Robust Farkas' Lemma for Uncertain Linear Systems with Applications*

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Abstract

We present a robust Farkas lemma, which provides a new generalization of the celebrated Farkas lemma for linear inequality systems to *uncertain* conical linear systems. We also characterize the robust Farkas lemma in terms of a generalized characteristic cone. As an application of the robust Farkas lemma we establish a characterization of uncertainty-immunized solutions of conical linear programming problems under uncertainty.

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1 Introduction

The celebrated Farkas lemma [9] states that for a given vector c and a given system of vectors $a_1, \ldots, a_m$, the linear function $c^T x$ is non-negative over the linear inequality system $a_1^T x \geq 0, \ldots, a_m^T x \geq 0$ means that the vector c can be expressed as a non-negative linear combination of the vectors $a_1, \ldots, a_m$. Symbolically, it describes that

$$[a_1^T x \geq 0, \ldots, a_m^T x \geq 0 \Rightarrow c^T x \geq 0] \Leftrightarrow [\exists \lambda_i \geq 0,\ c = \sum_{i=1}^{m} \lambda_i a_i].$$

The Farkas lemma [9] underpins the elegant and powerful duality theory of linear programming. It has undergone numerous generalizations [7, 8, 14, 19, 20] over a century. Its wide-ranging applications to optimization extend from linear programming and smooth optimization to modern areas of optimization such as semi-definite programming [23] and non-smooth optimization [18]. But, these generalizations [14, 17, 24] and applications

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have so far been limited mainly to systems without data uncertainty, despite the reality of data uncertainty in many real-world systems due to modelling or prediction errors [4].

The purpose of this paper is to present a new form of the Farkas lemma, called the robust Farkas lemma, for general uncertain conical linear systems and to derive a characterization of uncertainty-immunized solutions of conical linear programming problems under uncertainty.

In recent years, a great deal of attention has been focussed on optimization under uncertainty due to the importance of finding solutions to optimization problems that are affected by data uncertainty. Robust optimization methodology [2, 4, 10, 22] is a powerful approach for examining and solving optimization problems under data uncertainty. It treats data uncertainty as deterministic via bounded uncertainty sets and does not limit the data values to point estimates. For a detailed analysis of the robust optimization framework, see Ben-Tal and Nemirovski [2, 4], and El Ghaoui [10]. In this framework, how to characterize solutions that are immunized against data uncertainty has become a fundamental and critical question. These uncertainty-immunized solutions are called robust solutions of uncertain problems [1, 4]. Characterizations of such robust solutions for classes of uncertain linear programming problems have recently been given in Jeyakumar and Li [16].

More importantly, in many applications [4], optimization problems involving uncertain conical systems arise in the form of semi-definite programming problems [23] as well as semi-infinite programming problems [11, 12, 22]. As in optimization without data uncertainty [25], characterizing robust solutions of conical optimization problems under uncertainty requires a generalization of the Farkas lemma for uncertain conical linear systems.

In Section 2, we establish robust forms of the Farkas lemma for a general uncertain conical linear system. We then present a characterization of a robust Farkas lemma in terms of the closure of a convex cone, called robust characteristic cone, in Section 3. Finally, we provide an application of the robust Farkas lemma by giving a characterization of uncertainty-immunized solutions of conical linear programming problems under uncertainty.

2 Robust Farkas' Lemma

We begin this section by fixing notation and definitions that will be used throughout the paper. Let X, Y be locally convex Hausdorff spaces. The dual space of X (resp. Y) is denoted by X^* (resp. Y^*) which consists of all bounded linear functionals on Y. It is known that the space X^* endowed with the weak* topology is again a locally convex Hausdorff space. Let $L(X;Y)$ denote the set of all the continuous linear mappings from X to Y. Let $A \in L(X;Y)$ be a continuous linear mapping. Then the conjugate mapping of A is a continuous linear mapping from Y^* to X^* defined by

$$\langle A^*(y^*), x\rangle = \langle y^*, Ax\rangle \text{ for all } y^* \in Y^*, x \in X.$$

where $\langle \cdot, \cdot \rangle$ denotes the corresponding linear action between the dual pairs. For a set C in Y, the interior (resp. closure, convex hull, conical hull) of C is denoted by $\text{int}C$ (resp.

$\overline{C}$, co$C$, cone$C$). If $C \subseteq X^*$, then the weak* closure of C is denoted by $\overline{C}^{w^*}$. Let K be a closed cone in Y. Then the (positive) polar cone of K is defined by $K^+ := \{k^* \in Y^* : \langle k^*, k\rangle \geq 0\}$.

Theorem 2.1. *(Robust Farkas' Lemma). Let $\mathcal{U} \subseteq L(X;Y)$ and let $\mathcal{V} \subseteq X^*$ be closed convex uncertainty sets. Let S be a closed convex cone in Y. Then, the following statements are equivalent:*

(i) $\forall A \in \mathcal{U},\ Ax \in -S \ \Rightarrow\ \langle c, x\rangle \geq 0\ \forall c \in \mathcal{V}$.

(ii) $-\mathcal{V} \subseteq \overline{\mathrm{co} \bigcup_{A \in \mathcal{U}} A^*(S^+)}^{w^*}$.

Proof. [(i) $\Rightarrow$ (ii)] Suppose that (ii) does not hold. Then there exists $c \in \mathcal{V}$ such that

$$-c \notin \overline{\mathrm{co} \bigcup_{A \in \mathcal{U}} A^*(S^+)}^{w^*}.$$

Then, by the convex separation theorem, there exists $v \in X\backslash\{0\}$ such that

$$\langle u, v\rangle < -\langle c, v\rangle,\ \forall u \in \bigcup_{A \in \mathcal{U}} A^*(S^+).$$

As $F := \bigcup_{A \in \mathcal{U}} A^*(S^+)$ is a cone, we obtain that

$$\langle u, v\rangle \leq 0 < -\langle c, v\rangle,\ \forall u \in F.$$

This gives us that

$$\sup_{u \in F}\langle u, v\rangle \leq 0 \text{ and } \langle c, v\rangle < 0.$$

Now,

$$\sup_{A \in \mathcal{U}, a \in S^+}\langle Av, a\rangle = \sup_{A \in \mathcal{U}, a \in S^+}\langle A^*a, v\rangle = \sup_{u \in F}\langle u, v\rangle \leq 0 \quad \text{and} \quad \langle c, v\rangle < 0.$$

Hence, from the bipolar theorem, $\forall A \in \mathcal{U},\ Av \in -S$ and $\langle c, v\rangle < 0$. This contradicts (i).

[(ii) $\Rightarrow$ (i)] Let $x \in \mathbb{R}^n$ be such that $Ax \in -S\ \forall A \in \mathcal{U}$ and let $c \in \mathcal{V}$. Suppose that (ii) holds. Then

$$-c \in \overline{\mathrm{co} \bigcup_{A \in \mathcal{U}} A^*(S^+)}^{w^*}.$$

Then, there exists a net $\{c^s\} \subseteq \mathrm{co}\bigcup_{A \in \mathcal{U}} A^*(S^+)$ such that $c^s \to -c$. For each s, there exists an index set I_s with $|I_s| < +\infty$, $\lambda_i^s \in [0, 1]$, $i \in I_s$ with $\sum_{i \in I_s} \lambda_i^s = 1$, $A_i^s \in \mathcal{U}$, $x_i^{s*} \in S^+$, $i \in I_s$, such that

$$c^s = \sum_{i \in I_s} \lambda_i^s A_i^{s*}(x_i^{s*}).$$

So,

$$\langle c^s, x\rangle = \sum_{i \in I_s} \lambda_i^s \langle A_i^{s*}(x_i^{s*}), x\rangle = \sum_{i \in I_s} \lambda_i^s \langle x_i^{s*}, A_i^s(x)\rangle.$$

Since $A_i^s(x) \in -S$ and $x_i^{s*} \in S^+$, it follows that $\langle c^s, x\rangle \leq 0$. As $\langle c^s, x\rangle \to -\langle c, x\rangle$, we have $\langle c, x\rangle \geq 0$. Hence, (i) holds. □

Theorem 2.2. *(Robust Farkas Lemma with Interval Uncertainty) Let* $\underline{a}_i, \overline{a}_i \in \mathbb{R}^n$ *with* $\underline{a}_i \leq \overline{a}_i$, *for* $i = 0, 1, \ldots, m$. *Then, the following statements are equivalent:*

(i) $\forall a_1 \in [\underline{a}_1, \overline{a}_1],\ a_1^T x \leq 0, \ldots, \forall a_m \in [\underline{a}_m, \overline{a}_m],\ a_m^T x \leq 0 \Rightarrow a_0^T x \geq 0,\ \forall a_0 \in [\underline{a}_0, \overline{a}_0]$

(ii) $(\forall a_0 \in [\underline{a}_0, \overline{a}_0])\ (\exists \lambda \in \mathbb{R}^m_+)\ a_0 + \sum_{i=1}^m \lambda_i \overline{a}_i \geq 0$ *and* $a_0 + \sum_{i=1}^m \lambda_i \underline{a}_i \leq 0$.

Proof. Let $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, $S = \mathbb{R}^m_+$, $A = (a_1, a_2, \ldots, a_m)$, $\mathcal{U} = \prod_{i=1}^m [\underline{a}_i, \overline{a}_i]$ and $\mathcal{V} = [\underline{a}_0, \overline{a}_0]$. Then,

$$\begin{aligned}\bigcup_{A \in \mathcal{U}} A^*(S^+) &= \bigcup_{\lambda \in \mathbb{R}^m_+} \sum_{i=1}^m \lambda_i [\underline{a}_i, \overline{a}_i] \\ &= \bigcup_{\lambda \in \mathbb{R}^m_+} \sum_{i=1}^m \lambda_i ([\underline{a}_i^1, \overline{a}_i^1] \times [\underline{a}_i^2, \overline{a}_i^2] \times \ldots [\underline{a}_i^n, \overline{a}_i^n]).\end{aligned}$$

is a finitely generated convex cone and so is a closed convex cone. The conclusion now follows from Theorem 2.1. Note in this case that (ii) of Theorem 2.1 is equivalent to the condition that for each $a_0 \in [\underline{a}_0, \overline{a}_0]$ there exist $\lambda_i \geq 0, i = 1, \ldots, m$ such that $-a_0 = \sum_{i=1}^m \lambda_i d_i$, for some $d_i \in [\underline{a}_i, \overline{a}_i]$. This is, in turn, equivalent to (ii), as $a_0 + \sum_{i=1}^m \lambda_i \overline{a}_i \geq a_0 + \sum_{i=1}^m \lambda_i d_i = 0$ and $a_0 + \sum_{i=1}^m \lambda_i \underline{a}_i \leq a_0 + \sum_{i=1}^m \lambda_i d_i = 0$. □

A nonhomogeneous version of the robust Farkas lemma that is often used in applications to optimization is given in the following theorem.

Theorem 2.3. *(Robust non-homogeneous Farkas lemma) Let* $\mathcal{U} \subseteq L(X; Y) \times Y$ *be a closed convex uncertainty set and let* $\tilde{\mathcal{U}} = \{\tilde{A} \in L(X \times \mathbb{R}; Y \times \mathbb{R}) : \tilde{A}(x, t) = (Ax + tb, -t)$ *and* $(A, b) \in \mathcal{U}\}$. *Let* $c \in X^*$, $\alpha \in \mathbb{R}$ *and let* S *be a closed convex cone in* Y. *Then, the following statements are equivalent:*

(i) $\forall (A, b) \in \mathcal{U},\ Ax + b \in -S \ \Rightarrow\ \langle c, x \rangle + \alpha \geq 0$;

(ii) $-(c, \alpha) \in \overline{\mathrm{co} \bigcup_{\tilde{A} \in \tilde{\mathcal{U}}} \tilde{A}^*(S^+ \times \mathbb{R}_+)}^{w^*}$.

Proof. First of all, we show that (i) is equivalent to

(i') $\forall (A, b) \in \mathcal{U},\ Ax + tb \in -S,\ t \geq 0 \ \Rightarrow\ \langle c, x \rangle + t\alpha \geq 0$.

Clearly, (i$'$) $\Rightarrow$ (i). To see (i) $\Rightarrow$ (i$'$), we proceed by the method of contradiction and suppose that there exists $(x_0, t_0) \in X \times \mathbb{R}_+$ satisfies $Ax_0 + t_0 b \in -S$, $\forall (A, b) \in \mathcal{U}$ and $\langle c, x_0 \rangle + t_0 \alpha_0 < 0$. If $t_0 > 0$, then we have $A(\frac{x_0}{t_0}) + b \in -S$, $\forall (A, b) \in \mathcal{U}$ and $\langle c, \frac{x_0}{t_0} \rangle + \alpha_0 < 0$. This contradicts (i). If $t_0 = 0$, then $Ax_0 \in -S$, $\forall (A, b) \in \mathcal{U}$ and $\langle c, x_0 \rangle < 0$. Now, fix $\overline{x} \in X$ such that $\forall (A, b) \in \mathcal{U}$, $A\overline{x} + tb \in -S$ and consider $x_t = \overline{x} + tx_0$, $t \in \mathbb{R}$. Then,

$$Ax_t + tb = A(\overline{x} + tx_0) + tb = A\overline{x} + tb + tAx_0 \in -S,\ \forall\ (A, b) \in \mathcal{U}$$

and, as $t \to +\infty$,

$$\langle c, x_t \rangle + \alpha = \langle c, \overline{x} + tx_0 \rangle + \alpha \to -\infty.$$

This also contradicts (i), and so, (i) is equivalent to (i'). Now, define $\tilde{A} \in L(X \times \mathbb{R}; Y \times \mathbb{R})$ and $\tilde{c} \in X^* \times \mathbb{R}$ respectively by

$$\tilde{A}z = (Ax + tb, -t) \text{ and } \langle \tilde{c}, z \rangle = \langle c, x \rangle + t\alpha, \quad \forall z = (x,t) \in X \times \mathbb{R}.$$

Let $\mathcal{V} := \{(c,\alpha)\}$. Then (i') can be equivalently rewritten as

$$\forall \tilde{A} \in \tilde{\mathcal{U}},\ \tilde{A}z \in -(S \times \mathbb{R}_+) \ \Rightarrow\ \langle \tilde{c}, z \rangle \geq 0\ \forall \tilde{c} \in \mathcal{V}.$$

Therefore, by Theorem 2.1, we obtain that (i) is equivalent to (ii).

□

Note that our method of proof of Theorem 2.3 shows that it can be improved by allowing (c,α) also to be uncertain. However, we have given a form of nonhomogeneous robust Farkas' lemma with fixed data (c,α) because the subsequent study of robust solutions of uncertain optimization problems only requires such a form later in the paper.

3 Robust Characteristic Cones

In this Section, we establish a characterization of the robust Farkas lemma in terms of the closure of a characteristic convex cone, called robust characteristic cone. We also provide conditions guaranteeing the closure of the characteristic cone.

For $\mathcal{U} \subseteq L(X;Y) \times Y$ and a closed convex cone S of Y, we define the *robust characteristic cone* $C(\mathcal{U}, S)$ by

$$C(\mathcal{U}, S) = \mathrm{co} \bigcup_{\tilde{A} \in \tilde{\mathcal{U}}} \tilde{A}^*(S^+ \times \mathbb{R}_+),$$

where $\tilde{\mathcal{U}} = \{\tilde{A} \in L(\mathbb{R}^n \times \mathbb{R}; Y \times \mathbb{R}) : \tilde{A}(x,t) = (Ax + tb, -t) \text{ and } (A,b) \in \mathcal{U}\}$. We now derive a characterization of the robust non-homogeneous Farkas' lemma in terms of $C(\mathcal{U}, S)$.

Theorem 3.1. *Let $\mathcal{U} \subseteq L(X;Y) \times Y$ be a closed convex uncertainty set. Let S be a closed convex cone in Y. Then, the following statements are equivalent:*

(i) *For each $(c,\alpha) \in X^* \times \mathbb{R}$,*

$$[\forall (A,b) \in \mathcal{U},\ Ax + b \in -S \ \Rightarrow\ \langle c, x \rangle + \alpha \geq 0] \Leftrightarrow [-(c,\alpha) \in C(\mathcal{U}, S)].$$

(ii) *the convex cone $C(\mathcal{U}, S)$ is weak* closed.*

Proof. Fix an arbitrary $(c,\alpha) \in X^* \times \mathbb{R}$. From Theorem 2.3, we see that

$$[\forall (A,b) \in \mathcal{U},\ Ax + b \in -S \ \Rightarrow\ \langle c, x \rangle + \alpha \geq 0] \Leftrightarrow -(c,\alpha) \in \overline{\mathrm{co} \bigcup_{\tilde{A} \in \tilde{\mathcal{U}}} \tilde{A}^*(S^+ \times \mathbb{R}_+)}^{w^*}].$$

Therefore, (i) is equivalent to the condition that the convex cone

$$C(\mathcal{U}, S) = \text{co} \bigcup_{\tilde{A} \in \tilde{\mathcal{U}}} \tilde{A}^*(S^+ \times \mathbb{R}_+)$$

is weak* closed. □

A special case of Theorem 3.1, where $\mathcal{U}$ is a singleton, can be found in [15]. In this case, the cone $\bigcup_{\tilde{A} \in \tilde{\mathcal{U}}} \tilde{A}^*(S^+ \times \mathbb{R}_+)$ is convex (and so the convex hull in the definition of the characteristic cone or in Theorem 2.3 (ii) is superfluous). On the other hand, the following example shows that $\bigcup_{\tilde{A} \in \tilde{\mathcal{U}}} \tilde{A}^*(S^+ \times \mathbb{R}_+)$ is, in general, not necessarily convex.

Example 3.1. *Let $S = \mathbb{R}^2_+$ and let*

$$\mathcal{U} = \{(A, b) \in \mathbb{R}^{2\times 2} \times \mathbb{R} : A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix},\ a \in [-1, 1], b = 0\}.$$

Then, we have $\tilde{U} = \{\tilde{A} : \tilde{A}(x_1, x_2, t) = (ax_1, ax_2, -t), a \in [-1, 1]\}$ and so,

$$\begin{aligned} \bigcup_{\tilde{A} \in \tilde{\mathcal{U}}} \tilde{A}^*(S^+ \times \mathbb{R}_+) &= \bigcup_{a \in [-1,1]} \{(ax_1^*, ax_2^*, -r) : x_1^*, x_2^*, r \in \mathbb{R}_+\} \\ &= \Big((\mathbb{R}^2_+) \cup (-\mathbb{R}^2_+)\Big) \times (-\mathbb{R}), \end{aligned}$$

which is a nonconvex cone.

We show that the robust characteristic cone is closed whenever a Slater type condition holds.

Proposition 3.1. *Suppose that $X = \mathbb{R}^n$ and the closed convex uncertainty set $\mathcal{U}$ is compact and that there exists $x_0 \in \mathbb{R}^n$ such that*

$$Ax_0 + b \in -\text{int} S \ \forall\ (A, b) \in \mathcal{U}.$$

Then $C(\mathcal{U}, S)$ is closed.

Proof. First of all, we note that $\tilde{\mathcal{U}}$ is also a convex compact set. Let $c \in \overline{\text{co} \bigcup_{\tilde{A} \in \tilde{\mathcal{U}}} \tilde{A}^*(S^+ \times \mathbb{R}_+)}$. Then, there exists a sequence $\{c^k\} \subseteq \text{co} \bigcup_{\tilde{A} \in \tilde{\mathcal{U}}} \tilde{A}^*(S^+ \times \mathbb{R}_+)$ such that $c^k \to c$. Since $c^k \in \text{co} \bigcup_{\tilde{A} \in \tilde{\mathcal{U}}} \tilde{A}^*(S^+ \times \mathbb{R}_+)$, by Caratheodory's theorem, there exist $\lambda_i^k \in [0, 1]$ with $\sum_{i=1}^{n+1} \lambda_i^k = 1$, $\tilde{A}_i^{k*} \in \tilde{\mathcal{U}}$, $s_i^{k*} \in S^+ \times \mathbb{R}_+$, $i = 1, \ldots, n + 1$, such that

$$c^k = \sum_{i=1}^{n+1} \lambda_i^k \tilde{A}_i^{k*}(s_i^{k*}) \tag{3.1}$$

and $c^k \to c$. Now we divide the proof into two cases: Case 1, $\sum_{i=1}^{n+1} \|s_i^{k*}\|$ is bounded; Case 2, $\sum_{i=1}^{n+1} \|s_i^{k*}\|$ is unbounded.

Suppose that Case 1 holds. Then by passing to subsequence if necessary, we may assume that $\lambda_i^k \to \lambda_i$ with $\lambda_i \in [0, 1]$ and $\sum_{i=1}^{n+1} \lambda_i = 1$, $\tilde{A}_i^{k*} \to \tilde{A}_i^* \in \tilde{\mathcal{U}}$ and $s_i^{k*} \to s_i^* \in S^+$. Thus $c = \sum_{i=1}^{n+1} \lambda_i A_i^*(s_i^*) \in \text{co} \bigcup_{A \in \tilde{\mathcal{U}}} A^*(S^+)$, and so, the conclusion follows.

Suppose that Case 2 holds. Without loss of generality, we may assume that $\sum_{i=1}^{n+1} \|s_i^{k*}\| \to \infty$ as $k \to \infty$. By passing to subsequence, we may further assume that $\lambda_i^k \to \lambda_i$ with $\lambda_i \in [0,1]$ and $\sum_{i=1}^{n+1} \lambda_i = 1$, $\tilde{A}_i^{k*} \to \tilde{A}_i^* \in \tilde{\mathcal{U}}$ and

$$\frac{s_i^{k*}}{\sum_{i=1}^{n+1} \|s_i^{k*}\|} \to a_i^* \in S^+ \times \mathbb{R}_+, \; i = 1, \ldots, n+1,$$

with $(a_1^*, \ldots, a_{n+1}^*) \neq 0$. Dividing by $\sum_{i=1}^{n+1} \|s_i^{k*}\|$ on both sides of (3.1) and passing to limit, we have

$$0 = \sum_{i=1}^{n+1} \lambda_i \tilde{A}_i^*(a_i^*).$$

Now, by the assumption, there exists x_0 such that $Ax_0 + b \in -\mathrm{int}S \; \forall (A,b) \in \mathcal{U}$. Letting $u_0 = (x_0, 1)$ we see that

$$0 = \langle \sum_{i=1}^{n+1} \lambda_i \tilde{A}_i^*(a_i^{k*}), u_0 \rangle = \sum_{i=1}^{n+1} \lambda_i \langle a_i^*, \tilde{A}_i u_0 \rangle. \tag{3.2}$$

Note that, $\langle x, x^* \rangle < 0$ for all $x \in -\mathrm{int}K$ and for all $x^* \in K^+ \backslash \{0\}$ where K is a closed convex cone. This together with the fact that, for each $i = 1, \ldots, n+1$,

$$\tilde{A}_i u_0 = (A_i x_0 + b_i, -1) \in -\mathrm{int}(S \times \mathbb{R}_+).$$

gives us

$$\sum_{i=1}^{n+1} \lambda_i \langle a_i^*, \tilde{A}_i(u_0) \rangle < 0.$$

This contradicts (3.2), and so the conclusion follows. □

4 Characterizing Robust Solutions

In this section, as an application of the robust Farkas lemma, we derive a characterization of robust solutions of conical linear programming problems under uncertainty.

Consider

$$(CP) \quad \min_{x \in X} \quad \langle c, x \rangle \qquad \text{s.t.} \quad Ax + b \in -S,$$

where the data (A, b) is uncertain and they locate in the closed convex uncertainty set $\mathcal{U} \subseteq L(X;Y) \times Y$. The robust counterpart of (CP) can be formulated as follows:

$$(RCP) \quad \min_{x \in X} \quad \langle c, x \rangle \qquad \text{s.t.} \quad Ax + b \in -S, \; \forall (A,b) \in \mathcal{U}.$$

We may assume that the objective function is unaffected by data uncertainty. If the data c in the objective function is also uncertain and if c is in the uncertainty set $\mathcal{V}$, then the robust counterpart can be formulated as

$$\begin{array}{cl} \min\limits_{(x,t)\in X\times\mathbb{R}} & t \\ \text{s.t.} & \langle c, x\rangle \leq t,\ \forall c \in \mathcal{V} \\ & Ax + b \in -S,\ \forall (A,b) \in \mathcal{U}. \end{array}$$

This can then be expressed in the form of (RCP) as follows.

$$\begin{array}{cl} \min\limits_{(x,t)\in X\times\mathbb{R}} & t \\ \text{s.t.} & \overline{A}(x,t) + \overline{b} \in -(S\times\mathbb{R}_+), \end{array}$$

where $\overline{b} = (b,0) \in Y \times \mathbb{R}$, $\overline{A} \in L(X\times\mathbb{R}; Y\times\mathbb{R})$ is given by

$$\overline{A}(x,t) = (Ax, \langle c, x\rangle - t)$$

and $(\overline{A},\overline{b}) \in \overline{\mathcal{U}}$. The uncertainty set $\overline{\mathcal{U}} \subseteq L(X; Y\times\mathbb{R}) \times (Y\times\mathbb{R})$ is defined by

$$\overline{\mathcal{U}} = \{(\overline{A},\overline{b}) \in L(X\times\mathbb{R}; Y\times\mathbb{R}) \times (Y\times\mathbb{R}) : (A,b) \in \mathcal{U}, c \in \mathcal{V}\}.$$

Note that a robust feasible point $\overline{x}$ of (CP) is a feasible point of its robust counterpart. Moreover, $\overline{x}$ is said to be a robust solution of (CP) whenever $\overline{x}$ is a solution of its robust counterpart (RCP).

Theorem 4.1. *Let $\overline{x}$ be a robust feasible point of (CP). Suppose that the robust characteristic cone $C(\mathcal{U},S)$ is weak* closed. Then, $\overline{x}$ is a robust solution of (CP) if and only if there exist a finite index set I, $\lambda_i \in [0,1]$, $y_i^* \in S^+$ and $(A_i,b_i) \in \mathcal{U}$, $i \in I$ with $\sum_{i\in I}\lambda_i = 1$ such that*

$$c + \sum_{i\in I}\lambda_i A_i^*(y_i^*) = 0 \ \textit{and} \ \sum_{i\in I}\lambda_i\left(\langle y_i^*, A_i\overline{x} + b_i\rangle\right) = 0. \tag{4.3}$$

Proof. The point $\overline{x}$ is a robust solution of (CP) if and only if

$$\forall (A,b) \in \mathcal{U}\ Ax + b \in -S \quad\Rightarrow\quad \langle c, x\rangle \geq \langle c, \overline{x}\rangle.$$

Applying Theorem 2.3 and noting that $C(\mathcal{U},S)$ is weak* closed, we obtain that $\overline{x}$ is a solution of (RCP) if and only if

$$(-c, \langle c, \overline{x}\rangle) \in C(\mathcal{U},S) = \text{co}\bigcup_{\tilde{A}\in\tilde{\mathcal{U}}}\tilde{A}^*(S^+\times\mathbb{R}_+).$$

Equivalently, there exist a finite index I, $\lambda_i \in [0,1]$ with $\sum_{i\in I}\lambda_i = 1$, $(y_i^*, r_i) \in S^+\times\mathbb{R}_+$ and

$$\tilde{A}_i \in \tilde{\mathcal{U}} := \{\tilde{A} \in L(X\times\mathbb{R}; Y\times\mathbb{R}) : \tilde{A}(x,t) = (Ax + tb, -t),\ (A,b) \in \mathcal{U}\},\ i \in I,$$

such that $(-c, \langle c, \overline{x}\rangle) = \sum_{i\in I} \lambda_i \tilde{A}_i^*(y_i^*, r_i^*)$. So, there exists $(A_i, b_i) \in \mathcal{U}$ such that

$$\begin{aligned}(-c, \langle c, \overline{x}\rangle) &= \sum_{i\in I} \lambda_i \tilde{A}_i^*(y_i^*, r_i) \\ &= \sum_{i\in I} \lambda_i (A_i^*(y_i^*), \langle y_i^*, b_i\rangle - r_i),\end{aligned}$$

where $\tilde{A}_i^* \in L(Y^* \times \mathbb{R}; X^* \times \mathbb{R})$ and $\tilde{A}_i^*(y^*, r) = (A_i^* y^*, \langle y^*, b\rangle - r)$. Therefore, $\overline{x}$ is a robust solution of (CP) if and only there exist a finite index set I, $\lambda_i \in [0, 1]$ with $\sum_{i\in I} \lambda_i = 1$, $(y_i^*, r_i) \in S^+ \times \mathbb{R}_+$ and $(A_i, b_i) \in \mathcal{U}$ such that

$$c + \sum_{i\in I} \lambda_i A_i^*(y_i^*) = 0 \tag{4.4}$$

and

$$-\langle c, \overline{x}\rangle + \sum_{i\in I} \lambda_i \left(\langle y_i^*, b_i\rangle - r_i\right) = 0 \tag{4.5}$$

We now show that (4.4) and (4.5) are equivalent to (4.3). To see this, suppose that (4.4) and (4.5) hold. Then, we have

$$\langle c, \overline{x}\rangle + \sum_{i\in I} \lambda_i \langle y_i^*, A_i \overline{x}\rangle = \langle c, \overline{x}\rangle + \sum_{i\in I} \lambda_i \langle A_i^*(y_i^*), \overline{x}\rangle = 0$$

Adding this with (4.5), we obtain that

$$\sum_{i\in I} \lambda_i \left(\langle y_i^*, A_i \overline{x} + b_i\rangle - r_i\right) = 0$$

Noting that $A_i \overline{x} + b_i \in -S$, $y_i^* \in S^+$ and $r_i \in \mathbb{R}_+$, we obtain that

$$\sum_{i\in I} \lambda_i r_i = 0 \text{ and } \sum_{i\in I} \lambda_i \left(\langle y_i^*, A_i \overline{x} + b_i\rangle\right) = 0.$$

So, (4.3) follows.

Conversely, suppose that (4.3) holds. Then clearly (4.4) holds. To see (4.5), note from (4.4) that

$$-\langle c, \overline{x}\rangle - \sum_{i\in I} \lambda_i \langle y_i^*, A_i(\overline{x})\rangle = -\langle c, \overline{x}\rangle - \sum_{i\in I} \lambda_i \langle A_i^*(y_i^*), \overline{x}\rangle = 0.$$

Adding this with the second relation of (4.3), we obtain

$$-\langle c, \overline{x}\rangle + \sum_{i\in I} \lambda_i \langle y_i^*, b_i\rangle = 0.$$

Hence (4.5) holds with $r_i = 0$. □

Corollary 4.1. *For (CP), suppose that $X = \mathbb{R}^n$, $\mathcal{U}$ is compact and convex, and the following strong Slater condition holds:*

$$\exists x_0 \in \mathbb{R}^n,\ \ Ax_0 + b \in -\text{int}S\ \forall (A, b) \in \mathcal{U}.$$

Then, $\overline{x}$ is a robust solution of (CP) if and only if there exist a finite index set I, $\lambda_i \in [0,1]$, $y_i^* \in S^+$ and $(A_i, b_i) \in \mathcal{U}$, $i \in I$ with $\sum_{i\in I} \lambda_i = 1$ such that

$$c + \sum_{i\in I} \lambda_i A_i^*(y_i^*) = 0 \text{ and } \sum_{i\in I} \lambda_i \left(\langle y_i^*, A_i(\overline{x}) + b_i\rangle\right) = 0.$$

Proof. The conclusion follows from Proposition 3.1 and the preceding theorem. □

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