V}_i, i \in I} \max_{F \in S^m} \{-\mathrm{Tr}(A_0 F) : \mathrm{Tr}(A_i F) = c_i,\, i \in I \backslash \{0\},\, F \succeq 0\}. \end{aligned}$$

Proof. (i) Let $\{x^k\} \subset \mathbb{R}^{n+1}$ such that $\{x^k\} \subseteq D$, $x^k = (x_1^k, \dots, x_{n+1}^k)$ and $x^k \to x = (x_1, \dots, x_{n+1})$. Then, there exist $A_i^k \in \mathcal{V}_i$, $i \in I$, $F^k \succeq 0$ and $r^k \geq 0$ such that

$$x_i^k = -\mathrm{Tr}(A_i^k F_k), 1 \leq i \leq n, \tag{2.6}$$

and

$$x_{n+1}^k = \mathrm{Tr}(A_0^k F_k) + r^k. \tag{2.7}$$

As $\mathcal{V}_i$ is compact, by passing to a subsequence, if necessary, we may assume that $A_i^k \to A_i \in \mathcal{V}_i$. We first show that $\{\|F_k\|\}$ is a bounded sequence. Otherwise, without of loss of generality, we assume that $\|F_k\| \to +\infty$. Then, $\frac{F_k}{\|F_k\|} \to F \in S_+^m \backslash \{0\}$, where $S_+^m = \{A \in S^m \mid A \succeq 0\}$. Dividing both sides of (2.6) and (2.7) by $\|F_k\|$ and passing to limit, we obtain that

$$\mathrm{Tr}(A_i F) = 0, 1 \leq i \leq n, \text{ and } -\mathrm{Tr}(A_0 F) = \lim_{k \to \infty} \frac{r^k}{\|F_k\|} \geq 0.$$

Since $\mathcal{F}_{\succ} \neq \emptyset$, there exists $x^0 \in \mathbb{R}^n$ such that $A_0 + \sum_{i=1}^n x_i^0 A_i \succ 0$, $\forall\, A_i \in \mathcal{V}_i$, and

$$\operatorname{Tr}\left((A_0 + \sum_{i=1}^n x_i^0 A_i)F\right) = \operatorname{Tr}(A_0 F) + \sum_{i=1}^n x_i^0 \operatorname{Tr}(A_i F) \leq 0.$$

On the other hand, because $F \in S_+^m \backslash \{0\}$, $\operatorname{Tr}\left((A_0 + \sum_{i=1}^n x_i^0 A_i)F\right) > 0$. This is a contradiction, and so, $\{\|F_k\|\}$ is a bounded sequence. Hence, by (2.7), r^k is also bounded. Hence, by passing to subsequence, if necessary, we can assume that $F_k \to \overline{F}$ and $r^k \to \overline{r}$. Now, passing to limit in (2.6) and (2.7), we obtain that

$$x_i = -\operatorname{Tr}(A_i \overline{F}), 1 \leq i \leq n, \text{ and } x_{n+1} = \operatorname{Tr}(A_0 \overline{F}) + \overline{r}. \tag{2.8}$$

Thus, $x \in D$, and so, D is closed.

(ii). This will follow from (i) and the strong duality Theorem 2.2. □

We now provide two examples. The first example illustrate the situation where the robust Slater condition fails while the cone D is closed and convex and strong duality holds. The second example shows that, in general, the characteristic cone can be nonconvex.

Example 2.1. *Consider the following robust semi-definite linear programming problem:*

$$\begin{array}{lll} (E_1) & \inf_{x \in \mathbb{R}^2} & c_1 x_1 + c_2 x_2 \\ & s.t. & A_0 + x_1 A_1 + x_2 A_2 \succeq 0, \ \forall\ A_i \in \mathcal{V}_i,\ i = 0, 1, 2, \end{array}$$

where

$$\mathcal{V}_0 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}, \mathcal{V}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -a \\ 0 & -a & 0 \end{pmatrix} : a \in [0,1] \right\} \text{ and } \mathcal{V}_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}.$$

Clearly, for any $A_i \in \mathcal{V}_i$, $i = 0, 1, 2$,

$$A_0 + x_1 A_1 + x_2 A_2 = \begin{pmatrix} 1 + x_2 & 0 & 0 \\ 0 & 0 & -a x_1 \\ 0 & -a x_1 & 0 \end{pmatrix}.$$

Note that, any positive definite matrix must have positive diagonal elements. So, for each $A_i \in \mathcal{V}_i$, $i = 0, 1, 2$, $A_0 + x_1 A_1 + x_2 A_2$ is not positive definite, and so, the robust Slater condition fails. However,

$$\begin{aligned} D &= \bigcup_{A_i \in \mathcal{V}_i} \{(-\operatorname{Tr}(A_1 F), -\operatorname{Tr}(A_2 F), \operatorname{Tr}(A_0 F) + r) : F \succeq 0, r \geq 0\} \\ &= \bigcup_{a \in [0,1]} \{(-2a f_5, -f_1, f_1 + r) : F = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_2 & f_4 & f_5 \\ f_3 & f_5 & f_6 \end{pmatrix} \succeq 0, r \geq 0\} \\ &= \mathbb{R} \times \mathbb{R}_- \times \mathbb{R}_+, \end{aligned}$$

which is closed and convex.

On the other hand, the robust feasible set $\mathcal{F}$ *is given by*

$$\begin{aligned}
\mathcal{F} &= \{(x_1,x_2) : A_0 + x_1A_1 + x_2A_2 \succeq 0,\ \forall\ A_i \in \mathcal{V}_i,\ i=0,1,2\}\\
&= \{(x_1,x_2) : \begin{pmatrix} 1+x_2 & 0 & 0 \\ 0 & 0 & -ax_1 \\ 0 & -ax_1 & 0 \end{pmatrix} \succeq 0,\ \forall a \in [0,1]\}\\
&= \{(x_1,x_2) : x_2 \geq -1, x_1 = 0\}.
\end{aligned}$$

So, we have

$$\inf_{x\in\mathbb{R}^2}\{c_1x_1 + c_2x_2 : A_0 + x_1A_1 + x_2A_2 \succeq 0,\ \forall\, A_i \in \mathcal{V}_i,\ i=0,1,2\} = \begin{cases} -\infty, & if \quad c_2 < 0, \\ -c_2, & if \quad c_2 \geq 0. \end{cases}$$

Moreover, the optimistic counterpart *of its uncertain dual is*

$$\begin{aligned}
(ODP_1) \qquad & \max_{A_i\in\mathcal{V}_i, i=0,1,2}\ \max_{F\in S^3}\{-\mathrm{Tr}(A_0F) : \mathrm{Tr}(A_iF) = c_i,\ i=1,2,\ \ F \succeq 0\}\\
&= \max_{a\in[0,1]}\ \max_{F\in S^3}\{-f_1 : -2af_5 = c_1, f_1 = c_2, F = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_2 & f_4 & f_5 \\ f_3 & f_5 & f_6 \end{pmatrix} \succeq 0\}.
\end{aligned}$$

Note that $F \succeq 0$ *implies that* $f_1 \geq 0$*. So, if* $c_2 < 0$*, then the feasible set of* (ODP_1) *is empty and hence the value of it is* $-\infty$*. Moreover, if* $c_2 \geq 0$*, then the feasible set is nonempty and hence* $\max(ODP_1) = -c_2$*. Thus,*

$$\begin{aligned}
& \inf_{x\in\mathbb{R}^2}\{c_1x_1 + c_2x_2 : A_0 + x_1A_1 + x_2A_2 \succeq 0,\ \forall\, A_i \in \mathcal{V}_i,\ i=0,1,2\}\\
=\ & \max_{A_i\in\mathcal{V}_i, i=0,1,2}\ \max_{F\in S^3}\{-\mathrm{Tr}(A_0F) : \mathrm{Tr}(A_iF) = c_i,\ i=1,2,\ F \succeq 0\}\\
=\ & \begin{cases} -\infty, & if \quad c_2 < 0, \\ -c_2, & if \quad c_2 \geq 0. \end{cases}
\end{aligned}$$

Example 2.2. *Consider the following uncertain semi-definite linear program:*

$$\begin{aligned}
\min \quad & x_1 + x_2 \\
s.t. \quad & A_0 + x_1A_1 + x_2A_2 \succeq 0
\end{aligned}$$

where the matrix data A_0, A_1, A_2 *are uncertain and* $A_0 \in \mathcal{V}_0 = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right\}$, $A_1 \in \mathcal{V}_1 = \left\{\begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} : a \in [0,1]\right\}$ *and* $A_2 \in \mathcal{V}_2 = \left\{\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} : b \in [-1,0]\right\}$*. In this case, the characteristic cone* D *becomes*

$$\begin{aligned}
D &= \bigcup_{A_i\in\mathcal{V}_i}\{(-\mathrm{Tr}(A_1F), -\mathrm{Tr}(A_2F), \mathrm{Tr}(A_0F) + r) : F \succeq 0, r \geq 0\}\\
&= \bigcup_{a\in[0,1],\, b\in[-1,0]}\{(-2af_2, -2bf_2, r) : F = \begin{pmatrix} f_1 & f_2 \\ f_2 & f_3 \end{pmatrix} \succeq 0, r \geq 0\}\\
&= \bigcup_{a\in[0,1],\, b\in[-1,0]}\{(-2af_2, -2bf_2, r) : f_1 \geq 0, f_3 \geq 0, f_2^2 \leq f_1f_3, r \geq 0\}
\end{aligned}$$

It can be easily verified that $(0,2,0) \in D$ *(by letting* $a=0, b=-1, f_2=1,\ f_1=f_3=2$ *and* $r=0$*) and* $(2,0,0) \in D$ *(by letting* $a=1, b=0, f_2=-1,\ f_1=f_3=2$ *and* $r=0$*). However, we now show that their mid point* $(1,1,0) \notin D$*. Otherwise, we have* $2af_2=-1$ *and* $2bf_2=-1$ *for some* $a \in [0,1]$ *and* $b \in [-1,0]$*. This implies that* $a \neq 0,\ b \neq 0$ *and* $f_2 \neq 0$*. So, we see that* $a=b$*. This enforces that* $a=b=0$ *which is impossible. Thus, the characteristic cone is not convex.*

We now verify that the strong duality between the robust counterpart and the optimistic counterpart of the dual fails. To see this, note that the robust counterpart is

$$\begin{array}{ll} \min & x_1+x_2 \\ s.t. & A_0+x_1A_1+x_2A_2 \succeq 0 \ \forall \ A_i \in \mathcal{V}_i, i=0,1,2 \end{array}$$

This can equivalent written as

$$\begin{array}{ll} \min & x_1+x_2 \\ s.t. & \left(\begin{array}{cc} 0 & ax_1+bx_2 \\ ax_1+bx_2 & 0 \end{array}\right) \succeq 0 \ \forall \ a \in [0,1],\, b \in [-1,0]. \end{array}$$

Note that $(0,0)$ *is the only feasible point for the robust counterpart, so the optimal value is* 0*. On the other hand, the optimistic counterpart of the dual is*

$$\begin{array}{rl} & \max\limits_{A_i \in \mathcal{V}_i, i=0,1,2} \max\limits_{F \in S^m} \{-\mathrm{Tr}(A_0F) : \mathrm{Tr}(A_iF)=1,\ i=1,2,\ F=\left(\begin{array}{cc} f_1 & f_2 \\ f_2 & f_3 \end{array}\right) \succeq 0\} \\ = & \max\limits_{a\in[0,1], b\in[-1,0]} \max\limits_{F \in S^m} \{0 : 2af_1=1 \ \text{and}\ 2bf_2=1\}. \end{array}$$

It can be seen that there is no $a \in [0,1]$ *and* $b \in [-1,0]$ *such that* $2af_2=1$ *and* $2bf_2=1$*. So, the feasible set of the optimistic dual is empty and hence, strong duality fails between the robust counterpart and the optimistic dual.*

From the above example, we see that the characteristic cone can be nonconvex in general. A sufficient condition ensuring the convexity of the characteristic cone can be found in the appendix. Finally, it is worth noting that, like the robust counterparts (RP), the optimistic counterparts (ODP) can, in general, be difficult to solve for certain classes of uncertainties [4]. This prompts us to examine classes of robust SDP where optimistic counterparts are easily solvable in the next section.

3 Duality with Tractable Optimistic Dual

In this Section, we focus on classes of uncertain semi-definite linear programs for which optimistic counterparts can be computationally tractable and strong duality can be easily verified [*].

[*] It should be noted that the computational tractability of primal robust SDPs may also be checked independently for these classes. We do not intend to identify cases of robust SDPs where duality and optimistic counterparts are easily solvable but the robust counterparts are not necessarily tractable.

Let us first consider the uncertain semi-definite programming problem with **spectral norm uncertainty**:

$$(P_s) \qquad \inf_{x\in\mathbb{R}^n}\{c^Tx : A_0 + \sum_{i=1}^{n} x_iA_i \succeq 0\}$$

where $A_i \in \tilde{\mathcal{V}}_i := \{\overline{A}_i + \rho_i\Delta_i : \Delta_i \in S^m, \|\Delta_i\|_{\text{spec}} \leq 1\}$, $\overline{A}_i \in S^m$, $\rho_i \geq 0$ and $\|\Delta\|_{\text{spec}}$ denotes the square root of the largest eigenvalue of the matrix $\Delta^T\Delta$. In this case, the uncertainty set $\tilde{\mathcal{V}}_i$ is just a closed ball with center $\overline{A}_i$ and radius ρ_i in the matrix space S^m associated with the spectral norm.

The robust counterpart of this uncertain SDP problem is

$$(RP_s) \qquad \inf_{x\in\mathbb{R}^n}\{c^Tx : A_0 + \sum_{i=1}^{n} x_iA_i \succeq 0, \forall\, A_i \in \{\overline{A}_i + \rho_i\Delta_i : \Delta_i \in S^m, \|\Delta_i\|_{\text{spec}} \leq 1\}\}.$$

The optimistic counterpart of the dual of (RP_s) is

$$(ODP_s) \quad \max_{\|\Delta_i\|_{\text{spec}}\leq 1,\, i\in I}\ \max_{F\in S^m}\{-\text{Tr}\big((\overline{A}_0 + \rho_0\Delta_0)F\big) : \text{Tr}\big((\overline{A}_i + \rho_i\Delta_i)F\big) = c_i,\ i \in I\backslash\{0\},\ F \succeq 0\}.$$

In this case also we show that the robust characteristic cone D is a convex cone.

Proposition 3.1. *Let $\tilde{\mathcal{V}}_i = \{\overline{A}_i + \rho_i\Delta_i : \Delta_i \in S^m, \|\Delta\|_{\text{spec}} \leq 1\}$, $\overline{A}_i \in S^m$. Then the robust characteristic cone*

$$D := \bigcup_{A_i\in\tilde{\mathcal{V}}_i, i\in I}\{(-\text{Tr}(A_1F), \ldots, -\text{Tr}(A_nF), \text{Tr}(A_0F) + r) : F \succeq 0, r \geq 0\},$$

is convex.

Proof. First, we note that

$$\begin{aligned} D &= \bigcup_{A_i\in\tilde{\mathcal{V}}_i}\{(-\text{Tr}(A_1F), \ldots, -\text{Tr}(A_nF), \text{Tr}(A_0F) + r) : F \succeq 0, r \geq 0\} \\ &= \bigcup_{\|\Delta_i\|_{\text{spec}}\leq 1}\{(-\text{Tr}\big((\overline{A}_1 + \rho_1\Delta_1)F\big), \ldots, -\text{Tr}\big((\overline{A}_n + \rho_n\Delta_n)F\big), \text{Tr}\big((\overline{A}_0 + \rho_0\Delta_0)F\big) + r) \\ &\qquad\qquad : F \succeq 0, r \geq 0\}. \end{aligned}$$

To see the convexity of D, let $x = (x_1, \ldots, x_{n+1}) \in D$ and $y = (y_1, \ldots, y_{n+1}) \in D$. As D is a cone (by Lemma 2.1), it suffices to show that $x + y \in D$. Since $x, y \in D$, there exist $H_i, C_i \in \{\Delta : \|\Delta\|_{\text{spec}} \leq 1\}$, $i \in I$, $M, N \succeq 0$ and $r, s \geq 0$ such that

$$-x_i = \text{Tr}\big((\overline{A}_i + \rho_iH_i)M\big),\ 1 \leq i \leq n,\ x_{n+1} = \text{Tr}\big((\overline{A}_0 + \rho_0H_0)M\big) + r$$

and

$$-y_i = \text{Tr}\big((\overline{A}_i + \rho_iC_i)N\big),\ 1 \leq i \leq n,\ y_{n+1} = \text{Tr}\big((\overline{A}_0 + \rho_0C_0)N\big) + s.$$

New, we show that, for each fixed $F \succeq 0$,

$$[\text{Tr}\big((\overline{A}_i - \rho_iI_m)F\big), \text{Tr}\big((\overline{A}_i + \rho_iI_m)F\big)] = \bigcup_{\|\Delta_i\|_{\text{spec}}\leq 1}\{\text{Tr}\big((\overline{A}_i + \rho_i\Delta_i)F\big)\},\ i \in I. \qquad (3.9)$$

To see this, as $I_m, -I_m \in \{\Delta_i : \|\Delta_i\|_{\text{spec}} \le 1\}$, where I_m is the $(m \times m)$ identity matrix, and for each fixed $F \succeq 0$, $\Delta_i \mapsto \text{Tr}\big((\overline{A}_i + \rho_i \Delta_i)F\big)$ is a continuous function, it follows by the intermediate value theorem that, for each fixed $F \succeq 0$,

$$[\text{Tr}\big((\overline{A}_i - \rho_i I_m)F\big), \text{Tr}\big((\overline{A}_i + \rho_i I_m)F\big)] \subseteq \bigcup_{\|\Delta_i\|_{\text{spec}} \le 1} \{\text{Tr}\big((\overline{A}_i + \rho_i \Delta_i)F\big)\},\ i \in I.$$

To see the reverse inclusion, note that for each fixed $F \succeq 0$ (see [2, Proposition 2.1]),

$$\sup_{\|\Delta\|_{\text{spec}} \le 1} \text{Tr}(\Delta F) = \text{Tr}(F) \text{ and } \inf_{\|\Delta\|_{\text{spec}} \le 1} \text{Tr}(\Delta F) = -\text{Tr}(F).$$

So, the reverse inclusion follows as

$$\sup_{\|\Delta_i\|_{\text{spec}} \le 1} \{\text{Tr}\big((\overline{A}_i+\rho_i\Delta_i)F\big)\} = \text{Tr}\big((\overline{A}_i+\rho_i I_m)F\big) \text{ and } \inf_{\|\Delta_i\|_{\text{spec}} \le 1} \{\text{Tr}\big((\overline{A}_i+\rho_i\Delta_i)F\big)\} = \text{Tr}\big((\overline{A}_i-\rho_i I_m)F\big).$$

Therefore, (3.9) gives us that

$$-x_i \in [\text{Tr}\big((\overline{A}_i-\rho_i I_m)M\big), \text{Tr}\big((\overline{A}_i+\rho_i I_m)M\big)],\ 1 \le i \le n,\ x_{n+1}-r \in [\text{Tr}\big((\overline{A}_0-\rho_0 I_m)M\big), \text{Tr}\big((\overline{A}_0+\rho_0 I_m)M\big)]$$

and

$$-y_i \in [\text{Tr}\big((\overline{A}_i-\rho_i I_m)N\big), \text{Tr}\big((\overline{A}_i+\rho_i I_m)N\big)],\ 1 \le i \le n,\ y_{n+1}-s \in [\text{Tr}\big((\overline{A}_0-\rho_0 I_m)N\big), \text{Tr}\big((\overline{A}_0+\rho_0 I_m)N\big)].$$

Then,

$$-x_i - y_i \in [\text{Tr}\big((\overline{A}_i - \rho_i I_m)(M+N)\big), \text{Tr}\big((\overline{A}_i + \rho_i I_m)(M+N)\big)],\ 1 \le i \le n,$$

and

$$x_{n+1} + y_{n+1} - (r+s) \in [\text{Tr}\big((\overline{A}_0 - \rho_0 I_m)(M+N)\big), (\overline{A}_0 + \rho_0 I_m)(M+N)\big)].$$

Now, applying (3.9) we can find Δ_i with $\|\Delta\|_{\text{spec}} \le 1$ such that

$$-x_i-y_i = \text{Tr}\big((\overline{A}_i+\rho_i\Delta_i)(M+N)\big),\ 1 \le i \le n,\ x_{n+1}+y_{n+1}-(r+s) = \text{Tr}\big((\overline{A}_0+\rho_0\Delta_0)(M+N)\big).$$

Hence, $x + y \in D$. □

We finally establish that strong duality holds for (P_s) under the robust Slater condition and that the optimistic counterpart (ODP_s) is a single semi-definite linear program.

Theorem 3.1. *For (P_s), suppose that there exists $x^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$ such that*

$$A_0 + \sum_{i=1}^{n} x_i^0 A_i \succ 0\ \forall A_i \in \{\overline{A}_i + \rho_i \Delta_i : \|\Delta_i\|_{\text{spec}} \le 1\},\ i \in I \backslash \{0\}.$$

Then,

$$\begin{aligned} &\inf_{x \in \mathbb{R}^n} \{c^T x : A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0, \forall A_i \in \{\overline{A}_i + \rho_i \Delta_i : \|\Delta_i\|_{\text{spec}} \le 1\}\} \\ = &\max_{F \in S^m} \{-\text{Tr}\big((\overline{A}_0 - \rho_0 I)F\big) : \text{Tr}\big((\overline{A}_i - \rho_i I_m)F\big) \le c_i,\ i \in I\backslash\{0\} \\ &\qquad \text{Tr}\big((\overline{A}_i + \rho_i I_m)F\big) \ge c_i,\ i \in I\backslash\{0\},\ F \succeq 0\}. \end{aligned}$$

Proof. From Proposition 3.1 and Theorem 2.3, we see that the cone D is closed and convex. Then, by Theorem 2.1,

$$\begin{aligned}&\inf_{x\in\mathbb{R}^n}\{c^Tx : A_0+\sum_{i=1}^n x_iA_i\succeq 0,\,\forall\, A_i\in\{\overline{A}_i+\rho_i\Delta_i : \|\Delta_i\|_{\text{spec}}\le 1\}\}\\ =&\max_{\|\Delta_i\|_{\text{spec}}\le 1,\, i\in I}\max_{F\in S^m}\{-\text{Tr}\big((\overline{A}_0+\rho_0\Delta_0)F\big) : \text{Tr}\big((\overline{A}_i+\rho_i\Delta_i)F\big)=c_i,\, i\in I\backslash\{0\},\, F\succeq 0\}.\end{aligned}$$

Since, (3.9) holds for each $F\succeq 0$,

$$[\text{Tr}\big((\overline{A}_i-\rho_iI_m)F\big),\text{Tr}\big((\overline{A}_i+\rho_iI_m)F\big)]=\bigcup_{\|\Delta_i\|_{\text{spec}}\le 1}\{\text{Tr}\big((\overline{A}_i+\rho_i\Delta_i)F\big)\},\quad i\in I. \tag{3.10}$$

So, the optimistic counterpart can be simplified as

$$\begin{aligned}&\max_{\|\Delta_i\|_{\text{spec}}\le 1,\, i\in I}\max_{F\in S^m}\{-\text{Tr}\big((\overline{A}_0+\rho_0\Delta_0)F\big) : \text{Tr}\big((\overline{A}_i+\rho_i\Delta_i)F\big)=c_i,\, i\in I\backslash\{0\},\, F\succeq 0\}\\ =&\max_{\|\Delta_i\|_{\text{spec}}\le 1,\, i\in I\backslash\{0\}}\max_{F\in S^m}\{-\text{Tr}\big((\overline{A}_0-\rho_0I_m)F\big) : \text{Tr}\big((\overline{A}_i+\rho_i\Delta_i)F\big)=c_i,\, i\in I\backslash\{0\},\, F\succeq 0\}\\ =&\max_{F\in S^m}\{-\text{Tr}\big((\overline{A}_0-\rho_0I_m)F\big) : \text{Tr}\big((\overline{A}_i-\rho_iI_m)F\big)\le c_i,\, i\in I\backslash\{0\},\\ &\qquad\qquad \text{Tr}\big((\overline{A}_i+\rho_iI_m)F\big)\ge c_i,\, i\in I\backslash\{0\},\, F\succeq 0\}.\end{aligned}$$

Indeed, for each $F\succeq 0$ with $\text{Tr}\big((\overline{A}_i+\rho_i\Delta_i)F\big)=c_i$, we have

$$\text{Tr}\big((\overline{A}_i-\rho_iI_m)F\big)\le c_i \text{ and } \text{Tr}\big((\overline{A}_i+\rho_iI_m)F\big)\ge c_i,\, i\in I\backslash\{0\}.$$

Then,

$$\begin{aligned}&\max_{\substack{F\in S^m\\ \|\Delta_i\|_{\text{spec}}\le 1,\, i\in I\backslash\{0\}}}\{-\text{Tr}\big((\overline{A}_0-\rho_0I_m)F\big) : \text{Tr}\big((\overline{A}_i+\rho_i\Delta_i)F\big)=c_i,\, i\in I\backslash\{0\},\, F\succeq 0\}\\ \le&\max_{F\in S^m}\{-\text{Tr}\big((\overline{A}_0-\rho_0I_m)F\big) : \text{Tr}\big((\overline{A}_i-\rho_iI_m)F\big)\le c_i,\, i\in I\backslash\{0\},\\ &\qquad\qquad \text{Tr}\big((\overline{A}_i+\rho_iI_m)F\big)\ge c_i,\, i\in I\backslash\{0\},\, F\succeq 0\}.\end{aligned}$$

On the other hand, for any $F\succeq 0$ with

$$\text{Tr}\big((\overline{A}_i-\rho_iI_m)F\big)\le c_i \text{ and } \text{Tr}\big((\overline{A}_i+\rho_iI_m)F\big)\ge c_i,\, i\in I\backslash\{0\},$$

the equation (3.10) implies that there exist Δ_i with $\|\Delta_i\|_{\text{spec}}\le 1$ such that

$$\text{Tr}\big((\overline{A}_i+\rho_i\Delta_i)F\big)=c_i.$$

This gives us that

$$\begin{aligned}&\max_{\substack{F\in S^m\\ \|\Delta_i\|_{\text{spec}}\le 1,\, i\in I\backslash\{0\}}}\{-\text{Tr}\big((\overline{A}_0-\rho_0I_m)F\big) : \text{Tr}\big((\overline{A}_i+\rho_i\Delta_i)F\big)=c_i,\, i\in I\backslash\{0\},\, F\succeq 0\}\\ \ge&\max_{F\in S^m}\{-\text{Tr}\big((\overline{A}_0-\rho_0I_m)F\big) : \text{Tr}\big((\overline{A}_i-\rho_iI_m)F\big)\le c_i,\, i\in I\backslash\{0\}\\ &\qquad\qquad \text{Tr}\big((\overline{A}_i+\rho_iI_m)F\big)\ge c_i,\, i\in I\backslash\{0\},\, F\succeq 0\}.\end{aligned}$$

Hence, the equality holds. □

Remark 3.1. (Comparison with the known result) *It has been established (for example see [4, Chapter 8]) that the robust counterpart of the uncertain semi-definite linear programming problem with spectral norm uncertainty can be reformulated as a new semi-definite linear programming, and so, is also computational tractable. Our approach provides an alternative way for identifying this tractable class of robust semi-definite linear program via a dual approach.*

Remark 3.2. (Other simple tractable cases) *Consider the semi-definite linear programming model problem with affinely parameterized diagonal matrix data:*

$$(P_a) \qquad \inf_{x\in\mathbb{R}^n}\{c^Tx : A_0 + \sum_{i=1}^n x_iA_i \succeq 0\}$$

where for each $i = 0, 1, \ldots, n$,

$$A_i \in \overline{\mathcal{V}}_i = \{\mathrm{diag}\big(a_{i1}^{(0)}+\sum_{j=1}^k u_{i1}^{(j)}a_{i1}^{(j)}, \ldots, a_{im}^{(0)}+\sum_{j=1}^k u_{im}^{(j)}a_{im}^{(j)}\big) : u_{il} = (u_{il}^1, \ldots, u_{il}^k) \in \overline{U}_{il}, l = 1, \ldots, m\},$$

$a_{il}^{(j)} \in \mathbb{R}$ *for each* $l = 1, \ldots, m$ *and* $j = 1, \ldots, k$ *and* $\overline{U}_{il}$, $l = 1, \ldots, m$, *are convex compact sets in* $\mathbb{R}^k$*. Here,* $\mathrm{diag}(d_1, \ldots, d_n)$ *denotes a diagonal matrix with diagonal elements* $d_1, \ldots, d_n$*. The characteristic cone* D *collapses to*

$$\begin{aligned} D &= \bigcup_{u_{il}\in\overline{U}_{il}} \{\big(-\sum_{l=1}^m \lambda_l(a_{1l}^{(0)} + \sum_{j=1}^k u_{1l}^{(j)}a_{1l}^{(j)}), \ldots, -\sum_{l=1}^m \lambda_l(a_{nl}^{(0)} + \sum_{j=1}^k u_{nl}^{(j)}a_{nl}^{(j)}), \\ &\qquad r + \sum_{l=1}^m \lambda_l(a_{0l}^{(0)} + \sum_{j=1}^k u_{0l}^{(j)}a_{0l}^{(j)})\big) : \lambda_l \geq 0, r \geq 0\} \end{aligned}$$

Let $v_l = (u_{0l}^{(1)}, \ldots, u_{0l}^{(k)}, u_{1l}^{(1)}, \ldots, u_{1l}^{(k)}, u_{nl}^{(1)}, \ldots, u_{nl}^{(k)})$ *and*

$$g_l(x, v_l) = \big(a_{0l}^{(0)} + \sum_{j=1}^k u_{0l}^{(j)}a_{0l}^{(j)}\big) + \sum_{i=1}^n x_i\big(a_{il}^{(0)} + \sum_{j=1}^k u_{il}^{(j)}a_{il}^{(j)}\big).$$

Then, direct verification gives us that $D = \bigcup_{v_l\in\prod_{i=0}^n \mathcal{U}_{il}, \lambda_l\geq 0} \mathrm{epi}\big(\sum_{l=1}^m \lambda_l g_l(\cdot, v_l)\big)^*$ *where* $\mathrm{epi}f$ *is the epigraph of a function* f *given by* $\mathrm{epi}f = \{(x, s) : s \geq f(x)\}$ *and* f^* *denotes the usual convex conjugate of a convex function* f*. Clearly,* $g_l(\cdot, v_l)$ *is a linear function and* $g_l(x, \cdot)$ *is also linear. Thus [16, Proposition 2.3] implies that* $\bigcup_{v_l\in\prod_{i=0}^n \mathcal{U}_{il}, \lambda_l\geq 0} \mathrm{epi}\big(\sum_{l=1}^m \lambda_l g_l(\cdot, v_l)\big)^*$ *is convex, and so, the characteristic cone* D *is convex.*

(1) *If for each* $i = 0, 1, \ldots, n$, $l = 1, \ldots, m$, $\overline{\mathcal{U}}_{il} = \mathrm{co}\{z_{il}^{(1)}, \ldots, z_{il}^{(p)}\}$ *which is the convex hull of a given finitely generated scenarios* $z_{il}^{(t)} = (z_{il}^{(1t)}, \ldots, z_{il}^{(kt)}) \in \mathbb{R}^k$, $t = 1, \ldots, p$, *and* $p \in \mathbb{N}$, *then the characteristic cone* D *is closed and so, strong duality between the robust counterpart and the optimistic dual always holds (see [16, Theorem 4.1]). Moreover, the*

optimistic dual can be simplified as

$$\begin{aligned}
&\max_{A_i\in\overline{\mathcal{V}}_i,i\in I}\max_{F\in S^m}\{-\mathrm{Tr}(A_0F):\mathrm{Tr}(A_iF)=c_i,\ i\in I\backslash\{0\},\ F\succeq 0\}\\
=&\max_{u_{il}\in\overline{\mathcal{U}}_{il},\lambda_l\geq 0}\{-\sum_{l=1}^m\lambda_l(a_{0l}^{(0)}+\sum_{j=1}^k u_{0l}^{(j)}a_{0l}^{(j)}):\sum_{l=1}^m\lambda_l(a_{il}^{(0)}+\sum_{j=1}^k u_{il}^{(j)}a_{il}^{(j)})=c_i,i=1,\dots,n\}\\
=&\max_{\lambda_l\geq 0}\{-\sum_{l=1}^m\lambda_l(a_{0l}^{(0)}+\sum_{j=1}^k\sum_{t=1}^p\mu_{0l}^{(jt)}z_{0l}^{(jt)}a_{0l}^{(j)}):\\
&\sum_{l=1}^m\lambda_l(a_{il}^{(0)}+\sum_{j=1}^k\sum_{t=1}^p\mu_{il}^{(jt)}z_{il}^{(jt)}a_{il}^{(j)})=c_i,\sum_{t=1}^p\mu_{il}^{(jt)}=1,\mu_{il}^{(jt)}\geq 0\}
\end{aligned}$$

Letting $\lambda_{il}^{(jt)}=\lambda_l\mu_{0l}^{(jt)}\geq 0$*, then* $\sum_{t=1}^p\lambda_{il}^{(jt)}=\lambda_l$*,* $i=1,\dots,n$*,* $j=1,\dots,k$*,* $l=1,\dots,m$*, and the optimistic dual can be further simplified as*

$$\max_{\lambda_l\geq 0,\lambda_{il}^{(jt)}\geq 0}\{-\sum_{l=1}^m\lambda_l a_{0l}^{(0)}-\sum_{j=1}^k\sum_{t=1}^p\lambda_{0l}^{(jt)}z_{0l}^{(jt)}a_{0l}^{(j)}:\sum_{l=1}^m\lambda_l a_{il}^{(0)}+\sum_{j=1}^k\sum_{t=1}^p\lambda_{il}^{(jt)}z_{il}^{(jt)}a_{il}^{(j)}=c_i,\ \sum_{t=1}^p\lambda_{il}^{(jt)}=\lambda_l\}$$

which is a linear programming problem.

(2) *If for each* $i=0,1,\dots,n$*,* $l=1,\dots,m$*,* $\overline{U}_{il}=\{z\in\mathbb{R}^k:\|z\|\leq 1\}$ *and the robust Slater condition holds, strong duality between the robust counterpart and the optimistic dual always holds by Theorem 2.3. Moreover, the optimistic dual can be simplified as*

$$\begin{aligned}
&\max_{A_i\in\overline{\mathcal{V}}_i,i\in I}\max_{F\in S^m}\{-\mathrm{Tr}(A_0F):\mathrm{Tr}(A_iF)=c_i,\ i\in I\backslash\{0\},\ F\succeq 0\}\\
=&\max_{u_{il}\in\overline{\mathcal{U}}_{il},\lambda_l\geq 0}\{-\sum_{l=1}^m\lambda_l(a_{0l}^{(0)}+\sum_{j=1}^k u_{0l}^{(j)}a_{0l}^{(j)}):\sum_{l=1}^m\lambda_l(a_{il}^{(0)}+\sum_{j=1}^k u_{il}^{(j)}a_{il}^{(j)})=c_i,i=1,\dots,n\}\\
=&\max_{\|(u_{il}^{(1)},\dots,u_{il}^{(k)})\|\leq 1,\lambda_l\geq 0}\{-\sum_{l=1}^m\lambda_l(a_{0l}^{(0)}+\sum_{j=1}^k u_{0l}^{(j)}a_{0l}^{(j)}):\sum_{l=1}^m\lambda_l(a_{il}^{(0)}+\sum_{j=1}^k u_{il}^{(j)}a_{il}^{(j)})=c_i,i=1,\dots,n\}
\end{aligned}$$

Letting $\lambda_{il}^{(j)}=\lambda_l u_{il}^{(j)}$*, then* $\lambda_l\geq\|(\lambda_{il}^{(1)},\dots,\lambda_{il}^{(k)})\|$ *and the optimistic dual can be further simplified as*

$$\max_{\lambda_l\geq\|(\lambda_{il}^{(1)},\dots,\lambda_{il}^{(k)})\|}\{-\sum_{l=1}^m\lambda_l a_{0l}^{(0)}-\sum_{j=1}^k\lambda_{0l}^{(j)}a_{0l}^{(j)}:\sum_{l=1}^m\lambda_l a_{il}^{(0)}+\sum_{j=1}^k\lambda_{il}^{(j)}a_{il}^{(j)}=c_i,i=1,\dots,n\}$$

which is a second-order cone linear programming problem.

4 Conclusion and Further Research

In this paper we considered semi-definite linear programs subject to data uncertainty. We have presented a deterministic approach for studying such uncertain programs by restricting the uncertain data to closed and convex uncertainty sets. We have established necessary as well as sufficient conditions for strong duality for uncertain SDPs by proving strong duality between the associated deterministic counterparts, namely, the robust counterpart and the optimistic counterpart.

Our study presented in this paper raises interesting questions for further research. For instance, the tractability in robust semi-definite programming often relates to whether the robust counterpart of an uncertain semi-definite program is computationally tractable or not. On the other hand, from the dual point of view, the investigation of the tractability of optimistic counterpart of the dual of an uncertain semi-definite program would be of great interest, particularly, when strong duality is available. We have provided two classes of uncertain semi-definite linear programs for which strong duality holds under a robust Slater condition and the optimistic counterparts are computationally tractable semi-definite linear programs. It would be useful to examine this "dual tractability" for other specially structured problems with broad classes of uncertainty sets.

Especially, the duality approach, presented in this paper, could be applied to uncertain second-order cone programming problems. This would lead to identifying other broad classes of computationally tractable optimistic counterparts and will be presented elsewhere.

Appendix: Convexity of the Characteristic Cone

In this appendix, we provide a sufficient condition for the convexity of the characteristic cone. Consider the uncertain semi-definite programming model problem

$$(P_a) \quad \inf_{x \in \mathbb{R}^n} \{c^T x : A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0\}$$

where for each $i \in I$, the matrix A_i is uncertain and the uncertain data matrix is affinely parameterized, i.e., the matrix data A_i belongs to the uncertainty set

$$\overline{\mathcal{V}}_i = \{A_i^{(0)} + \sum_{j=1}^{k} u_i^j A_i^{(j)} : (u_i^1, \dots, u_i^k) \in U_i\},$$

where U_i, $i = 0, 1, \dots, n$, is a compact convex set in $\mathbb{R}^k$ and $A_i^{(j)} \in S^m$ $i = 0, 1, \dots, n$. The robust counterpart of this uncertain SDP problem is

$$(RP_a) \quad \inf_{x \in \mathbb{R}^n} \{c^T x : A_0 + \sum_{i=1}^{n} x_i A_i \succeq 0, \forall A_i \in \{A_i^{(0)} + \sum_{j=1}^{k} u_i^j A_i^{(j)} : (u_i^1, \dots, u_i^k) \in U_i\}\}.$$

The optimistic counterpart of its dual problem is

$$(ODP_a) \quad \max_{\substack{(u_i^1, \dots, u_i^k) \in U_i \\ i \in I}} \max_{F \in S^m} \quad \{-\mathrm{Tr}\big((A_0^{(0)} + \sum_{j=1}^{k} u_0^j A_0^{(j)})F\big) :$$
$$\mathrm{Tr}\big((A_i^{(0)} + \sum_{j=1}^{k} u_i^j A_i^{(j)})F\big) = c_i,\, i \in I \backslash \{0\},\, F \succeq 0\}.$$

We show in this case that the robust characteristic cone D is a convex cone.

Proposition 4.1. (Convexity of the Characteristic Cone: Sufficient Condition) *For each $i \in I$, $A_i \in \overline{\mathcal{V}}_i = \{A_i^{(0)} + \sum_{j=1}^k u_i^j A_i^{(j)} : (u_i^1, \ldots, u_i^k) \in U_i\}$, where U_i, $i = 0, 1, \ldots, n$, is a compact convex set in $\mathbb{R}^k$, $A_0^{(j)} \in S^m$ and $A_i^{(j)} \succeq 0$ $i = 1, \ldots, n$, $j = 1, \ldots, k$. Then the robust characteristic cone*

$$D := \bigcup_{A_i \in \overline{\mathcal{V}}_i, i \in I} \{(-\mathrm{Tr}(A_1 F), \ldots, -\mathrm{Tr}(A_n F), \mathrm{Tr}(A_0 F) + r) : F \succeq 0, r \geq 0\},$$

is convex.

Proof. Let $x, y \in D$ and let $\lambda \in [0, 1]$. As $x, y \in D$, there exist $H_i, C_i \in \overline{\mathcal{V}}_i$, $i \in I$, $M, N \succeq 0$ and $r, s \geq 0$ such that

$$x_i = -\mathrm{Tr}(H_i M),\ i \in I \backslash \{0\},\ x_{n+1} = \mathrm{Tr}(H_0 M) + r,$$

and

$$y_i = -\mathrm{Tr}(C_i N),\ i \in I \backslash \{0\},\ x_{n+1} = \mathrm{Tr}(C_0 N) + s.$$

As $H_i, C_i \in \overline{\mathcal{V}}_i$, there exist $(u_i^1, \ldots, u_i^k) \in U_i$ and $(v_i^1, \ldots, v_i^k) \in U_i$ such that, for each $i \in I$,

$$H_i = A_i^{(0)} + \sum_{j=1}^k u_i^j A_i^{(j)} \text{ and } C_i = A_i^{(0)} + \sum_{j=1}^k v_i^j A_i^{(j)}.$$

Now, fix an $i \in I \backslash \{0\}$. We see that

$$\begin{aligned}
\lambda x_i + (1-\lambda) y_i &= -\mathrm{Tr}(\lambda H_i M + (1-\lambda) C_i N) \\
&= -\mathrm{Tr}\Big(\lambda (A_i^{(0)} + \sum_{j=1}^k u_i^j A_i^{(j)}) M + (1-\lambda)(A_i^{(0)} + \sum_{j=1}^k v_i^j A_i^{(j)}) N\Big) \\
&= -\mathrm{Tr}\Big(A_i^{(0)}(\lambda M + (1-\lambda) N)\Big) \\
&\quad - \sum_{j=1}^k \Big(u_i^j \mathrm{Tr}\big(\lambda A_i^{(j)} M\big) + v_i^j \mathrm{Tr}\big((1-\lambda) A_i^{(j)} N\big)\Big)
\end{aligned}$$

Define, for each $i \in I$ and $j = 1, \ldots, k$,

$$w_i^j = \begin{cases} \frac{u_i^j \mathrm{Tr}\left(\lambda A_i^{(j)} M\right) + v_i^j \mathrm{Tr}\left((1-\lambda) A_i^{(j)} N\right)}{\mathrm{Tr}\left((\lambda M + (1-\lambda) N) A_i^{(j)}\right)} & \text{if } \mathrm{Tr}\big((\lambda M + (1-\lambda) N) A_i^{(j)}\big) \neq 0 \\ u_i^j & \text{if } \mathrm{Tr}\big((\lambda M + (1-\lambda) N) A_i^{(j)}\big) = 0. \end{cases}$$

Clearly, $(w_i^1, \ldots, w_i^k) \in U_i$, $i \in I$. Moreover, for each $i \in I \backslash \{0\}$ and $j = 1, \ldots, k$

$$w_i^j \mathrm{Tr}\big((\lambda M + (1-\lambda) N) A_i^{(j)}\big) = u_i^j \mathrm{Tr}\big(\lambda A_i^{(j)} M\big) + v_i^j \mathrm{Tr}\big((1-\lambda) A_i^{(j)} N\big).$$

(This equality follows trivially if $\mathrm{Tr}\big((\lambda M + (1-\lambda)N)A_i^{(j)}\big) \neq 0$. On the other hand, if $\mathrm{Tr}\big((\lambda M+(1-\lambda)N)A_i^{(j)}\big) = 0$, then $\mathrm{Tr}\big(\lambda A_i^{(j)} M\big) = \mathrm{Tr}\big((1-\lambda)A_i^{(j)} N\big) = 0$ as M, N and A_i^j, $i = 1,\ldots,k$, $j = 1 = 1,\ldots,m$ are all positive semi-definite matrices. Thus, the equality also follows in this case.) So,

$$\begin{aligned}
\lambda x_i + (1-\lambda)y_i &= -\mathrm{Tr}\Big(A_i^{(0)}(\lambda M + (1-\lambda)N)\Big) - \sum_{j=1}^{k} w_i^j \mathrm{Tr}\big((\lambda M + (1-\lambda N))A_i^{(j)}\big) \\
&= -\mathrm{Tr}\Big((A_i^{(0)} + \sum_{j=1}^{k} w_i^j A_i^{(j)})(\lambda M + (1-\lambda)N)\Big).
\end{aligned}$$

Similarly, we can also show that

$$\lambda x_{n+1} + (1-\lambda)y_{n+1} = \mathrm{Tr}\Big((A_0^{(0)} + \sum_{j=1}^{k} w_0^j A_0^{(j)})(\lambda M + (1-\lambda)N)\Big) + (\lambda r + (1-\lambda)s).$$

Note that $(w_i^1,\ldots,w_i^k) \in U_i$, $i \in I$, $\lambda M + (1-\lambda)N \succeq 0$ and $\lambda r + (1-\lambda)s \geq 0$. We see that

$$\lambda x + (1-\lambda)y \in D.$$

So, D is convex. □

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