A Note on Nonconvex Minimax Theorem with Separable Homogeneous Polynomials *

G. Y. Li †

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Abstract

The minimax theorem for a convex-concave bifunction is a fundamental theorem in optimization and convex analysis, and has a lot of applications in economics. In the last two decades, a nonconvex extension of this minimax theorem has been well studied under various generalized convexity assumptions. In this note, by exploiting the hidden convexity (joint range convexity) of separable homogeneous polynomials, we establish a nonconvex minimax theorem involving separable homogeneous polynomials. Our result complements the existing study of nonconvex minimax theorem by obtaining easily verifiable conditions for the nonconvex minimax theorem to hold.

Key words: Minimax theorem, Separable homogeneous polynomial, Generalized convexity, Joint range convexity.

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†Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia. E-mail: g.li@unsw.edu.au

1 Introduction

The minimax theorem for a convex-concave bifunction is a fundamental theorem in optimization and convex analysis, and has a lot of applications in economics. Extension of the classical minimax theorem to the nonconvex case has been well studied (for example, see [1,2,3]) in the last two decades, by imposing generalized convexity assumptions. However, much of the study has been devoted to obtaining more general relaxed conditions rather than explicit and easily verifiable conditions.

The purpose of this note is to provide a nonconvex minimax theorem with easily verifiable conditions. In particular, by exploiting the hidden convexity (joint range convexity) of separable homogeneous polynomials, we establish a nonconvex minimax theorem involving separable homogeneous polynomials. (Similar ideas along this line have been successfully employed to obtain theorems of the alternative for special nonconvex quadratic system; see [4,5]). Our result complements the existing study of nonconvex minimax theorem by obtaining easily verifiable conditions for the nonconvex minimax theorem to hold.

The organization of this paper is as follows. In Section 2, we establish the convexity of the joint range mapping of separable homogeneous polynomials. In Section 3, we provide a nonconvex minimax theorem involving separable homogeneous polynomials. Finally, as a direct application, we establish a zero duality gap result for nonconvex separable homogeneous polynomial programming with bounded constraints in Section 4.

2 Separable Homogeneous Polynomials: Joint Range Convexity

Firstly, $\mathbb{R}^m$ denotes the Euclidean space with dimension m. For each $x, y \in \mathbb{R}^m$, the inner product between x and y is defined by

$$\langle x, y\rangle = \sum_{i=1}^{m} x_i y_i \; x = (x_1, \dots, x_m) \text{ and } y = (y_1, \dots, y_m).$$

Recall that $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is said to be convex iff

$$f((1-\mu)x + \mu y) \leq (1-\mu)f(x) + \mu f(y), \forall \mu \in [0,1] \text{ and } x, y \in \mathbb{R}^m.$$

A set C is said to be convex iff $\mu c_1 + (1-\mu)c_2 \in C$, $\forall \mu \in [0,1]$ and $c_1, c_2 \in C$. We say f is a homogeneous polynomial with degree q iff f is a polynomial and $f(\alpha x) = \alpha^q f(x)$, $\forall \alpha \geq 0$, $x \in \mathbb{R}^m$. The function $f : \mathbb{R}^m \to \mathbb{R}$ is said to be a *separable and homogeneous polynomial* with degree q iff $f(x) = \sum_{j=1}^m f_j(x_j)$, $x = (x_1, \ldots, x_m)$ where each $f_j(\cdot)$ is a homogeneous polynomial with degree q on $\mathbb{R}$.

Let f_i, $i = 1, \ldots, p$, be (nonconvex) separable and homogeneous polynomials on $\mathbb{R}^m$ with degree q, where $q \in \mathbb{N}$. Let Δ be a compact box, i.e., $\Delta := \mathop{\mathbf{X}}_{j=1}^{m} \Delta_j$, where each Δ_j is an interval of $\mathbb{R}$. Consider the joint range mapping of $\{f_1, \ldots, f_p\}$ over Δ, defined by

$$R_\Delta(f_1, \ldots, f_p) := \{(f_1(x), \ldots, f_p(x)) : x \in \Delta\}.$$

Below, we present a lemma showing that $R_\Delta(f_1, \ldots, f_p)$ is always convex. This hidden convexity lemma will play an important role in our nonconvex minimax theorem.

Lemma 2.1. *Let Δ be a compact box in $\mathbb{R}^m$. Let f_i, $i = 1, \ldots, p$ be separable and homogeneous polynomials on $\mathbb{R}^m$ with degree q ($q \in \mathbb{N}$). Then,*

$$R_\Delta(f_1, \ldots, f_p) \textit{ is a convex set in } \mathbb{R}^p.$$

Proof. Since Δ is a compact box in $\mathbb{R}^m$, we can write $\Delta = \mathbf{X}_{j=1}^m \Delta_j$ where Δ_j, $j = 1, \ldots, m$ are intervals in $\mathbb{R}$. Moreover, noting that each f_i, $i = 1, \ldots, p$ is a separable and homogeneous polynomial on $\mathbb{R}^m$ with degree q, we can express

$$f_i(x) = \sum_{j=1}^m f_{ij}(x_j) \;\; \forall x = (x_1, \ldots, x_m),$$

where each $f_{ij} : \mathbb{R} \to \mathbb{R}$ is defined by $f_{ij}(x) := a_i^j x^q$ for some $a_i^j \in \mathbb{R}$, $i = 1, \ldots, p$ and $j = 1, \ldots, m$. Next, we first show that

$$R_\Delta(f_1, \ldots, f_p) = \sum_{j=1}^m \{(f_{1j}(x_j), \ldots, f_{pj}(x_j)) : x_j \in \Delta_j\}. \tag{1}$$

To see (1), take $(u_1, \dots, u_p) \in R_\Delta(f_1, \dots, f_p)$. Then, we have

$$(u_1, \dots, u_p) \in \{(f_1(x), \dots, f_p(x)) : x \in \mathop{\mathbf{X}}_{j=1}^{m} \Delta_j\},$$

and so, there exists $x = (x_1, \dots, x_m) \in \mathbf{X}_{j=1}^m \Delta_j$ such that

$$u_i = f_i(x) = \sum_{j=1}^{m} f_{ij}(x_j) \quad i = 1, \dots, p.$$

Thus, $(u_1, \dots, u_p) \in \sum_{j=1}^m \{(f_{1j}(x_j), \dots, f_{pj}(x_j)) : x_j \in \Delta_j\}$ and so,

$$R_\Delta(f_1, \dots, f_p) \subseteq \sum_{j=1}^{m} \{(f_{1j}(x_j), \dots, f_{pj}(x_j)) : x_j \in \Delta_j\}.$$

The converse inclusion can be verified in a similar way.

Now, by (1), it suffices to show that, for each $j = 1, \dots, m$,

$$\{(f_{1j}(z), \dots, f_{pj}(z)) : z \in \Delta_j\} \text{ is a convex set.} \tag{2}$$

(Indeed, suppose that (2) be true. Since the sum of convex sets is still a convex set, the conclusion follows by (1).) To see (2), fix an arbitrary $j \in \{1, \dots, m\}$. Since Δ_j is a convex compact set in $\mathbb{R}$, we may assume that $\Delta_j = [\alpha_j, \beta_j]$. Then,

$$\{(f_{1j}(z), \dots, f_{pj}(z)) : z \in \Delta_j\} = \{(a_1^j z^q, \dots, a_p^j z^q) : z \in [\alpha_j, \beta_j]\}.$$

Since $z \mapsto z^q$ is a continuous map in $\mathbb{R}$ and $[\alpha_j, \beta_j]$ is a compact connected set in $\mathbb{R}$,

$$C_j = \{z^q : z \in [\alpha_j, \beta_j]\}$$

is also a compact and connected set in $\mathbb{R}$. Thus, C_j is some compact interval in $\mathbb{R}$, $j = 1, \dots, m$. This, together with

$$\{(a_1^j z^q, \dots, a_p^j z^q) : z \in [\alpha_j, \beta_j]\} = \bigcup_{t \in C_j} t\{(a_1^j, \dots, a_p^j)\},$$

implies that $\{(a_1^j z^q, \dots, a_p^j z^q) : z \in [\alpha_j, \beta_j]\}$ is a convex set. Therefore, we see that, for each $j = 1, \dots, m$,

$$\{(f_{1j}(z), \dots, f_{pj}(z)) : z \in \Delta_j\} \text{ is a convex set.}$$

This proves (2) and completes the proof. □

Definition 2.1. *Let $q \in \mathbb{N}$. We define the set S_q which consisting of all homogeneous separable polynomial (up to a constant) as follows:*

$$S_q = \{f : f(x) = \sum_{j=1}^{m} a^j x_j^q + b,\ a^j, b \in \mathbb{R},\ j = 1, \dots, m\}.$$

Note that translation preserve the convexity. Thus, the following corollary follows immediately from the preceding lemma (Lemma 2.1).

Corollary 2.1. *Let Δ be a compact box in $\mathbb{R}^m$. Let $q \in \mathbb{N}$ and $f_i \in S_q$, $i = 1, \dots, p$. Then, we have*

$$R_\Delta(f_1, \dots, f_p) \text{ is a convex set in } \mathbb{R}^p.$$

3 Nonconvex Minimax Theorem

Using the joint range convexity of separable homogeneous polynomial, we now present our promised nonconvex minimax theorem. Our proof is along the similar line of the classical proof of minimax theorem for convex-concave bifunctions presented in [6]. However, for the convenience of the reader, we present a complete and self-contained proof here.

Theorem 3.1. *Let Δ be a compact box in $\mathbb{R}^m$. Let $q \in \mathbb{N}$ and let A be a convex subset of $\mathbb{R}^n$. Consider the bifunction $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ such that*

(1) for each fixed $y \in A$, $f(\cdot, y) \in S_q$;

(2) for each fixed $x \in \Delta$, $f(x, \cdot)$ is a convex function.

Then, we have

$$\inf_{y \in A} \max_{x \in \Delta} f(x, y) = \max_{x \in \Delta} \inf_{y \in A} f(x, y).$$

Proof. It suffices to show that

$$\inf_{y \in A} \max_{x \in \Delta} f(x, y) \leq \max_{x \in \Delta} \inf_{y \in A} f(x, y).$$

To see this, let $\max_{x \in \Delta} \inf_{y \in A} f(x, y) < \alpha$. Then, for each $x \in \Delta$, there exists $y_x \in A$ such that $f(x, y_x) < \alpha$. Since $f(\cdot, y_x)$ is continuous, there exists an open neighbourhood V_x of

x such that

$$f(u, y_x) < \alpha \text{ for all } u \in V_x. \tag{3}$$

Since Δ is compact and $\Delta \subseteq \bigcup_{x\in\Delta} V_x$, we can find $x_1, \ldots, x_p \in \Delta$ such that

$$\Delta \subseteq \bigcup_{i=1}^{p} V_{x_i}.$$

Let $y_i = y_{x_i}$ and consider the following set

$$C_1 := \operatorname{conv}\{(f(x, y_1) - \alpha, \ldots, f(x, y_p) - \alpha) : x \in \Delta\} \text{ and } C_2 = \mathbb{R}^p_+,$$

where $\operatorname{conv} P$ denotes the convex hull of the set P. It is clear that C_1, C_2 are both convex sets and $\operatorname{int} C_2 \neq \emptyset$. Next, we show that $C_1 \cap \operatorname{int} C_2 = \emptyset$. Otherwise, there exists $(u_1, \ldots, u_p) \in \operatorname{int} \mathbb{R}^p_+$ with

$$(u_1, \ldots, u_p) \in C_1 := \operatorname{conv}\{(f(x, y_1) - \alpha, \ldots, f(x, y_p) - \alpha) : x \in \Delta\}.$$

Thus, there exist $x \in \Delta$, $q \in \mathbb{N}$ and $\lambda_j \geq 0$, $j = 1, \ldots, q$ with $\sum_{j=1}^{q} \lambda_j = 1$ such that for each $i = 1, \ldots, p$,

$$0 < u_i = \sum_{j=1}^{q} \lambda_j (f(x_j, y_i) - \alpha) = \sum_{j=1}^{q} \lambda_j f(x_j, y_i) - \alpha. \tag{4}$$

Let $f_i(x) = f(x, y_i)$, $i = 1, \ldots, p$. Then by our assumption, each $f_i \in S_q$, $i = 1, \ldots, p$. This together with Corollary 2.1 implies that

$$R_\Delta(f_1, \ldots, f_p) := \{(f_1(x), \ldots, f_p(x)) : x \in \Delta\} \text{ is a convex set in } \mathbb{R}^p.$$

Note that, for each $j = 1, \ldots, q$,

$$\big(f(x_j, y_1), f(x_j, y_2) \ldots, f(x_j, y_p)\big) = \big(f_1(x_j), f_2(x_j), \ldots, f_p(x_j)\big) \in R_\Delta(f_1, \ldots, f_p).$$

Thus, we see that their convex combination

$$\sum_{j=1}^{q} \lambda_j \big(f(x_j, y_1), f(x_j, y_2). \ldots, f(x_j, y_p)\big) \in R_\Delta(f_1, \ldots, f_p),$$

and hence there exists $x_0 \in \Delta$ such that

$$\sum_{j=1}^{q} \lambda_j f(x_j, y_i) = f_i(x_0) = f(x_0, y_i),\ i = 1, \ldots, p.$$

This, together with (4), gives

$$f(x_0, y_i) > \alpha \text{ for all } i = 1, \ldots, p. \tag{5}$$

On the other hand, since $x_0 \in \Delta$ and $\Delta \subseteq \bigcup_{i=1}^{p} V_{x_i}$, there exists some $i_0 \in \{1, \ldots, p\}$ such that $x_0 \in V_{x_{i_0}}$. Let $y_{i_0} = y_{x_{i_0}}$. This together with (3) implies that

$$f(x_0, y_{i_0}) < \alpha.$$

This contradicts (5) and so, $C_1 \cap \operatorname{int} C_2 = \emptyset$.

Thus, from the convex separation theorem, we see that there exist $\mu_i \in \mathbb{R}$, $i = 1, \ldots, p$ with $\sum_{i=1}^{p} \mu_i = 1$ such that

$$\sum_{i=1}^{n} \mu_i \big(f(x, y_i) - \alpha \big) \leq \sum_{i=1}^{p} \mu_i u_i \text{ for all } u_i \geq 0 \text{ and for all } x \in \Delta.$$

By letting $u_i \to \infty$ if necessary, we see that each $\mu_i \geq 0$, $i = 1, \ldots, p$. This gives us that

$$\sum_{i=1}^{p} \mu_i f(x, y_i) \leq \alpha \text{ for all } x \in \Delta.$$

Let $y_0 := \sum_{i=1}^{p} \mu_i y_i \in A$ (thanks to the convexity of A). Then, as $f(x, \cdot)$ is convex for all $x \in \Delta$, we have

$$f(x, y_0) \leq \sum_{i=1}^{p} \mu_i f(x, y_i) \leq \alpha \text{ for all } x \in \Delta.$$

Thus,

$$\inf_{y \in A} \max_{x \in \Delta} f(x, y) \leq \max_{x \in \Delta} f(x, y_0) \leq \alpha.$$

So, the conclusion follows. □

Next, we provide three corollaries, which give easily verifiable conditions for minimax theorem to hold. In particular, the last one is known as the famous von-Neumann Minimax Theorem.

Corollary 3.1. *Let Δ be a compact box in $\mathbb{R}^m$. Let $q \in \mathbb{N}$ and let A be a convex subset of $\mathbb{R}^n$. Let $f_1 : \mathbb{R}^m \to \mathbb{R}$ be a separable and homogeneous polynomial with degree q, and let $f_2 : \mathbb{R}^n \to \mathbb{R}$ be an affine function. Then, we have*

$$\inf_{y \in A} \max_{x \in \Delta} f_1(x) f_2(y) = \max_{x \in \Delta} \inf_{y \in A} f_1(x) f_2(y).$$

Proof. Consider the bifunction $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, defined by

$$f(x, y) = f_1(x)f_2(y).$$

Note that, for each fixed $y \in \mathbb{R}^n$, $f(\cdot, y)$ is a homogeneous and separable polynomial with degree q, and for each fixed $x \in \mathbb{R}^m$, $f(x, \cdot)$ is an affine function. Thus, the conclusion follows from Theorem 3.1. □

Corollary 3.2. *Let Δ be a compact box in $\mathbb{R}^m$. Let $q \in \mathbb{N}$ and let A be a convex subset of $\mathbb{R}^n$. Let $f_1 : \mathbb{R}^m \to \mathbb{R}$ be a non-negative, separable and homogeneous polynomial with degree q and let $f_2 : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then, we have*

$$\inf_{y \in A} \max_{x \in \Delta} f_1(x)f_2(y) = \max_{x \in \Delta} \inf_{y \in A} f_1(x)f_2(y).$$

Proof. Consider the bifunction $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, defined by

$$f(x, y) = f_1(x)f_2(y).$$

Note that for each fixed $y \in \mathbb{R}^n$, $f(\cdot, y)$ is a homogeneous and separable polynomial with degree q and, for each fixed $x \in \mathbb{R}^m$, $f(x, \cdot)$ is a convex function (since f_1 is non-negative and f_2 is convex). Thus, the conclusion follows from Theorem 3.1. □

Corollary 3.3. *Let $m, n \in \mathbb{N}$. Let $\Delta = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : |x_i| \leq 1\}$ and let $U \in \mathbb{R}^{m \times n}$. Then, we have*

$$\inf_{y \in \mathbb{R}^n} \max_{x \in \Delta} \langle x, Uy \rangle = \max_{x \in \Delta} \inf_{y \in \mathbb{R}^n} \langle x, Uy \rangle.$$

Proof. Let $A = \mathbb{R}^n$. Consider the bifunction $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$, defined by

$$f(x, y) = \langle x, Uy \rangle;$$

for each fixed $y \in \mathbb{R}$, $f(\cdot, y)$ is a linear function and, for each fixed x, $f(x, \cdot)$ is also a linear function. Thus, the conclusion follows from Theorem 3.1, as any linear function is in particular convex and belongs to the set S_1. □

Next, we present an example illustrating Corollary 3.1.

Example 3.1. *Let* $m = 2$ *and* $n = 1$. *Let* $\Delta = [-1, 1] \times [-1, 1]$ *and* $A = \mathbb{R}$. *Consider the following bifunction* $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$,

$$f(x, y) := (x_1^4 - x_2^4)(y - 1).$$

Then, it can be verified that

$$\max_{(x_1,x_2)\in\Delta}(x_1^4 - x_2^4)(y - 1) = \ -(y - 1) \text{ for all } y \in [-1, 1],$$

and so,

$$\inf_{y\in A}\max_{x\in\Delta} f(x, y) = 0.$$

Moreover,

$$\inf_{y\in A}(x_1^4 - x_2^4)(y - 1) = \begin{cases} -\infty, & \text{if} \quad x_1^4 - x_2^4 < 0, |x_1| \leq 1, |x_2| \leq 1, \\ 0, & \text{if} \quad x_1^4 - x_2^4 = 0, |x_1| \leq 1, |x_2| \leq 1, \\ -\infty, & \text{if} \quad x_1^4 - x_2^4 > 0, |x_1| \leq 1, |x_2| \leq 1, \end{cases}$$

and hence

$$\max_{x\in\Delta}\inf_{y\in A} f(x, y) = 0.$$

Thus we see that

$$\inf_{y\in A}\max_{x\in\Delta} f(x, y) = \max_{x\in\Delta}\inf_{y\in A} f(x, y).$$

On the other hand, this equality can also be seen by Corollary 3.1, since, for each fixed $y \in \mathbb{R}$, $f(\cdot, y)$ *is a homogeneous and separable polynomial with degree* 4 *and, for each fixed* $x \in \mathbb{R}^2$, $f(x, \cdot)$ *is affine.*

4 Application

Consider the following nonconvex separable homogeneous polynomial programming with bounded box constraints:

$$(P) \ \min_{x\in\mathbb{R}^n} \ p(x) \quad \text{s.t} \quad x \in \mathop{\mathbf{X}}_{i=1}^{n}[-1, 1],$$

where p is a separable homogeneous nonconvex polynomial with degree $2q$ ($q \in \mathbb{N}$). In this section, as a direct application of our nonconvex minimax theorem, we obtain a zero duality gap result for problem (P). (For other approaches to establish zero duality gap result, one could consult [7,8,9,10,11,12])

Note that the constraint can be equivalently rewritten as

$$x_i^{2q} \le 1, \; i = 1, \dots, n.$$

Thus, the Lagrangian dual of (P) can be formulated as

$$(DP) \quad \sup_{y \in \mathbb{R}^n_+} \inf_{x \in \mathbb{R}^n} \{p(x) + \sum_{i=1}^n y_i(x_i^{2q} - 1)\}.$$

As a corollary of Theorem 3.1, we now show that zero duality gap holds between (P) and its Lagrangian dual (DP).

Theorem 4.1. *For the dual pair (P) and (DP), the following zero duality gap result holds*

$$\min_{x \in \mathbf{X}_{i=1}^n[-1,1]} p(x) = \sup_{y \in \mathbb{R}^n_+} \inf_{x \in \mathbb{R}^n} \{p(x) + \sum_{i=1}^n y_i(x_i^{2q} - 1)\}.$$

Proof. Let $A = \mathbb{R}^n_+$. For each $t > 1$, denote $\Delta_t = \mathbf{X}_{i=1}^n[-t,t]$. Consider the bifunction $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, defined by

$$f(x,y) = -p(x) - \sum_{i=1}^n y_i(x_i^{2q} - 1),$$

where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$. Clearly, for each fixed y, $f(\cdot, y) \in S_{2q}$ and for each fixed x, $f(x, \cdot)$ is affine (hence convex). Then, from Theorem 3.1, we have for each $t > 1$,

$$\inf_{y \in A} \max_{x \in \Delta_t} f(x,y) = \max_{x \in \Delta_t} \inf_{y \in A} f(x,y).$$

It can be verified that

$$\inf_{y \in A} \max_{x \in \Delta_t} f(x,y) = -\sup_{y \in \mathbb{R}^n_+} \min_{x \in \mathbf{X}_{i=1}^n[-t,t]} \{p(x) + \sum_{i=1}^n y_i(x_i^{2q} - 1)\}.$$

Moreover, for each $x \in \Delta_t = \mathbf{X}_{i=1}^n[-t,t]$,

$$\inf_{y \in A} f(x,y) = \inf_{y \in \mathbb{R}^n_+} \{-p(x) - \sum_{i=1}^n y_i(x_i^{2q} - 1)\} = \begin{cases} -p(x), & \text{if} \quad x \in \mathbf{X}_{i=1}^n[-1,1], \\ -\infty, & \text{else.} \end{cases}$$

Thus,

$$\max_{x\in\Delta_t}\inf_{y\in A} f(x,y) = \max_{x\in \mathbf{X}_{i=1}^n[-1,1]}\{-p(x)\}.$$

It follows that, for each $t > 1$,

$$\min_{x\in \mathbf{X}_{i=1}^n[-1,1]} p(x) = \sup_{y\in\mathbb{R}^n_+}\min_{x\in \mathbf{X}_{i=1}^n[-t,t]}\{p(x)+\sum_{i=1}^n y_i(x_i^{2q}-1)\}. \tag{6}$$

Let $p(x) = \sum_{i=1}^n a^i x_i^{2q}$. Note that, there exists $t_0 > 1$ such that, for each $y \in \mathbb{R}^n_+$ with $y_i \geq -a^i$,

$$\left(\operatorname{argmin}_{x\in\mathbb{R}^n}\{p(x)+\sum_{i=1}^n y_i(x_i^{2q}-1)\}\right)\cap\left(\mathop{\mathbf{X}}_{i=1}^n[-t_0,t_0]\right)\neq\emptyset,$$

and, if there exists some $i_0 \in \{1,\ldots,n\}$ such that $y_{i_0} < -a^{i_0}$, then

$$\inf_{x\in\mathbb{R}^n}\{p(x)+\sum_{i=1}^n y_i(x_i^{2q}-1)\} = -\infty.$$

Thus,

$$\sup_{y\in\mathbb{R}^n_+}\inf_{x\in\mathbb{R}^n}\{p(x)+\sum_{i=1}^n y_i(x_i^{2q}-1)\} = \sup_{y\in\mathbb{R}^n_+}\min_{x\in \mathbf{X}_{i=1}^n[-t_0,t_0]}\{p(x)+\sum_{i=1}^n y_i(x_i^{2q}-1)\},$$

and so, by (6), we have

$$\sup_{y\in\mathbb{R}^n_+}\inf_{x\in\mathbb{R}^n}\{p(x)+\sum_{i=1}^n y_i(x_i^{2q}-1)\} = \min_{x\in \mathbf{X}_{i=1}^n[-1,1]} p(x).$$

□

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