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Unified approach to some geometric results in variational analysis

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Abstract

Based on a study of a minimization problem, we present the following results applicable to possibly nonconvex sets in a Banach space: an approximate projection result, an extended extremal principle, a nonconvex separation theorem, a generalized Bishop–Phelps theorem and a separable point result. The classical result of Dieudonné (on separation of two convex sets in a finite-dimensional space) is also extended to a nonconvex setting.

Keywords: Abstract subdifferential; Outer ϵ-minimizer; Variational analysis; The Bishop–Phelps theorem; Dieudonné's separation result

1. Introduction

It is well known that separation theorems for convex sets play a fundamental role in the classical theory of functional analysis as well as in many aspects of nonlinear analysis and optimization. In particular, using separation theorems and convex approximation techniques, some

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problems with nonconvex and nonsmooth initial data can be solved efficiently. However, there is a large class of optimal control and optimization-related economic problems where the use of convex approximations are either impossible or do not lead to satisfactory results (cf. [14–16,21,28]). For two closed sets (not necessarily convex) with extremal property, Mordukhovich and Shao [17,18] established what is known as the extremal principle in Asplund spaces by using the fuzzy sum rule. Their work has led to an important progress in this topic and has found many applications in establishing optimality condition for nonconvex functions. For more detailed background information and motivations we refer the reader to the informative two-volume book [14,15]. In this paper, we attempt to unify and improve some geometric results in variational analysis. Much of our study here has been inspired by the works by Mordukhovich and his collaborators (see, in particular, [4,13–15,17,19]; see also [5,10] and Zhu [29]). Our analysis is based on the consideration of the following minimization problem:

$$\min_{x\in A} d\big(0, F(x)\big) \tag{MP}$$

in conjunction with the inclusion problem

$$\text{find } x \in A \text{ such that } F(x) = 0, \tag{IP}$$

where F is a mapping from a Banach space X to another Banach space Y and A is a closed subset of X. Clearly, if x is a solution of (IP) then x is also a solution of (MP). It is known that many optimization problems (the best approximation problem, the feasibility problem and the nonlinear least square problem) can be cast as (MP); see [7,22]. In terms of abstract subdifferential and normal cone, we provide a necessary condition for a point to be an outer ϵ-minimizer (see the definition given in Section 2) for the minimization problem (MP). By specializing in different types of F and A, we provide several nonconvex geometric consequences in Section 3, including an approximate projection result, an extended extremal principle, a nonconvex separation theorem, a generalized Bishop–Phelps theorem and a separable point result, which extend and improve the existing geometric results in variational analysis. In Section 4, some stronger results are reported under suitably strengthened assumptions; in particular we extend the classical result of Dieudonné (on separating two convex sets in a finite-dimensional space) to a nonconvex setting.

2. Preliminaries

Let X be a Banach space. We use $B_X(x,\epsilon)$ (respectively $\overline{B}_X(x,\epsilon)$) to denote the open (respectively closed) ball in X with center x and radius ϵ. We denote the unit sphere (respectively open unit ball, closed unit ball) in X by S_X (respectively B_X, $\overline{B}_X$). Given a subset A of X, we denote the interior (respectively topological boundary, topological closure, affine hull) of A by $\operatorname{int}(A)$ (respectively $\operatorname{bd} A$, $\overline{A}$, $\operatorname{aff}(A)$). As in [23], we denote the relative interior of A by $\operatorname{ri}(A)$, i.e.,

$$\operatorname{ri}(A) = \begin{cases} \{a \in A\colon\ \exists r > 0,\ B(a,r) \cap \operatorname{aff}(A) \subseteq A\} & \text{if } \operatorname{aff}(A) \text{ is closed},\\ \emptyset & \text{otherwise}. \end{cases} \tag{1}$$

When A is a subset of a Banach dual space, $\overline{A}^{w^*}$ denotes the weak*-closure of A. For a function $f: X \to \mathbb{R} \cup \{+\infty\}$, let $\operatorname{epi}(f)$ and $\operatorname{dom}(f)$ respectively denote the epigraph and the domain of f, that is,

$$\operatorname{epi}(f) := \big\{(x,t) \in X \times \mathbb{R}\colon\ f(x) \leqslant t\big\} \quad \text{and} \quad \operatorname{dom}(f) := \big\{x \in X\colon\ f(x) < +\infty\big\}.$$

For a subset A of X, let $d(\cdot, A)$ and $\delta(\cdot, A)$ respectively denote the distance function and the indicator function of A, that is

$$d(x,A)=\inf\bigl\{\|x-a\|\colon a\in A\bigr\}\quad\text{and}\quad \delta(x,A)=\begin{cases}0 & \text{if } x\in A,\\ \infty & \text{otherwise.}\end{cases}$$

Sometimes, we also write $d_A(x)$ for $d(x,A)$ and $\delta_A(x)$ for $\delta(x,A)$. As usual let X^* denote the Banach dual space of X; for $x\in X$, $x^*\in X^*$ we sometimes write $\langle x^*, x\rangle$ for $x^*(x)$. For Banach spaces $X_1, X_2, \ldots, X_m$, let $\prod_{i=1}^m X_i$ denote the product space which is also a Banach space under the following "l^1-norm": for any $(x_1, x_2, \ldots, x_m)\in\prod_{i=1}^m X_i$

$$\bigl\|(x_1,\ldots,x_m)\bigr\| = \sum_{i=1}^m \|x_i\|.$$

For any $A_i\subseteq X_i$ $(i=1,2,\ldots,m)$ and $x:=(x_1,\ldots,x_m)\in\prod_{i=1}^m X_i$, we have the following relation:

$$d\left(x,\prod_{i=1}^m A_i\right)=\sum_{i=1}^m d(x_i,A_i)\quad\text{and}\quad \delta\left(x,\prod_{i=1}^m A_i\right)=\sum_{i=1}^m \delta(x_i,A_i). \tag{2}$$

We identify x^* with $(x_1^*,\ldots,x_m^*)$ when $x^*\in(\prod_{i=1}^m X_i)^*$, $x_i^*\in X_i^*$ and

$$\bigl\langle x^*,(x_1,\ldots,x_m)\bigr\rangle=\sum_{i=1}^m\bigl\langle x_i^*,x_i\bigr\rangle\quad\text{for all } (x_1,\ldots,x_m)\in\prod_{i=1}^m X_i. \tag{3}$$

Note that $(\prod_{i=1}^m X_i, \|\cdot\|)^* = (\prod_{i=1}^m X_i^*, \|\cdot\|_\infty)$ is also a Banach space where the corresponding dual norm on $\prod_{i=1}^m X_i^*$ is defined as $\|(x_1^*, x_1^*, \ldots, x_m^*)\|_\infty = \max_{1\leqslant i\leqslant m}\|x_i^*\|$ for any $(x_1^*, x_2^*, \ldots, x_m^*)\in\prod_{i=1}^m X_i^*$. Consequently, the closed unit ball in $\prod_{i=1}^m X_i^*$ is the Cartesian product $\prod_{i=1}^m \overline{B}_{X_i^*}$ defined by $\prod_{i=1}^m \overline{B}_{X_i^*} = \overline{B}_{X_1^*}\times\cdots\times\overline{B}_{X_m^*}$. Given two Banach spaces X, Y, we use $L(X;Y)$ to denote the Banach space of all continuous linear operators from X to Y. For $F: X\to Y$ and $x\in X$, we say that F is (Gateaux) differentiable at x with derivative $\nabla F(x)\in L(X;Y)$ if

$$\lim_{t\downarrow 0}\frac{F(x+th)-F(x)-\langle\nabla F(x), th\rangle}{t}=0\quad\forall h\in X.$$

If $X=\prod_{i=1}^m X_i$, we use $\nabla_i F(x)$ to denote the ith partial derivative of F at x which is defined to be an element of $L(X_i;Y)$ such that the following holds:

$$\lim_{t\to 0}\frac{F(x+th^i)-F(x)-\langle\nabla_i F(x), th_i\rangle}{t}=0\quad\forall h_i\in X_i, \tag{4}$$

where h^i is the element in $\prod_{i=1}^m X_i$ defined by $h^i = (t_1, t_2, \dots, t_m)$ with $t_i = h_i$ and $t_j = 0$ for all $j \neq i$. Thus, if F is differentiable at x then $\langle \nabla_i F(x), h_i \rangle = \langle \nabla F(x), h^i \rangle$, $\forall h_i \in X_i$. Consequently, we have

$$\sum_{i=1}^{m} \langle \nabla_i F(x), h_i \rangle = \langle \nabla F(x), (h_1, \dots, h_m) \rangle \quad \text{for all } (h_1, \dots, h_m) \in \prod_{i=1}^{m} X_i. \tag{5}$$

We say that $F : X \to Y$ admits a strict derivative at x, an element of $L(X; Y)$, denoted by $D_s F(x)$, provided that the following holds:

$$\lim_{\substack{x' \to x \\ t \downarrow 0}} \frac{F(x' + th) - F(x') - \langle D_s F(x), th \rangle}{t} = 0 \quad \forall h \in X,$$

and provided that the convergence is uniform for h in compact sets. From the definition, it is clear that if F is strictly differentiable at x then F is differentiable at x and $\nabla F(x) = D_s F(x)$. We recall that, if F is strictly differentiable at x, then F is Lipschitz near x (cf. [6, Proposition 2.2.1]). In what follows we denote by $\mathcal{X}$ a class of some Banach spaces such that $\prod_{i=1}^m X_i \in \mathcal{X}$ for any $X_i \in \mathcal{X}$ $(i = 1, \dots, m)$, for instance, the class of all smooth Banach spaces, the class of all reflexive Banach spaces, or the class of all Asplund spaces. For X in $\mathcal{X}$, $\Gamma(X)$ denotes the set of all lower semicontinuous functions from X to $\mathbb{R} \cup \{+\infty\}$. We consider an abstract subdifferential ∂_a associated with the pair $\{\mathcal{X}, \Gamma\}$ as a mapping which associates to any X in $\mathcal{X}$, $f \in \Gamma(X)$, $x \in X$ a subset $\partial_a f(x)$ of X^* such that it satisfies the following properties (P1)–(P7):

- (P1) Let $X \in \mathcal{X}$. If $f \in \Gamma(X)$ is convex and $x \in \mathrm{dom}(f)$, then $\partial_a f(x)$ coincides with the subdifferential $\partial f(x)$ of f at x in convex analysis.
- (P2) Let $X \in \mathcal{X}$ and $x \in X$. Then we have $\partial_a f(x) = \partial_a g(x)$ for any $f, g \in \Gamma(X)$ if they coincide near x.
- (P3) Let $X_i \in \mathcal{X}$ $(i = 1, 2, \dots, m)$ and $X = \prod_{i=1}^m X_i$. If $f \in \Gamma(X)$ is given by

$$f(x) = \sum_{i=1}^{m} f_i(x_i) \quad \forall x = (x_1, \dots, x_m) \in X,$$

  where each $f_i \in \Gamma(X_i)$ $(i = 1, 2, \dots, m)$, then $\partial_a f(x) \subseteq \partial_a f_1(x_1) \times \cdots \times \partial_a f_m(x_m)$ for all $x \in X$.
- (P4) Let $X, Y \in \mathcal{X}$. If $F : X \to Y$ is strictly differentiable, then $\partial_a \|F(\cdot)\|(x) \subseteq \{y^* \circ \nabla F(x) : y^* \in \partial_a \| \cdot \|(F(x))\}$.
- (P5) Let $X \in \mathcal{X}$. For any closed set A of X, it holds that $\partial_a d(x, A) \subseteq N_a(x, A)$ for all $x \in A$ where $N_a(x, A)$ denotes the abstract normal cone of A at x and is defined by $N_a(x, A) = \emptyset$ if $x \notin A$ and

$$N_a(x, A) = \partial_a \delta_A(x) \quad \text{if } x \in A. \tag{6}$$

- (P6) Let $X \in \mathcal{X}$. If $f \in \Gamma(X)$ and f attains a global minimum at $x \in \mathrm{dom}(f)$, then we have $0 \in \partial_a f(x)$.

(P7) (*Fuzzy sum rule.*) Let $X \in \mathcal{X}$. Let $f_1, f_2 \in \Gamma(X)$, $x \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$ and $x^* \in \partial_a(f_1 + f_2)(x)$. If f_1 or f_2 is locally Lipschitz near x then for any $\epsilon > 0$ there exist $x_1, x_2 \in B(x, \epsilon)$ such that $|f_i(x_i) - f_i(x)| < \epsilon$ $(i = 1, 2)$ and $x^* \in \partial_a f(x_1) + \partial_a f_2(x_2) + \epsilon B_{X^*}$.

Remark 2.1. Let $X, Y \in \mathcal{X}$ and $F : X \to Y$ be strictly differentiable. From (P1) and [23, Corollary 2.4.16], we know that for any $y \in Y$,

$$\partial_a \|\cdot\|(y) = \partial \|\cdot\|(y) = \{y^* \in \overline{B}_{Y^*}\colon \langle y^*, y\rangle = \|y\|\}. \tag{7}$$

Thus it follows from (P4) that

$$\partial_a \|F(\cdot)\|(x) \subseteq \{y^* \circ \nabla F(x)\colon\ y^* \in \overline{B}_{Y^*}, \langle y^*, F(x)\rangle = \|F(x)\|\}. \tag{8}$$

(If $F(x) \neq 0$, then the closed unit ball $\overline{B}_{Y^*}$ in (8) can be replaced by the unit sphere S_{Y^*}.)

Remark 2.2. Let $X \in \mathcal{X}$. Let f be a locally Lipschitz function on X and A be a closed subset of X. Let $x \in \mathrm{dom}\, f \cap A$ and let $x^* \in \partial_a(f + \delta_A)(x)$. By (6) and (P7), we know that for any $\epsilon > 0$ there exist $x_1 \in B(x, \epsilon)$, $x_2 \in A \cap B(x, \epsilon)$ such that $|f(x_1) - f(x)| < \epsilon$, $x_1^* \in \partial_a f(x_1)$, $x_2^* \in N_a(x_2, A)$ and $x^* - (x_1^* + x_2^*) \in \epsilon B_{X^*}$.

Remark 2.3. Let $X_i \in \mathcal{X}$ and let A_i be a closed subset of X_i $(i = 1, 2, \ldots, m)$. By (2), (3), (6) and property (P3) we know that for any $x = (x_1, \ldots, x_m) \in \prod_{i=1}^m X_i$

$$\partial_a d\left(\cdot, \prod_{i=1}^m A_i\right)(x) \subseteq \prod_{i=1}^m \partial_a d(\cdot, A_i)(x_i) \quad \text{and} \quad N_a\left(x, \prod_{i=1}^m A_i\right) \subseteq \prod_{i=1}^m N_a(x_i, A_i). \tag{9}$$

Remark 2.4. Let $\rho > 0$ and $X \in \mathcal{X}$. Let A be a closed subset of X and $\bar{x} \in A$. For any $x \in A \cap B_X(\bar{x}, \rho)$, it is easy to verify that $\delta_A(\cdot) = \delta_{A \cap \overline{B}_X(\bar{x}, \rho)}(\cdot)$ near x and hence it follows from (P2), (6) that $N_a(x, A) = N_a(x, A \cap \overline{B}_X(\bar{x}, \rho))$.

Remark 2.5. Let $X \in \mathcal{X}$ and let A be a closed subset of X with $x \in A$. If $N_a(x, A) \neq \{0\}$ then $x \in \mathrm{bd}\, A$. Indeed, if $x \in \mathrm{int}\, A$, then $\delta_A(\cdot) = 0$ on a neighborhood of x. Thus properties (P1) and (P2) imply that $N_a(x, A) = \partial_a \delta_A(x) = \{0\}$.

Remark 2.6. Let $X \in \mathcal{X}$ and let A be a closed subset of X. For any $x \in A$, it is easy to verify that x is a minimum point of $d(\cdot, A)$ hence it follows from (P6) that $0 \in \partial_a d(\cdot, A)(x)$.

An abstract subdifferential ∂_a is said to be complete if the following additional conditions are satisfied:

(P4$^+$) Let $X, Y, Z \in \mathcal{X}$. If $G : X \to Y$ is locally Lipschitz and $F : Y \to Z$ is strictly differentiable, then $\partial_a(G \circ F)(x) \subseteq \{y^* \circ \nabla F(x)\colon\ y^* \in \partial_a G(F(x))\}$.

(P7$^+$) (*Exact sum rule*) Let $X \in \mathcal{X}$ and let $f_1, f_2 \in \Gamma(X)$, $x \in \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$. If f_1 and f_2 are both locally Lipschitz, then $\partial_a(f_1 + f_2)(x) \subseteq \partial_a f_1(x) + \partial_a f_2(x)$.

(P8) Let $X \in \mathcal{X}$ and let $f \in \Gamma(X)$, $x \in X$ and $\alpha > 0$. Then $\partial_a(\alpha f)(x) = \alpha \partial_a f(x)$.

(P9) Let $X \in \mathcal{X}$ and $x \in X$. Suppose that $f \in \Gamma(X)$ is locally Lipschitz near x. Then for any nets (generalized sequences) $\{x_n\}$, $\{x_n^*\}$ such that $x_n \to x$, $x_n^* \in \partial_a f(x_n)$ and $x_n^* \to_{w^*} x^*$, we have $x^* \in \partial_a f(x)$, where $\to_{w^*}$ denotes the convergence with respect to the weak* topology.

For example, consider the following cases:

(C1) $\mathcal{X}$ is the class of all Banach spaces and ∂_a is the Clarke–Rockafellar subdifferential ∂_c;
(C2) $\mathcal{X}$ is the class of all β-smooth Banach spaces and ∂_a is the corresponding viscosity subdifferential ∂_β;
(C3) $\mathcal{X}$ is the class of all Asplund spaces and ∂_a is the limiting subdifferential ∂_L;
(C4) $\mathcal{X}$ is the class of all weakly compact generated Asplund spaces and ∂_a is the limiting subdifferential ∂_L;
(C5) $\mathcal{X}$ is the class of all Asplund spaces and ∂_a is the Fréchet subdifferential ∂_F.

It is known that ∂_a is an abstract subdifferential in each of the above 5 cases, and it is complete in each of the cases (C1) and (C4) (cf. [2,6,14,17]).

3. Fuzzy results in Banach spaces

3.1. Outer ϵ-minimizers and separation

In this section, we study (MP) and (IP) defined as in the introduction. We begin with the following definition.

Definition 3.1. Let $X, Y \in \mathcal{X}$. Let F be a mapping from X to Y and let A be a closed subset of X. Consider the minimization problem (MP) and the inclusion problem (IP). We say that $\bar{x} \in A$ is an ϵ-minimizer of (MP) provided that $\epsilon > 0$ and

$$\|F(\bar{x})\| < d\big(0, F(A)\big) + \epsilon^2. \tag{10}$$

Moreover, $\bar{x}$ is called an outer ϵ-minimizer of (MP) if it is an ϵ-minimizer of (MP) and

$$0 \notin F\big(A \cap B_X(\bar{x}, \epsilon)\big) \tag{11}$$

(that is, each point in $B_X(\bar{x}, \epsilon)$ is not a solution of (IP)).

Our analysis is based on the following result providing a necessary condition for outer ϵ-minimizers of (MP).

Theorem 3.2. *Let $X, Y \in \mathcal{X}$ and let $F : X \to Y$ be a strictly differentiable mapping. Suppose that $\bar{x}$ is an outer ϵ-minimizer of the minimization problem* (MP) *for some $\epsilon > 0$. Then the following assertions hold*:

(i) *There exist* $u \in A \cap \overline{B}_X(\bar{x}, \epsilon)$, $v \in \overline{B}_X(\bar{x}, \epsilon)$ *and* $y^* \in S_{Y^*}$ *such that*

$$\langle y^*, F(v)\rangle = \|F(v)\| \neq 0, \tag{12}$$

$$-y^* \circ \nabla F(v) \in N_a(u, A) + \epsilon \overline{B}_{X^*}. \tag{13}$$

(ii) *Suppose the abstract subdifferential* ∂_a *is complete and that* F *is Lipschitz on* X *with rank* L. *Then there exist* $x \in A \cap \overline{B}_X(\bar{x}, \epsilon)$ *and* $y^* \in S_{Y^*}$ *such that* $\langle y^*, F(x)\rangle = \|F(x)\| \neq 0$ *and*

$$-y^* \circ \nabla F(x) \in (L+\epsilon)\partial_a d(\cdot, A)(x) + \epsilon \overline{B}_{X^*} \tag{14}$$

(*and hence* (12) *and* (13) *hold with* $u = v = x$).

Proof. (i) By the given assumption, (10) and (11) hold. Define a lower semicontinuous function $f : X \to [0, \infty]$ by $f(\cdot) = \delta_{A \cap \overline{B}_X(\bar{x}, \epsilon)}(\cdot) + \|F(\cdot)\|$. Then it follows from (10) that

$$\begin{aligned} f(\bar{x}) = \|F(\bar{x})\| &< d\big(0, F(A)\big) + \epsilon^2 \\ &= \inf\big\{\|F(x)\|:\ x \in A\big\} + \epsilon^2 \\ &\leqslant \inf\big\{\|F(x)\|:\ x \in A \cap \overline{B}_X(\bar{x}, \epsilon)\big\} + \epsilon^2 \\ &= \inf_{x \in X} f(x) + \epsilon^2. \end{aligned}$$

Denoting $\alpha = [f(\bar{x}) - \inf_{x \in X} f(x)]^{1/2}$, we have $\alpha < \epsilon$ and $f(\bar{x}) = \inf_{x \in X} f(x) + \alpha^2$. Hence, by the Ekeland variational principle (cf. [23, Theorem 1.4.1]), there exists a vector $x \in A$ with

$$\|x - \bar{x}\| \leqslant \alpha \tag{15}$$

such that x is a minimal point of the function $g : X \to [0, \infty]$ defined by

$$\begin{aligned} g(\cdot) &= f(\cdot) + \alpha\ \|\cdot - x\| \\ &= \delta_{A \cap \overline{B}_X(\bar{x}, \epsilon)}(\cdot) + \|F(\cdot)\| + \alpha\ \|\cdot - x\|. \end{aligned} \tag{16}$$

Note that $x \in B(\bar{x}, \epsilon)$ since $\alpha < \epsilon$ and thanks to (15). It follows from (11) that $\|F(x)\| > 0$ and so $\|F(\cdot)\| > 0$ on $B(x, \eta)$ for some $\eta \in (0, \epsilon - \alpha)$. Hence, by Remark 2.1, for any $z \in B(x, \eta)$

$$\partial_a \|F(\cdot)\|(z) \subseteq \big\{y^* \circ \nabla F(z):\ y^* \in S_{Y^*}, \langle y^*, F(z)\rangle = \|F(z)\|\big\}. \tag{17}$$

Since $B(x, \eta) \subseteq B(\bar{x}, \epsilon)$ (by (15) and the fact that $\alpha + \eta < \epsilon$), we also have from (P2), (P5) and Remark 2.4 that for any $z \in B(x, \eta)$,

$$\partial_a \delta_{A \cap \overline{B}(\bar{x}, \epsilon)}(z) = \partial_a \delta_A(z) = N_a(z, A). \tag{18}$$

On the other hand, by (P6) and (P7) and Remark 2.2, there exist $u, v, w \in B(x, \eta)$ with $u \in A$ such that

$$0 \in \partial_a \delta_{A \cap \overline{B}_X(\bar{x}, \epsilon)}(u) + \partial_a \|F(\cdot)\|(v) + \partial_a\big(\alpha\|\cdot - x\|\big)(w) + \eta B_{X^*}.$$

It follows from (17) and (18) that $0 = u^* + y^* \circ \nabla F(v) + w^* + x^*$ for some $u^* \in N_a(u, A)$, $y^* \in S_{Y^*}$ with $\langle y^*, F(v)\rangle = \|F(v)\|$, $w^* \in \partial_a(\alpha\ \|\cdot - x\|)(w)$ and $x^* \in \eta B_{X^*}$. By (P1) and (7), $\|w^* + x^*\| \leqslant \alpha + \eta < \epsilon$ and so

$$-y^* \circ \nabla F(v) = u^* + w^* + x^* \in N_a(u, A) + \epsilon B_{X^*}.$$

Thus (13) holds. Also, from the first relation in (17) and $v \in B(x, \eta)$, we see that (12) holds and so does the conclusion of (i).

(ii) We suppose ∂_a is complete and that F is Lipschitz on X with rank L. Recalling that $\alpha < \epsilon$ and that x is a minimizer of g defined in (16), we have $\|F(x)\| \leqslant \|F(\cdot)\| + \alpha\|\cdot - x\| \leqslant \|F(\cdot)\| + \epsilon\|\cdot - x\|$ on $A \cap \overline{B}(\bar{x}, \epsilon)$. Noting that $\|F(\cdot)\| + \epsilon\ \|\cdot - x\|$ is Lipschitz on X with rank $L + \epsilon$, it follows from [6, Proposition 2.4.2] that the function $h : X \to [0, +\infty]$ defined by $h(\cdot) := \|F(\cdot)\| + \epsilon\ \|\cdot - x\| + (L + \epsilon)d(\cdot, A \cap \overline{B}_X(\bar{x}, \epsilon))$ attains global minimum at x over X. Hence, noticing that $F(x) \neq 0$, by properties (P6), (P7$^+$), (P8) and Remark 2.1, we obtain

$$0 \in (L + \epsilon)\partial_a d\big(\cdot, A \cap \overline{B}_X(\bar{x}, \epsilon)\big)(x) + y^* \circ \nabla F(x) + \epsilon\partial_a\big(\|\cdot - x\|\big)(x)$$

for some $y^* \in S_{Y^*}$ satisfying $\langle y^*, F(x)\rangle = \|F(x)\| \neq 0$. Thus we have $-y^* \circ \nabla F(x) \in (L + \epsilon)\partial_a d(\cdot, A \cap \overline{B}_X(\bar{x}, \epsilon))(x) + \epsilon B_{X^*}$, because by (P1) we have $\partial_a(\|\cdot - x\|)(x) = \partial(\|\cdot - x\|)(x) \subseteq \overline{B}_{X^*}$. By (15) and $\alpha < \epsilon$, x is in the interior of the ball $\overline{B}_X(\bar{x}, \epsilon)$. This yields that $\partial_a d(\cdot, A \cap \overline{B}_X(\bar{x}, \epsilon))(x) = \partial_a d(\cdot, A)(x)$ thanks to property (P2). Hence

$$-y^* \circ \nabla F(x) \in (L + \epsilon)\partial_a d(\cdot, A)(x) + \epsilon\overline{B}_{X^*}. \tag{19}$$

This completes the proof. □

Theorem 3.3. *Let $X_i \in \mathcal{X}$ $(i = 1, \ldots, m)$ and $Y \in \mathcal{X}$. Let $X = \prod_{i=1}^m X_i$ and $F : X \to Y$ be strictly differentiable. Let $A := \prod_{i=1}^m A_i$ where each A_i is a closed subset of X_i and let $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \in A$. Suppose that $\bar{x}$ is an outer ϵ-minimizer for the corresponding minimization problem* (MP). *Then the following assertions hold*:

(i) *There exist $y^* \in S_{Y^*}$, $u = (u_1, \ldots, u_m) \in A \cap \overline{B}_{\prod_{i=1}^m X_i}(\bar{x}, \epsilon)$ and $v = (v_1, \ldots, v_m) \in \overline{B}_{\prod_{i=1}^m X_i}(\bar{x}, \epsilon)$ such that*

$$\big\langle y^*, F(v)\big\rangle = \big\|F(v)\big\| \neq 0, \tag{20}$$

$$-y^* \circ \nabla_i F(v) \in N_a(u_i, A_i) + \epsilon\overline{B}_{X_i^*} \quad (i = 1, \ldots, m), \tag{21}$$

where $\nabla_i F$ denotes the ith partial derivative of F defined as in (4).

(ii) *Suppose that the abstract subdifferential ∂_a is complete and that F is Lipschitz on X with rank L. Then there exist $x = (x_1, \ldots, x_m) \in A \cap \overline{B}_{\prod_{i=1}^m X_i}(\bar{x}, \epsilon)$ and $y^* \in S_{Y^*}$ with $\langle y^*, F(x)\rangle = \|F(x)\| \neq 0$ such that*

$$-y^* \circ \nabla_i F(x) \in (L + \epsilon)\partial_a d(\cdot, A_i)(x_i) + \epsilon\overline{B}_{X_i^*}. \tag{22}$$

Proof. (i) By Theorem 3.2(i), there exist $y^* \in S_{Y^*}$, $u = (u_1, \ldots, u_m) \in A \cap \overline{B}_{\prod_{i=1}^m X_i}(\bar{x}, \epsilon)$ and $v = (v_1, \ldots, v_m) \in \overline{B}_{\prod_{i=1}^m X_i}(\bar{x}, \epsilon)$ such that (20) and the following (23) hold:

$$-y^* \circ \nabla F(v) \in N_a\left(u, \prod_{i=1}^m A_i\right) + \epsilon \overline{B}_{X^*}. \tag{23}$$

By (3), (5), (9) and $\overline{B}_{X^*} = \prod_{i=1}^m \overline{B}_{X_i^*}$ (23) implies that

$$\left(-y^* \circ \nabla_1 F(v), \ldots, -y^* \circ \nabla_m F(v)\right) \in \prod_{i=1}^m N_a(u_i, A_i) + \epsilon \prod_{i=1}^m \overline{B}_{X_i^*}.$$

Thus (21) holds so does the conclusion of (i).

(ii) Assume that the abstract subdifferential ∂_a is complete and that F is Lipschitz on X with rank L. Then by Theorem 3.2(ii), there exist $x = (x_1, \ldots, x_m) \in A \cap \overline{B}_{\prod_{i=1}^m X_i}(\bar{x}, \epsilon)$ and $y^* \in S_{Y^*}$ with $\langle y^*, F(x)\rangle = \|F(x)\| \neq 0$ such that the following (24) hold:

$$-y^* \circ \nabla F(x) \in (L + \epsilon)\partial_a d\left(\cdot, \prod_{i=1}^m A_i\right)(x) + \epsilon \overline{B}_{X^*}. \tag{24}$$

Thus, $(-y^* \circ \nabla_1 F(x), \ldots, -y^* \circ \nabla_m F(x)) \in (L + \epsilon) \prod_{i=1}^m \partial_a d(\cdot, A_i)(x_i) + \epsilon \prod_{i=1}^m \overline{B}_{X_i^*}$ as in the proof of (i). Therefore (22) is seen to hold. □

3.2. Approximate projection results in Banach spaces

This subsection is devoted to establish some approximate projection results. The approximate projection results for a single closed set were first established in [25,27]. Let $X \in \mathcal{X}$ and let X^m denote the product of m copies of X. Let $Y = X$, $w = (w_1, \ldots, w_m) \in X^m$ and let $F : X^m \to X$ be defined by

$$F(x_1, x_2, \ldots, x_m) = \sum_{i=1}^m (x_i - w_i). \tag{25}$$

Then F is affine and Lipschitz with rank 1. It is easy to verify that for any $x \in X^m$, $\nabla F(x)$ is the linear map $(h_1, \ldots, h_m) \mapsto \sum_{i=1}^m h_i$ and $\nabla_i F(x)$ is the identity map I on X for all i. Moreover, we have $F^{-1}(0) = \{(x_1, \ldots, x_m) \in \prod_{i=1}^m X\colon \sum_{i=1}^m x_i = \sum_{i=1}^m w_i\}$.

Theorem 3.4. *Let $X \in \mathcal{X}$ and let $A_1, \ldots, A_m$ be closed subsets of X. Let $w \in X \setminus (\sum_{i=1}^m A_i)$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_m) \in \prod_{i=1}^m A_i$ and let $\epsilon > 0$ be such that*

$$\left\| w - \sum_{i=1}^m \bar{x}_i \right\| < d\left(w, \sum_{i=1}^m A_i\right) + \epsilon^2. \tag{26}$$

Then the following assertions hold:

(i) *There exist* $u_i \in A_i \cap \overline{B}_X(\bar{x}_i, \epsilon)$ *and* $x^* \in X^*$ *with* $\|x^*\| = 1$ *such that*

$$\left\langle x^*, w - \sum_{i=1}^m u_i \right\rangle \geqslant \left\| w - \sum_{i=1}^m u_i \right\| - 4\epsilon$$

and

$$x^* \in \bigcap_{i=1}^m \big(N_a(u_i, A_i) + \epsilon \overline{B}_{X^*}\big). \tag{27}$$

(ii) *If we further assume that* ∂_a *is complete, then there exist* $x_i \in A_i \cap \overline{B}_X(\bar{x}_i, \epsilon)$ *and* $x^* \in X^*$ *with* $\|x^*\| = 1$ *such that* $\langle x^*, w - \sum_{i=1}^m x_i \rangle = \|w - \sum_{i=1}^m x_i\|$ *and*

$$x^* \in \bigcap_{i=1}^m \big((1+\epsilon)\partial_a d(\cdot, A_i)(x_i) + \epsilon \overline{B}_{X^*}\big).$$

Proof. (i) Let $A = \prod_{i=1}^m A_i$, $w_i = w/m$ $(i = 1, \ldots, m)$ and let F be defined by (25), that is

$$F(x_1, \ldots, x_m) = \left(\sum_{i=1}^m x_i\right) - w \quad \text{for all } (x_1, \ldots, x_m) \in X^m.$$

Then it is easily verified that $d(0, F(A)) = d(w, \sum_{i=1}^m A_i)$. Consequently (26) can be rewritten as

$$\big\|F(\bar{x})\big\| < d\big(0, F(A)\big) + \epsilon^2. \tag{28}$$

Moreover, one has

$$F^{-1}(0) \cap \prod_{i=1}^m A_i = \left\{(x_1, \ldots, x_m) \in \prod_{i=1}^m A_i \colon \sum_{i=1}^m x_i = w\right\} = \emptyset, \tag{29}$$

thanks to the assumption that $w \notin \sum_{i=1}^m A_i$. This implies that, for the present F and A, the problem (IP) does not have any solution and so any ϵ-minimizer of the problem (MP) is automatically an outer ϵ-minimizer. Hence $\bar{x}$ is an outer ϵ-minimizer of the corresponding minimization problem (MP). Applying Theorem 3.3(i) there exist $y^* \in S_{X^*}$, $u = (u_1, \ldots, u_m) \in (\prod_{i=1}^m A_i) \cap \overline{B}_{X^m}(\bar{x}, \epsilon)$ and $v = (v_1, \ldots, v_m) \in \overline{B}_{X^m}(\bar{x}, \epsilon)$ such that

$$\left\langle y^*, \sum_{i=1}^m v_i - w \right\rangle = \left\| w - \sum_{i=1}^m v_i \right\| \tag{30}$$

and

$$-y^* \in N_a(u_i, A_i) + \epsilon \overline{B}_{X^*} \quad (i = 1, \ldots, m).$$

Thus (27) follows by taking $x^* = -y^*$. Moreover, from (30), $y^* \in S_{X^*}$ and the triangle inequality we see that

$$\begin{aligned}
\left\langle x^*, w - \sum_{i=1}^m u_i \right\rangle &= \left\langle y^*, \sum_{i=1}^m u_i - w \right\rangle \\
&= \left\langle y^*, \sum_{i=1}^m v_i - w \right\rangle + \sum_{i=1}^m \langle y^*, u_i - v_i \rangle \\
&= \left\| w - \sum_{i=1}^m v_i \right\| + \sum_{i=1}^m \langle y^*, u_i - v_i \rangle \\
&\geqslant \left\| w - \sum_{i=1}^m u_i \right\| - \sum_{i=1}^m \|u_i - v_i\| + \sum_{i=1}^m \langle y^*, u_i - v_i \rangle \\
&\geqslant \left\| w - \sum_{i=1}^m u_i \right\| - 2\sum_{i=1}^m \|u_i - v_i\| \\
&\geqslant \left\| w - \sum_{i=1}^m u_i \right\| - 2\sum_{i=1}^m \big(\|u_i - \bar{x}_i\| + \|v_i - \bar{x}_i\|\big) \\
&\geqslant \left\| w - \sum_{i=1}^m u_i \right\| - 4\epsilon.
\end{aligned}$$

(ii) We further assume that ∂_a is complete. Since F is Lipschitz on X with rank 1, by Theorem 3.3(ii) there exist $x = (x_1, \dots, x_m) \in (\prod_{i=1}^m A_i) \cap \overline{B}_{X^m}(\bar{x}, \epsilon)$ and $y^* \in S_{X^*}$ with $\langle y^*, \sum_{i=1}^m x_i - w\rangle = \|w - \sum_{i=1}^m x_i\|$ such that

$$-y^* \in (1+\epsilon)\partial_a d(\cdot, A)(x_i) + \epsilon \overline{B}_{X^*} \quad (i = 1, \dots, m).$$

Thus (ii) is shown by taking $x^* = -y^*$. This completes the proof. □

Corollary 3.1. *(See [27].) Let $X \in \mathcal{X}$ and let A be a closed subset of X. Let $w \in X \setminus A$. Then for any $\epsilon \in (0, 1)$, there exists $u \in \operatorname{bd} A$ and $x^* \in X^*$ with $\|x^*\| = 1$ such that $\langle x^*, w - u\rangle \geqslant (1-\epsilon)\|w - u\|$ and*

$$x^* \in N_a(u, A) + \epsilon \overline{B}_{X^*}. \tag{31}$$

Proof. Let $\epsilon \in (0, \min\{1, d(w, A)/4\})$. Choose $x \in A$ such that $\|w - x\| < d(w, A) + \epsilon^4$. Applying Theorem 3.4(i) to the tuple $\{1, w, x, \epsilon^2\}$ in place of $\{m, w, \bar{x}_1, \epsilon\}$, we obtain $u \in A \cap \overline{B}(x, \epsilon^2)$ and $x^* \in X^*$ with $\|x^*\| = 1$ such that (31) and the following (32) hold:

$$\langle x^*, w - u \rangle \geqslant \|w - u\| - 4\epsilon^2. \tag{32}$$

This together with our choice of ϵ give that

$$\begin{aligned}\langle x^*, w-u\rangle &\geqslant \|w-u\| - 4\epsilon^2\\ &= (1-\epsilon)\|w-u\| + \epsilon\|w-u\| - 4\epsilon^2\\ &\geqslant (1-\epsilon)\|w-u\| + \epsilon d(w,A) - 4\epsilon^2\\ &\geqslant (1-\epsilon)\|w-u\|.\end{aligned}$$

Finally, from (31) $N_a(u,A) \neq \{0\}$, and hence Remark 2.5 implies that $u \in \mathrm{bd}(A)$. This completes the proof. □

Corollary 3.2. *Let X and $\{A_i\}_{i=1}^m$ be as in Theorem* 3.4. *Let $\bar{x}_i \in A_i$ be such that*

$$\sum_{i=1}^m \bar{x}_i \in \mathrm{bd}\Bigg(\sum_{i=1}^m A_i\Bigg). \tag{33}$$

Then the following assertions hold:

(i) *For any $\epsilon > 0$ there exist $x_i \in A_i \cap \overline{B}_X(\bar{x}_i, \epsilon)$ and $x^* \in X^*$ with $\|x^*\| = 1$ such that*

$$x^* \in \bigcap_{i=1}^m \big[N_a(x_i, A_i) + \epsilon \overline{B}_{X^*}\big]. \tag{34}$$

(ii) *If ∂_a is assumed to be complete, then* (34) *can be strengthened to the following form*:

$$x^* \in \bigcap_{i=1}^m \big[(1+\epsilon)\partial_a d(\cdot, A_i)(x_i) + \epsilon \overline{B}_{X^*}\big].$$

Proof. By (33), for any $\epsilon > 0$ there exists $w \in X$ such that $w \notin \sum_{i=1}^m A_i$ and $\|w - \sum_{i=1}^m \bar{x}_i\| < \epsilon^2$. Then the conclusion follows by applying Theorem 3.4. □

3.3. Separation results in Banach spaces

In this subsection, we consider another case when $m \geqslant 2$, $Y = X^m$, $X_i = X$ and $F : X^m \to X^m$ is defined by

$$F(x_1, \ldots, x_m) = (0, x_2 - x_1, x_3 - x_1, \ldots, x_m - x_1). \tag{35}$$

It is clear that F is continuous, linear (and hence strictly differentiable with $\nabla F(x) = F$ for each x). In addition, it can be verified that F is Lipschitz with rank $m-1$, that is

$$\big\|F(x) - F(x')\big\| \leqslant (m-1)\|x - x'\| \quad \text{for all } x, x' \in X^m. \tag{36}$$

Moreover, it is routine to verify that

$$\sum_{i=1}^m \nabla_i F(x) = 0. \tag{37}$$

Thus by (5) we have for each $y^* \in (X^m)^*$ that

$$\begin{aligned}\big\langle y^*\circ F, x\big\rangle = \big\langle y^*\circ \nabla F(x), x\big\rangle &= \sum_{i=1}^{m}\big\langle y^*\circ \nabla_i F(x), x_i\big\rangle \\ &= \sum_{i=2}^{m}\big\langle y^*\circ \nabla_i F(x), x_i\big\rangle + \left\langle y^*\circ\left(-\sum_{i=2}^{m}\nabla_i F(x)\right), x_1\right\rangle \\ &= \sum_{i=2}^{m}\big(y^*\circ \nabla_i F(x)\big)(x_i - x_1) \\ &\leqslant \Big(\max_{2\leqslant i\leqslant m}\big\|y^*\circ \nabla_i F(x)\big\|\Big)\cdot \sum_{i=2}^{m}\|x_i - x_1\| \\ &\leqslant \Big(\max_{2\leqslant i\leqslant m}\big\|y^*\circ \nabla_i F(x)\big\|\Big)\big\|F(x)\big\|. \end{aligned} \tag{38}$$

Definition 3.5. (See [26].) Let $\{A_i\}_{i\in I}$ be a collection of closed subsets of X for some index set I. The non-intersection index for $\{A_i\}_{i\in I}$ is defined by

$$\gamma(A_i; I) := \inf\left\{\sum_{i\in I}\|x_i - x_1\|\colon\ x_i\in A_i\right\}.$$

The following result is established in some special case such as when $\mathcal{X}$ is the class of all Banach spaces and ∂_a is the Clarke–Rockafellar subdifferential ∂_c (see [26]).

Theorem 3.6. *Let $X\in\mathcal{X}$ and let $I=\{1,2,\dots,m\}$. Let A_i $(i\in I)$ be closed subsets of X with $\bigcap_{i\in I}A_i=\emptyset$. Let $\bar{x}=(\bar{x}_1,\dots,\bar{x}_m)\in\prod_{i=1}^{m}A_i$ and $\epsilon>0$ be such that*

$$\sum_{i\in I}\|\bar{x}_i - \bar{x}_1\| < \gamma(A_i; I) + \epsilon^2. \tag{39}$$

Then the following assertions hold:

(i) *There exist $u_i\in A_i\cap \overline{B}_X(\bar{x}_i,\epsilon)$ and $x_i^*\in X^*$ such that*

$$\sum_{i=1}^{m}x_i^* = 0, \qquad \sum_{i=1}^{m}\big\|x_i^*\big\| = 1 \tag{40}$$

and

$$x_i^*\in N_a(u_i, A_i) + \epsilon\overline{B}_{X_i^*}\quad (i=1,2,\dots,m). \tag{41}$$

(ii) *Suppose the abstract subdifferential ∂_a is complete. Then there exist $x_i\in A_i\cap\overline{B}_X(\bar{x}_i,\epsilon)$, $x_i^*\in X^*$, $K\in[\frac{m-1+\epsilon}{m(m-1)}, m-1+\epsilon]$ satisfying* (40) *and*

$$x_i^*\in K\partial_a d(\cdot, A_i)(x_i) + \epsilon\overline{B}_{X_i^*}\quad (i=1,2,\dots,m). \tag{42}$$

Proof. (i) Let F be defined by (35) and let $A = \prod_{i=1}^m A_i$. Then it is easy to verify that $\|F(\bar{x})\| = \sum_{i\in I} \|\bar{x}_i - \bar{x}_1\|$ and that $d(0, F(A)) = \gamma(A_i; I)$; consequently (39) can be rewritten as

$$\|F(\bar{x})\| < d\big(0, F(A)\big) + \epsilon^2. \tag{43}$$

Moreover, one can verify easily that

$$F^{-1}(0) \cap \prod_{i=1}^m A_i = \left\{(x, x, \ldots, x) \in X^m\colon\ x \in \bigcap_{i=1}^m A_i\right\} = \emptyset, \tag{44}$$

thanks to the assumption $\bigcap_{i=1}^m A_i = \emptyset$. By (43) and (44), it is easy to see that, for the present F and A, $\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m)$ is an outer ϵ-minimizer of the minimization problem (MP). By Theorem 3.3(i), there exists $y^* \in (X^m)^*$ with $\|y^*\| = 1$, $u = (u_1, u_2, \ldots, u_m) \in A \cap \prod_{i=1}^m \overline{B}_{X_i}(\bar{x}, \epsilon)$, $v = (v_1, v_2, \ldots, v_m) \in \prod_{i=1}^m \overline{B}_{X_i}(\bar{x}, \epsilon)$ such that

$$-y^* \circ \nabla_i F(v) \in N_a(u_i, A_i) + \epsilon \overline{B}_{X_i^*} \quad (i = 1, 2, \ldots, m), \tag{45}$$

$$\big\langle y^*, F(v)\big\rangle = \big\|F(v)\big\| \neq 0. \tag{46}$$

On the other hand, by (38) (applied to v in place of x), we have

$$\big\langle y^*, F(v)\big\rangle \leqslant \max_{2\leqslant i\leqslant m} \big\|y^* \circ \nabla_i F(v)\big\| \cdot \big\|F(v)\big\|. \tag{47}$$

Combining (46) and (47), we have $\sum_{i=1}^m \|y^* \circ \nabla_i F(v)\| \geqslant \max_{2\leqslant i\leqslant m} \|y^* \circ \nabla_i F(v)\| \geqslant 1$. Let $\alpha = \sum_{i=1}^m \|y^* \circ \nabla_i F(v)\|$ and $x_i^* = -y^* \circ \nabla_i F(v)/\alpha$. Then $\alpha \geqslant 1$ and $\sum_{i=1}^m \|x_i^*\| = 1$. Moreover, by (37) we have $\sum_{i=1}^m y^* \circ \nabla_i F(v) = 0$, and so $\sum_{i=1}^m x_i^* = 0$. Thus (40) holds. Since $\alpha \geqslant 1$, (45) implies that $x_i^* \in N_a(u_i, A_i) + \epsilon \overline{B}_{X_i^*}$ $(i = 1, 2, \ldots, m)$. Thus (41) holds and the proof of (i) is completed.

(ii) Suppose that ∂_a is complete. By Theorem 3.3(ii) and (36), there exist $x = (x_1, \ldots, x_m) \in A \cap \overline{B}_{\prod_{i=1}^m X_i}(\bar{x}, \epsilon)$ and $y^* \in S_{Y^*}$ such that

$$\big\langle y^*, F(x)\big\rangle = \big\|F(x)\big\| \neq 0 \tag{48}$$

and $-y^* \circ \nabla_i F(x) \in (m - 1 + \epsilon)\, \partial_a d(\cdot, A_i)(x_i) + \epsilon \overline{B}_{X_i^*}$. Using (48) in place of (46), we can show as before that $\beta := \sum_{i=1}^m \|y^* \circ \nabla_i F(x)\| \geqslant 1$. In addition, since $\|y^*\| = 1$ and F is Lipschitz with rank $m - 1$, we have $\beta \leqslant m(m - 1)$. Let $x_i^* = -y^* \circ \nabla_i F(x)/\beta$ and $K := \frac{m-1+\epsilon}{\beta}$. Then we have $K \in [\frac{m-1+\epsilon}{m(m-1)}, m - 1 + \epsilon]$. Moreover (40) and the following (49) hold:

$$x_i^* \in K\, \partial_a d_{A_i}(x) + \frac{\epsilon}{\beta} \overline{B}_{X_i^*} \subseteq K\, \partial_a d_{A_i}(x) + \epsilon \overline{B}_{X_i^*}. \tag{49}$$

This completes the proof. □

As a consequence, we present the following corollary. Part (i) of it is known as the extended extremal principle (see Mordukhovich et al. [19] in the special case when $\mathcal{X}$ is the class of all Asplund spaces and ∂_a is the Fréchet subdifferential). To begin with, we recall the definition of extremal point.

Definition 3.7. Let $X \in \mathcal{X}$ and $m \geqslant 2$. Let $S_i : M_i \to 2^X$ $(i = 1, 2, \dots, m)$ denote m multifunctions from metric spaces (M_i, d_i) into X. We say that $\bar{x} \in X$ is an extremal point of the system $(S_1, S_2, \dots, S_m)$ at $(\bar{s}_1, \bar{s}_2, \dots, \bar{s}_m)$, provided that $\bar{x} \in S_1(\bar{s}_1) \cap S_2(\bar{s}_2) \cap \dots \cap S_m(\bar{s}_m)$ and that there exists $\rho > 0$ such that for any $\epsilon > 0$ there exists $(s_1, s_2, \dots, s_m) \in M_1 \times \dots \times M_m$ with $d_i(s_i, \bar{s}_i) \leqslant \epsilon$ and $d(\bar{x}, S_i(s_i)) \leqslant \epsilon$ such that

$$B(\bar{x}, \rho) \cap \left(\bigcap_{i=1}^{m} S_i(s_i) \right) = \emptyset.$$

Corollary 3.3. *Let $(\bar{s}_1, \dots, \bar{s}_m) \in \prod_{i=1}^m M_i$ and let $\bar{x}$ be an extremal point of the system $(S_1, \dots, S_m)$ at $(\bar{s}_1, \dots, \bar{s}_m)$. Suppose that each S_i is closed-valued near $\bar{s}_i$. Then the following assertions hold*:

(i) *For any $\epsilon > 0$ there exist $s_i \in M_i$ with $d_i(s_i, \bar{s}_i) \leqslant \epsilon$, $x_i \in S_i(s_i) \cap \overline{B}_X(\bar{x}, \epsilon)$ and $x_i^* \in X^*$ such that*

$$x_i^* \in N_a\big(x_i, S_i(s_i)\big) + \epsilon \overline{B}_{X_i^*}, \tag{50}$$

$$\sum_{i=1}^{m} x_i^* = 0 \quad \textit{and} \quad \sum_{i=1}^{m} \|x_i^*\| = 1. \tag{51}$$

(ii) *Suppose that abstract subdifferential ∂_a is complete. Then* (50) *can be strengthened to the following form*: *there exists $K \in [\frac{1}{m}, m]$ such that*

$$x_i^* \in K \partial_a d\big(\cdot, S_i(s_i)\big)(x_i) + \epsilon \overline{B}_{X_i^*}.$$

Proof. (i) Let $I = \{1, \dots, m\}$ and take $\rho > 0$ satisfying the properties stated in Definition 3.7. By the assumption and considering smaller ρ if necessary, we can assume that each $S_i(s)$ is closed whenever $d_i(s, \bar{s}_i) \leqslant \rho$ $(i = 1, 2, \dots, m)$. Take ϵ such that $0 < \epsilon < \min\{\rho, 1\}$. Then there exists a corresponding $(s_1, s_2, \dots, s_m) \in M_1 \times M_2 \times \dots \times M_m$ with

$$d_i(s_i, \bar{s}_i) < \frac{\epsilon^2}{8m^2} < \epsilon \quad \text{and} \quad d\big(\bar{x}, S_i(s_i)\big) < \frac{\epsilon^2}{8m^2} \tag{52}$$

such that $B(\bar{x}, \rho) \cap (\bigcap_{i=1}^m S_i(s_i)) = \emptyset$. Thus letting

$$A_i := S_i(s_i) \cap \overline{B}(\bar{x}, \rho) \tag{53}$$

we have $\bigcap_{i=1}^m A_i = \emptyset$. Moreover, by our choice of ρ and ϵ, it is easy to check that each A_i is closed. By (52), one can choose $\hat{x}_i \in S_i(s_i)$ such that $\|\hat{x}_i - \bar{x}\| < \frac{\epsilon^2}{8m^2}$. Since $\epsilon < \min\{1, \rho\}$, it follows that $\hat{x}_i \in A_i$ and $\sum_{i \in I} \|\hat{x}_i - \hat{x}_1\| \leqslant \sum_{i \in I} \|\hat{x}_i - \bar{x}\| + m\|\hat{x}_1 - \bar{x}\| < \epsilon^2/4$. Therefore we obtain $\sum_{i \in I} \|\hat{x}_i - \hat{x}_1\| < \gamma(A_i; I) + \epsilon^2/4$. Applying Theorem 3.6(i) to the tuple $(\{\hat{x}_i\}, \{A_i\}, \epsilon/2)$ in place of $\{\{\bar{x}_i\}, \{A_i\}, \epsilon\}$, there exist $x_i \in A_i \cap B_{X_i}(\hat{x}_i, \epsilon/2)$ and $x_i^* \in X^*$ with $x_i^* \in N_a(x_i, A_i) + \frac{\epsilon}{2} \overline{B}_{X_i^*}$ such that $\sum_{i=1}^m x_i^* = 0$ and $\sum_{i=1}^m \|x_i^*\| = 1$. In particular, (51) is satisfied. To finish the proof of (i), it remains to verify that

$$\|x_i - \bar{x}\| \leqslant \epsilon \quad \text{and} \quad N_a(x_i, A_i) = N_a\big(x_i, S_i(s_i)\big). \tag{54}$$

By the triangle inequality and our choice of ϵ, we obtain that for each i

$$\|x_i - \bar{x}\| \leqslant \|x_i - \hat{x}_i\| + \|\hat{x}_i - \bar{x}\| < \epsilon < \rho.$$

It follows from (53) and Remark 2.4 that (54) holds as required.

(ii) Suppose that ∂_a is complete. Applying Theorem 3.6(ii) to the tuple $\{\{\hat{x}_i\}, \{A_i\}, \epsilon/2\}$ in place of $\{\{\bar{x}_i\}, \{A_i\}, \epsilon\}$, there exist $K \in [\frac{m-1+\epsilon}{m(m-1)}, m-1+\epsilon] \subseteq [\frac{1}{m}, m]$, $x_i \in A_i \cap B_{X_i}(\hat{x}_i, \epsilon/2)$ and $x_i^* \in X^*$ such that (51) holds and $x_i^* \in K\partial_a d(\cdot, A_i)(x_i) + \frac{\epsilon}{2}\overline{B}_{X_i^*}$. Similar to the proof of part (i), one can show that $\|x_i - \bar{x}\| < \epsilon < \rho$ and hence that $\partial_a d(\cdot, A_i)(x_i) = \partial_a d(\cdot, S_i(s_i))(x_i)$. Therefore the tuple $(\{x_i\}, \{x_i^*\}, \{s_i\})$ has the desired properties stated in (ii). □

If X is reflexive and each A_i is weakly closed, then the preceding theorem can be extended to the case involving infinitely many sets.

Theorem 3.8. *Let $X \in \mathcal{X}$ and suppose that X is reflexive. Let J be an arbitrary index set and $\{A_i\colon i \in J\}$ be a family of weakly closed subsets of X with empty intersection. Let $\{\bar{x}_i\colon i \in J\}$ be elements in X such that $\bar{x}_i \in A_i$ $(i \in J)$ and*

$$\sum_{i \in J} \|\bar{x}_i - \bar{x}_1\| < \gamma(A_i; J) + \epsilon^2 < \infty, \tag{55}$$

where ϵ is a positive constant. Then there exist $x_i \in A_i \cap B_X(\bar{x}_i, \epsilon)$ and $x_i^* \in X^*$ such that

$$x_i^* \in N_a(x_i, A_i) + \epsilon \overline{B}_{X_i^*} \quad (i \in J), \tag{56}$$

$$\sum_{i \in J} x_i^* = 0 \quad \text{and} \quad \sum_{i \in J} \|x_i^*\| = 1. \tag{57}$$

Proof. Fix $\rho \in (\epsilon, \infty)$ and define $P_i := A_i \cap \overline{B}_X(\bar{x}_i, \rho)$ $(i \in J)$. Since X is reflexive and P_i is weakly closed and bounded, P_i is weakly compact in X for each $i \in J$. Since $\bigcap_{i \in J} A_i = \emptyset$, $\bigcap_{i \in J} P_i = \emptyset$ and it follows that $\bigcap_{i \in I} P_i = \emptyset$ for some finite subset I of J. Note that

$$\gamma(A_i; I) + \sum_{i \in J \setminus I} \|\bar{x}_i - \bar{x}_1\| \geqslant \gamma(A_i; J).$$

It follows from (55) and $P_i \subseteq A_i$ $(i \in J)$ that

$$\begin{aligned} \sum_{i \in J} \|\bar{x}_i - \bar{x}_1\| &= \sum_{i \in I} \|\bar{x}_i - \bar{x}_1\| - \sum_{i \in J \setminus I} \|\bar{x}_i - \bar{x}_1\| \\ &< \gamma(A_i; J) + \epsilon^2 - \sum_{i \in J \setminus I} \|\bar{x}_i - \bar{x}_1\| \\ &\leqslant \gamma(A_i; I) + \epsilon^2 \\ &\leqslant \gamma(P_i; I) + \epsilon^2. \end{aligned} \tag{58}$$

Applying Theorem 3.6(i) to $\{P_i\}_{i\in I}$ in place of $\{A_i\}_{i\in I}$, there exist $x_i \in P_i \cap B_X(\bar{x}_i, \epsilon)$ and $x_i^* \in X^*$ with $x_i^* \in N_a(x_i, P_i) + \epsilon \overline{B}_{X_i^*}$ $(i \in I)$ such that $\sum_{i\in I} x_i^* = 0$ and $\sum_{i\in I} \|x_i^*\| = 1$. Moreover, since $\epsilon < \rho$, x_i is in the interior of $\overline{B}_{X_i}(x_i, \rho)$ and so $N_a(x_i, P_i) = N_a(x_i, A_i)$ by Remark 2.4. Thus (56) and (57) are satisfied if we further define, for $i \in J \setminus I$, $x_i = \bar{x}_i$ and $x_i^* = 0$. □

3.4. Extension of the Bishop–Phelps theorem

The famous Bishop–Phelps theorem (cf. [20]) can be stated as follows. If A is a closed convex subset of a Banach space X, then the support points of A are dense in $\operatorname{bd} A$ and the support functionals of A are dense in the barrier cone $\operatorname{barr}(A)$ of A, where $\operatorname{barr}(A)$ is defined by

$$\operatorname{barr}(A) = \left\{x^* \in X^*\colon \sup_{a\in A}\langle x^*, a\rangle < +\infty\right\}. \tag{59}$$

Note that $x \in A$ is a support point of A with a support functional x^* if and only if $x^* \in N(x, A) \setminus \{0\}$ where $N(x, A)$ denotes the (convex) normal cone of A at x. The following Theorem 3.10 can be regarded as a nonconvex extension of the Bishop–Phelps theorem and is to be established via a lemma of independent interest. Recall that, given a proper lower semicontinuous function $f : X \to \mathbb{R} \cup \{+\infty\}$, the conjugate function of f is defined by

$$f^*(x^*) = \sup_{x\in X}\{\langle x^*, x\rangle - f(x)\}.$$

Lemma 3.9. *Let $X \in \mathcal{X}$ and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Then the following assertions hold*:

(i) $\operatorname{dom}(\partial_a f)$ *is dense in* $\operatorname{dom}(f)$.
(ii) $\mathrm{R}(\partial_a f)$ *is dense in* $\operatorname{dom}(f^*)$ *where* $\mathrm{R}(\partial_a f) := \bigcup_{x\in X} \partial_a f(x)$.

Proof. (i) Let $x \in \operatorname{dom} f$. By the lower semicontinuity of f, there exists $\eta > 0$ such that f is bounded below on $\overline{B}(x, \eta)$. Consider any $\epsilon \in (0, \eta)$. For each $n \in \mathbb{N}$, define a function $g_n : X \to \mathbb{R} \cup \{+\infty\}$ by $g_n(\cdot) = f(\cdot) + \delta_{\overline{B}(x,\eta)}(\cdot) + n\|\cdot - x\|$. Each g_n is also lower semicontinuous and bounded below. Choose $\bar{x}_n \in X$ such that

$$g_n(\bar{x}_n) < \inf_{x\in X} g_n(x) + \epsilon^2/4.$$

By the Ekeland variational principle, there exists $x_n \in \overline{B}(x, \eta)$ with $\|x_n - \bar{x}_n\| \leqslant \epsilon/2$ such that $g_n(\cdot) + (\epsilon/2)\|\cdot - x_n\|$ attains its minimum at x_n, i.e.

$$g_n(x_n) \leqslant g_n(a) + (\epsilon/2)\|a - x_n\| \quad \text{for all } a \in X. \tag{60}$$

We claim that

$$\liminf_{n\to\infty} \|x_n - x\| = 0. \tag{61}$$

Granting this there then exists some n_0 such that $x_{n_0} \in B(x, \epsilon/2) \subseteq B(x, \eta)$. Hence (60) and the definition of g_{n_0} imply that x_{n_0} is a local minimizer of

$$f(\cdot) + n_0\|\cdot - x\| + (\epsilon/2)\|\cdot - x_{n_0}\|.$$

It follows from (P2), (P6) and (P7) that there exist $y_1, y_2, y_3 \in B(x_{n_0}, \epsilon/2)$ such that $0 \in \partial_a f(y_1) + \partial_a(n_0\|\cdot - x\|)(y_2) + \partial_a(\frac{\epsilon}{2}\|\cdot - x_{n_0}\|)(y_3) + \frac{\epsilon}{2}B_{X^*}$. In particular, we have $y_1 \in \operatorname{dom}(\partial_a f) \cap B(x_{n_0}, \epsilon/2) \subseteq \operatorname{dom}(\partial_a f) \cap B(x, \epsilon)$ and hence (i) holds. Now we turn to the proof of (61). Suppose on the contrary that there exists $\alpha > 0$ such that

$$\|x_n - x\| \geqslant \alpha \quad \text{for all } n \in \mathbb{N}. \tag{62}$$

Substituting $a = x$ in (60) and taking into account of the definition of g_n, we obtain

$$f(x_n) + n\|x_n - x\| \leqslant f(x) + (\epsilon/2)\|x - x_n\| \quad \text{for all } n \in \mathbb{N}.$$

This together with (62) yield that

$$f(x_n) \leqslant f(x) - (n - \epsilon/2)\|x - x_n\| \leqslant f(x) - (n - \epsilon/2)\alpha \quad \text{for all } n \in \mathbb{N}.$$

In particular, we have $f(x_n) \to -\infty$ as $n \to \infty$. This is impossible since $x_n \in \overline{B}(x, \eta)$ and f is bounded below on $\overline{B}(x, \eta)$. Thus (61) holds.

(ii) Let $\epsilon > 0$ and $x^* \in \operatorname{dom}(f^*)$. By the definition of conjugate function, it follows that $\sup_{x \in X}\{\langle x^*, x\rangle - f(x)\} < \infty$. Define g by $g(\cdot) := -\langle x^*, \cdot\rangle + f(\cdot)$. It follows that g is lower semicontinuous and bounded below. Choose $x_0 \in X$ such that $g(x_0) \leqslant \inf_{a \in X} g(a) + \epsilon^2/4$. By the Ekeland variational principle, there exists $x_1 \in \overline{B}(x_0, \epsilon/2)$ such that x_1 is a minimal point of h defined by $h(\cdot) := g(\cdot) + (\epsilon/2)\|\cdot - x_1\|$. By (P1), (P6), (P7) and Remark 2.2, there exist $x_2 \in B(x_1, \epsilon/2)$ and $x_3 \in B(x_1, \epsilon/2)$ such that $0 \in -x^* + \partial_a f(x_2) + \partial_a(\frac{\epsilon}{2}\|\cdot - x_1\|)(x_3) + \frac{\epsilon}{2}B_{X^*}$. Hence from Remark 2.1, there exists $y^* \in \partial_a f(x_2)$ such that $\|x^* - y^*\| \leqslant \epsilon$. Thus (ii) holds and this completes the proof. □

Theorem 3.10. *Let $X \in \mathcal{X}$ and let A be a nonempty proper closed subset of X. Then the following assertions hold*:

(i) *The set* $P := \{x \in \operatorname{bd} A\colon\ N_a(x, A) \neq \{0\}\}$ *is dense in* $\operatorname{bd} A$.
(ii) *Let* $K := \bigcup_{x \in P} N_a(x, A)$. *Then K is dense in* $\operatorname{barr}(A)$, *i.e., for any* $x^* \in \operatorname{barr}(A)$ *and* $\epsilon > 0$ *there exists* $y^* \in K$ *such that* $\|x^* - y^*\| \leqslant \epsilon$.

Proof. (i) In view of Remark 2.5, the proof of (i) is immediate by applying Corollary 3.2 with $m = 2$, $A_1 = A$ and $A_2 = \{0\}$.

(ii) Since A is a proper subset, $\operatorname{bd} A \neq \emptyset$ and hence $P \neq \emptyset$ thanks to (i). Noting that $0 \in N_a(x, A)$ for all $x \in A$ (since x is a minimizer of $\delta_A(\cdot)$ for all $x \in A$ and thanks to (P6)), it follows that $0 \in K$. Fix $x^* \in \operatorname{barr}(A)$ and without loss of generality, we may assume $x^* \neq 0$. Consider the indicator function δ_A of A and note that $\operatorname{dom}(\delta_A^*) = \operatorname{barr}(A)$. Thus we have in particular that $x^* \in \operatorname{dom}\delta_A^*$. Hence, for any $\epsilon \in (0, \|x^*\|)$, one can apply Lemma 3.9(ii) to δ_A in place of f and we conclude that there exist $x \in X$ and $y^* \in \partial_a \delta_A(x) = N_a(x, A)$ such that $\|x^* - y^*\| \leqslant \epsilon$. Since $\epsilon < \|x^*\|$, we have $y^* \in N_a(x, A) \setminus \{0\}$. This together with Remark 2.5 imply that $x \in P$ and so $y^* \in K$. This completes the proof. □

Remark 3.11. The nonconvex Bishop–Phelps programme began from the seminal paper [17,18] of Mordukhovich and Shao where they established Theorem 3.10(i) in Asplund spaces. In the Banach space setting, Theorem 3.10(i) is known when ∂_a is the Clarke–Rockafellar subdifferential ∂_c (cf. [27]). To the best of our knowledge, the result given in part (ii) of Theorem 3.10 is new even for a restricted class $\mathcal{X}$ of (Banach or Asplund) spaces.

3.5. Separate point theorem in Banach spaces

This subsection is devoted to establish a separate point theorem and related results in Banach spaces.

Definition 3.12. Let $X \in \mathcal{X}$ and $m \geqslant 1$. Let $A_1, \dots, A_m$ be closed subsets of X and let $\epsilon > 0$. We say that $\bar{x} := (\bar{x}_1, \dots, \bar{x}_m) \in X^m$ is an ϵ-separable point of the system $\{A_1, \dots, A_m\}$ provided that $A_i \cap B_X(\bar{x}_i, \epsilon^2/m) \neq \emptyset$ $(i = 1, \dots, m)$ and $\sum_{i=1}^m \bar{x}_i \notin \sum_{i=1}^m A_i$.

Theorem 3.13. *Let $\bar{x} \in X^m$ be an ϵ-separable point of the system $\{A_1, \dots, A_m\}$ for some $\epsilon > 0$. Then the following assertions hold*:

(i) *There exist $x_i \in A_i \cap B_X(\bar{x}_i, \varepsilon)$ and $x^* \in X^*$ with $\|x^*\| = 1$ such that*

$$x^* \in \bigcap_{i=1}^m \big(N_a(x_i, A_i) + \varepsilon \overline{B}_{X^*}\big). \tag{63}$$

(ii) *If we further assume that ∂_a is complete, then* (63) *can be strengthened to the following form*:

$$x^* \in \bigcap_{i=1}^m \big[(1+\epsilon)\partial_a d(\cdot, A_i)(x_i) + \epsilon \overline{B}_{X^*}\big].$$

Proof. (i) Let $A := \prod_{i=1}^m A_i$ and let $F : X^m \to X$ be defined by $F(x_1, \dots, x_m) = \sum_{i=1}^m (x_i - \bar{x}_i)$. Since $\bar{x}$ is a ϵ-separable point of the system $\{A_1, \dots, A_m\}$. We see that $\bar{x}$ is an outer ϵ-minimizer of the corresponding minimization problem (MP). Applying Theorem 3.3(i) there exist $y^* \in S_{X^*}$, $x = (x_1, \dots, x_m) \in (\prod_{i=1}^m A_i) \cap \overline{B}_{X^m}(\bar{x}, \epsilon)$ such that

$$-y^* \in N_a(x_i, A_i) + \epsilon \overline{B}_{X^*} \quad (i = 1, \dots, m).$$

Thus (63) follows by taking $x^* = -y^*$.

(ii) We further assume that ∂_a is complete. Since F is Lipschitz on X with rank 1. The conclusion follows immediately by applying Theorem 3.3(ii). □

As a consequence, we present the following corollary. The part (i) of it was proved by Zhu in [29] in the special case when $\mathcal{X}$ is the class of all β-smooth Banach spaces and ∂_a is the corresponding viscosity subdifferential. To begin with we recall the following definition of separable points (cf. [29]) for set-valued maps.

Definition 3.14. Let $X \in \mathcal{X}$ and $m \geqslant 1$. Let $S_i : M_i \to 2^X$ $(i = 1, \dots, m)$ be multifunctions from metric spaces M_i with metrics d_i into a Banach space X. We say that $(\bar{x}_1, \dots, \bar{x}_m) \in X^m$ is a separable point of the system $(S_1, S_2, \dots, S_m)$ at $(\bar{s}_1, \bar{s}_2, \dots, \bar{s}_m)$ provided that $(\bar{x}_1, \dots, \bar{x}_m) \in S_1(\bar{s}_1) \times S_2(\bar{s}_2) \times \cdots \times S_m(\bar{s}_m)$ and that there exists $\rho > 0$ with the following property: for any $\epsilon > 0$ there exists $s_i \in M_i$ $(i = 1, \dots, m)$ such that $d_i(s_i, \bar{s}_i) < \epsilon$, $S_i(s_i) \cap B_X(\bar{x}_i, \epsilon) \neq \emptyset$ and

$$\sum_{i=1}^{m} \bar{x}_i \notin \sum_{i=1}^{m} \big[S_i(s_i) \cap \overline{B}_X(\bar{x}_i, \rho)\big].$$

Corollary 3.4. *Let $(\bar{s}_1, \dots, \bar{s}_m) \in \prod_{i=1}^{m} M_i$ and let $(\bar{x}_1, \dots, \bar{x}_m) \in X^m$ be an separable point of the system $(S_1, S_2, \dots, S_m)$ at $(\bar{s}_1, \bar{s}_2, \dots, \bar{s}_m)$. Suppose that each S_i is closed-valued near $\bar{s}_i$. Then the following assertions hold*:

(i) *For any $\epsilon > 0$ there exist $s_i \in B_{M_i}(\bar{s}_i, \epsilon)$, $x_i \in S_i(s_i) \cap B_X(\bar{x}_i, \epsilon)$ and $x^* \in X^*$ with $\|x^*\| = 1$ such that*

$$x^* \in \bigcap_{i=1}^{m} \big[N_a\big(x_i, S_i(s_i)\big) + \epsilon \overline{B}_{X^*}\big]. \tag{64}$$

(ii) *If we further assume that ∂_a is complete, then* (64) *can be strengthened to the following form*:

$$x^* \in \bigcap_{i=1}^{m} \big[(1+\epsilon)\partial_a d\big(\cdot, S_i(s_i)\big)(x_i) + \epsilon \overline{B}_{X^*}\big].$$

Proof. (i) Take $\rho > 0$ with the properties stated in Definition 3.14. By the assumption and considering smaller ρ if necessary, we can assume that each $S_i(s)$ is closed whenever $d_i(s, \bar{s}_i) \leqslant \rho$ $(i = 1, 2, \dots, m)$. Consider any ϵ such that $0 < \epsilon < \min\{\rho, 1\}$. Then there exists a corresponding $(s_1, s_2, \dots, s_m) \in M_1 \times \cdots \times M_m$ with $d_i(s_i, \bar{s}_i) \leqslant \frac{\epsilon^2}{m} < \epsilon$ such that

$$S_i(s_i) \cap B_X\left(\bar{x}_i, \frac{\epsilon^2}{m}\right) \neq \emptyset \quad \text{and} \quad \sum_{i=1}^{m} \bar{x}_i \notin \sum_{i=1}^{m} \big[S_i(s_i) \cap \overline{B}_X(\bar{x}_i, \rho)\big]. \tag{65}$$

Define $A_i := S_i(s_i) \cap \overline{B}_X(\bar{x}_i, \rho)$. We see that A_i is closed (by our choice of ϵ and ρ) and $\bar{x}$ is an ϵ-separable point of the system $\{A_1, \dots, A_m\}$. Applying Theorem 3.13(i) there exist $x_i \in A_i \cap B_{X_i}(\bar{x}_i, \epsilon)$ and $x^* \in X^*$ with $\|x^*\| = 1$ such that $x^* \in \bigcap_{i=1}^{m} [N_a(x_i, A_i) + \epsilon \overline{B}_{X^*}]$. Moreover, since $\|x_i - \bar{x}_i\| < \epsilon < \rho$. This and Remark 2.4 imply that $N_a(x_i, A_i) = N_a(x_i, S_i(s_i))$. Thus (i) is seen to hold.

(ii) We further assume that ∂_a is complete. Applying Theorem 3.13(ii), there exist $x_i \in A_i \cap B_{X_i}(\bar{x}_i, \epsilon)$ and $x^* \in X^*$ with $\|x^*\| = 1$ such that

$$x^* \in \bigcap_{i=1}^{m} \big[(1+\epsilon)\partial_a d(\cdot, A_i)(x_i) + \epsilon \overline{B}_{X^*}\big].$$

Similar to the proof of (i), one can show that $\partial_a d(\cdot, A_i)(x_i) = \partial_a d(\cdot, S_i(s_i))(x_i)$. Thus the tuple $(\{x_i\}, \{x_i^*\}, \{s_i\})$ satisfies the conclusion of part (ii). $\square$

4. Sharper versions

4.1. Bishop–Phelps type

This section is devoted to give some sharper results of Section 3 (but under stronger assumptions). The first one is on a condition slightly stronger than that of "sequentially normally compact" introduced by Mordukhovich et al. (cf. [9,17]), while the second one is on a condition slightly weaker than that of the concept "closedness of the multifunction $N_a(\cdot, A)$" (cf. [6, Corollary, p. 54]).

Definition 4.1. Let $X \in \mathcal{X}$ and let A be a closed subset in X. We say that A is ∂_a-normally compact at some given point $x \in A$ if the following implication holds for any nets (generalized sequences) $\{x_n\}$, $\{x_n^*\}$:

$$x_n \xrightarrow{A} x, \quad x_n^* \in N_a(x_n, A), \quad x_n^* \to_{w^*} 0 \quad \Rightarrow \quad x_n^* \to 0.$$

Definition 4.2. Let $X \in \mathcal{X}$ and let A be a closed subset in X. We say that A is sequentially ∂_a-normally closed at some given point $x \in A$ if the following implication holds for any sequences $\{x_n\}$, $\{x_n^*\}$:

$$x_n \xrightarrow{A} x, \quad x_n^* \in N_a(x_n, A), \quad x_n^* \to x^* \quad \Rightarrow \quad x^* \in N_a(x, A).$$

Remark 4.3. It is known that A is sequentially ∂_a-normally closed at any point $x \in A$ if A is convex.

For the remainder of this paper, we assume that the abstract subdifferential ∂_a is complete.

Lemma 4.4. *Let $X \in \mathcal{X}$. Let $A_1, \ldots, A_m$ be closed subsets in X and $\bar{x}_i \in A_i$ be such that $\sum_{i=1}^m \bar{x}_i \in \mathrm{bd}(\sum_{i=1}^m A_i)$. Suppose that for some $i_0 \in \{1, \ldots, m\}$, A_{i_0} is ∂_a-normally compact at $\bar{x}_{i_0}$. Then there exists $x^* \in X^* \setminus \{0\}$ such that $x^* \in \bigcap_{i=1}^m \partial_a d(\cdot, A_i)(\bar{x}_i)$.*

Proof. Let $\{\epsilon_k\}$ be a sequence of positive real numbers such that $\epsilon_k \to 0$. By Corollary 3.2, for each k, there exists $x_k^i \in A_i \cap B(\bar{x}_i, \epsilon_k)$, $x_k^* \in X^*$ with $\|x_k^*\| = 1$ such that $x_k^* \in \bigcap_{i=1}^m((1 + \epsilon_k)\partial_a d(\cdot, A_i)(x_k^i) + \epsilon_k \overline{B}_{X^*})$. Thus there exists $u_k^{i*} \in (1 + \epsilon_k)\partial_a d(\cdot, A_i)(x_k^i)$, $v_k^{i*} \in \epsilon_k \overline{B}_{X^*}$ such that $x_k^* = u_k^{i*} + v_k^{i*}$ $(i = 1, \ldots, m)$. By the Alaoglu theorem (and by passing to subnets if necessary), we may assume that $x_k^* \to_{w^*} x^*$. Since $v_k^{i*} \to 0$ $(i = 1, \ldots, m)$, this implies that

$$u_k^{i*} \to_{w^*} x^* \quad (i = 1, \ldots, m). \tag{66}$$

Since $u_k^{i*} \in (1 + \epsilon_k)\partial_a d(\cdot, A_i)(x_k^i)$ and $x_k^i \to \bar{x}_i$ for each i, it follows from property (P9) that $x^* \in \bigcap_{i=1}^m \partial_a d(\cdot, A_i)(\bar{x}_i)$. To finish the proof, it suffices to show $x^* \neq 0$. Suppose $x^* = 0$. Then (66) implies in particular that $u_k^{i_0*} \to_{w^*} 0$. Since $u_k^{i_0*} \in (1 + \epsilon_k)\partial_a d(\cdot, A_{i_0})(x_k^{i_0}) \subseteq N_a(x_k^{i_0}, A_{i_0})$ (by property (P5)), $x_k^{i_0} \in A_{i_0} \cap B(\bar{x}_{i_0}, \epsilon_k)$, $\epsilon_k \to 0$ and A_{i_0} is ∂_a-normally compact at $\bar{x}_{i_0}$, it follows that $u_k^{i_0*} \to 0$. However this is impossible since $\|u_k^{i_0*}\| = \|x_k^* - v_k^{i_0*}\| \geqslant \|x_k^*\| - \|v_k^{i_0*}\| \geqslant 1 - \epsilon_k \to 1$ as $k \to \infty$. This completes the proof. $\square$

Theorem 4.5. *Let $X \in \mathcal{X}$. Let A be a nonempty proper closed subset of X. Suppose that A is ∂_a-normally compact at x for all $x \in \operatorname{bd} A$. Then the following assertions hold*:

(i) $P = \operatorname{bd} A$, *where* $P := \{x \in \operatorname{bd} A\colon\ N_a(x, A) \neq \{0\}\}$.
(ii) *Suppose A is compact and sequentially ∂_a-normally closed at any $x \in \operatorname{bd} A$. Then* $K = \operatorname{barr}(A) = X^*$, *where* $K := \bigcup_{x \in P} N_a(x, A)$ *and* $\operatorname{barr}(A)$ *is defined by* (59).

Proof. (i) Fix an arbitrary $\bar{x} \in \operatorname{bd} A$. Since $\bar{x} + 0 \in \operatorname{bd}(A + \{0\})$, one can apply Lemma 4.4 to $\{A, \{0\}\}$ in place of $\{A_1, A_2\}$ and so there exists $x^* \in X^* \setminus \{0\}$ such that $x^* \in \partial_a d(\cdot, A)(\bar{x})$. Since $\bar{x}$ is an arbitrary element in $\operatorname{bd} A$, this implies that

$$\operatorname{bd} A = \bigl\{x \in \operatorname{bd} A\colon\ \partial_a d(\cdot, A)(x) \neq \{0\}\bigr\}.$$

Noting that $\{x \in \operatorname{bd} A\colon\ \partial_a d(\cdot, A)(x) \neq \{0\}\} \subseteq P$ (by property (P5) and Remark 2.5) and $P \subseteq \operatorname{bd}(A)$, it follows that the conclusion of (i) holds.

(ii) Since A is compact, $\operatorname{barr}(A) = X^* \supseteq K$. In view of Theorem 3.10, it remains to show that K is closed. Take a sequence $x_n^* \in K$ such that $x_n^* \to x^*$. By the definition of K, there exists $x_n \in P$ such that $x_n^* \in N_a(x_n, A)$. From the compactness of P and part (i) of this theorem, we assume without lose of generality that $x_n \to x$ for some $x \in \operatorname{bd} A = P$. This together with the sequentially ∂_a-normally closedness of A give that $x^* \in N_a(x, A)$. Hence K is closed and this completes the proof. □

Below, we give some criteria ensuring the ∂_a-normal compactness. Following [3], we say that a closed subset A of X is compactly epi-Lipschitzian at $\bar{x} \in A$ if there exist $\eta > 0$, a compact set $K \subseteq X$ and open sets Ω, U respectively containing $\bar{x}$ and 0 such that

$$A \cap \Omega + \lambda U \subseteq A + \lambda K \quad \forall \lambda \in (0, \eta).$$

Theorem 4.6. *Let $X \in \mathcal{X}$ and let A be a closed subset of X. Consider the following statements*:

(i) *The set A is ∂_a-normally compact at $\bar{x}$ for all $\bar{x} \in A$.*
(ii) *For any $\bar{x} \in A$, there exist $\rho > 0$ and a compact subset K in X such that for all $x \in A \cap B(\bar{x}, \rho)$*

$$N_a(x, A) \subseteq \Bigl\{x^* \in X^*\colon\ \|x^*\| \leqslant \sup_{k \in K} \langle x^*, k\rangle\Bigr\}. \tag{67}$$

(iii) *A is a convex set with* $\operatorname{ri}(A) \neq \emptyset$ *such that* $\operatorname{aff}(A)$ *is of finite codimension.*
(iv) *A is a convex set and there exists $x \in A$ such that $(A - x)^\circ$ is weak* locally compact, where* $(A - x)^\circ = \{x^* \in X^*\colon\ \langle x^*, a - x\rangle \leqslant 0,\ \forall a \in A\}$.
(v) *X is finite-dimensional.*
(vi) *A is a convex set with* $\operatorname{int}(A) \neq \emptyset$.
(vii) *A is a polyhedron.*

Then the following implications hold:

$$\begin{array}{ccccccc} \text{(vi)} & \Rightarrow & \text{(iii)} & \Rightarrow & \text{(ii)} & \Rightarrow & \text{(i)} \\ & & \Uparrow & & \Uparrow & & \\ \text{(vii)} & \Rightarrow & \text{(iv)} & & \text{(v)} & & \end{array}$$

Proof. (vi) $\Rightarrow$ (iii). It is easy to verify that $\operatorname{aff}(A) = X$ if $\operatorname{int}(A) \neq \emptyset$. Thus (vi) $\Rightarrow$ (iii) holds.

(iii) $\Rightarrow$ (ii). Suppose (iii) holds. Then, by [3, Theorem 2.5], A is compactly epi-Lipschitzian at $\bar{x}$ for any $\bar{x} \in A$. Thus using [12, Proposition 3.7], we obtain that for any $\bar{x} \in A$, there exist $\rho > 0$, $\epsilon > 0$ and a compact set K_1 in X such that for all $x \in B(\bar{x}, \rho) \cap A$

$$N(x, A) \subseteq \Big\{x^* \in X^*\colon \epsilon\|x^*\| \leqslant \sup_{k \in K_1} \langle x^*, k\rangle\Big\}, \tag{68}$$

where $N(\cdot, A)$ is the usual (convex) normal cone of A. Thus (67) holds with $K = K_1/\epsilon$, thanks to (P1).

(ii) $\Rightarrow$ (i). Let $\bar{x} \in A$. Take ρ and K satisfying the corresponding properties in (ii). Let $\{x_n\} \subseteq A$ and $\{x_n^*\} \subseteq X^*$ be nets with $x_n^* \in N_a(x_n, A)$ such that $x_n \to \bar{x}$ and $x_n^* \to_{w^*} 0$. We assume without loss of generality that $x_n \in A \cap B(\bar{x}, \rho)$ for all n. Then by our choice of ρ and K, one has that $\|x_n^*\| \leqslant \sup_{k \in K} \langle x_n^*, k\rangle$. Pick $k_1, \ldots, k_m \in K$ such that $K \subseteq \bigcup_{i=1}^m B(k_i, 1/2)$. Then

$$\begin{aligned} \|x_n^*\| &\leqslant \sup\Big\{\langle x_n^*, k\rangle\colon k \in \bigcup_{i=1}^m B(k_i, 1/2)\Big\} \\ &\leqslant \max_{1 \leqslant i \leqslant m} \langle x_n^*, k_i\rangle + \frac{1}{2}\|x_n^*\|. \end{aligned}$$

It follows that $\|x_n^*\| \leqslant 2\max_{1 \leqslant i \leqslant m} \langle x_n^*, k_i\rangle \to 0$. This means that A is ∂_a-normally compact at $\bar{x}$.

(iv) $\Rightarrow$ (iii). This is shown in [3, Theorem 2.5] (recall (1)).

(vii) $\Rightarrow$ (iv). Let A be a polyhedron and take the following form

$$A = \big\{x\colon \langle a_i^*, x\rangle \leqslant b_i,\ i = 1, 2, \ldots, m\big\}.$$

It is clear that $\operatorname{ri}(A) \neq \emptyset$. Moreover, for any $x \in \operatorname{ri}(A)$, let $I(x)$ denotes the set of all active indices at x, i.e., $I(x) = \{i \in \{1, \ldots, m\}\colon \langle a_i^*, x\rangle = b_i\}$. Observing that $(A - x)^\circ = \{0\}$ for all $x \in \operatorname{int}(A)$, we may assume without loss of generality that $x \in \operatorname{bd} A$. Then by a standard result in convex analysis (cf. [11, Lemma 2.1]) we obtain

$$(A - x)^\circ = \overline{\Big\{\sum_{i \in I(x)} \lambda_i a_i^*\colon \lambda_i \geqslant 0\Big\}}.$$

It follows that $(A - x)^\circ$ is a closed convex cone of finite dimension hence weak* locally compact. Thus (iv) holds.

(v) ⇒ (ii). Since X is of finite dimension, then the corresponding dual space X^* is also of finite dimension. Thus for all $x \in A \cap B(\bar{x}, \rho)$

$$N_a(x, A) \subseteq X^* = \Big\{x^* \in X^*\colon \ \|x^*\| \leqslant \sup_{k \in \overline{B}_X} \langle x^*, k\rangle\Big\}.$$

Thus (ii) holds as $\overline{B}_X$ is compact by (v). This completes the proof. □

4.2. Separation type

Using the preceding results, we now give the following sharper version of separation theorem type under some strengthened assumptions.

Theorem 4.7. *Let $X \in \mathcal{X}$. Let A_1 be a closed convex subset of X with* $\mathrm{ri}(A_1) \neq \emptyset$ *and let A_2 be a closed subset of X. Suppose that* $\mathrm{aff}\, A_1$ *is finite-codimensional and* $\mathrm{ri}(A_1) \cap A_2 = \emptyset$. *Let $\bar{x} \in A_1 \cap A_2$. Then there exists $x^* \in X^*$ with $\|x^*\| = 1$ such that*

$$x^* \in N_a(\bar{x}, A_2) \quad \text{and} \quad \langle x^*, \bar{x}\rangle = \inf_{x \in A_1} \langle x^*, x\rangle. \tag{69}$$

Proof. By the implication (iii) ⇒ (i) in Theorem 4.6, the given assumptions ensure that A_1 is ∂_a-normally compact at each of its element. Let $M_1 = \mathbb{R}^1$, $M_2 = \{0\}$ and $x_0 \in \mathrm{ri}(A_1)$. We may assume without loss of generality that $x_0 = 0$ (replace A_1, A_2 by $A_1 - x_0$, $A_2 - x_0$ if necessary). Define the multifunction $S_i : M_i \to 2^X$ $(i = 1, 2)$ as follows: $S_1(t) = tA_1$ $(\forall t \in \mathbb{R}^1)$ and $S_2(0) = A_2$. Note that $\bar{x} \in S_1(1) \cap S_2(0)$ and that $S_1(t) \subseteq \mathrm{ri}(A_1)$ for each $t \in [0, 1)$ (cf. [1, Lemma 3.1]). It follows from the assumption $\mathrm{ri}(A_1) \cap A_2 = \emptyset$ that $S_1(t) \cap S_2(0) = \emptyset$. Then $\bar{x}$ is an extremal point for (S_1, S_2) at $(1, 0)$. Applying $m = 2$, $\bar{s}_1 = 1$, $\bar{s}_2 = 0$ and $\epsilon = \frac{1}{n}$ in Corollary 3.3(ii), for any $n \in \mathbb{N}$ there exist $t_n \in (1 - 1/n, 1 + 1/n)$, $K_n \in [\frac{1}{2}, 2]$, $x_{1n} \in S_1(t_n) \cap B_X(\bar{x}, 1/n)$, $x_{2n} \in A_2 \cap B_X(\bar{x}, 1/n)$ such that $x^*_{1n} \in K_n \partial_a d(\cdot, S_1(t_n))(x_{1n}) + \frac{1}{n} B_{X^*}$, $x^*_{2n} \in K_n \partial_a d(\cdot, A_2)(x_{2n}) + \frac{1}{n} B_{X^*}$ and

$$x^*_{1n} + x^*_{2n} = 0, \qquad \|x^*_{1n}\| + \|x^*_{2n}\| = 1. \tag{70}$$

By the Alaoglu theorem and by passing to subnets if necessary, we may assume that $x^*_{2n} \to_{w^*} x^*$ (hence $x^*_{1n} \to_{w^*} -x^*$) and $K_n \to K \in [1/2, 2]$. This together with $x^*_{2n} \in K_n \partial_a d(\cdot, A_2)(x_{2n}) + \frac{1}{n} B_{X^*}$, $x_{2n} \to \bar{x}$ and property (P5), (P9) give that $x^* \in K \partial_a d(\cdot, A_2)(\bar{x}) \subseteq N_a(\bar{x}, A_2)$. On the other hand, since $S_1(t_n) = t_n A_1$ is a convex set, by property (P1) we have $N_a(x_{1n}, S_1(t_n)) = N(x_{1n}, S_1(t_n)) = N(x_{1n}/t_n, A_1)$, where $N(\cdot, A_1)$ is the usual (convex) normal cone of A_1. Since $x^*_{1n} \in K_n \partial_a d(\cdot, S_1(t_n))(x_{1n}) + \frac{1}{n} B_{X^*}$, this together with (P5) imply that

$$x^*_{1n} \in N(x_{1n}/t_n, A_1) + \frac{1}{n} B_{X^*}. \tag{71}$$

Since $x_{1n} \to \bar{x}$, $t_n \to 1$ and $x^*_{1n} \to_{w^*} -x^*$, it follows that $-x^* \in N(\bar{x}, A_1) = \{a^*\colon \langle a^*, x - \bar{x}\rangle \leqslant 0, \forall x \in A_1\}$. Therefore $\langle x^*, \bar{x}\rangle = \inf_{x \in A_1} \langle x^*, x\rangle$. To finish the proof, it remains to show that $x^* \neq 0$. Suppose $x^* = 0$. Then $x^*_{1n} \to_{w^*} 0$. By (71) there exists $\bar{x}^*_n \in N(x_{1n}/t_n, A_1)$ such that

$$x^*_{1n} \in \bar{x}^*_n + \frac{1}{n} B_{X^*} \quad \text{and} \quad \bar{x}^*_n \to_{w^*} 0. \tag{72}$$

This together with $x_{1n}/t_n \to \bar{x}$ (since $t_n \to 1$ and $x_{1n} \to \bar{x}$) imply that $\bar{x}_n^* \to 0$ as A_1 is ∂_a-normally compact. Hence $x_{1n}^* \to 0$ by the first relation in (72). However, this contradicts to (70) and completes the proof. □

Using the preceding theorem, we now establish the following three interesting corollaries (the first two of which require no proof).

Corollary 4.1. *Let $X \in \mathcal{X}$. Let A_1 be a closed convex subset of X with* $\mathrm{int}(A_1) \neq \emptyset$ *and A_2 be a closed subset of X. Suppose that* $\mathrm{int}(A_1) \cap A_2 = \emptyset$. *Let $\bar{x} \in A_1 \cap A_2$. Then there exists $x^* \in X^*$ with* $\|x^*\| = 1$ *such that* (69) *holds.*

Remark 4.8. This corollary was proved in [24, Lemma 2.1] in the special case when $\mathcal{X}$ is the class of all Banach spaces and ∂_a is the Clarke–Rockafellar subdifferential ∂_c.

Corollary 4.2. *Let X be a finite-dimensional space in $\mathcal{X}$. Let A_1 be a closed convex subset of X and let A_2 be a closed subset of X. Suppose that* $\mathrm{ri}(A_1) \cap A_2 = \emptyset$. *Let $\bar{x} \in A_1 \cap A_2$. Then there exists $x^* \in X^*$ with* $\|x^*\| = 1$ *such that* (69) *holds.*

Corollary 4.3. *Let $X \in \mathcal{X}$. Let A_1 be a polyhedron in X and let A_2 be a closed subset of X. Let $\bar{x} \in A_1 \cap A_2$. Suppose that* $\mathrm{ri}(A_1) \cap A_2 = \emptyset$. *Then there exists $x^* \in X^*$ with* $\|x^*\| = 1$ *such that* (69) *holds.*

Proof. By the assumption and the implication (vi) ⇒ (iii) in Theorem 4.6, we see that the conditions in Theorem 4.7 are fulfilled and hence the conclusion follows by applying Theorem 4.7. □

Finally we present a Dieudonné's separation theorem type result in finite-dimensional Banach spaces but applicable to possibly nonconvex sets. Following [21], we define the horizon cone A^∞ of a closed set A by $A^\infty = \{v : \exists \lambda_k \to 0, a_k \in A$ such that $\lambda_k a_k \to v\}$. When A is convex, this definition coincides with the usual recession cone in convex analysis (cf. [21, Theorem 3.6]).

Theorem 4.9. *Let X be a finite-dimensional space in $\mathcal{X}$. Let A_1, A_2 be two closed subsets of X such that $A_1 \cap A_2 = \emptyset$ and $A_1^\infty \cap A_2^\infty = \{0\}$. Then there exist $x_1 \in A_1$, $x_2 \in A_2$ and $x^* \in X^*$ with $\|x^*\| = 1$ such that $x^* \in N_a(x_1, A_1) \cap (-N_a(x_2, A_2))$ and*

$$\langle x^*, x_2 - x_1 \rangle = \|x_2 - x_1\| > 0. \tag{73}$$

Proof. Define $A = A_1 - A_2$. Then $0 \notin A$. We prove that A is a closed subset of X. Let $\bar{x} \in \overline{A}$, and let $\{x_n\}$ be a sequence in A such that $x_n \to x$ where each $x_n = x_n^1 - x_n^2$ for some $x_n^1 \in A_1$ and $x_n^2 \in A_2$. We claim that $\{\|x_n^1\| + \|x_n^2\|\}$ is bounded. Granting this and since X is finite-dimensional, it is easy to verify that $x \in A$. Let us suppose on the contrary that the sequence $\{\|x_n^1\| + \|x_n^2\|\}$ is unbound, i.e., there exists a subsequence n_k such that $\|x_{n_k}^1\| + \|x_{n_k}^2\| \to \infty$ as $k \to \infty$. Noting that $\{\frac{x_{n_k}^1}{\|x_{n_k}^1\| + \|x_{n_k}^2\|}\}$ and $\{\frac{x_{n_k}^2}{\|x_{n_k}^1\| + \|x_{n_k}^2\|}\}$ are bounded sequences, by passing to subsequence if necessary, it follows that

$$a_k^1 := \frac{x_{n_k}^1}{\|x_{n_k}^1\| + \|x_{n_k}^2\|} \to a_1 \in A_1^\infty \quad \text{and} \quad a_k^2 := \frac{x_{n_k}^2}{\|x_{n_k}^1\| + \|x_{n_k}^2\|} \to a_2 \in A_2^\infty.$$

Since $\|a_k^1\| + \|a_k^2\| = 1$, this implies that

$$\|a_1\| + \|a_2\| = 1. \tag{74}$$

Moreover, since

$$0 = \lim_{k\to\infty} \frac{x}{\|x_{n_k}^1\| + \|x_{n_k}^2\|} = \lim_{k\to\infty} \frac{x_{n_k}^1}{\|x_{n_k}^1\| + \|x_{n_k}^2\|} - \lim_{k\to\infty} \frac{x_{n_k}^2}{\|x_{n_k}^1\| + \|x_{n_k}^2\|} = a_1 - a_2,$$

we see that $a_1 = a_2 \in A_1^\infty \cap A_2^\infty$. This is not possible by (74) and the assumption that $A_1^\infty \cap A_2^\infty = \{0\}$. This completes the proof that A is closed. Since A is finite-dimensional and $0 \notin A$, it follows that there exist $x_1 \in A_1$, $x_2 \in A_2$ such that $\|x_1 - x_2\| = d(0, A) > 0$. Let $\{\epsilon_k\}$ be a sequence of positive real numbers such that $\epsilon_k \to 0$. Applying Theorem 3.4(ii) to the tuple $\{2, 0, x_1, -x_2, A_1, -A_2\}$ in place of $\{m, w, \bar{x}_1, \bar{x}_2, A_1, A_2\}$, for each $k \in \mathbb{N}$ there exists $x_1^k \in A_1 \cap B_X(x_1, \epsilon_k)$, $x_2^k \in A_2 \cap B_X(x_2, \epsilon_k)$, $x_k^* \in X^*$ with $\|x_k^*\| = 1$ such that

$$x_k^* \in \big((1+\epsilon_k)\partial_a d(\cdot, A_1)\big(x_1^k\big) + \epsilon_k \overline{B}_{X^*}\big) \cap \big((1+\epsilon_k)\partial_a d(\cdot, -A_2)\big(-x_2^k\big) + \epsilon_k \overline{B}_{X^*}\big) \tag{75}$$

and

$$\big\langle x_k^*, x_2^k - x_1^k\big\rangle = \big\|x_2^k - x_1^k\big\|. \tag{76}$$

Since $\|x_k^*\| = 1$, by passing to subsequence if necessary, we may assume that $x_k^* \to x^*$ for some $x^* \in X^*$ such that $\|x^*\| = 1$. Letting $k \to \infty$ in (75) and noting that $\partial_a d(\cdot, -A_2)(-x_2^k) \subseteq -\partial_a d(\cdot, A_2)(x_2^k)$ (by applying (P4$^+$) to $G = d(\cdot, A_2)$, $F = -I$ and $x = -x_2^k$ where I denotes the identity map from X to X), it follows from (P9) and (P5) that

$$x^* \in \partial_a d(\cdot, A_1)(x_1) \cap -\partial_a d(\cdot, A_2)(x_2) \subseteq N_a(x_1, A_1) \cap \big(-N_a(x_2, A_2)\big). \tag{77}$$

Finally, since $x_k^* \to x^*$, $x_i^k \to x_i$ $(i = 1, 2)$, (73) follows by letting $k \to \infty$ in (76). This completes the proof. □

Corollary 4.4. *(See [8].) Let X be a finite-dimensional space in $\mathcal{X}$. Let A_1, A_2 be two closed convex subsets of X such that $A_1 \cap A_2 = \emptyset$ and $A_1^\infty \cap A_2^\infty = \{0\}$. Then there exists $x^* \in X^*$ with $\|x^*\| = 1$ such that*

$$\sup_{x\in A_1} \big\langle x^*, x\big\rangle < \inf_{x\in A_2} \big\langle x^*, x\big\rangle.$$

Proof. From the preceding theorem, there exists $x_1 \in A_1$, $x_2 \in A_2$ and $x^* \in X^*$ with $\|x^*\| = 1$ such that $x^* \in N_a(x_1, A_1) \cap (-N_a(x_2, A_2))$ and $\langle x^*, x_2 - x_1\rangle = \|x_2 - x_1\| > 0$. Noting that $N_a(x, A_i) = N(x, A_i)$ $(i = 1, 2)$ where $N(x, A_i)$ is the usual (convex) normal cone, it follows that

$$\sup_{x\in A_1} \big\langle x^*, x\big\rangle = \big\langle x^*, x_1\big\rangle < \big\langle x^*, x_2\big\rangle = \inf_{x\in A_2} \big\langle x^*, x\big\rangle.$$

This completes the proof. □

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