Characterizing Robust Solution Sets of Convex Programs under Data Uncertainty*

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Abstract This paper deals with convex optimization problems in the face of data uncertainty within the framework of robust optimization. It provides various properties and characterizations of the set of all robust optimal solutions of the problems. In particular, it provides generalizations of the constant subdifferential property as well as the constant Lagrangian property for solution sets of convex programming to robust solution sets of uncertain convex programs. The paper shows also that the robust solution sets of uncertain convex quadratic programs and sum-of-squares convex polynomial programs under some commonly used uncertainty sets of robust optimization can be expressed as conic representable sets. As applications, it derives robust optimal solution set characterizations for uncertain fractional programs. The paper presents several numerical examples illustrating the results.

Keywords Convex optimization problems with data uncertainty · robust optimization · optimal solution set · uncertain convex quadratic programs · uncertain sum-of-squares convex polynomial programs

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1 Introduction

The characterizations of the optimal solution sets of mathematical programming problems are important to our understanding of the behaviour of solution methods for mathematical programs that have multiple optimal solutions. These characterizations are well known for various classes of mathematical programs (see [1-11]) and they assume perfect information (that is, precise values for the input quantities or data of the programs).

However, in reality, it is common that the input data associated with the objective function and the constraints of programs are uncertain or incomplete due to prediction or measurement errors [12, 13]. In this paper, we study the problem of characterizing the set of robust optimal solutions of uncertain convex programs. This is done by examining the set of optimal solutions of their robust counterparts (see (3) in Section 2). In recent years, issues related to characterizations of optimal solutions, duality properties and computational tractability of the robust counterparts have been extensively studied in the literature (see [12-19] and other references therein).

The purpose of this work is two-fold: its first goal is to derive some properties and characterizations of the robust solution sets of uncertain convex programs under suitable conditions. In particular, we provide generalizations of the constant subdifferential property as well as the constant Lagrangian property for solution sets of convex programming to robust solution sets of uncertain convex programs. Its second aim is to examine special classes of uncertain convex programs for which the robust solution sets can be described as conic representable sets. The significance of conic representable robust solution sets is that they can be further studied using conic programming such as semidefinite programming. For properties and applications of conic representable sets, see [20]. We show that the robust solution sets of convex quadratic programs and sum-of-squares convex (in short, SOS-convex) polynomial programs [21-23] under some commonly used uncertainty sets of robust optimization, such as the ellipsoidal, scenario and spectral norm uncertainties, can be expressed as conic representable sets.

The outline of the paper is as follows. Section 2 gives preliminary results involving existence of Lagrange multipliers for the robust counterparts of the given uncertain convex programs. Section 3 presents various characterizations of the robust solution sets of uncertain convex programs. Section 4 provides characterizations of the robust solution set of an uncertain convex quadratic program where the objective function has spectral norm uncertainty whereas the constraints have either ellipsoidal or scenario data uncertainty. Section 5 examines the robust solution set for the class of uncertain SOS-convex polynomial programs for which the solution set is described in terms of sums of squares polynomial representations. Section 6 develops characterizations of robust solution sets of uncertain fractional programming problems. Section 7 provides a conclusion of the work presented and outlines further research on the topic area of the work.

2 Preliminaries

We begin this section by fixing notation and definitions. Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean space with dimension n. The inner product $\langle \cdot, \cdot \rangle$ is defined on $\mathbb{R}^n$. The norm of $x \in \mathbb{R}^n$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. The non-negative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$ and is defined by $\mathbb{R}^n_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0\}$. The closed (resp. open) interval between $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$ is denoted by $[\alpha, \beta]$ (resp. $]\alpha, \beta[$). For a set A in $\mathbb{R}^n$, the interior (resp. relative interior, closure, convex hull) of A is denoted by $\text{int}A$ (resp. $\text{ri}S$, $\text{cl}A$, $\text{conv}A$). We say A is convex whenever $\mu a_1 + (1-\mu) a_2 \in A$ for all $\mu \in [0,1]$, $a_1, a_2 \in A$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex iff $f((1-\mu)x + \mu y) \leq (1-\mu) f(x) + \mu f(y)$ for all $\mu \in [0,1]$ and for all $x, y \in \mathbb{R}^n$. The function f is said to be concave on $\mathbb{R}^n$ whenever $-f$ is convex on $\mathbb{R}^n$. Let A be a closed and convex set in $\mathbb{R}^n$. The indicator function δ_A respect to a set A, is defined by

$$\delta_A(x) := \begin{cases} 0, & \text{if } x \in A, \\ +\infty, & \text{otherwise.} \end{cases}$$

The (convex) normal cone of A at a point $x \in \mathbb{R}^n$ is defined as

$$N_A(x) := \begin{cases} \{y \in \mathbb{R}^n : \langle y, a - x \rangle \leq 0 \text{ for all } a \in A\}, & \text{if } x \in A, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We use S^n to denote the space of $(n \times n)$ symmetric matrices. For $A \in S^n$, $A \succeq 0$ (resp., $A \succ 0$) means that A is positive semi-definite (resp., definite). The $(n \times n)$ identity matrix is denoted by I_n. For a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$, we use ∇f to denote the gradient of f. Let $C \subseteq \mathbb{R}^q$. If $f : \mathbb{R}^n \times C \to \mathbb{R}$ is continuously differentiable, we use $\nabla_x f$ to denote the gradient of f with respect to the first variable. Let f be a continuous and convex functions on $\mathbb{R}^n$. The (convex) subdifferential of f at $x \in \mathbb{R}^n$ is defined by

$$\partial f(x) := \{z \in \mathbb{R}^n : \langle z, y - x\rangle \le f(y) - f(x)\, \forall\, y \in \mathbb{R}^n\}.$$

Moreover, for a function $f : \mathbb{R}^n \times C \to \mathbb{R}$ such that $f(\cdot, u)$ is convex for all fixed $u \in C$, we use $\partial_x f(\cdot, u)$ to denote the subdifferential of f with respect to the first variable.

Lemma 2.1. *Let $\mathcal{U}$ be a convex compact set in $\mathbb{R}^{q_0}$. Let $f : \mathbb{R}^n \times \mathbb{R}^{q_0} \to \mathbb{R}$ be a function such that for each fixed $u \in \mathcal{U}$, $f(\cdot, u)$ is a convex function on $\mathbb{R}^n$ and for each fixed $x \in \mathbb{R}^n$, $f(x, \cdot)$ is a concave function on $\mathbb{R}^{q_0}$. Let $\tilde{f}(x) = \max_{u\in\mathcal{U}} f(x, u)$. Then, the set $\bigcup_{u\in\mathcal{U}}\{\partial_x f(\bar{x}, u) : f(\bar{x}, u) = \tilde{f}(\bar{x})\}$ is closed and convex, and*

$$\partial \tilde{f}(\bar{x}) = \bigcup_{u\in\mathcal{U}}\{\partial_x f(\bar{x}, u) : f(\bar{x}, u) = \tilde{f}(\bar{x})\}.$$

Proof. To show convexity, let $a_1, a_2 \in \bigcup_{u\in\mathcal{U}}\{\partial_x f(\bar{x}, u) : f(\bar{x}, u) = \tilde{f}(\bar{x})\}$ and let $\mu \in [0, 1]$. Then, for $i = 1, 2$, there exist $u_i \in \mathcal{U}$ such that $f(\bar{x}, u_i) = \tilde{f}(\bar{x})$ and $a_i \in \partial_x f(\bar{x}, u_i)$. It then follows from the concavity of $f(\bar{x}, \cdot)$ that

$$f(\bar{x}, \mu u_1 + (1-\mu)u_2) \ge \mu f(\bar{x}, u_1) + (1-\mu) f(\bar{x}, u_2) = \tilde{f}(\bar{x}).$$

Note that $\tilde{f}(\bar{x}) = \max_{u\in\mathcal{U}} f(\bar{x}, u)$. This implies that

$$f(\bar{x}, \mu u_1 + (1-\mu)u_2) = \mu f(\bar{x}, u_1) + (1-\mu) f(\bar{x}, u_2) = \tilde{f}(\bar{x}). \tag{1}$$

As $a_i \in \partial_x f(\bar{x}, u_i)$, for each $z \in \mathbb{R}^n$

$$\langle a_i, z - \bar{x}\rangle \le f(z, u_i) - f(\bar{x}, u_i).$$

So, we have, for each $z \in \mathbb{R}^n$,

$$\begin{aligned}\langle \mu a_1 + (1-\mu)a_2, z - \bar{x}\rangle &\le \Big(\mu f(z, u_1) + (1-\mu) f(z, u_2)\Big) - \Big(\mu f(\bar{x}, u_1) + (1-\mu) f(\bar{x}, u_2)\Big)\\ &\le f(z, \mu u_1 + (1-\mu)u_2) - f(\bar{x}, \mu u_1 + (1-\mu)u_2),\end{aligned}$$

where the last inequality follows from (1). Thus, $\bigcup_{u\in\mathcal{U}}\{\partial_x f(\bar{x}, u) : f(\bar{x}, u) = \tilde{f}(\bar{x})\}$ is convex.

To see $\bigcup_{u\in\mathcal{U}}\{\partial_x f(\bar{x}, u) : f(\bar{x}, u) = \tilde{f}(\bar{x})\}$ is closed, let $a_n \in \bigcup_{u\in\mathcal{U}}\{\partial_x f(\bar{x}, u) : f(\bar{x}, u) = \tilde{f}(\bar{x})\}$ with $a_n \to a$. Then, there exist $u_n \in \mathcal{U}$ such that $a_n \in \partial_x f(\bar{x}, u_n)$ with $f(\bar{x}, u_n) = \tilde{f}(\bar{x})$, and so,

$$\langle a_n, y - \bar{x}\rangle \le f(y, u_n) - f(\bar{x}, u_n) \text{ for all } y \in \mathbb{R}^n.$$

As $\mathcal{U}$ is compact, we may assume that $u_n \to \bar{u} \in \mathcal{U}$. By passing to the limit, we have $f(\bar{x}, \bar{u}) = \tilde{f}(\bar{x})$ and

$$\langle a, y - \bar{x} \rangle \leq f(y, \bar{u}) - f(\bar{x}, \bar{u}) \text{ for all } y \in \mathbb{R}^n.$$

So, $a \in \bigcup_{u \in \mathcal{U}} \{\partial_x f(\bar{x}, u) : f(\bar{x}, u) = \tilde{f}(\bar{x})\}$, and hence, $\bigcup_{u \in \mathcal{U}} \{\partial_x f(\bar{x}, u) : f(\bar{x}, u) = \tilde{f}(\bar{x})\}$ is closed.

The subdifferential equation now follows from the subdifferential rule for maximum functions [24, 25] that

$$\partial \tilde{f}(\bar{x}) = \text{clconv} \bigcup_{u \in \mathcal{U}} \{\partial_x f(\bar{x}, u) : f(\bar{x}, u) = \tilde{f}(\bar{x})\},$$

where clconv(A) denotes the closure of the convex hull of a set A. □

Consider the convex optimization problem

$$\min_{x \in \mathbb{R}^n} \{f(x) \;:\; g_i(x) \leq 0,\, i = 1, \ldots, m\}, \tag{2}$$

where f and g_i, $i = 1, \ldots, m$, are convex functions on $\mathbb{R}^n$. This problem assumes perfect information (that is, precise values for the input quantities or data), have been extensively studied in the literature. In particular, many characterizations of the optimal solution sets of mathematical programming problems, including models of the form (P_0), have been given (see [1-11]) due to their role in our understanding of the behaviour of solution methods for mathematical programs that have multiple optimal solutions.

However, in reality, it is common that the input data associated with the objective function and the constraints of (2) are uncertain or incomplete due to prediction or measurement errors [12, 13]. The model (2) in the face of data uncertainty in the objective and constraint functions can be captured by the following parameterized model

$$(P) \qquad \min_{x \in \mathbb{R}^n} \{f(x, u) \;:\; g_i(x, v_i) \leq 0,\, i = 1, \ldots, m\},$$

where u and v_i are uncertain parameters and they belong to the specified convex and compact uncertainty sets $\mathcal{U} \subset \mathbb{R}^{q_0}$ and $\mathcal{V}_i \subset \mathbb{R}^q$, respectively.

Assumption 2.1. *Throughout this section, we assume that* $f : \mathbb{R}^n \times \mathbb{R}^{q_0} \to \mathbb{R}$ *is a continuous function on* $\mathbb{R}^n \times \mathbb{R}^{q_0}$ *such that for each fixed* $u \in \mathcal{U} \subseteq \mathbb{R}^{q_0}$, $f(\cdot, u)$ *is a convex function on* $\mathbb{R}^n$ *and* g_i:$\mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ *is a continuous function such that for each fixed* $v_i \in \mathcal{V}_i \subseteq \mathbb{R}^q$, $g_i(\cdot, v_i)$ *is a convex function.*

We now study the problem of characterizing the set of robust optimal solutions of (P) in terms of a given solution point. This is done by examining the set of optimal solutions of the robust counterpart of (P) which can be formulated as the robust convex optimization problem:

$$(RP) \qquad \min_{x\in\mathbb{R}^n} \quad \max_{u\in\mathcal{U}} f(x,u) \quad \text{s.t.} \quad g_i(x,v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i=1,\dots,m. \quad (3)$$

Definition 2.1. (Robust feasible sets) *We define the robust feasible set of* (P) *by*

$$F := \{x \in \mathbb{R}^n : g_i(x,v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i=1,\dots,m\}.$$

Definition 2.2. (Robust solution sets) *The vector* $\bar{x} \in F$ *is a robust solution of* (P) *whenever it is a solution of the robust counterpart. The robust solution set* S *of* (P) *is the set which consists of all the robust solutions of (P) and is given by*

$$S = \{x \in F : \max_{u\in\mathcal{U}} f(x,u) \leq \max_{u\in\mathcal{U}} f(y,u), \forall y \in F\}.$$

In recent years, issues related to characterizing solution points of (RP), duality properties of (RP) and computational tractability of (RP) have been extensively studied in the literature (see [12-18] and other reference therein).

As a consequence of the preceding Lemma we obtain the following multiplier characterization for a robust solution which plays a key role in deriving characterizations of robust solution sets.

Proposition 2.1. (Necessary and Sufficient Condition for Robust Solution) *For problem (P), let* F *be the robust feasible set and let* S *be the robust solution set. Let* $\bar{x} \in F$*. Suppose that for each fixed* $x \in \mathbb{R}^n$*,* $f(x,\cdot)$ *and* $g_i(x,\cdot)$ *are concave functions and that there exists* $x_0 \in \mathbb{R}^n$ *such that* $g_i(x_0,v_i) < 0,\ \forall v_i \in \mathcal{V}_i,\ i=1,\dots,m$*. Then,* $\bar{x}$ *is a robust solution (that is,* $\bar{x} \in S$*) if and only if there exist* $\bar{\lambda}_i \geq 0$*,* $\bar{u} \in \mathcal{U}$ *and* $\bar{v}_i \in \mathcal{V}_i$ *such that*
$f(\bar{x},\bar{u}) = \max_{u\in\mathcal{U}} f(\bar{x},u)$ *and*

$$0 \in \partial_x f(\bar{x},\bar{u}) + \sum_{i=1}^{n} \bar{\lambda}_i \partial_x g_i(\bar{x},\bar{v}_i),\ \ \bar{\lambda}_i g_i(\bar{x},\bar{v}_i) = 0. \quad (4)$$

Proof. $[\Rightarrow]$ Let $\bar{x} \in S$. Define $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ by $\tilde{f}(x) = \max_{u\in\mathcal{U}} f(x,u)$ for all $x \in \mathbb{R}^n$. As $\mathcal{U}$ is compact and $f(\cdot,u)$ is continuous and convex for each fixed u, $\tilde{f}$ is a real-valued convex function. So, $\tilde{f}$ is continuous and convex. As $\bar{x} \in S$,

$$0 \in \partial(\tilde{f} + \delta_F)(\bar{x}) = \partial\tilde{f}(\bar{x}) + N_F(\bar{x}),$$

where δ_F is the indicator function with respect to the set F, $N_F(\bar{x})$ is the normal cone of the set F at $\bar{x}$ and the last equality follows as $\tilde{f}$ is continuous. From Lemma 2.1, we have

$$0 \in \bigcup_{u\in\mathcal{U}} \{\partial_x f(\bar{x},u) : f(\bar{x},u) = \tilde{f}(\bar{x})\} + N_F(\bar{x}).$$

To finish the proof, it suffices to show that

$$N_F(\bar{x}) \subseteq \bigcup_{v_i\in\mathcal{V}_i,\lambda_i\geq 0} \{\sum_{i=1}^{n} \lambda_i a_i : a_i \in \partial_x g_i(\bar{x},v_i), \lambda_i g_i(\bar{x},v_i) = 0\}.$$

To see this, let $a \in N_F(\bar{x})$. Then, $h(x) := -\langle a, x\rangle$ attains its minimum over F. Note that

$$F = \{x : g_i(x,v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1,\dots,m\} = \{x : \max_{v_i\in\mathcal{V}_i} g_i(x,v_i) \leq 0, i = 1,\dots,m\}.$$

Then, our assumption, together with the Lagrangian duality [25], implies that

$$\begin{aligned} -\langle a,\bar{x}\rangle = \inf_{x\in F}\{-\langle a,x\rangle\} &= \max_{\lambda_i\geq 0} \inf_{x\in\mathbb{R}^n} \{-\langle a,x\rangle + \sum_{i=1}^{m} \lambda_i \max_{v_i\in\mathcal{V}_i} g_i(x,v_i)\} \\ &= \max_{\lambda_i\geq 0} \inf_{x\in\mathbb{R}^n} \max_{v_i\in\mathcal{V}_i}\{-\langle a,x\rangle + \sum_{i=1}^{m} \lambda_i g_i(x,v_i)\}. \end{aligned}$$

As each $\mathcal{V}_i$ is compact, for each fixed $v_i \in \mathcal{V}_i \subseteq \mathbb{R}^q$, $g_i(\cdot,v_i)$ is a continuous convex function and for each fixed $x \in \mathbb{R}^n$, $g_i(x,\cdot)$ is a continuous concave function, the convex-concave minimax theorem [26] implies that

$$\begin{aligned} \max_{\lambda_i\geq 0} \inf_{x\in\mathbb{R}^n} \max_{v_i\in\mathcal{V}_i}\{-\langle a,x\rangle + \sum_{i=1}^{m} \lambda_i g_i(x,v_i)\} &= \max_{\lambda_i\geq 0} \max_{v_i\in\mathcal{V}_i} \inf_{x\in\mathbb{R}^n} \{-\langle a,x\rangle + \sum_{i=1}^{m} \lambda_i g_i(x,v_i)\} \\ &= \max_{\lambda_i\geq 0, v_i\in\mathcal{V}_i} \inf_{x\in\mathbb{R}^n} \{-\langle a,x\rangle + \sum_{i=1}^{m} \lambda_i g_i(x,v_i)\}. \end{aligned}$$

So, there exist $\lambda_i \geq 0, v_i \in \mathcal{V}_i$ such that

$$-\langle a,\bar{x}\rangle \leq -\langle a,x\rangle + \sum_{i=1}^{m} \lambda_i g_i(x,v_i) \text{ for all } x \in \mathbb{R}^n.$$

Letting $x = \bar{x}$, we have $\sum_{i=1}^{m} \lambda_i g_i(\bar{x},v_i) \geq 0$. Note that $\lambda_i \geq 0$ and $\bar{x} \in F$. So, $\sum_{i=1}^{m} \lambda_i g_i(\bar{x},v_i) = 0$. Then, $q(x) := -\langle a,x\rangle + \sum_{i=1}^{m} \lambda_i g_i(x,v_i)$ attains its minimum at $\bar{x}$, and so,

$$a \in \bigcup_{v_i\in\mathcal{V}_i,\lambda_i\geq 0} \{\sum_{i=1}^{m} \lambda_i a_i : a_i \in \partial_x g_i(\bar{x},v_i), \lambda_i g_i(\bar{x},v_i) = 0\}.$$

Hence, (4) holds.

[$\Leftarrow$]. This implication holds by the standard sufficient optimality arguments of convex programming using convexity. $\square$

The following example illustrates that, if the concavity assumption with respect to the uncertainty parameter is dropped, the above existence result multipliers and uncertainty parameters may fail.

Example 2.1. *(***Failure of multiplier characterization without concavity***) Consider the robust optimization problem*

$$\min \quad \max_{u\in[0,1]}(x-u)^2 \ \ s.t. \ \ x \in \mathbb{R}.$$

Note that any convex function attains its maximum over a polytope at some extreme points, $\max_{u\in[0,1]}(x-u)^2 = \max\{x^2,(x-1)^2\}$*. So, the robust solution set* $S = \{1/2\}$*.*

On the other hand, let $\bar{x} = 1/2$ *and* $f(x,u) = (x-u)^2$*. Let* $g_1(x) \equiv -1$*. Then, the robust feasible set* $F = \mathbb{R} = \{x : g_1(x) \le 0\}$*. We note that the strict feasibility condition is always satisfied. Take* $\bar{\lambda}_1 = 0$ *and let* $\bar{u} \in \mathcal{U} = [0,1]$ *with* $f(\bar{x},\bar{u}) = \max_{u\in\mathcal{U}} f(\bar{x},u)$*. Then,* $\bar{u} \in \{0,1\}$ *and so,*

$$\partial_x f(\bar{x},\bar{u}) = \nabla_x f(\bar{x},\bar{u}) \in \{-1,1\}.$$

So, $0 \notin \partial_x f(\bar{x},\bar{u})$*. Thus, the above multiplier characterization fails. Finally, we observe that* $u \mapsto f(x,u)$ *is not concave.*

3 Characterizations of Solution Sets

In this Section, we present various characterizations of robust solution sets in terms of a given robust solution point of the given problem. We begin by deriving basic properties of the subdifferential of the objective function on the solution set. Note that, in the uncertainty free case, the subdifferential of the objective function is constant on the relative interior of its solution set. A generalization of this result for convex optimization problem in the face of data uncertainty is presented below.

For a given point $x \in \mathbb{R}^n$, let $A(x) := \bigcup_{u\in U(x)}\{\partial_x f(x,u)\}$, where $U(x) = \{\bar{u} : f(x,\bar{u}) = \max_{u\in\mathcal{U}} f(x,u)\}$. We first start with a simple fact which states that $A(x)$ is a constant over the relative interior of the robust solution set S.

Lemma 3.1. *(***Generalized constant subdifferential property***) For problem (P), let* S *be the robust solution set, and suppose that for each fixed* $x \in \mathbb{R}^n$*,* $f(x,\cdot)$ *and* $g_i(x,\cdot)$ *are concave functions. Then* $A(x_1) = A(x_2)$ *for any* $x_1, x_2 \in \mathrm{ri}S$*. Moreover, we have* $A(x) \subseteq A(x')$ *for any* $x \in \mathrm{ri}S$ *and* $x' \in S$*.*

Proof. Fix any $x_1, x_2 \in \mathrm{ri}S$. Let $\tilde{f}(x) = \max_{u\in\mathcal{U}} f(x,u)$ and let $F = \{x : g_i(x, v_i) \le 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \dots, m\}$. Then, the robust solution set S is the solution set of the following nonsmooth convex optimization problem

$$\min\ \tilde{f}(x) \quad \text{s.t.} \quad x \in F.$$

From Mangasarian [7], we see that $\partial f(x)$ is a constant over $\mathrm{ri}S$, and so, $\partial\tilde{f}(x_1) = \partial\tilde{f}(x_2)$. Thus, the conclusion follows from Lemma 2.1.

To see the second assertion, let $x \in \mathrm{ri}S$ and $x' \in S$. Then, for any $\lambda \in]0,1[$, $\lambda x + (1-\lambda)x' \in \mathrm{ri}S$. So, the first assertion implies that $A(x) = A(\lambda x + (1-\lambda)x')$ for any $\lambda \in]0,1[$. Let $w \in A(x)$ and let $\lambda_n \to 0$. Define $x_n = \lambda_n x + (1-\lambda_n)x'$. Then, $x_n \to x'$ and $w \in A(x) = A(x_n)$. So, there exist $u_n \in \mathcal{U}$ such that $f(x_n, u_n) = \max_{u\in\mathcal{U}} f(x_n, u)$ and $w \in \partial_x f(x_n, u_n)$. As $\mathcal{U}$ is compact, by passing to subsequence, we may assume $u_n \to \bar{u} \in \mathcal{U}$. Then,

$$\langle w, z - x_n\rangle \le f(z, u_n) - f(x_n, u_n) \text{ for all } z \in \mathbb{R}^n$$

Letting $n \to \infty$, we have $f(x', \bar{u}) = \max_{u\in\mathcal{U}} f(x', u)$ and

$$\langle w, z - x'\rangle \le f(z, \bar{u}) - f(x', \bar{u}) \text{ for all } z \in \mathbb{R}^n.$$

So, $w \in A(x')$. Thus, the conclusion follows. □

The following simple example illustrates that the generalized constant subdifferential property cannot be extended to the whole robust solution set in general.

Example 3.1. *Consider the robust optimization problem*

$$\min_{x\in\mathbb{R}} \max_{u_1+u_2+u_3=1, u_i\ge 0} u_1(x-1) + u_2(-x+1) + u_3.$$

Note that any linear function attains its maximum over a polytope at some extreme points of the polytope. So, we have $\max_{u_1+u_2+u_3=1,u_i\ge 0}\big(u_1(x-1)+u_2(-x+1)+u_3\big) = \max\{(x-1), -(x-1), 1\} = \max\{|x-1|, 1\}$. *Then, the robust solution set* $S = \{x : |x-1| \le 1\}$. *Take* $x_1 = 0$ *and* $x_2 = 2$. *Then,* $x_1, x_2 \in S$. *Direct verification shows that* $A(x_1) = [-1, 0]$ *and* $A(x_2) = [0, 1]$. *Thus,* $A(x_1) \ne A(x_2)$.

Note that the generalized constant subdifferential property yields the classical result for uncertainty free case that the gradient of the objective function is a constant over the solution set for a smooth convex optimization problem.

Corollary 3.1. *[7] For problem (P) with $\mathcal{U}$ and $\mathcal{V}_i$ are singleton sets, let the objective function f be continuously differentiable and let S_0 be the solution set. Then, ∇f is a constant over S_0.*

Proof. Let $\mathcal{U}$ and $\mathcal{V}_i$ be singleton sets. Then, the above lemma implies that ∇f is a constant over $\mathrm{ri} S_0$. Now, take any point $x \in S_0$ and $a \in \mathrm{ri} S_0$. Then, for any $\lambda \in]0,1[$, $\lambda a + (1-\lambda)x \in S_0$ and so, $\nabla f(\lambda a + (1-\lambda)x) = \nabla f(a)$. Letting $\lambda \to 0$, as f is continuously differentiable, we have $\nabla f(x) = \nabla f(a)$. Thus, the conclusion follows. □

In the following proposition, we now obtain a basic robust solution set characterization. In the uncertainty free case, this result collapses to [7, Theorem 1a].

Proposition 3.1. (Basic robust solution set characterization) *For problem (P), let F be the robust feasible set and let S be the robust solution set. Suppose that for each fixed $x \in \mathbb{R}^n$, $f(x,\cdot)$ and $g_i(x,\cdot)$ are concave functions. Let $a \in S$. Then,*

$$S = \{x \in F \ \ : \ \ \langle w_a, x-a \rangle = 0 \text{ for some } w_a \in A(x) \cap A(a)\},$$

where $A(x) := \bigcup_{u \in U(x)} \{\partial_x f(x,u)\}$ and $U(x) = \{\bar{u} \in \mathcal{U} : f(x,\bar{u}) = \max_{u \in \mathcal{U}} f(x,u)\}$.

Proof. [$\subseteq$] Let $x \in S$. Clearly, $x \in F$. Fix $\tilde{x} \in \mathrm{ri} S$. We first see that $A(\tilde{x}) \neq \emptyset$. Indeed, as $f(\cdot,\cdot)$ is continuous and $f(\cdot,u)$ is convex for any fixed $u \in U$, we see that $\partial_x f(\tilde{x},u) \neq \emptyset$ for all $u \in U$. So, $A(\tilde{x}) := \bigcup_{u \in U(\tilde{x})} \{\partial_x f(\tilde{x},u)\} \neq \emptyset$. Take $w \in A(\tilde{x})$. We note that $w \in A(\tilde{x}) \subseteq A(x) \cap A(a)$. As $\tilde{x} \in \mathrm{ri} S$, $\tilde{x} = \lambda x + (1-\lambda)y$ for some $y \in S$ and $\lambda \in]0,1[$. Since $w \in A(\tilde{x})$, there exists $\tilde{u} \in \mathcal{U}$ such that $f(\tilde{x},\tilde{u}) = \max_{u \in \mathcal{U}} f(\tilde{x},u)$ and $w \in \partial_x f(\tilde{x},\tilde{u})$. So,

$$(1-\lambda)\langle w, x-y \rangle = \langle w, x-\tilde{x} \rangle \leq \max_{u \in \mathcal{U}} f(x,u) - f(\tilde{x},\tilde{u}) = 0$$

and

$$\lambda \langle w, y-x \rangle = \langle w, y-\tilde{x} \rangle \leq \max_{u \in \mathcal{U}} f(y,u) - f(\tilde{x},\tilde{u}) = 0.$$

As $\lambda \in]0,1[$, this implies that $\langle w, x-y \rangle \leq 0$ and $\langle w, y-x \rangle \leq 0$. So, we have $\langle w, y-x \rangle = 0$, and hence $\langle w, x-\tilde{x} \rangle = (1-\lambda)\langle w, x-y \rangle = 0$. Similarly, as $a \in S$, we can show that $\langle w, a-\tilde{x} \rangle = 0$. So, $\langle w, x-a \rangle = \langle w, x-\tilde{x} \rangle - \langle w, a-\tilde{x} \rangle = 0$.

[$\supseteq$] Take $x \in F$ with $\langle w_a, x-a \rangle = 0$ for some $w_a \in A(x) \cap A(a)$. As $w_a \in A(x)$, there exists $\bar{u} \in \mathcal{U}$ with $f(x,\bar{u}) = \max_{u \in \mathcal{U}} f(x,u)$ and $w_a \in \partial_x f(x,\bar{u})$. Then, we have

$$0 = \langle w_a, a-x \rangle \leq f(a,\bar{u}) - f(x,\bar{u}) \leq \max_{u \in \mathcal{U}} f(a,u) - \max_{u \in \mathcal{U}} f(x,u).$$

This implies that $\max_{u\in\mathcal{U}} f(x,u) \le \max_{u\in\mathcal{U}} f(a,u)$, and so, $x \in S$. $\square$

It is worth noting that the concavity of $f(x,\cdot)$ is often automatically satisfied for robust optimization problems, where the data uncertainty is affinely parameterized.

For problem (P), let F be the robust feasible set and let S be the robust solution set. Let $a \in S$. Let $\lambda_i^a \ge 0$, $(u^a, v_i^a) \in \mathcal{U} \times \mathcal{V}_i$, $i = 1, \dots, m$, satisfy

$$0 \in \partial_x f(a, u^a) + \sum_{i=1}^m \lambda_i^a \partial_x g_i(a, v_i^a),\ \ \lambda_i^a g_i^a(a, v_i) = 0 \text{ and } f(a, u^a) = \max_{u\in\mathcal{U}} f(a,u). \tag{5}$$

We define the Lagrangian function $L_a(\cdot, \lambda^a, u^a, v^a)$ by

$$L_a(x, \lambda^a, u^a, v^a) = f(x, u^a) + \sum_{i=1}^m \lambda_i^a g_i(x, v_i^a) \text{ for all } x \in \mathbb{R}^n.$$

Theorem 3.1. (Constant Lagrangian over the robust solution set) *For problem (P), let F be the robust feasible set and let S be the robust solution set. Suppose that for each fixed $x \in \mathbb{R}^n$, $f(x,\cdot)$ and $g_i(x,\cdot)$ are concave functions and there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0, v_i) < 0$, $\forall v_i \in \mathcal{V}_i$, $i = 1, \dots, m$. Let $a \in S$, and let $\lambda_i^a \ge 0$, $u^a \in \mathcal{U}$ and $v_i^a \in \mathcal{V}_i$ satisfy (5). Then, for each $x \in S$, $\lambda_i^a g_i(x, v_i^a) = 0$, $f(x, u^a) = \max_{u\in\mathcal{U}} f(x,u)$ and $L_a(\cdot, \lambda^a, u^a, v^a)$ is a constant on S.*

Proof. As $a \in S$, Proposition 2.1 shows us that $0 \in \partial L_a(a, \lambda^a, u^a, v^a)$. So, the definition of convex subdifferential shows that for all $x \in \mathbb{R}^n$,

$$0 \le L_a(x, \lambda^a, u^a, v^a) - L_a(a, \lambda^a, u^a, v^a).$$

So, for all $x \in \mathbb{R}^n$,

$$f(x, u^a) + \sum_{i=1}^m \lambda_i^a g_i(x, v_i^a) \ge f(a, u^a) + \sum_{i=1}^m \lambda_i^a g_i(a, v_i^a) = f(a, u^a) = \max_{u\in\mathcal{U}} f(a,u), \tag{6}$$

where the last equality follows from the multiplier characterization of a robust solution. Note that for each $x \in S$, $\max_{u\in\mathcal{U}} f(x,u) = \max_{u\in\mathcal{U}} f(a,u)$. So, for each $x \in S$,

$$\sum_{i=1}^m \lambda_i^a g_i(x, v_i^a) \ge 0.$$

For each $x \in S$, we have $g_i(x, v_i^a) \le 0$. It then follows that $\lambda_i^a g_i(x, v_i^a) = 0$, $i = 1, \dots, m$.

To see the second assertion, from (6) and $\lambda_i^a g_i(x, v_i^a) = 0$, $i = 1, \dots, m$, we have

$$\max_{u\in\mathcal{U}} f(x,u) \ge f(x, u^a) \ge \max_{u\in\mathcal{U}} f(a,u).$$

Note that $\max_{u\in\mathcal{U}} f(x,u) = \max_{u\in\mathcal{U}} f(a,u)$ (as $x \in S$). It follows that $f(x,u^a) = \max_{u\in\mathcal{U}} f(x,u)$.

To see the last assertion, we only need to notice, for each $x \in S$

$$L_a(x,\lambda^a,u^a,v^a) = f(x,u^a) + \sum_{i=1}^{m} \lambda_i^a g_i(x,v_i^a) = \max_{u\in\mathcal{U}} f(x,u) = \max_{u\in\mathcal{U}} f(a,u) = f(a,u^a),$$

which is a constant. □

Theorem 3.2. (Multiplier characterization of robust solution set) *For problem (P), let F be the robust feasible set and let S be the robust solution set. Suppose that for each fixed $x \in \mathbb{R}^n$, $f(x,\cdot)$ and $g_i(x,\cdot)$ are concave functions, and there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0,v_i) < 0$, $\forall v_i \in \mathcal{V}_i$, $i = 1,\ldots,m$. Let $a \in S$, and let $\lambda_i^a \geq 0$, $u^a \in \mathcal{U}$ and $v_i^a \in \mathcal{V}_i$ be the multiplier and the uncertainty parameters associated with a. Then,*

$$\begin{aligned} S = \{x \in F \ :\ & \lambda_i^a g_i(x,v_i^a) = 0, i = 1,\ldots,m, f(x,u^a) = \max_{u\in\mathcal{U}} f(x,u), \\ & \exists\, w_a \in \partial_x f(x,u^a) \cap \partial_x f(a,u^a),\ \langle w_a, x-a\rangle = 0\}. \end{aligned}$$

Proof. [$\subseteq$] Let $x \in S$. Clearly $x \in F$. From the multiplier characterization, there exist $\lambda_i^a \geq 0$, $u^a \in \mathcal{U}_a$ and $v_i^a \in \mathcal{V}_i$ such that

$$0 \in \partial_x f(a,u^a) + \sum_{i=1}^{m} \lambda_i^a \partial_x g_i(a,v_i^a),\ \ \lambda_i^a g_i(a,v_i^a) = 0 \text{ and } f(a,u^a) = \max_{u\in\mathcal{U}} f(a,u).$$

Then, there exist $w_a \in \partial_x f(a,u^a)$ and $z_a \in \sum_{i=1}^{m} \lambda_i^a \partial_x g_i(a,v_i^a)$ such that $w_a + z_a = 0$. As $z_a \in \sum_{i=1}^{m} \lambda_i^a \partial_x g_i(a,v_i^a)$, we have

$$\langle z_a, x-a\rangle \leq \sum_{i=1}^{m} \lambda_i^a g_i(x,v_i^a) - \sum_{i=1}^{m} \lambda_i^a g_i(a,v_i^a).$$

From the preceding theorem and noting that $x \in S$ and $a \in S$, we have $f(x,u^a) = \max_{u\in\mathcal{U}} f(x,u)$ and $\lambda_i^a g_i(x,v_i^a) = \lambda_i^a g_i(a,v_i^a) = 0, i = 1,\ldots,m$. It then follows that $\langle z_a, x-a\rangle \leq 0$. So, $w_a + z_a = 0$ implies that

$$\langle w_a, x-a\rangle \geq 0. \tag{7}$$

On the other hand, as $w_a \in \partial_x f(a,u^a)$,

$$\langle w_a, x-a\rangle \leq f(x,u^a) - f(a,u^a) \leq \max_{u\in\mathcal{U}} f(x,u) - \max_{u\in\mathcal{U}} f(a,u) = 0, \tag{8}$$

where the last equality follows from the fact that $x, a \in S$. Combining (7) and (8), we have

$$\langle w_a, x-a\rangle = 0. \tag{9}$$

We now show $w_a \in \partial_x f(x, u^a)$. To see this, for any $y \in \mathbb{R}^n$, we have

$$\langle w_a, y-x\rangle = \langle w_a, y-a\rangle + \langle w_a, a-x\rangle = \langle w_a, y-a\rangle \leq f(y,u^a) - f(a,u^a) = f(y,u^a) - f(x,u^a),$$

where the second equality follows from (9), the first inequality is from $w_a \in \partial_x f(a, u^a)$ and the last equality follows by the fact that $x \in S$ and $a \in S$ (and so, $f(a, u^a) = \max_{u\in\mathcal{U}} f(a, u) = \max_{u\in\mathcal{U}} f(x, u) = f(x, u^a)$). Thus, $w_a \in \partial_x f(x, u^a)$. Therefore,

$$\begin{aligned} S \subseteq \{x \in F \quad : \quad & \lambda_i^a g_i(x, v_i^a) = 0, i = 1, \ldots, m, \\ & \exists\, w_a \in \partial_x f(x, u^a) \cap \partial_x f(a, u^a),\ \langle w_a, x-a\rangle = 0\}. \end{aligned}$$

[$\supseteq$] Let $x \in F$ be such that $\lambda_i^a g_i(x, v_i^a) = 0, i = 1, \ldots, m,$ $f(x, u^a) = \max_{u\in\mathcal{U}} f(x, u)$ and there exists $w_a \in \partial_x f(x, u^a) \cap \partial_x f(a, u^a)$, such that $\langle w_a, x-a\rangle = 0$. Then, we see that

$$0 = \langle w_a, a-x\rangle \leq f(a, u^a) - f(x, u^a),$$

where the last inequality follows from the fact that $w_a \in \partial_x f(x, u^a)$. So,

$$\max_{u\in\mathcal{U}} f(x, u) = f(x, u^a) \leq f(a, u^a) = \max_{u\in\mathcal{U}} f(a, u).$$

Note that $a \in S$ and $x \in F$. This implies that $x \in S$. □

Corollary 3.2. (Robust solution set for uncertain smooth convex optimization) *For problem (P), let F be the robust feasible set and let S be the robust solution set. Suppose that $f(\cdot, u)$ is differentiable for each fixed $u \in \mathbb{R}^{q_0}$ and that, for each fixed $x \in \mathbb{R}^n$, $f(x, \cdot)$ and $g_i(x, \cdot)$ are concave functions, and there exists $x_0 \in \mathbb{R}^n$ such that*

$$g_i(x_0, v_i) < 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m.$$

Let $a \in S$, and let $\lambda_i^a \geq 0$, $u^a \in \mathcal{U}$ and $v_i^a \in \mathcal{V}_i$ be the multiplier associated with a. Then, we have

$$\begin{aligned} S = \{x \in F \quad : \quad & \lambda_i^a g_i(x, v_i^a) = 0, i = 1, \ldots, m, f(x, u^a) = \max_{u\in\mathcal{U}} f(x, u), \\ & \nabla_x f(x, u^a) = \nabla_x f(a, u^a),\ \langle \nabla_x f(a, u^a), x-a\rangle = 0\}. \end{aligned}$$

Proof. As $f(\cdot, u)$ are differentiable for each fixed $u \in \mathbb{R}^{q_0}$, for any $w_a \in \partial_x f(x, u^a) \cap \partial_x f(a, u^a)$ with $\langle w_a, x-a\rangle = 0$, we have

$$w_a = \nabla_x f(x, u^a) = \nabla_x f(a, u^a).$$

Thus the conclusion follows from the preceding theorem. □

Before we end this section, let us illustrate our robust solution set characterization via an example.

Example 3.2. *Consider the following uncertain linear programming problem*

$$\begin{aligned}
\min_{(x_1,x_2)\in\mathbb{R}^2} \quad & u_1x_1 + u_2x_2 \\
s.t. \quad & x_1 + \alpha_1 x_2 \le 0, \\
& -x_1 + \alpha_2 \le 0, \qquad (10)
\end{aligned}$$

where the uncertain coefficients $(u_1, u_2) \in \{(1,1) + (\alpha,\beta) : \|(\alpha,\beta)\| \le 1\}$, $\alpha_1 \in [0,1]$ *and* $\alpha_2 \in [-1,0]$. *Its robust counterpart is the following robust linear programming problem*

$$\begin{aligned}
\min_{(x_1,x_2)\in\mathbb{R}^2} \max_{(u_1,u_2)\in\mathcal{U}} \quad & u_1x_1 + u_2x_2 \\
s.t. \quad & x_1 + \alpha_1 x_2 \le 0, \, \forall \alpha_1 \in [0,1], \\
& -x_1 + \alpha_2 \le 0, \, \forall \alpha_2 \in [-1,0],
\end{aligned}$$

where $\mathcal{U} = \{(1,1)+(\alpha,\beta) : \|(\alpha,\beta)\| \le 1\}$, $\mathcal{V}_1 = \{1\}\times[0,1]\times\{0\}$ *and* $\mathcal{V}_2 = \{-1\}\times\{0\}\times[-1,0]$. *This problem can be equivalently rewritten as follows*

$$\begin{aligned}
\min_{x=(x_1,x_2)\in\mathbb{R}^2} \max_{u=(u_1,u_2)\in\mathcal{U}} \quad & f(x,u) = \langle u, x\rangle \\
s.t. \quad & g_1(x,(b_1,\gamma_1)) = b_1^T x + \gamma_1 \le 0, \, \forall (b_1,\gamma_1) \in \mathcal{V}_1, \\
& g_2(x,(b_2,\gamma_2)) = b_2^T x + \gamma_2 \le 0, \, \forall (b_2,\gamma_2) \in \mathcal{V}_2.
\end{aligned}$$

Let F *be the robust feasible set of* (10). *It can be verified that*

$$F = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 \ge 0, x_1 + x_2 \le 0\}.$$

Note that, for all $(x_1,x_2) \in F$,

$$\max_{(u_1,u_2)\in\mathcal{U}} \{u_1x_1 + u_2x_2\} = x_1 + x_2 + \|(x_1,x_2)\| \ge x_2 + (-x_2) = 0.$$

It follows that the robust solution set $S = \{(x_1,x_2) : x_1 = 0$ *and* $x_2 \le 0\}$. *Let* $\tilde{x} = (1,-2)$. *We observe that the strict feasibility condition is satisfied at* $\tilde{x}$ *as*

$$g_i(\tilde{x},(b_i,\gamma_i)) < 0, \;\; \forall \; (b_i,\gamma_i) \in \mathcal{V}_i, \;\; i = 1,2.$$

Let $a = (0,0) \in S$. *Let* $u^a = (u_1^a, u_2^a) = (1,0)$, $\lambda_1^a = 0$, $\lambda_2^a = 1$, $v_1^a = (b_1^a, \gamma_1^a)$ *with* $\gamma_1^a = 0$, $b_1^a = (1,1)$ *and* $v_2^a = (b_2^a, \gamma_2^a)$ *with* $b_2^a = (-1,0)$ *and* $\gamma_2^a = 0$. *Then, we have*

$$(0,0) = (u_1^a, u_2^a) + \lambda_1^a(1, v_1^a) + \lambda_2^a(-1,0) = \nabla_x f(a,u^a) + \sum_{i=1}^n \lambda_i^a \nabla_x g_i(a, v_i^a),$$

$\lambda_i^a g_i(a, v_i^a) = 0$ *and* $f(a, u^a) = \max_{u\in\mathcal{U}} f(a, u)$. *It can be verified that*

$$\nabla_x f(x, u^a) = \nabla_x f(a, u^a) = u^a = (1, 0),$$

and so,

$$\begin{aligned} & \{x \in F : \lambda_i^a\big(\langle b_i^a, x\rangle + \gamma_i^a\big) = 0, i = 1, 2, \langle u^a, x\rangle = \max_{u\in\mathcal{U}}\langle u, x\rangle, \langle u_a, x - a\rangle = 0\} \\ = & \{x \in F : x_1 = 0, x_1 = x_1 + x_2 + \|(x_1, x_2)\|\} \\ = & \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = 0, x_2 \le 0\} = S. \end{aligned}$$

This verifies our robust solution set characterization.

4 Solution Sets of Uncertain Convex Quadratic Programs

In this section, we examine robust solution sets of uncertain convex quadratic programs under various classes of commonly used uncertainty sets and describe the structure of the solution sets. Consider the following robust convex quadratic optimization problem:

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & \max\limits_{A\in\mathcal{U}}\{\frac{1}{2}\langle x, Ax\rangle \; + \langle h, x\rangle\} \\ \text{s.t.} & \langle b_i, x\rangle + \gamma_i \le 0, \; \forall (b_i, \gamma_i) \in \mathcal{V}_i, \; i = 1, \ldots, m, \end{array}$$

where, $\mathcal{U} \subseteq S^n$ and $\mathcal{V}_i \subseteq \mathbb{R}^n \times \mathbb{R}$ are convex compact uncertainty sets. Note that this robust quadratic optimization problem is indeed the robust counterpart of the following uncertain quadratic optimization problem

$$(QP) \qquad \min_{x\in\mathbb{R}^n} f(x, A) \quad \text{s.t.} \quad g_i(x, v_i) \le 0, \; i = 1, \ldots, m,$$

where $A \in \mathcal{U}$, $v_i := (b_i, \gamma_i) \in \mathcal{V}_i$, $i = 1, \ldots, m$,

$$f(x, A) = \frac{1}{2}\langle x, Ax\rangle + \langle h, x\rangle,$$

and

$$g_i(x, v_i) = \langle b_i, x\rangle + \gamma_i, \quad v_i = (b_i, \gamma_i), \; i = 1, \ldots, m.$$

We now obtain simplified robust solution set characterizations for various commonly used data uncertainties.

Ellipsoidal constraint data uncertainty

Consider the quadratic convex program under ellipsoidal constraint data uncertainty where $\mathcal{U}$ is the spectral norm uncertainty set $\mathcal{U}^{\rm spec}$ given by

$$\mathcal{U}^{\rm spec} := \{A_0 + M : \ M \in S^n, \ A + M \succeq 0, \ \|M\|_{\rm spec} \leq \rho\} \tag{11}$$

with $\rho \geq 0$ and $A_0 \in S^n$ with $A_0 \succeq 0$, and $\mathcal{V}_i$ is the ellipsoidal uncertainty set $\mathcal{V}_i^e$ given by

$$\mathcal{V}_i^e = \{b_i^0 + \sum_{l=1}^{q} \beta_i^l b_i^l : \|(\beta_i^1, \ldots, \beta_i^q)\| \leq 1\} \times [\underline{\gamma}_i, \overline{\gamma}_i], \tag{12}$$

with $b_i^l \in \mathbb{R}^n$, $l = 0, 1, \ldots, q$ and $\underline{\gamma}_i, \overline{\gamma}_i \in \mathbb{R}$. In this case, the robust quadratic convex program under ellipsoidal constraint data uncertainty [12] is given by

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^n} & \max\limits_{A\in\mathcal{U}^{\rm spec}} \{\frac{1}{2}\langle x, Ax\rangle + \langle h, x\rangle\} \\ \text{s.t.} & \langle b_i, x\rangle + \gamma_i \leq 0, \ \forall (b_i, \gamma_i) \in \mathcal{V}_i^e, \ i = 1, \ldots, m, \end{array}$$

Now, we see that the robust solution set of the quadratic convex program under ellipsoidal constraint data uncertainty can be described in terms of the feasible set of a second-order cone programming problem.

Theorem 4.1. (Convex QP under ellipsoidal constraint data uncertainty) *For problem (QP) with $\mathcal{U} = \mathcal{U}^{\rm spec}$ and $\mathcal{V}_i = \mathcal{V}_i^e$, $i = 1, \ldots, m$, let F be the robust feasible set and let S be the robust solution set. Suppose that there exists x_0 such that*

$$\langle b_i, x_0\rangle + \gamma_i < 0, \ \forall (b_i, \gamma_i) \in \mathcal{V}_i^e, \ i = 1, \ldots, m.$$

Let $a \in S$. Then, we have

$$\begin{array}{rcl} S = \{x \in \mathbb{R}^n & : & \|\big(\langle b_i^1, x\rangle, \ldots, \langle b_i^q, x\rangle\big)\| \leq -\langle b_i^0, x\rangle - \overline{\gamma}_i \\ & & (A_0 + \rho I_n)(x - a) = 0, \ \langle (A_0 + \rho I_n)a + h, x - a\rangle = 0\}. \end{array}$$

Proof. Identify S^n with $\mathbb{R}^{\frac{n(n+1)}{2}}$ and consider

$$f(x, A) = \frac{1}{2}\langle x, Ax\rangle + \langle h, x\rangle$$

and

$$g_i(x, (b_i, \gamma_i)) = \langle b_i, x\rangle + \gamma_i.$$

Clearly, $f(\cdot, A)$ is differentiable and convex for all $A \in \mathcal{U}^{\text{spec}}$ and $f(x, \cdot)$ is affine for each $x \in \mathbb{R}^n$; $g_i(\cdot, (b_i, \gamma_i))$ is affine for all $(b_i, \gamma_i) \in \mathcal{V}_i^e$ and $g_i(x, \cdot)$ is affine for each $x \in \mathbb{R}^n$. Then applying Proposition 2.1 gives us that there exist $\lambda_i^a \geq 0$, $A^a \in \mathcal{U}^{\text{spec}}$ and $(b_i^a, \gamma_i^a) \in \mathcal{V}_i^e$ such that

$$0 = (A_0 + \rho I_n)a + h + \sum_{i=1}^{n} \lambda_i^a b_i^a \text{ and } \lambda_i^a \Big(\langle b_i^a, a \rangle + \gamma_i^a \Big) = 0, i = 1, \dots, m.$$

So, it follows from Corollary 3.2 that

$$\begin{aligned} S = \{x \in F \quad : \quad & \lambda_i^a g_i(x, (b_i^a, \gamma_i^a)) = 0, i = 1, \dots, m, f(x, A^a) = \max_{A \in \mathcal{U}^{\text{spec}}} f(x, A), \\ & \nabla_x f(x, A^a) = \nabla_x f(a, A^a), \langle \nabla_x f(a, A^a), x - a \rangle = 0\}. \end{aligned}$$

Note that $\max_{A \in \mathcal{U}^{\text{spec}}} f(x, A)$ is attained at $A^a = A_0 + \rho I_n$. Then, $u^a = A_0 + \rho I_n$.

$$\begin{aligned} S = \{x \in F \quad : \quad & \lambda_i^a g_i(x, (b_i^a, \gamma_i^a)) = 0, i = 1, \dots, m, \\ & (A_0 + \rho I_n)(x - a) = 0, \langle (A_0 + \rho I_n)a + h, x - a \rangle = 0\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} F &= \{x : \langle b_i, x \rangle + \gamma_i \leq 0, \ \forall (b_i, \gamma_i) \in \mathcal{V}_i^e, \ i = 1, \dots, m\} \\ &= \{x : \langle b_i^0 + \sum_{l=1}^{q} \beta_i^l b_i^l, x \rangle + \gamma_i \leq 0 \ \forall \|(\beta_i^1, \dots, \beta_i^q)\| \leq 1 \text{ and } \gamma_i \in [\underline{\gamma}_i, \overline{\gamma}_i]\} \\ &= \{x : \langle b_i^0, x \rangle + \|\big(\langle b_i^1, x \rangle, \dots, \langle b_i^q, x \rangle\big)\| + \overline{\gamma}_i \leq 0\} \end{aligned}$$

Now, S can be expressed as

$$\begin{aligned} S = \{x \in \mathbb{R}^n \quad : \quad & \langle b_i^0, x \rangle + \|\big(\langle b_i^1, x \rangle, \dots, \langle b_i^q, x \rangle\big)\| + \overline{\gamma}_i \leq 0 \\ & \lambda_i^a \big(\langle b_i^a, x \rangle + \gamma_i^a \big) = 0, \ i = 1, \dots, m, \\ & (A_0 + \rho I_n)(x - a) = 0, (\langle A_0 + \rho I_n)a + h, x - a \rangle = 0\}. \end{aligned}$$

Finally, to see the conclusion, we only need to show, for any $x \in F$ with

$$\langle (A_0 + \rho I_n)a + h, x - a \rangle = 0,$$

we have

$$\lambda_i^a \big(\langle b_i^a, x \rangle + \gamma_i^a \big) = 0, i = 1, \dots, m$$

To see this, take any $x \in F$. Then, we have $\langle b_i^a, x \rangle + \gamma_i^a \leq 0$. This together with $\lambda_i^a \geq 0$ implies that

$$\lambda_i^a \big(\langle b_i^a, x \rangle + \gamma_i^a \big) \leq 0, i = 1, \dots, m. \tag{13}$$

Now,

$$0 = \langle (A_0 + \rho I_n)a + h + \sum_{i=1}^{m} \lambda_i^a b_i^a, x - a\rangle = \langle \sum_{i=1}^{m} \lambda_i^a b_i^a, x - a\rangle.$$

So,

$$\sum_{i=1}^{m} \lambda_i^a \big(\langle b_i^a, x\rangle + \gamma_i^a\big) = \sum_{i=1}^{m} \lambda_i^a \big(\langle b_i^a, a\rangle + \gamma_i^a\big) = 0.$$

This together with (13) implies that $\lambda_i^a \big(\langle b_i^a, x\rangle + \gamma_i^a\big) = 0, i = 1, \dots, m.$ □

In the uncertainty free case, that is, $\rho = 0$ and $b_i^l = 0$, $l = 1, \dots, q$, our result collapses to the solution set characterization of uncertainty-free convex quadratic optimization problem given in [7].

Remark 4.1. *Note that Theorem 4.1 shows that the robust solution set of an uncertain convex quadratic optimization problem under ellipsoidal data uncertainty is either an empty set or a conic representable set in the sense that it can be described as the feasible set of a suitable second order cone programming problem.*

Scenario constraint data uncertainty

Consider the quadratic convex program under scenario constraint data uncertainty [13, 12] where $\mathcal{U}$ is the spectral norm uncertainty set $\mathcal{U}^{\rm spec}$ given by

$$\mathcal{U}^{\rm spec} := \{A_0 + M : \ M \in S^n, \ A + M \succeq 0, \ \|M\|_{\rm spec} \leq \rho\}$$

with $\rho \geq 0$ and $A_0 \in S^n$ with $A_0 \succeq 0$, and $\mathcal{V}_i$ is the scenario uncertainty set $\mathcal{V}_i^s$ given by

$$\mathcal{V}_i^s = {\rm co}\{(b_i^1, \gamma_i^1), \dots, (b_i^{p_i}, \gamma_i^{p_i})\},$$

with $(b_i^l, \gamma_i^l) \in \mathbb{R}^n \times \mathbb{R}$, $l = 1, \dots, p_i$. In this case, the robust quadratic convex program under scenario constraint data uncertainty is given by

$$\begin{array}{ll} \min\limits_{x \in \mathbb{R}^n} & \max\limits_{A \in \mathcal{U}^{\rm spec}} \{\frac{1}{2}\langle x, Ax\rangle + \langle h, x\rangle\} \\ \text{s.t.} & \langle b_i, x\rangle + \gamma_i \leq 0, \ \forall (b_i, \gamma_i) \in \mathcal{V}_i^s, \ i = 1, \dots, m, \end{array}$$

Now, we see that the robust solution set of the quadratic convex program under scenario constraint data uncertainty can be expressed as a polyhedral set.

Theorem 4.2. (Convex QP under scenario data uncertainty) *For problem* (QP) *with* $\mathcal{U} = \mathcal{U}^{\rm spec}$ *and* $\mathcal{V}_i = \mathcal{V}_i^s$, $i = 1, \ldots, m$, *let* F *be the robust feasible set and let* S *be the robust solution set. Suppose that there exists* x_0 *such that* $b_i, x_0 + \gamma_i < 0$, $\forall (b_i, \gamma_i) \in \mathcal{V}_i^s$, $i = 1, \ldots, m$. *Let* $a \in S$. *Then, we have*

$$\begin{aligned} S = \{x \in \mathbb{R}^n \quad : \quad & \langle b_i^l, x\rangle + \gamma_i^l \leq 0,\ l = 1, \ldots, p_i,\ i = 1, \ldots, m, \\ & (A_0 + \rho I_n)(x - a) = 0,\ \langle (A_0 + \rho I_n)a + h, x - a\rangle = 0\}. \end{aligned}$$

Proof. Identify S^n with $\mathbb{R}^{\frac{n(n+1)}{2}}$, and consider

$$f(x, A) = \frac{1}{2}\langle x, Ax\rangle + \langle h, x\rangle$$

and

$$g_i(x, (b_i, \gamma_i)) = \langle b_i, x\rangle + \gamma_i,\ \ i = 1, \ldots, m.$$

Using similar method of proof as in Theorem 4.1, we see that there exist $\lambda_i^a \geq 0$, $A^a \in \mathcal{U}^{\rm spec}$ and $(b_i^a, \gamma_i^a) \in \mathcal{V}_i^s$ such that

$$0 = (A_0 + \rho I_n)a + h + \sum_{i=1}^{n} \lambda_i^a b_i^a,$$

$$\lambda_i^a \bigg(\langle b_i^a, a\rangle + \gamma_i^a \bigg) = 0, i = 1, \ldots, m$$

and

$$\begin{aligned} S = \{x \in F \quad : \quad & \lambda_i^a g_i(x, (b_i^a, \gamma_i^a)) = 0, i = 1, \ldots, m, \\ & (A_0 + \rho I_n)(x - a) = 0,\ \langle (A_0 + \rho I_n)a + h, x - a\rangle = 0\}. \end{aligned}$$

Note that

$$\begin{aligned} F &= \{x : \langle b_i, x\rangle + \gamma_i \leq 0\ \forall (b_i, \gamma_i) \in {\rm co}\{(b_i^1, \gamma_i^1), \ldots, (b_i^{p_i}, \gamma_i^{p_i})\}, i = 1, \ldots, m\} \\ &= \{x : \langle b_i^l, x\rangle + \gamma_i^l \leq 0,\ l = 1, \ldots, p_i,\ i = 1, \ldots, m\}, \end{aligned}$$

where the second equality follows by the fact that $(b_i, \gamma_i) \mapsto g_i(x, (b_i, \gamma_i))$ is affine and the maximum of $\max_{(b_i, \gamma_i) \in \mathcal{V}_i^s} g_i(x, (b_i, \gamma_i))$ is attained at some extreme point of $\mathcal{V}_i^s$. Similar to the proof as in Theorem 4.1, we can show that for any $x \in F$ with $\langle (A_0 + \rho I_n)a + h, x - a\rangle = 0$, we have $\lambda_i^a \big(\langle b_i^a, x\rangle + \gamma_i^a \big) = 0, i = 1, \ldots, m.$ $\square$

Remark 4.2. *Theorem 4.2 shows that the robust solution set of an uncertain convex quadratic optimization problem under scenario data uncertainty is either an empty set or a polyhedral set.*

Now, we obtain a characterization of the boundedness of the robust solution set of the quadratic convex program under scenario constraint data uncertainty. Recall that, for a closed and convex set A, its recession cone A^∞ is defined by

$$A^\infty := \{d : x + \gamma d \in A \text{ for all } \gamma \geq 0, x \in A\}.$$

Recall also that a closed convex set A is bounded if and only if its recession cone $A^\infty = \{0\}$.

Corollary 4.1. (Boundedness of the robust solution set) *For problem (QP) with $\mathcal{U} = \mathcal{U}^{\rm spec}$ and $\mathcal{V}_i = \mathcal{V}_i^s$, $i = 1, \ldots, m$, let F be the robust feasible set and let S be the robust solution set. Suppose that there exists x_0 such that $\langle b_i, x_0\rangle + \gamma_i < 0,\ \forall (b_i, \gamma_i) \in \mathcal{V}_i^s$, $i = 1, \ldots, m$ and $S \neq \emptyset$. Then, the robust solution set S is bounded if and only if*

$$\{d \in \mathbb{R}^n \ : \ \langle b_i^l, d\rangle \leq 0,\, l = 1, \ldots, p_i,\, i = 1, \ldots, m, (A_0 + \rho I_n)d = 0,\, \langle h, d\rangle = 0\} = \{0\}.$$

Proof. Let $a \in S$. From the preceding theorem, we have

$$\begin{aligned} S = \{x \in \mathbb{R}^n \ : \ & \langle b_i^l, x\rangle + \gamma_i^l \leq 0,\, l = 1, \ldots, p_i,\, i = 1, \ldots, m, \\ & (A_0 + \rho I_n)(x - a) = 0,\, \langle (A_0 + \rho I_n)a + h, x - a\rangle = 0\}. \end{aligned}$$

Note that S is bounded if and only its recession cone $S^\infty = \{0\}$ and

$$\begin{aligned} S^\infty &= \{d \in \mathbb{R}^n : \langle b_i^l, d\rangle \leq 0,\, l = 1, \ldots, p_i,\, i = 1, \ldots, m, \\ &\qquad (A_0 + \rho I_n)d = 0,\, \langle (A_0 + \rho I_n)a + h, d\rangle = 0\} \\ &= \{d \in \mathbb{R}^n : \langle b_i^l, d\rangle \leq 0,\, l = 1, \ldots, p_i,\, i = 1, \ldots, m, \\ &\qquad (A_0 + \rho I_n)d = 0,\, \langle h, d\rangle = 0\}. \end{aligned}$$

Thus, the conclusion follows. □

5 Uncertain Sum-of-squares convex Polynomial Optimization Problems

In this Section, we establish robust solution set characterization for an uncertain sum-of-squares convex (in short, SOS-convex) polynomial optimization problem. As a special case, we show that the robust solution set of a quadratically constrained quadratic optimization problem under scenario uncertainty can be described as a semidefinite representable set.

We recall that a real polynomial f on $\mathbb{R}^m$ is sum of squares if there exist real polynomials f_j, $j = 1, \ldots, r$, such that $f(x) = \sum_{j=1}^{r} f_j^2(x)$ for all $x \in \mathbb{R}^m$. The set consisting of all sum of squares real polynomials is denoted by Σ^2. Moreover, the set consisting of all sum of squares real polynomials with degree at most d is denoted by Σ_d^2. One of the interesting and important features of a sum-of-squares polynomial is that checking a polynomial is sum of squares or not is equivalent to solving a linear matrix inequality problem.

Definition 5.1. (SOS-Convexity [27, 28]) *A real polynomial f on $\mathbb{R}^n$ is called SOS-convex if $\sigma(x, y) := f(x) - f(y) - \nabla f(y), (x - y)$ is a sum of squares polynomial.*

Clearly, a SOS-convex polynomial is convex. However, the converse is not true. Thus, there exists a convex polynomial which is not SOS-convex [27]. It is known that any convex quadratic function and any convex separable polynomial is a SOS-convex polynomial. Moreover, a SOS-convex polynomial can be non-quadratic and non-separable. For instance, $f(x) = x_1^8 + x_1^2 + x_1x_2 + x_2^2$ is an SOS-convex polynomial, which is non-quadratic and non-separable.

Consider the following uncertain SOS-convex polynomial programming problem

$$(PP) \qquad \min_{x \in \mathbb{R}^n} f(x, u) \quad \text{s.t.} \quad g_i(x, v_i) \leq 0,\ i = 1, \ldots, m,$$

where $u \in \mathcal{U}^s$, $v_i \in \mathcal{V}_i^s$, and $\mathcal{U}^s$ and $\mathcal{V}_i^s$ are scenario uncertainty sets, that is,

$$\mathcal{U}^s = \text{co}\{u_1, \ldots, u_{p_0}\} \text{ and } \mathcal{V}_i^s = \text{co}\{v_i^1, \ldots, v_i^{p_i}\}.$$

Here $f : \mathbb{R}^n \times \mathbb{R}^{q_0} \to \mathbb{R}$ is a function such that for each fixed $u \in \mathcal{U} \subseteq \mathbb{R}^{q_0}$, $f(\cdot, u)$ is a SOS-convex polynomial with degree at most d on $\mathbb{R}^n$; for each fixed $x \in \mathbb{R}^n$, $f(x, \cdot)$ is an affine function on $\mathbb{R}^q$, $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ is a function such that for each fixed $v_i \in \mathcal{V}_i \subseteq \mathbb{R}^q$, $g_i(\cdot, v_i)$ is a SOS-convex polynomial with degree at most d and for each fixed $x \in \mathbb{R}^n$, $g_i(x, \cdot)$ is an affine function. The robust counterpart of the above uncertain SOS-convex polynomial programming problem can be given by

$$\min_{x \in \mathbb{R}^n} \max_{u \in \mathcal{U}^s} f(x, u) \quad \text{s.t.} \quad g_i(x, v_i) \leq 0,\ \forall v_i \in \mathcal{V}_i^s,\ i = 1, \ldots, m.$$

Theorem 5.1. *For problem (PP), let F be the robust feasible set and let S be the robust solution set. Suppose that there exists x_0 such that $g_i(x, v_i) < 0,\ \forall v_i \in \mathcal{V}_i,\ i = 1, \ldots, m$. Let $a \in S$, and let $\lambda_i^a \geq 0$, $u^a \in \mathcal{U}^s$ and $v_i^a \in \mathcal{V}_i^s$ be the multipliers associated with a. Then,*

$$\begin{aligned} S = \{z \in \mathbb{R}^n \ \ : \ \ & g_i(z, v_i^l) \leq 0, l = 1, \ldots, p_i,\ i = 1, \ldots, m, \\ & f(\cdot, u^a) + \sum_{i=1}^m \lambda_i^a g_i(\cdot, v_i^a) - \max_{1 \leq i \leq p_0} f(z, u_i) \in \Sigma_d^2\}. \end{aligned}$$

Proof. Let $z \in S$. Clearly, $z \in F$. Note that any affine function attains its maximum on a compact polytope at some extreme points of the polytope. As for each fixed $x \in \mathbb{R}^n$, $g_i(x, \cdot)$ is an affine function, we obtain that $\max_{v_i \in \mathcal{V}_i^s} g_i(z, v_i) = \max_{1 \leq l \leq p_i} g_i(z, v_i^l)$, $l = 1, \ldots, p_i$, $i = 1, \ldots, m$. This implies that

$$F = \{z : g_i(z, v_i) \leq 0, \forall\, v_i \in \mathcal{V}_i^s\} = \{z : g_i(z, v_i^l) \leq 0, l = 1, \ldots, p_i,\ i = 1, \ldots, m\}.$$

We now show that $f(\cdot, u^a) + \sum_{i=1}^m \lambda_i^a g_i(\cdot, v_i^a) - \tilde{f}(z) \in \Sigma_d^2$. To see this, note that $a \in S$ and $\lambda_i^a \geq 0$, $u^a \in \mathcal{U}$ and $v_i^a \in \mathcal{V}_i$ are the multipliers associated with a. This shows that

$$0 = \nabla_x f(a, u^a) + \sum_{i=1}^m \lambda_i^a \nabla_x g_i(a, v_i^a) \text{ and } \lambda_i^a g_i(a, v_i^a) = 0, i = 1, \ldots, m.$$

As $f(\cdot, u^a)$ and $g_i(\cdot, v^a)$ are SOS-convex polynomials and $\lambda_i^a \geq 0$,

$$h(x) := f(x, u^a) + \sum_{i=1}^m \lambda_i^a g_i(x, v_i^a) - f(a, u^a)$$

is also a SOS-convex polynomial with degree at most d. Note that h attains its global minimum at a with $\nabla h(a) = 0$ and $h(a) = 0$. From the definition of SOS-convex polynomial

$$\sigma(x, y) := h(x) - h(y) - \langle \nabla h(y), x - y \rangle$$

is a sum-of-squares polynomial. In particular, letting $y = a$, we see that $h(x) = \sigma(x, a)$ is also a sum-of-squares polynomial. Note that, for each $x \in \mathbb{R}^n$, $\max_{u \in \mathcal{U}^s} f(x, u) = \max_{1 \leq i \leq p_0} f(x, u_i)$ (as $f(x, \cdot)$ is affine and $\mathcal{U}^s = \text{co}\{u_1, \ldots, u_{p_0}\}$ is a polytope). So, we have

$$\max_{1 \leq i \leq p_0} f(a, u_i) = \max_{u \in \mathcal{U}^s} f(a, u) = \max_{u \in \mathcal{U}^s} f(z, u) = \max_{1 \leq i \leq p_0} f(z, u_i),$$

where the second equality follows by the fact that $z \in S$ and $a \in S$. This implies that

$$\begin{aligned} f(x, u^a) + \sum_{i=1}^m \lambda_i^a g_i(x, v_i^a) - \max_{1 \leq i \leq p_0} f(z, u_i) &= h(x) + (f(a, u^a) - \max_{1 \leq i \leq p_0} f(z, u_i)) \\ &= h(x) + (\max_{u \in \mathcal{U}^s} f(a, u) - \max_{1 \leq i \leq p_0} f(z, u_i)) = h(x) \end{aligned}$$

is a sum-of-squares polynomial. Moreover, note that $f(\cdot, u^a)$ is a SOS-convex polynomial with degree at most d on $\mathbb{R}^n$, $g_i(\cdot, v_i^a)$ is a SOS-convex polynomial with degree at most d and $\max_{1\le i\le p_0} f(z, u_i)$ is a constant. So, h is a sum-of-squares polynomial with degree at most d.

Conversely, let $z \in F$ with $f(\cdot, u^a) + \sum_{i=1}^m \lambda_i^a g_i(\cdot, v_i^a) - \max_{1\le i\le p_0} f(z, u_i) \in \Sigma_d^2$. Note that any sum-of-squares polynomial must take non-negative value. So,

$$\begin{aligned} 0 \le f(a, u^a) + \sum_{i=1}^m \lambda_i^a g_i(a, v_i^a) - \max_{1\le i\le p_0} f(z, u_i) &\le \max_{1\le i\le p_0} f(a, u_i) - \max_{1\le i\le p_0} f(z, u_i) \\ &= \max_{u\in\mathcal{U}^s} f(a, u) - \max_{u\in\mathcal{U}^s} f(z, u). \end{aligned}$$

This together with $z \in F$ shows that $z \in S$. □

We present an example to illustrate the robust solution set characterization of an uncertain SOS-convex polynomial program.

Example 5.1. *Consider the following uncertain SOS-convex polynomial optimization problem*

$$\min_{(x_1,x_2)\in\mathbb{R}^2} \{x_1^4 + \alpha_1 x_1 + \alpha_2 x_2\} \quad s.t. \ \beta x_1 + 1 \le 0,$$

where the uncertain parameters $\alpha_1 \in [-1, 1]$, $\alpha_2 \in [0, 1]$ *and* $\beta \in [-2, -1]$. *Its robust counterpart is given by*

$$\min_{(x_1,x_2)\in\mathbb{R}^2} \max_{\alpha_1\in[-1,1],\alpha_2\in[0,1]} \{x_1^4 + \alpha_1 x_1 + \alpha_2 x_2\} \quad s.t. \quad \beta x_1 + 1 \le 0, \forall \beta \in [-2, -1].$$

Let $f(x, u) = x_1^4 + u_1 x_1 + u_2 x_2$. *Then,* $f(\cdot, u)$ *is a separable convex polynomial (and so, SOS-convex) and* $f(x, \cdot)$ *is affine. Let* $g_1(x, v_1) = v_1^1 x_1 + v_1^2 x_2 + 1$. *Then,* $g_1(\cdot, v_1)$ *is affine and* $g_1(x, \cdot)$ *is affine. This robust optimization problem can be written as*

$$\min_{x\in\mathbb{R}^2} \max_{u\in\mathcal{U}} f(x, u) \quad s.t. \quad g_1(x, v_1) \le 0, \forall v_1 \in \mathcal{V}_1.$$

where $\mathcal{U} = \mathrm{co}\{(-1, 0), (1, 0), (-1, 1), (1, 1)\}$ *and* $\mathcal{V}_1 = \mathrm{co}\{(-2, 0), (-1, 0)\}$. *This problem is equivalent to*

$$\min_{(x_1,x_2)\in\mathbb{R}^2} \{x_1^4 + |x_1| + \max\{x_2, 0\}\} \quad s.t. \quad -x_1 + 1 \le 0.$$

and so, $F = \{(x_1, x_2) : x_1 \ge 1\}$ *and* $S = \{(x_1, x_2) : x_1 = 1, x_2 \le 0\}$.

To verify our robust solution characterization, let $a = (1,0) \in S$, $u^a = (1,0)$, $v^a = (-1,0)$ *and* $\lambda^a = 5$. *Then, we have*

$$(0,0)^T = \nabla_x f(a,u^a) + \lambda^a \nabla_x g_1(a,v^a) = (5,0)^T + 5(-1,0)^T$$

and

$$\lambda^a g_1(a,v^a) = 0.$$

Note that $\mathcal{U} = \mathrm{co}\{u_1,\ldots,u_4\}$ *with* $u_1 = (-1,0), u_2 = (1,0), u_3 = (-1,1)$ *and* $u_4 = (1,1)$. *So,*

$$\begin{aligned} & \{z \in \mathbb{R}^n : g_1(z,v_1^l) \le 0, l = 1,2,\ f(\cdot,u^a) + \lambda^a g_1(\cdot,v^a) - \max_{1\le i\le 4} f(z,u_i) \in \Sigma_d^2\} \\ = \ & \{z \in \mathbb{R}^2 : z_1 \ge 1, h - (z_1^4 + |z_1| + \max\{z_2,0\}) \in \Sigma_4^2\}, \end{aligned} \tag{14}$$

where $h(x_1,x_2) = x_1^4 - 4x_1 + 5$. *Note that for any* $z_1 = 1$ *and* $z_2 \le 0$, $z_1^4 + |z_1| + \max\{z_2,0\} = 2$, *and so,*

$$h - (z_1^4 + |z_1| + \max\{z_2,0\}) = x_1^4 - 4x_1 + 3 = (x_1^2 - 1)^2 + 2(x_1 - 1)^2 \in \Sigma_4^2.$$

Moreover, for any $(z_1,z_2) \in ([1,+\infty) \times \mathbb{R}) \backslash (\{1\} \times (-\infty,0])$, $z_1^4 + |z_1| + \max\{z_2,0\} > 2$, *and so,*

$$h(1,0) - (z_1^4 + |z_1| + \max\{z_2,0\}) = 2 - (z_1^4 + |z_1| + \max\{z_2,0\}) < 0,$$

This shows that for any $(z_1,z_2) \in ([1,+\infty) \times \mathbb{R}) \backslash (\{1\} \times (-\infty,0])$,

$$h - (z_1^4 + |z_1| + \max\{z_2,0\}) \notin \Sigma_4^2.$$

Therefore, (14) implies that

$$\begin{aligned} & \{z \in \mathbb{R}^n : g_1(z,v_1^l) \le 0, l = 1,2,\ f(\cdot,u^a) + \lambda^a g_1(\cdot,v^a) - \tilde{f}(z) \in \Sigma_d^2\} \\ = \ & \{z \in \mathbb{R}^2 : z_1 \ge 1, h - (z_1^4 + |z_1| + \max\{z_2,0\}) \in \Sigma_4^2\} \\ = \ & \{z \in \mathbb{R}^2 : z_1 = 1, z_2 \le 0\} = S. \end{aligned}$$

This verifies our robust solution characterization.

Consider the following quadratic convex program with quadratic constraint under scenario data uncertainty

$$\min_{x\in\mathbb{R}^n} \frac{1}{2}\langle x, Ax\rangle + \langle h, x\rangle \quad \text{s.t.} \quad \frac{1}{2}\langle x, B_i x\rangle + \langle b_i, x\rangle + \gamma_i \le 0, i = 1,\ldots,m,$$

where the data $(A,h) \in S^n \times \mathbb{R}^n$ and $(B_i, b_i, \gamma_i) \in S^n \times \mathbb{R}^n \times \mathbb{R}$ are uncertain, $(A,h) \in \mathcal{U}^s$, $(B_i, b_i, \gamma_i) \in \mathcal{V}_i^s$ and $\mathcal{U}^s$, $\mathcal{V}_i^s$ are the scenario data uncertainty sets given by

$$\mathcal{U}^s := \mathrm{co}\{(A_1, h_1), \ldots, (A_{p_0}, h_{p_0})\} \text{ and } \mathcal{V}_i^s = \mathrm{co}\{(B_i^1, b_i^1, \gamma_i^1), \ldots, (B_i^{p_i}, b_i^{p_i}, \gamma_i^{p_i})\},$$

with $(A_i, h_i) \in S^n \times \mathbb{R}^n$ and $(B_i^l, b_i^l, \gamma_i^l) \in S^n \times \mathbb{R}^n \times \mathbb{R}$, $l = 1, \ldots, p_i$ and A_i and B_i^l are positive semidefinite matrices. Define

$$f(x,u) = \frac{1}{2}\langle x, Ax\rangle + \langle h, x\rangle, \quad u = (A,h),$$

and

$$g_i(x, v_i) = \frac{1}{2}\langle x, B_i x\rangle + \langle b_i, x\rangle + \gamma_i, \quad v_i = (B_i, b_i, \gamma_i).$$

The the above quadratic convex program with quadratic constraint under scenario data uncertainty can be written as a form of (PP)

$$(QQP_s) \qquad \min_{x \in \mathbb{R}^n} f(x,u) \quad \text{s.t.} \quad g_i(x, v_i) \le 0, i = 1, \ldots, m,$$

where $u = (A,h) \in \mathcal{U}^s$ and $v_i = (B_i, b_i, \gamma_i) \in \mathcal{V}_i^s$. The robust counterpart of quadratic convex program with quadratic constraint under scenario data uncertainty can be given by

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \max_{(A,h) \in \mathcal{U}^s} \{\frac{1}{2}\langle x, Ax\rangle + \langle h, x\rangle\} \\ \text{s.t.} \quad & \frac{1}{2}\langle x, B_i x\rangle + \langle b_i, x\rangle + \gamma_i \le 0, \ \forall (B_i, b_i, \gamma_i) \in \mathcal{V}_i^s, \ i = 1, \ldots, m. \end{aligned}$$

In this case we see that the solution set of quadratic convex program with quadratic constraint under scenario data uncertainty can be described by a semi-definite representable set.

To do this, we first introduce some definitions and present a simple fact which will be used later on. For any $q \in \mathbb{N}$ and $(u,r) \in \mathbb{R}^q \times \mathbb{R}$, we define $\|(u,r)\|_2 := \sqrt{\langle u, u\rangle + r^2}$. Then, it is known that [29], for any $(u,r) \in \mathbb{R}^q \times \mathbb{R}$ with $q \in \mathbb{N}$, we have

$$\begin{aligned} \|u\|^2 + 2r \le 0 \quad & \Leftrightarrow \quad \sqrt{\langle u, u\rangle + (1 + \frac{r}{2})^2} \le 1 - \frac{r}{2} \\ & \Leftrightarrow \quad \|(u, 1 + \frac{r}{2})\|_2 \le 1 - \frac{r}{2}. \end{aligned} \tag{15}$$

Corollary 5.1. *For (QQP_s), let F be the robust feasible set and let S be the robust solution set. Suppose that there exists x_0 such that*

$$\frac{1}{2}\langle x_0, (B_i^l)x_0\rangle + \langle b_i^l, x_0\rangle + \gamma_i^l < 0, l = 1, \ldots, p_i, \ i = 1, \ldots, m.$$

Let $a \in S$, and let $\lambda_i^a \geq 0$, $(A^a, h^a) \in \mathcal{U}^s$ and $(B_i^a, b_i^a, \gamma_i^a) \in \mathcal{V}_i^s$ be the multipliers associated with a. Then, we have

$$
\begin{aligned}
S = \{z \in \mathbb{R}^n : \exists t \in \mathbb{R} \quad s.t. \quad & \|(L_i^l x, 1 + \frac{\gamma_i^l + \langle b_i^l, x\rangle}{2})\|_2 \leq 1 - \frac{\gamma_i^l + \langle b_i^l, x\rangle}{2}, l = 1, \ldots, p_i,\, i = 1, \ldots, m, \\
& \begin{pmatrix} A^a & h^a \\ (h^a)^T, & -2t \end{pmatrix} + \sum_{i=1}^m \lambda_i^a \begin{pmatrix} B_i^a & (b_i^a) \\ (b_i^a)^T, & 2\gamma_i^a \end{pmatrix} \succeq 0, \\
& \|(M_i x, 1 + \frac{t - \langle h_i, x\rangle}{2})\|_2 \leq 1 - \frac{t - \langle h_i, x\rangle}{2}, i = 1, \ldots, p_0.\}
\end{aligned}
$$

where $L_i^l \in \mathbb{R}^{s_i^l \times n}$ is a matrix satisfying $B_i^l = (L_i^l)^T L_i^l$, $s_i^l \in \mathbb{N}$, $l = 1, \ldots, p_i$, $i = 1, \ldots, m$ and $M_i \in \mathbb{R}^{r_i \times n}$ is a matrix satisfying $A_i = M_i^T M_i$, $r_i \in \mathbb{N}$, $i = 1, \ldots, m$.

Proof. Consider

$$f(x, (A, h)) = \frac{1}{2}\langle x, Ax\rangle + \langle h, x\rangle \text{ and } g_i(x, (B_i, b_i, \gamma_i)) = \frac{1}{2}\langle x, B_i x\rangle + \langle b_i, x\rangle + \gamma_i.$$

Then, the preceding theorem implies that

$$
\begin{aligned}
S = \{z \in \mathbb{R}^n \quad : \quad & \frac{1}{2}\langle x, B_i^l x\rangle + \langle b_i^l, x\rangle + \gamma_i^l \leq 0, l = 1, \ldots, p_i,\, i = 1, \ldots, m, \\
& \frac{1}{2}\langle x, A^a x\rangle + \langle h^a, x\rangle + \sum_{i=1}^m \lambda_i^a (\frac{1}{2}\langle x, B_i^a x\rangle + \langle b_i^a, x\rangle + \gamma_i^a) - \max_{1 \leq i \leq p_0} \{\frac{1}{2}\langle z, A_i z\rangle + \langle h_i, z\rangle\} \in \Sigma_d^2\}.
\end{aligned}
$$

The robust solution set S can be equivalently rewritten as

$$
\begin{aligned}
S = \{z \in \mathbb{R}^n : \exists\, t \in \mathbb{R}^n \quad \text{s.t.} \quad & \frac{1}{2}\langle x, B_i^l x\rangle + \langle b_i^l, x\rangle + \gamma_i^l \leq 0, l = 1, \ldots, p_i,\, i = 1, \ldots, m, \\
& \frac{1}{2}\langle x, A^a x\rangle + \langle h^a, x\rangle + \sum_{i=1}^m \lambda_i^a (\frac{1}{2}\langle x, B_i^a x\rangle + \langle b_i^a, x\rangle + \gamma_i^a) - t \in \Sigma_d^2\}, \\
& \frac{1}{2}\langle x, A_i x\rangle + \langle h_i, x\rangle \leq t, i = 1, \ldots, p_0\}.
\end{aligned}
$$

Letting $B_i^l = (L_i^l)^T L_i^l$ where $L_i^l \in \mathbb{R}^{s_i^l \times n}$ for some $s_i^l \in \mathbb{N}$ and $A_i = M_i^T M_i$ where $M_i \in \mathbb{R}^{r_i \times n}$ for some $r_i \in \mathbb{N}$, then $\frac{1}{2}\langle x, (B_i^l)x\rangle + \langle b_i^l, x\rangle + \gamma_i^l \leq 0$ is equivalent to $\|L_i^l x\|^2 + 2(\gamma_i^l + \langle b_i^l, x\rangle) \leq 0$. Applying (15) with $u = L_i^l x$ and $r = \gamma_i^l + \langle b_i^l, x\rangle$, we see that $\frac{1}{2}\langle x, (B_i^l)x\rangle + \langle b_i^l, x\rangle + \gamma_i^l \leq 0$ which can be further equivalently rewritten as

$$\|(L_i^l x, 1 + \frac{\gamma_i^l + \langle b_i^l, x\rangle}{2})\|_2 \leq 1 - \frac{\gamma_i^l + \langle b_i^l, x\rangle}{2}, l = 1, \ldots, p_i,\, i = 1, \ldots, m. \tag{16}$$

Similarly, $\max_{(A,h) \in \mathcal{U}^s} \{\frac{1}{2}\langle x, Ax\rangle + \langle h, x\rangle\} \leq t$ is equivalent to $\frac{1}{2}\langle x, A_i x\rangle + \langle h_i, x\rangle - t \leq 0$ for all $i = 1, \ldots, p_0$ which can be equivalently rewritten as

$$\|(M_i x, 1 + \frac{t - \langle h_i, x\rangle}{2} + 1)\|_2 \leq 1 - \frac{t - \langle h_i, x\rangle}{2}, i = 1, \ldots, p_0.$$

Thus, the conclusion follows by noting that

$$\frac{1}{2}\langle x, A^a x\rangle + \langle h^a, x\rangle + \sum_{i=1}^{m} \lambda_i^a \Big(\frac{1}{2}\langle x, B_i^a x\rangle + \langle b_i^a, x\rangle + \gamma_i^a\Big) - t \in \Sigma_d^2$$

$$\Leftrightarrow \quad \begin{pmatrix} A^a & h^a \\ (h^a)^T & -2t \end{pmatrix} + \sum_{i=1}^{m} \lambda_i^l \begin{pmatrix} B_i^a & (b_i^a) \\ (b_i^a)^T & 2\gamma_i^a \end{pmatrix} \succeq 0.$$

□

Remark 5.1. *As $\|x\| \leq t$ is equivalent to $\begin{pmatrix} tI_n & x \\ x^T & t \end{pmatrix} \succeq 0$, the above corollary shows that the robust solution set of the quadratic programming problem with quadratic constraint under scenario data uncertainty can be written as the projection of a set described by linear matrix inequalities (which is often referred as semi-definite representable set). More generally, noting that any lower level set of SOS-convex inequality is semi-definite representable [28], Theorem 5.1 shows that robust solution set of the SOS-convex polynomial programming problem under scenario data uncertainty set is also semi-definite representable.*

6 Solution Sets of Uncertain Fractional Programs

The uncertain fractional programming problem can be captured by the following parameterized problem:

$$(FP) \qquad \min_{x\in\mathbb{R}^n} \frac{f(x,u)}{h(x,w)} \quad \text{s.t.} \quad g_i(x,v_i) \leq 0, i = 1,\ldots,m,$$

where u, w, v_i are uncertain parameters and they belong to the corresponding convex and compact uncertainty sets $\mathcal{U} \subseteq \mathbb{R}^{q_0}$, $\mathcal{W} \subseteq \mathbb{R}^{q_1}$ and $\mathcal{V}_i \subseteq \mathbb{R}^q$. Note that $f : \mathbb{R}^n \times \mathbb{R}^{q_0} \to \mathbb{R}$ is a continuous function such that for each fixed $u \in \mathcal{U}$, $f(\cdot, u)$ is a convex function on $\mathbb{R}^n$; for each fixed $x \in \mathbb{R}^n$, $f(x, \cdot)$ is a concave function on $\mathbb{R}^q$. Moreover $g_i : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$ is a continuous function such that $g_i(\cdot, v_i)$ is a convex function and for each fixed $x \in \mathbb{R}^n$, $g_i(x, \cdot)$ is a concave function. Finally, for each fixed $v_i \in \mathcal{V}_i$, $h : \mathbb{R}^n \times \mathbb{R}^{q_1} \to \mathbb{R}$ is a continuous function such that for each fixed $w \in \mathcal{W} \subseteq \mathbb{R}^{q_1}$, $h(\cdot, w)$ is a concave function on $\mathbb{R}^n$; for each fixed $x \in \mathbb{R}^n$, $h(x, \cdot)$ is a convex function on $\mathbb{R}^{q_1}$.

Its robust counterpart can be formulated as

$$\min_{x\in\mathbb{R}^n} \frac{\max_{u\in\mathcal{U}} f(x,u)}{\min_{w\in\mathcal{W}} h(x,w)} \quad \text{s.t.} \quad g_i(x,v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1,\ldots,m.$$

The robust feasible set of (FP) is denoted by F, and is given by

$$F = \{x \in \mathbb{R}^n : g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m\},$$

Moreover, the robust solution set of (FP) is denoted by S, and is defined by

$$S = \{x \in F : \frac{\max_{u \in \mathcal{U}} f(x, u)}{\min_{w \in \mathcal{W}} h(x, w)} \leq \frac{\max_{u \in \mathcal{U}} f(y, u)}{\min_{w \in \mathcal{W}} h(y, w)} \, \forall \, y \in F\}.$$

We assume that $f(x, u) \geq 0$ and $h(x, w) > 0$ for all $x \in F$, $u \in \mathcal{U}$ and $w \in \mathcal{W}$. In the case where $h(\cdot, w)$ are all affine functions for all $w \in \mathcal{W}$, the condition $f(x, u) \geq 0$ for all $x \in F$ and $u \in \mathcal{U}$ can be dropped.

Proposition 6.1. *For problem (FP), let F be the robust feasible set and let S be the robust solution set. Suppose that $f(x, u) \geq 0$ and $h(x, w) > 0$ for all $x \in F$, $u \in \mathcal{U}$ and $w \in \mathcal{W}$. Let a be a robust solution of (FP), that is, $a \in S$. Then, there exist $\lambda_i^a \geq 0$, $(u^a, w^a) \in \mathcal{U} \times \mathcal{W}$ and $v_i^a \in \mathcal{V}_i$ such that*

$$0 \in q(a)\partial_x f(a, u^a) + p(a)\partial_x(-h)(a, u^a) + \sum_{i=1}^{n} \lambda_i^a \partial_x g_i(a, v_i^a), \ \lambda_i^a g_i^a(a, v_i) = 0$$

and

$$q(a)f(x, u^a) - p(a)h(x, w^a) = \max_{(u,w) \in \mathcal{U} \times \mathcal{W}} \{q(a)f(a, u) - p(a)h(a, w)\}.$$

Proof. As $a \in S$, we see that a is a solution of the following robust convex optimization problem:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \max_{(u,w) \in \mathcal{U} \times \mathcal{W}} \{q(a)f(x, u) - p(a)h(x, w)\} \\ \text{s.t.} \quad & g_i(x, v_i) \leq 0, \ \forall v_i \in \mathcal{V}_i, \ i = 1, \ldots, m, \end{aligned}$$

where

$$q(a) = \min_{w \in \mathcal{W}} h(a, w) \text{ and } p(a) = \max_{u \in \mathcal{U}} f(a, u).$$

Define $\tilde{f}(x, u, w) := q(a)f(x, u) - p(a)h(x, w)$. From our assumption, it is clear that $\tilde{f}(\cdot, u, w)$ is continuous convex for any $(u, w) \in \mathcal{U} \times \mathcal{W}$ and $\tilde{f}(x, \cdot, \cdot)$ is continuous concave for any $x \in \mathbb{R}^n$. So, the conclusion follows from Proposition 2.1 with f replaced by $\tilde{f}$. □

Theorem 6.1. (Robust solution set of uncertain fractional program) *For problem (FP), let F be the robust feasible set and let S be the robust solution set. Suppose that there exists $x_0 \in \mathbb{R}^n$ such that $g_i(x_0, v_i) < 0$, $\forall v_i \in \mathcal{V}_i$, $i = 1, \ldots, m$. Let a be a robust solution of*

(FP), that is, $a \in S$. Let $\lambda_i^a \geq 0$, $(u^a, w^a) \in \mathcal{U}$ and $v_i^a \in \mathcal{V}_i$ be the multiplier associated with a. Then,

$$S = \{x \in F \ : \ \lambda_i^a g_i(x, v_i^a) = 0, i = 1, \ldots, m,$$
$$q(a)f(x, u^a) - p(a)h(x, w^a) = \max_{(u,w) \in \mathcal{U} \times \mathcal{W}} \{q(a)f(x, u) - p(a)h(x, w)\}$$
$$\exists\, w_a \in \big(q(a)\partial_x f(x, u) + p(a)\partial_x(-h)(x, w)\big) \cap \big(q(a)\partial_x f(a, u) + p(a)\partial_x(-h)(a, w)\big),$$
$$\langle w_a, x - a\rangle = 0\}.$$

Proof. Define $\tilde{f}(x, u, w) := q(a)f(x, u) - p(a)h(x, w)$. From our assumption, it is clear that $\tilde{f}(\cdot, u, w)$ is continuous convex for any $(u, w) \in \mathcal{U} \times \mathcal{W}$ and $\tilde{f}(x, \cdot, \cdot)$ is continuous concave for any $x \in \mathbb{R}^n$. So, the conclusion follows from Theorem 3.2 with f replaced by $\tilde{f}$. □

7 Conclusions

Robust optimization has emerged as a powerful approach for dealing with data uncertainty and it treats uncertainty as deterministic, but does not limit data values to point estimates. In this framework, one associates with the uncertain optimization problem its *robust counterpart*, where the uncertain constraints are enforced for every possible value of the data within their prescribed uncertainty sets.

Recent research in robust convex optimization theory has focused on characterizing robust solution points of convex optimization problems in the face of data uncertainty. In this paper, we established simple properties and characterizations of robust solution sets of uncertain convex optimization problems by way of characterizing solution sets of the robust counterpart of the uncertain optimization problems. In particular, we presented generalizations of the constant subdifferential property as well as the constant Lagrangian property for solution sets of convex programming to robust solution sets of uncertain convex programs. We provided various characterizations of robust solution sets of uncertain convex quadratic programs and SOS-convex polynomial programs, under commonly used uncertainty sets of robust optimization, such as the ellipsoidal, scenario and spectral norm uncertainties. We also gave classes of uncertain convex programs where the solution sets can be expressed as conic representable sets.

An interesting open problem is to find robust solutions of hard uncertain bi-level optimization problems by way of studying the conic representability, in particular semidefinite representability, of robust solution sets, described in Section 5. Indeed, the structure of the robust solution set of the inner uncertain bi-level problem can often be useful in studying the outer bi-level problem because in the outer bi-level problem, optimization is carried out on the robust solution set of the uncertain inner problem. In particular, the semidefinite or conic representable robust solution sets of uncertain inner problems can be further studied using semidefinite programming or conic programming respectively. This will form an interesting topic of future research in robust optimization.

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