On the asymptotic well behaved functions and global error bound for convex polynomials*

Guoyin Li [‡]

December 11, 2009

Abstract

In this paper, we present some new and tractable sufficient conditions for convex asymptotic well behaved (AWB) functions. Moreover, we establish several Lipschitz/Hölder type global error bound results for single convex polynomial and for function which can be expressed as maximum of finitely many nonnegative convex polynomials. An advantage of our approach is that the corresponding Hölder exponent in our Hölder type global error bound results can be determined explicitly.

1 Introduction

Consider the following minimization problem

$$(P)\ \inf f = \inf\{f(x) : x \in \mathbb{R}^m\}$$

where f is a proper, lower semicontinuous and convex function. We are interested in finding a minimizing sequence, that is, a sequence $\{x_k\}$ satisfying $\lim_{k\to\infty} f(x_k) \to \inf f$. Usually the sequence $\{x_k\}$ is generated by some first-order numerical algorithm in solving (P) and hence is typically a stationary sequence, i.e. $d(0, \partial f(x_k)) \to 0$ where ∂f is the convex subdifferential of f. Therefore, from both the theoretical and computational view point, it is desire to have: every stationary sequence is indeed a minimizing sequence. However, as point out by [1], the above implication is false in general. In order to identify this property, Auslender etal. (cf. [1]) introduced the following definition of asymptotic well behaved function: A convex function $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is said to be asymptotic well behaved (AWB) (in some other reference, AWB are also referred to be "well-posed" for example see [10, 18]) if every stationary sequence $\{x_k\}$ of f is a minimizing sequence, that is,

$$d(0, \partial f(x_k)) \to 0 \quad \Rightarrow \quad f(x_k) \to \inf f.$$

The theory of asymptotic well behaved function has attracted many researchers (see [1, 10, 33] and the reference therein). For instance, Auslender etal. show that weakly coercive convex functions are AWB. Later on, Lewis and Pang [18] (see also [1]) relates the theory of AWB functions with the well-studied concept "error bound". More explicitly, they show that (1) a convex AWB function satisfying the Slater condition has a Lipschitz type global error bound; (2) a nonnegative convex function which has a Lipschitz type global error bound is AWB. It is known that the study of error bounds has grown significantly and has found many important applications (see [1, 18, 27, 33] for excellent survey). In particular, they have been used in sensitivity analysis of linear programming/linear complementary problem. Moreover, they have also been used as termination criteria for decent algorithms.

[‡]Department of Applied Mathematics, University of New South Wales, Sydney 2052, Australia. Email: g.li@unsw.edu.au

The purpose of this paper is of two fold: firstly, we provide some new and tractable sufficient conditions for AWB functions. Secondly, we employee the techniques and results from the theory of AWB functions to study the global error bound for convex polynomials.

In this paper, much of our study on error bound is in the sprit of [5, 8] and is motivated from the recent work on extension of Frank-Wolfe Theorem [4, 26]. More explicitly, we achieve our error bound results by examining recession properties of convex polynomials (some other approaches and related references for studying error bound can be found in [6, 7, 13, 18, 19, 20, 27, 31, 32]). Interestingly, several Lipschitz/Hölder type global error bound results are presented for single convex polynomial and for functions which can be expressed as maximum of finitely many nonnegative convex polynomials. As we will see later, an advantage of our approach is that the corresponding Hölder exponent (in the Hölder type global error bound result) can be determined explicitly.

The organization of this paper is as follows. In section 2, we present some definitions and some basic results on convex functions and polynomials. In Section 3, we provide some new sufficient conditions for a function to be asymptotic well behaved (AWB). Finally, in section 4, we establish some Lipschitz/Hölder type global error bound results for convex polynomials.

2 Preliminaries

Throughout this paper, $\mathbb{R}^m$ denotes Euclidean space with dimension m. The corresponding inner product (resp. norm) in $\mathbb{R}^m$ is defined by $\langle x, y\rangle = x^Ty$ for any $x, y \in \mathbb{R}^m$ (resp. $\|x\| := x^Tx$, [†] for any $x \in \mathbb{R}^m$). We use $\mathbb{B}(x,\epsilon)$ (resp. $\overline{\mathbb{B}}(x,\epsilon)$) to denote the open (resp. closed) ball with center x and radius ϵ. For a set A in $\mathbb{R}^m$, the interior (resp. relative interior, closure, convex hull, affine hull) of A is denoted by $\mathrm{int}A$ (resp. $\mathrm{ri}A$, $\overline{A}$, $\mathrm{co}A$, $\mathrm{aff}A$). The kernel of A is denoted by $A^\perp$ and is defined as $A^\perp := \{d : a^Td = 0, \forall a \in A\}$. For a function f on $\mathbb{R}^m$, the effective domain and the epigraph are respectively defined by $\mathrm{dom}f := \{x \in \mathbb{R}^m : f(x) < +\infty\}$ and $\mathrm{epi}f := \{(x, r) \in \mathbb{R}^m \times \mathbb{R} : f(x) \le r\}$. We say f is proper if $f(x) > -\infty$ for all $x \in \mathbb{R}^m$ and $\mathrm{dom}f \neq \emptyset$. For each $\epsilon \in \mathbb{R}$, we use $[f \le \epsilon]$ (resp. $[f = \epsilon]$) to denote the level set $\{x \in \mathbb{R}^m : f(x) \le \epsilon\}$ (resp. $\{x : f(x) = \epsilon\}$). Moreover, if $\liminf_{x'\to x} f(x') \ge f(x)$ for all $x \in \mathbb{R}^m$, we say f is a lower semicontinuous function. A function f is called a proper lower semicontinuous separable function on $\mathbb{R}^m$ if

$$f(x) = \sum_{l=1}^{m} g_l(x_l) \quad \forall\, x = (x_1, \ldots, x_m) \in \mathbb{R}^m \tag{2.1}$$

for some proper lower semicontinuous functions g_l on $\mathbb{R}$ $(1 \le l \le m)$. It is clear that any affine function is a separable function. A function $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is said to be convex if for all $\mu \in [0, 1]$

$$f((1-\mu)x + \mu y) \le (1-\mu)f(x) + \mu f(y) \text{ for all } x, y \in \mathbb{R}^m.$$

Let f be a proper lower semicontinuous convex functions on $\mathbb{R}^m$. The (convex) subdifferential of f at $x \in \mathbb{R}^m$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in \mathbb{R}^m : \langle x^*, y - x\rangle \le f(y) - f(x)\, \forall\, y \in \mathbb{R}^m\}, & \text{if } x \in \mathrm{dom}f, \\ \emptyset, & \text{otherwise.} \end{cases} \tag{2.2}$$

As usual, for any proper lower semicontinuous convex function f on $\mathbb{R}^m$, its conjugate function $f^* : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ is defined by $f^*(x^*) = \sup\limits_{x\in\mathbb{R}^m} \{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in \mathbb{R}^m$. The definition of f^* entails that $\langle x^*, x\rangle \le f^*(x^*) + f(x)$ (Young's inequality) for any $x, x^* \in \mathbb{R}^m$. Moreover, we have the following Young's equality

$$x^* \in \partial f(x) \;\Leftrightarrow\; \langle x^*, x\rangle = f^*(x^*) + f(x).$$

[†]Sometimes, we also use $\|x\|_{\mathbb{R}^m}$ (in stead of $\|x\|$), if the underlying space need to be specified

Let f be a proper function on $\mathbb{R}^m$. Its associated recession function f^∞ is defined by

$$f^\infty(v) = \liminf_{t\to\infty,\, v'\to v} \frac{f(tv')}{t} \text{ for all } v \in \mathbb{R}^m. \tag{2.3}$$

If f is further assumed to be lower semicontinuous and convex, one has (cf. [1, Proposition 2.5.2])

$$f^\infty(v) = \lim_{t\to\infty} \frac{f(x+tv) - f(x)}{t} = \sup_{t>0} \frac{f(x+tv) - f(x)}{t} \text{ for all } x \in \mathrm{dom} f. \tag{2.4}$$

Remark 2.1. *Let f be a proper lower semicontinuous convex function with* $\inf f > -\infty$. *From the first relation of (2.4), it is easy to see that*

$$f^\infty(v) \geq 0 \text{ for all } v \in \mathbb{R}^m. \tag{2.5}$$

Remark 2.2. *From Proposition 3 (and the remark of this Proposition) in [9], if $\{v : f^\infty(v) < 0\} \neq \emptyset$, then $0 \notin \overline{\mathrm{dom} f^*}$ and f is AWB (indeed, in this case, no stationary sequence exists).*

Next, we recall an important characterization of AWB function (see [1, Theorem 4.2.2]) which is useful for our later analysis.

Lemma 2.1. *Let $f : \mathbb{R}^m \to \mathbb{R}$ be a proper lower semicontinuous convex function such that $f(x) > \inf f \Rightarrow x \in \mathrm{ri}\,\mathrm{dom} f$. Then f is AWB if and only if all stationary sequence $\{x_n\}$ with $f(x_n)$ bounded above satisfy $f(x_n) \to \inf f$.*

As usual, we say $f : \mathbb{R}^m \to \mathbb{R}$ is a (real) polynomial if there exists $r \in \mathbb{N}$ such that

$$f(x) = \sum_{0\leq|\alpha|\leq r} \lambda_\alpha x^\alpha$$

where $\lambda_\alpha \in \mathbb{R}$, $x = (x_1, \ldots, x_m)$, $x^\alpha := x_1^{\alpha_1} \cdots x_m^{\alpha_m}$, $\alpha_i \in \mathbb{N} \cup \{0\}$ and $|\alpha| := \sum_{j=1}^m \alpha_j$. The corresponding constant r is called the degree of f and is denoted by $\deg(f)$. Moreover, we say $f : \mathbb{R}^m \to \mathbb{R}$ is an (real) analytic function if it can be represented locally on X by a convergent infinite power series, i.e. for all $\overline{x} = (\overline{x}_1, \ldots, \overline{x}_m) \in \mathbb{R}^m$ there exists a neighborhood $N(\overline{x})$ of $\overline{x}$ such that for all $x = (x_1, \ldots, x_m) \in N(\overline{x})$

$$f(x) = \sum_{|\alpha|=0}^{\infty} \lambda_\alpha (x - \overline{x})^\alpha,$$

where $\lambda_\alpha \in \mathbb{R}$, $(x - x_0)^\alpha := (x_1 - \overline{x}_1)^{\alpha_1} \cdots (x_m - \overline{x}_m)^{\alpha_m}$ $\alpha_i \in \mathbb{N} \cup \{0\}$ and $|\alpha| := \sum_{j=1}^m \alpha_j$.

Following [11, 24], a set $C \subseteq \mathbb{R}^m$ is said to be

(i) semianalytic, if for any $x \in \mathbb{R}^m$, there exists a neighborhood U of x such that

$$C \cap U = \bigcup_{i=1}^{l} \bigcap_{j=1}^{s} \{x \in U : f_{ij}(x) = 0,\, g_{ij}(x) < 0\}$$

for some integers l, s and some real analytic functions f_{ij}, g_{ij} on $\mathbb{R}^m$ $(1 \leq i \leq l,\, 1 \leq j \leq s)$;

(ii) subanalytic if for any $x \in C$, there exist a neighborhood U of x and a bounded semianalytic set $Z \subseteq \mathbb{R}^{m+p}$ such that $C \cap U = \{x \in \mathbb{R}^m : (x, y) \in Z \text{ for some } y \in \mathbb{R}^p\}$.

Moreover, a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be subanalytic if its graph $\mathrm{gph} f := \{(x, f(x)) : x \in \mathbb{R}^m\}$ is subanalytic.

We summarize below some basic properties of subanalytic sets and subanalytic functions

(P1) Any analytic function is subanalytic (hence any polynomial is subanalytic).

(P2) (cf. [11, (p3) P.597]) $d(\cdot, C)$ is a subanalytic function if C is a subanalytic set where $d(x, C) :=$

$\inf_{c\in C}\|x-c\|$ for any $x\in\mathbb{R}^m$.
(P3) (cf. [11, (p5) and (p8) P.598] If f, g are subanalytic functions on $\mathbb{R}^m$ and $\lambda\in\mathbb{R}$ then $f+g$ (resp. λf, $\max\{f,g\}$, fg) is subanalytic.
(P4) (cf. [11, (p4) P.597]) If f is subanalytic and $\lambda\in\mathbb{R}$, then the sets $[f=\lambda]$, $[f<\lambda]$ and $[f\leq\lambda]$ are subanalytic.
(P5) (Łojasiewicz's inequality, cf. [24, Theorem 2.1.1]) If ϕ,ψ are nonnegative continuous subanalytic functions on compact subanalytic set $K\subseteq\mathbb{R}^m$ such that $\emptyset\neq\phi^{-1}(0)\subseteq\psi^{-1}(0)$ then there exist constants $\mu>0$ and $\delta\in(0,1]$ such that

$$\psi(x)\leq\mu\phi(x)^\delta \text{ for all } x\in K.$$

Remark 2.3. *As pointed out by [24], the corresponding exponent δ in the Lojasiewicz's inequality is hard to determine and is typically unknown.*

Remark 2.4. *Let f be a continuous subanalytic functions with $\mathrm{argmin}f\neq\emptyset$ and let $\lambda\in\mathbb{R}$. Applying (P5) with $\phi=f-\inf f$ and $\psi=d(\cdot,\mathrm{argmin}f)$, it follows from (P2) and (P4) that for any compact set K there exist $\mu>0$ and $\delta\in(0,1]$ such that*

$$d(x,\mathrm{argmin}\,f)\leq\mu(f(x)-\inf f)^\delta \textit{ for all } x\in K.$$

Finally, we present two basic properties of convex polynomials. The first one of them is from [26, Corollary 4.1 and Lemma 2.4] and [1, Propsition 3.2.1]. The second one is standard and may be known somewhere. However, we cannot find it in an appropriate reference. Hence we include the proof in the appendix for completeness.

Lemma 2.2. *Let f be a convex polynomial on $\mathbb{R}^m$. Then the following statements hold.*
(1) *Suppose that $\inf f>-\infty$. Then f attains its minimum on $\mathbb{R}^m$.*
(2) *Suppose that f is a constant along $\{tv:t\geq 0\}$ for some $v\in\mathbb{R}^m$. Then f is a constant along any half-line $\{x+tv:t\geq 0\}$ $(x\in\mathbb{R}^m)$.*
(3) *Let $v\in\mathbb{R}^m$ be such that $f^\infty(v)=0$. Then $f(x+tv)=f(x)$ for all $t\in\mathbb{R}$ and for all $x\in\mathbb{R}^m$.*

Lemma 2.3. *Let f_i, $i=1,\dots,r$, be convex polynomials on $\mathbb{R}^m$ with $\inf f_i>-\infty$ and let $f=\max_{1\leq i\leq r}f_i$. Let $v\in\mathbb{R}^m$ be such that $f^\infty(v)=0$. Then $f(x+tv)=f(x)$ for all $t\in\mathbb{R}$ and for all $x\in\mathbb{R}^m$.*

3 Asymptotic well behaved function

In [1], Auslender shows that the class of weakly coercive functions, i.e. a proper, lower semicontinuous and convex function f satisfying $0\in\mathrm{ridom}f^*$, is a subclass of AWB functions. Unfortunately, this condition, on the one hand is not easy to verify in general (computation of f^* is equivalent to solving a global optimization problem), and on the other hand may not be satisfied even when f is a two dimensional continuous convex function (as shown in the following examples).

Example 3.1. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by*

$$f(x)=e^{x_1}+x_2^2\ \ \big(x=(x_1,x_2)\big). \tag{3.1}$$

It is clear that f is a differentiable convex function. Note that $\partial f(x)=\nabla f(x)=(e^{x_1},2x_2)$ and $\inf f=0$. It is easy to see that f is AWB. However, it can be verify that $\mathrm{dom}f^*=[0,+\infty)\times\mathbb{R}$. Thus, f is not weakly coercive.

Example 3.2. *Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by*

$$f(x)=(x_1+x_2)^4+x_2. \tag{3.2}$$

It can be verified that f is a convex AWB function. (*Indeed,note that* $\nabla f(x) = (4(x_1+x_2)^3, 4(x_1+x_2)^3+1)$ *It follows that no stationary sequence exists. Thus f is AWB.) On the other hand, since* $\inf f = -\infty$*, we see that* $0 \notin \mathrm{dom} f^*$. *Thus f is not weakly coercive.*

The preceding observations/examples prompt us to examine new and tractable sufficient conditions for convex asymptotic well behaved (AWB) functions. Next, we show that f is AWB if it satisfies either one of the following three conditions:

(1) f is a separable convex function;

(2) f is a convex polynomial;

(3) f can be expressed as maximum of finitely many bounded below convex polynomials.

(Indeed, it can be easily verified that the function defined in Example 3.1 is a separable convex function and the function defined in Example 3.2 is a convex polynomial.)

3.1 Separable case

In this subsection, we show that a separable convex function is AWB. To establish the desire result, we need the following two lemmas for preparation (where the first one is from [1, Theorem 4.2.4]).

Lemma 3.1. *Let* $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous convex function with* $\inf f > -\infty$. *Suppose that there exist* $M > 0$ *and* $\delta > 0$ *such that* $f^*(x^*) \le M$ *for all* $x^* \in \mathrm{dom} f^* \cap [-\delta, \delta]$ *and f is locally conical in the sense that* $\mathbb{R}_+ \mathrm{dom} f^* \cap [-\delta, \delta] \subseteq \mathrm{dom} f^* \cap [-\delta, \delta]$. *Then f is AWB.*

Lemma 3.2. *Let* $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ *be a proper lower semicontinuous convex function. Then f is AWB.*

Proof. First of all, we note that, without loss of generality, we may assume that $\mathrm{dom} f^*$ contains at least two points and hence $\mathrm{int}\,\mathrm{dom} f^* \neq \emptyset$ (otherwise, $\mathrm{Im}\partial f := \bigcup\{x^* : x^* \in \partial f(x) \text{ for some } x \in \mathbb{R}^m\} \subseteq \mathrm{dom} f^*$ is a single point or empty set. Thus f is clearly AWB.). We divide our proof into the following 4 cases.
Case 1. $0 \notin \overline{\mathrm{dom} f^*}$. In this case, there exists $r > 0$ such that

$$[-r, r] \cap \mathrm{dom} f^* = \emptyset. \tag{3.3}$$

Let $x_n \in \mathbb{R}$ and $x_n^* \in \partial f(x_n)$. Since $\mathrm{Im}\partial f \subseteq \mathrm{dom} f^*$, $x_n^* \in \mathrm{dom} f^*$. It follows from (3.3) that $\inf_{n\in\mathbb{N}} \|x_n^*\| \ge r$. This implies that no stationary sequence exists and hence the conclusion of this lemma is true in this case.
Case 2. $0 \in \overline{\mathrm{dom} f^*} \backslash \mathrm{dom} f^*$. In this case, we have $\inf f = -\infty$ (since $0 \notin \mathrm{dom} f^*$). Since $\mathrm{dom} f^*$ is a convex set in $\mathbb{R}$, without loss generality, we may assume that $\mathrm{dom} f^* = (0, \alpha)$ for some $\alpha \in (0, +\infty]$. Let $\{x_n\}$ be a stationary sequence. Then there exists $x_n^* \in \partial f(x_n)$ such that $x_n^* > 0$ and $x_n^* \to 0$. From the Young's equality we have

$$\langle x_n^*, x_n \rangle = f(x_n) + f^*(x_n^*). \tag{3.4}$$

We now proceed by contradiction and suppose $\limsup_{n\to\infty} f(x_n) = r$ where $r \in \mathbb{R} \cup \{+\infty\}$. Note that f^* is a lower semicontinuous convex function on $\mathbb{R}$ with $\mathrm{dom} f^* = (0, \alpha)$ and hence continuous on $[0, \alpha)$ (cf. [33, Proposition 2.1.6]). * This implies that $f^*(x_n^*) \to f^*(0) = -\inf f = +\infty$. It follows from (3.4) that

$$\limsup_{n\to\infty} \langle x_n^*, x_n \rangle = +\infty.$$

Without loss of generality, we assume that $x_n \to +\infty$ and $x_n > 0$ for all n. Note that $f^*(x_n^*) \to f^*(0) = +\infty$. It follows that

$$\limsup_{n\to\infty} \frac{f^*(x_n^*)}{x_n} \ge 0. \tag{3.5}$$

*Here, f^* is continuous at 0 means for any $x_n^* \in (0, \alpha)$ with $x_n^* \to 0$ one has $f^*(x_n^*) \to f^*(0)$.

On the other hand, since $x_n^* \in \partial f(x_n)$, we have $x_n \in \partial f^*(x_n^*)$. Thus,

$$\langle x_n, x^* - x_n^* \rangle \le f^*(x^*) - f^*(x_n^*) \text{ for all } x^* \in (0, \alpha). \tag{3.6}$$

Define $\beta = \alpha/2$ if $\alpha < +\infty$ and $\beta = 1$ if $\alpha = +\infty$. Substituting $x^* = \beta$ in (3.6), we have

$$x_n(\beta - x_n^*) \le f^*(\beta) - f^*(x_n^*). \tag{3.7}$$

Note that $x_n > 0$ for all n, $x_n \to +\infty$ and $x_n^* \to 0$. Dividing x_n on both sides in (3.7) and passing to the lower limit, it follows from (3.5) that

$$0 < \beta \le \liminf_{n\to\infty} \frac{-f^*(x_n^*)}{x_n} = -\limsup_{n\to\infty} \frac{f^*(x_n^*)}{x_n} \le 0.$$

This is impossible and finish the proof of this case.

Case 3. $0 \in \mathrm{dom} f^* \backslash \mathrm{int}\,\mathrm{dom} f^*$. In this case, without loss generality, we may assume that $\mathrm{dom} f^* = [0, \alpha)$ for some $\alpha \in (0, +\infty]$. Since f^* is continuous on $[0, \alpha)$, there exist $M > 0$ and $\delta \in (0, \alpha)$ such that $f^*(x^*) \le M$ for all $x^* \in \mathrm{dom} f^* \cap [-\delta, \delta]$. Moreover, it is clear that $\mathbb{R}_+ \mathrm{dom} f^* \cap [-\delta, \delta] = \mathrm{dom} f^* \cap [-\delta, \delta]$ and hence f is locally conical. Note that $\inf f > -\infty$ (since $0 \in \mathrm{dom} f^*$). Thus, Lemma 3.1 implies that the conclusion is true in this case.

Case 4. $0 \in \mathrm{int}\,\mathrm{dom} f^*$. In this case, f is weakly coercive. Thus the conclusion follows from [1, Corollary 4.2.1]. □

The following example shows that f might not be AWB when f is a two dimensional convex function.

Example 3.3. *(cf. [1, 28]) Let* $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ *be defined by*

$$f(x_1, x_2) = \begin{cases} \frac{x_1^2}{2x_2}, & \text{if } x_2 > 0, \\ 0 & \text{if } (x_1, x_2) = (0, 0), \\ +\infty & \text{else.} \end{cases} \tag{3.8}$$

It can be verified that f is a proper, lower semicontinuous and convex function with $\inf f = 0$*. Consider $x_n = (n, n^2)$. Then one has $f(x_n) = 1/2$ and $\partial f(x_n) = \nabla f(x_n) = (1/n, -1/2n^2) \to 0$. Thus f is not an AWB function.*

Theorem 3.1. *Let f be a proper lower semicontinuous separable convex function on* $\mathbb{R}^m$*. Then f is AWB.*

Proof. Since f is a proper lower semicontinuous separable convex function, we may assume that

$$f(x) = \sum_{l=1}^m g_l(x_l) \text{ for all } x = (x_1, \ldots, x_m) \in \mathbb{R}^m.$$

for some proper lower semicontinuous convex function g_l on $\mathbb{R}$ $(1 \le l \le m)$. Let $\{x_n\}$ be a stationary sequence of f and denote $x_n = (x_{n1}, \ldots, x_{nm})$. Then there exists $x_n^* \in \partial f(x_n)$ such that $x_n^* = (x_{n1}^*, \ldots, x_{nm}^*) \to 0$. Note that

$$\partial f(x_n) = \partial g_1(x_{n1}) \times \partial g_2(x_{n2}) \cdots \times \partial g_m(x_{nm}).$$

It follows that $x_{nl}^* \in \partial g_l(x_{nl})$ and $x_{nl}^* \to 0$ for all $1 \le l \le m$. This implies that for each fixed $1 \le l \le m$, x_{nl}^* is a stationary sequence of g_l. Since g_l is a proper lower semicontinuous function on $\mathbb{R}$, Lemma 3.2 implies that

$$g_l(x_{nl}) \to \inf g_l \text{ for all } 1 \le l \le m.$$

It follows that

$$f(x_n) = \sum_{l=1}^m g_l(x_{nl}) \to \sum_{l=1}^m \inf_{x_l \in \mathbb{R}} g_l(x_l) = \inf_{x=(x_1,\ldots,x_m)\in\mathbb{R}^m} \sum_{l=1}^m g_l(x_l) = \inf f.$$

This completes the proof. □

3.2 Single convex polynomial case

In this subsection, we show that any convex polynomial is AWB.

Theorem 3.2. *Let f be a convex polynomial on $\mathbb{R}^m$. Then, f is AWB.*

Proof. Let f be a convex polynomial on $\mathbb{R}^m$. To see the conclusion, we proceed by contradiction and assume that f is not AWB. Note that f is continuous, and so, the following implication always holds

$$f(x) > \inf f \Rightarrow x \in \operatorname{ri}\operatorname{dom} f.$$

By Lemma 2.1, there exists a stationary sequence $\{x_n\}$ satisfying $\{f(x_n)\}$ is bounded above and

$$\liminf_{n\to\infty} f(x_n) > \inf f. \tag{3.9}$$

Since $\{f(x_n)\}$ is a bounded above stationary sequence, $\nabla f(x_n) \to 0$ and there exists $r \in \mathbb{R}$ such that $f(x_n) \le r$ for all n. For each fixed n, choose

$$y_n \in \operatorname{argmin}\{\|x\| : f(x) = f(x_n) \text{ and } \nabla f(x) = \nabla f(x_n)\}.$$

If $\{y_n\}$ is bounded, then by passing to the subsequence if necessary, we may assume that $y_n \to y$. Since $f(x_n) = f(y_n)$ and $\nabla f(y_n) = \nabla f(x_n) \to 0$, it follows from the continuity of f and ∇f that $\liminf_{n\to\infty} f(x_n) = \liminf_{n\to\infty} f(y_n) = f(y)$ and $\nabla f(y) = 0$. Hence f attains the minimum at y (since f is convex), and so, $\liminf_{n\to\infty} f(x_n) = \inf f$. This contradicts (3.9). Therefore we may assume that $\|y_n\| \to \infty$. By passing to subsequence if necessary, we have

$$\frac{y_n}{\|y_n\|} \to v. \tag{3.10}$$

Note that $f(y_n) = f(x_n) \le r$. It follows from (2.3) that

$$f^\infty(v) = \liminf_{t\to\infty,\, v'\to v} \frac{f(tv')}{t} \le \liminf_{n\to\infty} \frac{f(\|y_n\| \frac{y_n}{\|y_n\|})}{\|y_n\|} = \liminf_{n\to\infty} \frac{f(y_n)}{\|y_n\|} \le 0.$$

If $f^\infty(v) < 0$, then f is AWB by Remark 2.2. Thus, we may assume without loss of generality that $f^\infty(v) = 0$. By Lemma 2.2(3), $f(x+tv) = f(x)$ for any $x \in \mathbb{R}^m$ and for any $t \in \mathbb{R}$. Fix an arbitrary $n \in \mathbb{N}$. Define $z_n = y_n - v$. Then, one has

$$f(z_n) = f(y_n) = f(x_n) \text{ and } f(z_n + tu) = f\big((y_n + tu) - v\big) = f(y_n + tu) \text{ for all } u \in \mathbb{R}^m.$$

This implies that for any $u \in \mathbb{R}^m$

$$\langle \nabla f(z_n), u\rangle = \lim_{t\to 0} \frac{f(z_n + tu) - f(z_n)}{t} = \lim_{t\to 0} \frac{f(y_n + tu) - f(y_n)}{t} = \langle \nabla f(y_n), u\rangle = \langle \nabla f(x_n), u\rangle.$$

It follows that $\nabla f(x_n) = \nabla f(z_n)$. Thus we have

$$z_n \in \{x : f(x) = f(x_n) \text{ and } \nabla f(x) = \nabla f(x_n)\}.$$

From our choice of y_n, we have $\|y_n\| \le \|z_n\|$. This implies that

$$\begin{aligned}
\|y_n\| \le \|z_n\| = \|y_n - v\| &= \|(1 - \frac{1}{\|y_n\|})y_n + (\frac{y_n}{\|y_n\|} - v)\| \\
&\le (1 - \frac{1}{\|y_n\|})\|y_n\| + \|\frac{y_n}{\|y_n\|} - v\| \\
&= \|y_n\| + \big(\|\frac{y_n}{\|y_n\|} - v\| - 1\big).
\end{aligned}$$

Henceforth, one has

$$\|\frac{y_n}{\|y_n\|} - v\| \ge 1.$$

Letting $n \to \infty$ and noting (3.10), one has $0 \ge 1$ which is impossible. This finishes the proof. □

3.3 Maximum of bounded below convex polynomials case

In this subsection, we show that f is AWB if f can be expressed as maximum of finitely many bounded below convex polynomials, i.e. $f = \max_{1 \le i \le r} f_i$ where each f_i is a convex polynomial with $\inf f_i > -\infty$, $i = 1, \ldots, r$.

As preparation, we first establish the following two results (which might be of some independent interest) on estimating the nonsmooth slope of f in terms of the corresponding function value, where the nonsmooth slope of f is denoted by m_f and is defined by

$$m_f(x) := \inf\{\|x^*\| : x^* \in \partial f(x)\} \text{ for all } x \in \mathbb{R}^m.$$

It should be note that the first result valid for function $f = \max_{1 \le i \le r} f_i$ for some convex polynomials f_i such that f has a compact level set. In contrast, the second result relax the assumption "f has a compact level set" but further assume that each f_i is bounded below, i.e., $\inf f_i > -\infty$.

Lemma 3.3. *Let $\lambda \in \mathbb{R}$ and $r \in \mathbb{N}$. Let f_i, $i = 1, \ldots, r$, be convex polynomials on $\mathbb{R}^m$ and let $f = \max_{1 \le i \le r} f_i$. Suppose that $[f \le \lambda]$ is a nonempty compact set. Then, there exist $\tau, \gamma > 0$ such that*

$$m_f(x) \ge \tau(f(x) - \inf f)^\gamma \text{ for all } x \in [f \le \lambda]. \tag{3.11}$$

Proof. First of all, we note that f is a continuous convex function. Since $[f \le \lambda]$ is compact, it follows that $\text{argmin} f \ne \emptyset$. In particular, we have $\inf f > -\infty$. To see (3.11), let $x \in [f \le \lambda]$ and $x^* \in \partial f(x)$. Let $x_0 \in \text{argmin} f$ be such that $\|x - x_0\| = d(x, \text{argmin} f)$. Then, by the Cauchy Schwartz inequality and the definition of convex subdifferential, we have

$$\|x^*\| d(x, \text{argmin} f) \ge \langle x^*, x - x_0 \rangle \ge f(x) - f(x_0) = f(x) - \inf f.$$

Thus, for all $x \in [f \le \lambda]$ and $x^* \in \partial f(x)$, we obtain that

$$\|x^*\| d(x, \text{argmin} f) \ge f(x) - \inf f. \tag{3.12}$$

On the other hand, since each f_i ($i = 1, \ldots, r$) is a polynomial, we have $f = \max_{1 \le i \le r} f_i$ is a continuous subanalytic function. It follows from Remark 2.4 and the compactness of $[f \le \lambda]$ that there exist $\mu > 0$ and $\delta \in (0, 1]$ such that

$$d(x, \text{argmin} f) \le \mu[f(x) - \inf f]^\delta \text{ for all } x \in [f \le \lambda]. \tag{3.13}$$

Combining (3.12) and (3.13), we obtain that for all $x \in [f \le \lambda]$

$$\|x^*\| \ge \tau[f(x) - \inf f]^\gamma \text{ for all } x^* \in \partial f(x).$$

where $\tau = \mu^{-1}$ and $\gamma = 1 - \delta$. If $\delta \in (0, 1)$, then (3.11) follows by taking infimum for all $x^* \in \partial f(x)$. On the other hand, if $\delta = 1$ (and so, $\gamma = 0$), then we have $m_f(x) \ge \tau$ for all $x \in [f \le \lambda]$. Note that $0 \le f(x) - \inf f \le M$ for all $x \in [f \le \lambda]$ where $M := \lambda - \inf f + 1 > 0$. Therefore, we have

$$m_f(x) \ge \tau \ge \frac{\tau}{M}(f(x) - \inf f) \text{ for all } x \in [f \le \lambda].$$

Therefore, (3.11) also follows in this case and this completes the proof. □

Proposition 3.1. *Let $\lambda \in \mathbb{R}$ and $r \in \mathbb{N}$. Let f_i, $i = 1, \ldots, r$, be convex polynomials on $\mathbb{R}^m$ with $\inf f_i > -\infty$ and let $f = \max_{1 \le i \le r} f_i$. Suppose that $[f \le \lambda] \ne \emptyset$. Then, there exist $\tau, \gamma > 0$ such that*

$$m_f(x) \ge \tau(f(x) - \inf f)^\gamma \text{ for all } x \in [f \le \lambda]. \tag{3.14}$$

Proof. We shall prove Proposition 3.1 by induction on m (the dimension of the underlying space).

Suppose that $m = 1$. If $[f \le \lambda]$ is bounded (and hence compact), then (3.14) holds by Lemma 3.3. Thus, we may assume that $[f \le \lambda]$ is unbounded. Then, there exists a sequence $x_n \in [f \le \lambda]$ with $\|x_n\| \to +\infty$. By passing to subsequence, we may assume that $x_n/\|x_n\| \to v$ for some $v \in \mathbb{R}^m$ with $\|v\| = 1$. It follows from (2.3) that

$$f^\infty(v) = \liminf_{t\to\infty,\, v'\to v} \frac{f(tv')}{t} \le \liminf_{n\to\infty} \frac{f(\|x_n\| \frac{x_n}{\|x_n\|})}{\|x_n\|} = \liminf_{n\to\infty} \frac{f(x_n)}{\|x_n\|} \le 0.$$

Note that $\inf f_i > -\infty$ (and hence $\inf f > -\infty$). It follows from Remark 2.1 that $f^\infty(v) = 0$. This together with Lemma 2.3 implies that $f(x+tv) = f(x)$ for all $x \in \mathbb{R}$ and $t \in \mathbb{R}$. Since $v \neq 0$ and $m = 1$, it follows that f is a constant function. Thus Proposition 3.1 holds with $m = 1$ as the right hand side of (3.14) identically equals 0.

Now suppose that the statement of Proposition 3.1 is true with $m \le s$ ($s \in \mathbb{N}$). We then consider the case when $m = s+1$. From Lemma 3.3, without loss of generality, we may assume that $[f \le \lambda]$ is a unbounded set. Then, there exists a sequence $x_n \in [f \le \lambda]$ with $\|x_n\| \to +\infty$. By passing to subsequence, we may assume that $x_n/\|x_n\| \to v$ for some $v \in \mathbb{R}^{s+1}$ with $\|v\| = 1$. It follows from (2.3) again that $f^\infty(v) \le 0$. Note that $\inf f_i > -\infty$ (and hence $\inf f > -\infty$). It follows from Remark 2.1 that $f^\infty(v) = 0$. This together with Lemma 2.3 implies that

$$f(x+tv) = f(x) \text{ for all } x \in \mathbb{R}^{s+1} \text{ and } t \in \mathbb{R}. \tag{3.15}$$

Let $A = v^\perp := \{d \in \mathbb{R}^{s+1} : d^T v = 0\}$. Since $v \neq 0$, $\dim A = s$. Thus, there exists a full rank matrix $Q \in \mathbb{R}^{(s+1)\times s}$ such that $\{Qz : z \in \mathbb{R}^s\} = A$. Let $h_i : \mathbb{R}^s \to \mathbb{R}$ be defined by $h_i(z) = f_i(Qz)$, $i = 1, \dots, r$. It is clear that each h_i is a convex polynomial on $\mathbb{R}^s$, $i = 1, \dots, r$. Moreover, we see that

$$\inf h_i = \inf_{z\in\mathbb{R}^s} f_i(Qz) = \inf_{x\in A} f_i(x) \ge \inf f_i > -\infty.$$

Define $h : \mathbb{R}^s \to \mathbb{R}$ by $h(z) = \max_{1\le i\le r} h_i(z)$ for all $z \in \mathbb{R}^s$. It follows that, for all $z \in \mathbb{R}^s$,

$$h(z) = \max_{1\le i\le r} h_i(z) = \max_{1\le i\le r} f_i(Qz) = f(Qz). \tag{3.16}$$

Note that $\mathbb{R}^{s+1} = A \oplus \mathrm{span}\{v\}$ where $\oplus$ denotes the direct sum and $\mathrm{span}\{v\} := \{tv : t \in \mathbb{R}\}$. For any $x \in \mathbb{R}^{s+1}$, we can decompose it as

$$x = \mathrm{Pr}_A(x) + \mathrm{Pr}_{\mathrm{span}\{v\}}(x) \tag{3.17}$$

where Pr is the usual Euclidean projection. Thus, from (3.15), we see that $f(x) = f(\mathrm{Pr}_A(x))$ for all $x \in \mathbb{R}^{s+1}$. This together with (3.16) implies that

$$\inf f = \inf_{x\in A} f(x) = \inf_{x=Qz,\, z\in\mathbb{R}^s} f(x) = \inf_{z\in\mathbb{R}^s} f(Qz) = \inf h. \tag{3.18}$$

Now, since $h = \max_{1\le i\le r} h_i$ where each h_i is a convex polynomial on $\mathbb{R}^s$ with $\inf h_i > -\infty$, the induction hypothesis implies that there exist $\tau_1, \gamma_1 > 0$ such that

$$m_h(z) \ge \tau_1 (h(z) - \inf h)^{\gamma_1} \text{ for all } z \in [h \le \lambda], \tag{3.19}$$

where $m_h(z) := \inf\{\|z^*\| : z^* \in \partial h(z)\}$. We now show that (3.14) holds with $\tau := \tau_1 \|Q^T\|^{-1}$ and $\gamma := \gamma_1$, where $Q^T \in \mathbb{R}^{s\times(s+1)}$ is the transpose of Q and $\|Q^T\| := \sup\{\|Q^T x\|_{\mathbb{R}^s} : \|x\| = 1, x \in \mathbb{R}^{s+1}\} > 0$ (as Q is of full rank). To see this, fix an arbitrary $x_0 \in [f \le \lambda]$. Let $u_0 = \mathrm{Pr}_A(x_0)$. Since $u_0 \in A = \{Qz : z \in \mathbb{R}^s\}$, there exists $z_0 \in \mathbb{R}^s$ such that $u_0 = Qz_0$. Then, from (3.16) and $f(x) = f(\mathrm{Pr}_A(x))$ for all $x \in \mathbb{R}^{s+1}$, we have

$$h(z_0) = f(Qz_0) = f(u_0) = f(x_0). \tag{3.20}$$

Noting that $x_0 \in [f \le \lambda]$, this implies that $z_0 \in [h \le \lambda]$. It then follows from (3.19) and (3.20) that

$$m_h(z_0) \ge \tau_1(h(z_0) - \inf h)^{\gamma_1} = \tau_1(f(x_0) - \inf h)^{\gamma_1} = \tau_1(f(x_0) - \inf f)^{\gamma_1} \tag{3.21}$$

where the last equality follows from (3.18). Next, we claim that

$$\{Q^T x^* : x^* \in \partial f(x_0)\} \subseteq \partial h(z_0). \tag{3.22}$$

Granting this, we have

$$\begin{array}{rcl} m_h(z_0) & = & \inf\{\|z^*\| : z^* \in \partial h(z_0)\} \\ & \le & \inf\{\|Q^T x^*\| : x^* \in \partial f(x_0)\} \\ & \le & \inf\{\|Q^T\| \cdot \|x^*\| : x^* \in \partial f(x_0)\} \\ & = & \|Q^T\|\, m_f(x_0), \end{array}$$

Recall that $\|Q^T\| > 0$. This together with (3.21) implies that

$$m_f(x_0) \ge \|Q^T\|^{-1}\tau_1(f(x_0) - \inf f)^{\gamma_1} = \tau(f(x_0) - \inf f)^{\gamma}.$$

Therefore, (3.14) follows, and so, Proposition 3.1 holds with $m = s + 1$.

To see (3.22), let $x^* \in \partial f(x_0)$. Since $f(x_0 + tv) = f(x_0)$ for any $t \in \mathbb{R}$, we obtain that $\langle x^*, v\rangle = 0$. Let $x_0 = \mathrm{Pr}_A(x_0) + \mathrm{Pr}_{\mathrm{span}\{v\}}(x_0)$ (see (3.17)). Then $\langle x^*, \mathrm{Pr}_{\mathrm{span}\{v\}}(x_0)\rangle = 0$ and so, $\langle x^*, x_0\rangle = \langle x^*, \mathrm{Pr}_A(x_0)\rangle$. It follows from $Qz_0 = u_0 = \mathrm{Pr}_A(x_0)$ that for all $z \in \mathbb{R}^s$

$$\begin{array}{rcl} \langle Q^T x^*, z - z_0\rangle = \langle x^*, Qz - Qz_0\rangle & = & \langle x^*, Qz - \mathrm{Pr}_A(x_0)\rangle \\ & = & \langle x^*, Qz - x_0\rangle. \end{array}$$

Note that $x^* \in \partial f(x_0)$, and so, $\langle x^*, Qz - x_0\rangle \le f(Qz) - f(x_0)$. It follows from (3.20) that

$$\langle Q^T x^*, z - z_0\rangle \le f(Qz) - f(x_0) = h(z) - h(z_0) \quad \text{for all } z \in \mathbb{R}^s.$$

Thus, $Q^T x^* \in \partial h(z_0)$ and hence (3.22) holds. This completes the proof. □

Now we are ready to state our main result in this subsection.

Theorem 3.3. *Let $r \in \mathbb{N}$. Let f_i, $i = 1, \ldots, r$, be convex polynomials such that* $\inf f_i > -\infty$ *and let $f = \max_{1 \le i \le r} f_i$. Then, f is AWB.*

Proof. Note that f is continuous and so, the following implication always holds:

$$f(x) > \inf f \;\Rightarrow\; x \in \mathrm{ri}\,\mathrm{dom} f.$$

Thus, from Lemma 2.1, it suffices to show that any bounded above stationary sequence is indeed a minimizing sequence. To see this, let $\{x_n\}$ be a stationary sequence satisfying $\{f(x_n)\}_{n \in \mathbb{N}}$ is bounded above. Then, there exist $\lambda \in \mathbb{R}$ and $x_n^* \in \partial f(x_n)$ $(n \in \mathbb{N})$ such that $f(x_n) \le \lambda$ for all $n \in \mathbb{N}$ and $x_n^* \to 0$. In particular, we have $[f \le \lambda] \ne \emptyset$. Applying Proposition 3.1, we obtain that there exist $\tau, \gamma > 0$ such that

$$m_f(x) \ge \tau(f(x) - \inf f)^{\gamma} \text{ for all } x \in [f \le \lambda],$$

where $m_f(x) := \inf\{\|x^*\| : x^* \in \partial f(x)\}$. Note that $x_n \in [f \le \lambda]$ for all $n \in \mathbb{N}$, $x_n^* \in \partial f(x_n)$ and $x_n^* \to 0$, and so, $m_f(x_n) \to 0$. It follows that $f(x_n) \to \inf f$. Thus, the conclusion follows. □

4 Application: global error bound results

In this section, we apply our results and techniques (developed in the last sections) to study the global error bound results. To begin with, we formally recall the definition of error bounds.

Definition 4.1. *Let f be a proper lower semicontinuous convex function on $\mathbb{R}^m$. We say f has a*
(1) *Lipschitz type global error bound if there exists $\tau > 0$ such that*

$$d(x, [f \leq 0]) \leq \tau [f(x)]_+ \text{ for all } x \in \mathbb{R}^m,$$

where $[\alpha]_+$ denotes the number $\max\{\alpha, 0\}$.
(2) *Hölder type global error bound if there exist $\tau, \delta > 0$ such that*

$$d(x, [f \leq 0]) \leq \tau\big([f(x)]_+ + [f(x)]_+^\delta\big) \text{ for all } x \in \mathbb{R}^m$$

(The corresponding δ satisfying the preceding inequality is called the Hölder exponent).
(3) *Hölder type local error bound if for any compact set K, there exist $\tau, \delta > 0$ (depend on K) such that*

$$d(x, [f \leq 0]) \leq \tau\big([f(x)]_+ + [f(x)]_+^\delta\big) \text{ for all } x \in K.$$

To avoid the triviality, throughout this section, we assume that $\emptyset \neq [f \leq 0] \neq \mathbb{R}^m$.

4.1 Lipschitz type global error bound results

In this subsection, as an application of our new sufficient conditions for AWB function, we study the Lipschitz type global error bound results. To do this, we recall some definitions as well as some known results. Let f be a proper lower semicontinuous convex function on $\mathbb{R}^m$. We say f satisfy the Slater condition if

$$\text{there exists } x_0 \in \mathbb{R}^m \text{ such that } f(x_0) < 0.$$

Mangasarian showed that f has a Lipschitz type global error bound if f satisfy the Slater condition and an asymptotic constraint qualification. Later, Luo and Luo [22] showed that the asymptotic constraint qualification is superfluous if f can be expressed as maximum of finitely many convex quadratic functions. Next, as a consequence of Theorem 3.1 and Theorem 3.3, we present some other cases where the asymptotic constraint qualification is also superfluous. To do this, we need the following Lemma which comes from [33, Theorem 3.10.10] (see also [18, Corollary 1(c)] and [16, Theorem 3.3]). Here we state it in a form which is convenient to us.

Lemma 4.1. *Let f be a proper lower semicontinuous convex function on $\mathbb{R}^m$. Then f is AWB is equivalent to*

$$l_f(\lambda) := \inf\{\frac{f(x) - \lambda}{d(x, [f \leq \lambda])} : x \in [f > \lambda]\} > 0 \text{ for all } \lambda \in [\inf f, +\infty)$$

Theorem 4.1. *Let f be a proper lower semicontinuous function on $\mathbb{R}^m$ satisfying the Slater condition. Suppose that either one of the following conditions holds:*

(1) *f is a separable convex function.*

(2) *f is a convex polynomial.*

(3) *there exist $r \in \mathbb{N}$ and convex polynomials f_i, $i = 1, \ldots, r$, with* $\inf f_i > -\infty$ *such that* $f = \max_{1 \leq i \leq r} f_i$.

Then f has a Lipschitz type global error bound, i.e., there exists $\tau > 0$ such that

$$d(x, [f \leq 0]) \leq \tau [f(x)]_+ \text{ for all } x \in \mathbb{R}^m.$$

Proof. To see the conclusion, it suffices to show that there exists $\tau > 0$ such that

$$d(x, [f \le 0]) \le \tau f(x) \text{ for any } x \in [f > 0]. \tag{4.1}$$

To see (4.1), suppose that any one of the condition (1), (2) and (3) holds. Then f is a convex AWB function by Theorem 3.1, Theorem 3.2 and Theorem 3.3. Note that $\inf f \le 0$ (since f satisfy the Slater condition). Applying Lemma 4.1 with $\lambda = 0$, we see that

$$l_f(0) := \inf\{\frac{f(x)}{d(x, [f \le 0])} : x \in [f > 0]\} > 0$$

Thus, (4.1) holds with $\tau = l_f(0)^{-1}$. This completes the proof. □

Next, we present an example showing that our condition (1), (2) and (3) in the preceding theorem cannot be dropped. This example was first given by V.M. Shironin in [29] showing that a multifunction M defined by $M(\lambda) = \{x : f(x) \le \lambda\}$ might be discontinuous in the Hausdorff sense if $f = \max_{1 \le i \le r} f_i$ for some convex polynomials f_i (some related discussion and example can be found in [16, Note of Proposition 3.10] and [2]) †.

Example 4.1. *Let $f_1, f_2 : \mathbb{R}^4 \to \mathbb{R}$ be defined by $f_1(x_1, x_2, x_3, x_4) = x_1$ and*

$$f_2(x_1, x_2, x_3, x_4) = x_1^{16} + x_2^8 + x_3^6 + x_1 x_2^3 x_3^3 + x_1^2 x_2^4 x_3^2 + x_2^2 x_3^4 + x_1^4 x_3^4 + x_1^4 x_2^6 + x_1^2 x_2^6 + x_1^2 + x_2^2 + x_3^2 - x_4$$

Define $f = \max\{f_1, f_2\}$. It is clear that f is not separable and is not a polynomial. Note that

$$f(-n, 0, 0, n^{16} + n^2 + n) = -n \to -\infty \text{ as } n \to \infty. \tag{4.2}$$

Thus, f is not bounded below and hence f cannot be expressed into maximum of finitely many bounded below convex polynomials. Moreover, Slater condition holds (see (4.2)). It has been shown by V.M. Shironin that (cf [29] and [2]): f_1, f_2 are convex polynomials and for any $\lambda > \mu$

$$\rho([f \le \lambda], [f \le \mu]) := \sup_{x \in [f \le \lambda]} d(x, [f \le \mu]) = +\infty. \tag{4.3}$$

Next, we show that f do not have Hölder type global error bound (and hence f do not have Lipschitz type global error bound). To see this, we proceed by contradiction, i.e., there exist $\tau, \delta > 0$ such that

$$d(x, [f \le 0]) \le \tau\big([f(x)]_+ + [f(x)]_+^{\delta}\big) \text{ for all } x \in \mathbb{R}^m.$$

Setting $\lambda = 1$ and $\mu = 0$ in (4.3), it follows that there exists $\{x_n\}_{n \in \mathbb{N}}$ with $f(x_n) \le 1$ such that $d(x_n, [f \le 0]) \to +\infty$. Therefore, we have

$$d(x_n, [f \le 0]) \le \tau\big([f(x_n)]_+ + [f(x_n)]_+^{\delta}\big) \le 2\tau \text{ and } d(x_n, [f \le 0]) \to +\infty.$$

This is impossible, and so, f do not have Hölder type global error bound.

4.2 Hölder type global error bound results

In this subsection, we devote ourselves to the study of Hölder type global error bound result for convex polynomials. In [22, 23, 30], they showed that Hölder type global error bound result holds if f can be expressed as maximum of finitely many convex quadratic functions. However, little is known if we go beyond the convex quadratic case. In fact, for general quadratic cases (not necessarily convex), [22, 23] gave an example showing that, in general, only Hölder type local error bound result can be achieved

†The author would like to thank one of the referees for suggesting this example

(but the corresponding Hölder exponent is unknown in general). According to [27] "we know that many systems will have global Höderian error bounds with a certain non-unit exponent. It would be useful to obtain bounds for these exponents in extended cases because such information will produce substantial benefits." As we will see later, our approach has the advantage that the corresponding Hölder exponent can be determined explicitly.

To do this, we need the following notation and Lemmas. The first lemma is a simple property for polynomial functions and has been mentioned (without proof) in [1]. Its proof is standard and hence is omitted here. The second lemma is an effective inequality for single polynomial with isolated zero. This inequality plays a key role in our later analysis.

Definition 4.2. *Let $m, d \in \mathbb{N}$. Define $\kappa(m,d) := (d-1)^m + 1$. Moreover, we also define $\Gamma_d(\mathbb{R}^m) = \{f : f$ is a convex polynomial on $\mathbb{R}^m$ with degree no larger than $d\}$.*

Lemma 4.2. *Let f be an polynomial function on $\mathbb{R}^m$. Let x_1, x_2 be two points in $\mathbb{R}^m$. If f is constant on $C := [x_1, x_2]$. Then f is constant on $\mathrm{aff}C$.*

Lemma 4.3. *(cf. [12, Theorem 3]) Let $d \in \mathbb{N}$ and let f be a polynomial with degree no larger than d. Suppose that $f(0) = 0$ and there exists $\epsilon_0 > 0$ such that $f(x) > 0$ for all $x \in \mathbb{B}(0, \epsilon_0)\backslash\{0\}$. Then there exist constants $\tau, \epsilon > 0$ such that*

$$\tau\|x\|^{\kappa(m,d)} \leq f(x) \text{ for all } x \in \mathbb{B}(0, \epsilon). \tag{4.4}$$

Lemma 4.4. *(cf. [10, Corollary 2.1] and [11, Proposition 6.1.3]).) Let f be a continuous convex function on $\mathbb{R}^m$ and let $\delta, \gamma > 0$. Suppose that there exists $\mu > 0$ such that*

$$d(x, [f \leq 0]) \leq \mu[f(x)]_+^\delta \text{ for all } x \in [f \leq \gamma]. \tag{4.5}$$

Then there exists $\tau > 0$ such that

$$d(x, [f \leq 0]) \leq \tau\big([f(x)]_+ + [f(x)]_+^\delta\big) \text{ for all } x \in \mathbb{R}^m.$$

4.2.1 Single convex polynomial case

In this part, we establish the Hölder type global error bound for single convex polynomial. We begin with the following lemma.

Lemma 4.5. *Let $d \in \mathbb{N}$ and let $f \in \Gamma_d(\mathbb{R}^m)$ with $\inf f = 0$. Suppose that $[f \leq 0]$ is compact. Then there exists $\mu > 0$ such that*

$$d(x, [f \leq 0]) \leq \mu f(x)^{\kappa(m,d)^{-1}} \text{ for all } x \in [f \leq 1] \tag{4.6}$$

Proof. We first show that $[f \leq 0]$ is a single point. Suppose that it is not true. Then there exist x_1, x_2 ($x_1 \neq x_2$) such that $x_1, x_2 \in [f \leq 0]$. By the convexity of f, $[x_1, x_2] \subseteq [f \leq 0] = [f = 0]$ (as $\inf f = 0$). Now, by Lemma 4.2, we see that $[f = 0]$ contains $\mathrm{aff}([x_1, x_2])$, which contradicts to the fact that $[f \leq 0]$ is compact. Thus, we see that $[f \leq 0]$ is a single point, say $[f \leq 0] = \{x_0\}$. Define f_0 by $f_0(x) = f(x+x_0)$ for all $x \in \mathbb{R}^m$. Then $f_0(0) = \inf f = 0$ and $f_0(x) > 0$ for all $x \neq 0$. Applying Lemma 4.3 to f_0, we obtain $\tau > 0$ and $\epsilon > 0$ such that $\tau\|x\|^{\kappa(m,d)} \leq f_0(x)$ for all $x \in \mathbb{B}(0, \epsilon)$. This implies that

$$\tau\|x\|^{\kappa(m,d)} \leq f(x + x_0) \text{ for all } x \in \mathbb{B}(0, \epsilon).$$

It follows that $\tau\|x - x_0\|^{\kappa(m,d)} \leq f(x)$ for all $x \in \mathbb{B}(x_0, \epsilon)$. Since $[f \leq 0] = \{x_0\}$, we can find $\delta \in (0, 1)$ such that $[f \leq \delta] \subseteq B(x_0, \epsilon)$ ‡. Then, we have

$$d(x, [f \leq 0]) = \|x - x_0\| \leq \tau^{-\kappa(m,d)^{-1}} f(x)^{\kappa(m,d)^{-1}} \text{ for all } x \in [f \leq \delta]. \tag{4.7}$$

‡Otherwise, there exist $\delta_n \to 0$ and $x_n \in \mathbb{R}^m$ such that $x_n \in [f \leq \delta_n]\backslash\mathbb{B}(x_0, \epsilon)$. Without loss generality, we may assume that $x_n \in [f \leq 1]$ for all $n \in \mathbb{N}$. Since $[f \leq 0] = \mathrm{argmin} f$ is compact and f is a convex polynomial, f is level bounded in the sense that $[f \leq \lambda]$ is bounded for any $\lambda \geq \inf f$ (see [1, Proposition 3.13]) and so, $[f \leq 1]$ is also compact. Thus, we may further assume that $x_n \to x$. By passing to limit, we see that $x \in [f \leq 0]$ and $\|x - x_0\| \geq \epsilon > 0$. However, this contradicts the fact that $[f \leq 0] = \{x_0\}$

On the other hand, define $\overline{f} = f - \delta$. Note that $\overline{f}$ is a convex polynomial satisfying $[\overline{f} < 0] \neq \emptyset$. It then follows from Theorem 4.1 (ii) that there exists $\tau_1 > 0$ such that

$$d(x, [f \leq \delta]) = d(x, [\overline{f} \leq 0]) \leq \tau_1 \max\{\overline{f}(x), 0\} \leq \tau_1 f(x) \text{ for all } x \in \mathbb{R}^m. \tag{4.8}$$

Now, let $x \in [f > \delta] \cap [f \leq 1]$ and let $\overline{x} \in [f \leq \delta]$ be such that $\|x - \overline{x}\| = d(x, [f \leq \delta])$. Since $\overline{x} \in [f \leq \delta] \subseteq B(x_0, \epsilon)$ and $f(x) > \delta$, we see that $\|\overline{x} - x_0\| \leq \epsilon \leq \frac{\epsilon f(x)}{\delta}$. It follows from the triangle inequality that

$$\begin{aligned} d(x, [f \leq 0]) = \|x - x_0\| &\leq \|x - \overline{x}\| + \|\overline{x} - x_0\| \\ &= d(x, [f \leq \delta]) + \|\overline{x} - x_0\| \\ &\leq d(x, [f \leq \delta]) + \frac{\epsilon f(x)}{\delta}. \end{aligned}$$

This together with (4.8) implies that

$$d(x, [f \leq 0]) \leq (\tau_1 + \frac{\epsilon}{\delta}) f(x) \leq (\tau_1 + \frac{\epsilon}{\delta}) f(x)^{\kappa(m,d)^{-1}} \text{ for all } x \in [f > \delta] \cap [f \leq 1], \tag{4.9}$$

where the last inequality holds as $\kappa(m, d) \geq 1$. Combining (4.7) with (4.9) and noting that $\delta \in (0, 1)$, we see that

$$d(x, [f \leq 0]) \leq \mu f(x)^{\kappa(m,d)^{-1}} \text{ for all } x \in [f \leq 1]$$

where $\mu = \max\{\tau^{-\kappa(m,d)^{-1}}, \tau_1 + \frac{\epsilon}{\delta}\}$. □

Next, we show that the assumption "$[f \leq 0]$ is compact" in Lemma 4.5 can be relaxed.

Lemma 4.6. *Let $d \in \mathbb{N}$ and let $f \in \Gamma_d(\mathbb{R}^m)$ with* $\inf f = 0$. *Then there exists $\mu > 0$ such that*

$$d(x, [f \leq 0]) \leq \mu f(x)^{\kappa(m,d)^{-1}} \text{ for all } x \in [f \leq 1] \tag{4.10}$$

Proof. Since $\inf f = 0$, we have $[f \leq 0] = [f = 0]$. We now prove Lemma 4.6 by induction on m (the dimension of the underlying space).

Suppose that $m = 1$. We may assume without loss of generality that $[f \leq 0]$ is a single point. (Otherwise, there exist $x_1, x_2 \in [f \leq 0]$ with $x_1 \neq x_2$. Then, the convexity of f entails that $[x_1, x_2] \subseteq [f \leq 0] = [f = 0]$. Thus, from Lemma 4.2, we see that $f(x) = 0$ for all $x \in \text{aff}[x_1, x_2]$. Since $m = 1$, we see that f is a function identically equals 0. Thus, (4.10) follows as the left hand side identically equals 0). Now, since $[f \leq 0]$ is a single point, we have in particular that $[f \leq 0]$ is compact. Thus, from Lemma 4.5, we see that the conclusion is true for any $f \in \Gamma_d(\mathbb{R}^m)$ with $\inf f = 0$ when $m = 1$.

Now suppose that the conclusion is true for any $f \in \Gamma_d(\mathbb{R}^m)$ with $\inf f = 0$ whenever $m \leq s$ ($s \in \mathbb{N}$). We then consider the case for $m = s + 1$. From Lemma 4.5, we may assume that $[f \leq 0]$ is unbounded. Let $x_n \in [f \leq 0]$ be such that $\|x_n\| \to +\infty$. By passing to subsequence, $x_n/\|x_n\| \to v$ for some $v \in \mathbb{R}^{s+1}$ with $\|v\| = 1$. It follows from (2.3) that $f^\infty(v) \leq 0$. Moreover, since $\inf f = 0 > -\infty$, $f^\infty(w) \geq 0$ for all $w \in \mathbb{R}^{s+1}$ (see Remark 2.1). It follows that $f^\infty(v) = 0$. Thus Lemma 2.2(3) implies that

$$f(x + tv) = f(x) \text{ for any } x \in \mathbb{R}^{s+1} \text{ and for any } t \in \mathbb{R}. \tag{4.11}$$

Define $A = v^\perp := \{d \in \mathbb{R}^{s+1} : d^T v = 0\}$. Since $v \neq 0$, we have $\dim A = s$. Thus, there exists a full rank matrix $Q \in \mathbb{R}^{(s+1)\times s}$ such that $\{Qz : z \in \mathbb{R}^s\} = A$. Note that $\mathbb{R}^{s+1} = A \oplus \text{span}\{v\}$ where $\oplus$ denotes the direct sum and $\text{span}\{v\} := \{tv : t \in \mathbb{R}\}$. For any $x \in \mathbb{R}^{s+1}$, one has

$$x = \text{Pr}_A(x) + \text{Pr}_{\text{span}\{v\}}(x) \tag{4.12}$$

where Pr is the usual Euclidean projection. From (4.11), we see that $f(x) = f(\mathrm{Pr}_A(x))$ for all $x \in \mathbb{R}^{s+1}$. Defining $h : \mathbb{R}^s \to \mathbb{R}$ be defined by $h(z) := f(Qz)$, it follows that $h \in \Gamma_d(\mathbb{R}^s)$ and

$$\inf h = \inf_{z\in\mathbb{R}^s} f(Qz) = \inf_{x\in A} f(x) = \inf f = 0.$$

Now, we see that $h \in \Gamma_d(\mathbb{R}^s)$ with $\inf h = 0$. Thus, the induction hypothesis implies that there exists $\mu_1 > 0$ such that

$$d(z, [h \le 0]) \le \mu_1 h(z)^{\kappa(s,d)^{-1}} \text{ for all } z \in [h \le 1]. \tag{4.13}$$

Next, we show that (4.10) holds with $\mu = \mu_1\|Q\|$. To see this, let $x \in [f \le 1]$. Since Q is of full rank and $f(x + tv) = f(x)$ for any $x \in \mathbb{R}^{s+1}$ and for any $t \in \mathbb{R}$, the definitions of h and (4.12) entail that [§]

$$[f \le 1] = Q([h \le 1]) + \mathrm{span}\{v\} \text{ and } [f \le 0] = Q([h \le 0]) + \mathrm{span}\{v\}. \tag{4.14}$$

Since $x \in [f \le 1]$, it follows that there exist $z_0 \in [h \le 1]$ and $q \in \mathbb{R}$ such that $x = Qz_0 + qv$. This together with the second relation of (4.14) and (4.13) implies that

$$\begin{aligned} d(x, [f \le 0]) &= d\big(Qz_0 + qv, Q([h \le 0]) + \mathrm{span}\{v\}\big) \\ &\le d\big(Qz_0, Q([h \le 0])\big) \\ &\le \|Q\| d(z_0, [h \le 0]) \\ &\le \mu_1\|Q\| h(z_0)^{\kappa(s,d)^{-1}}, \end{aligned}$$

where $\|Q\| = \sup\{\|Qz\|_{\mathbb{R}^{s+1}} : \|z\| = 1, z \in \mathbb{R}^s\}$. Note that $f(x) = f(Qz_0 + qv) = f(Qz_0) = h(z_0)$ (see (4.11)), $x \in [f \le 1]$ and $\kappa(s, d) \le \kappa(s + 1, d)$. It follows that

$$d(x, [f \le 0]) \le \mu_1\|Q\| f(x)^{\kappa(s,d)^{-1}} \le \mu_1\|Q\| f(x)^{\kappa(s+1,d)^{-1}}.$$

Thus (4.10) holds with $\mu = \mu_1\|Q\|$ when $m = s + 1$. This completes the proof. ☐

As a consequence of Lemma 4.6 and Lemma 4.4, we have the following Hölder type global error bound result for single convex polynomial. In the special case when $d = 2$, i.e. f is a convex quadratic function (and hence $\kappa(m, d) = 2$ for all $m \in \mathbb{N}$), this result has been presented in [21] and [25].

Theorem 4.2. *Let $d \in \mathbb{N}$ and let $f \in \Gamma_d(\mathbb{R}^m)$. Then there exists $\tau > 0$ such that*

$$d(x, [f \le 0]) \le \tau\big([f(x)]_+ + [f(x)]_+^{\kappa(m,d)^{-1}}\big) \text{ for all } x \in \mathbb{R}^m. \tag{4.15}$$

Proof. We split the proof into two cases. *Case 1* $\inf f = 0$ and *Case 2* $\inf f < 0$.

Suppose that Case 1 holds. From Lemma 4.6 and $\inf f = 0$, we see that $d(x, [f \le 0]) \le \tau f(x)^{\kappa(m,d)^{-1}} = \tau[f(x)]_+^{\kappa(m,d)^{-1}}$ for all $x \in [f \le 1]$. Thus, the conclusion follows from lemma 4.4.

Suppose that Case 2 holds. Then there exists $x_0 \in \mathbb{R}^m$ such that $f(x_0) < 0$, i.e., the Slater condition holds. From Theorem 4.1 (2), there exists $\tau_0 > 0$ such that

$$d(x, [f \le 0]) \le \tau_0[f(x)]_+ \text{ for all } x \in \mathbb{R}^m. \tag{4.16}$$

It follows that

$$d(x, [f \le 0]) \le \tau_0[f(x)]_+ \le \tau_0\big([f(x)]_+ + [f(x)]_+^{\kappa(m,d)^{-1}}\big) \text{ for all } x \in \mathbb{R}^m.$$

Thus (4.15) holds with $\tau = \tau_0$.

Therefore, the conclusion follows by combining Case 1 and Case 2. ☐

[§]For example, to see the first relation of (4.14), let $z \in [h \le 1]$ and $t \in \mathbb{R}$. Then, from (4.11), we see that $f(Qz + tv) = f(Qz) = h(z) \le 1$. Thus, $Q([h \le 1]) + \mathrm{span}\{v\} \subseteq [f \le 1]$. To see the converse inclusion, let $x \in [f \le 1]$. From (4.12), $x = \mathrm{Pr}_A(x) + \mathrm{Pr}_{\mathrm{span}\{v\}}(x)$. Since $\mathrm{Pr}_A(x) \in A = \{Qz : z \in \mathbb{R}^s\}$, there exists $z_0 \in \mathbb{R}^s$ such that $\mathrm{Pr}_A(x) = Qz_0$. Thus, we have $x = Qz_0 + \mathrm{Pr}_{\mathrm{span}\{v\}}(x)$. Note that $\mathrm{Pr}_{\mathrm{span}\{v\}}(x) \in \mathrm{span}\{v\}$. To finish the proof, it suffices to show that $z_0 \in [h \le 1]$. To see this, since $f(x) = f(\mathrm{Pr}_A(x))$, we have $h(z_0) = f(Qz_0) = f(\mathrm{Pr}_A(x)) = f(x) \le 1$. Thus the conclusion follows. The proof of the second relation is similar.

In particular, we have the following corollary.

Corollary 4.1. *Let $d \in \mathbb{N}$ and let $f \in \Gamma_d(\mathbb{R}^m)$. Then there exists $\tau > 0$ such that*

$$d(x, \text{argmin} f) \leq \tau\big([f(x) - \inf f]_+ + [f(x) - \inf f]_+^{\kappa(m,d)^{-1}}\big) \text{ for all } x \in \mathbb{R}^m. \tag{4.17}$$

Proof. Without loss of generality, we may assume that $\inf f > -\infty$ (Otherwise, the right-hand side of (4.17) equals $+\infty$ and hence (4.17) follows). It follows from Lemma 2.2(1) that $\text{argmin} f \neq \emptyset$. Let $x_0 \in \text{argmin} f$ and let $\alpha = f(x_0) = \inf f$. Define $g = f - \alpha$. It is clear that $g \in \Gamma_d(\mathbb{R}^m)$ and $[g \leq 0] = \text{argmin} f \neq \emptyset$. Thus (4.17) follows by the preceding Theorem. □

4.2.2 Nonnegative convex polynomials case

In this part, for a nonnegative convex polynomial system $\{f_1, \ldots, f_r\}$ (i.e., each f_i is a nonnegative convex polynomial), we establish that Hölder type global error bound holds for $f = \max_{1 \leq i \leq r} f_i$ with Hölder exponent $\kappa(m,d)^{-1}$. In the special case when $d = 2$, i.e. each f_i is a convex quadratic function, this result has been presented in [23].

Theorem 4.3. *Let $d, r \in \mathbb{N}$ and let $f_i \in \Gamma_d(\mathbb{R}^m)$ $(i = 1, \ldots, r)$ with $\inf f_i \geq 0$. Let $f = \max_{1 \leq i \leq r} f_i$. Then there exists a constant $\tau > 0$ such that*

$$d(x, [f \leq 0]) \leq \tau\big([f(x)]_+ + [f(x)]_+^{\kappa(m,d)^{-1}}\big) \text{ for all } x \in \mathbb{R}^m. \tag{4.18}$$

Proof. From Lemma 4.4 and $\inf f \geq 0$, it suffices to show that there exists $\mu > 0$ such that

$$d(x, [f \leq 0]) \leq \mu f(x)^{\kappa(m,d)^{-1}} \text{ for all } x \in [f \leq 1/r]. \tag{4.19}$$

To see this, since $\inf f \geq 0$ (thanks to $\inf f_i \geq 0$) and $[f \leq 0] \neq \emptyset$ (by our convention), we observe that $\inf f = 0$ and $\inf f_i = 0$, $i = 1, \ldots, r$. Define $h : \mathbb{R}^m \to \mathbb{R}$ by $h(x) = \sum_{i=1}^r f_i(x)$. It is clear that $h \in \Gamma_d(\mathbb{R}^m)$ and $\inf h = 0$ (as $f_i(x) \leq h(x) \leq r f(x)$ for all $x \in \mathbb{R}^m$ and for any $i = 1, \ldots, r$). Thus Lemma 4.6 implies that there exists $\tau_0 > 0$ such that

$$d\big(x, [h \leq 0]\big) \leq \tau_0 h(x)^{\kappa(m,d)^{-1}} \text{ for all } x \in [h \leq 1].$$

Notice that $h(x) = \sum_{i=1}^r f_i(x) \leq r f(x)$, and so, $[f \leq 1/r] \subseteq [h \leq 1]$. It follows that

$$d\big(x, [h \leq 0]\big) \leq \mu f(x)^{\kappa(m,d)^{-1}} \text{ for all } x \in [f \leq 1/r]$$

where $\mu = \tau_0 r^{\kappa(m,d)^{-1}}$. Finally, since $[h \leq 0] = \bigcap_{1 \leq i \leq r} [f_i \leq 0] = [f \leq 0] \neq \emptyset$, we see that (4.19) holds, and so, the conclusion follows immediately. □

The following example shows that the nonnegative assumption on f_i cannot be dropped.

Example 4.2. *Let $r \in \mathbb{N}$ and let $f_1, f_2, f_3 : \mathbb{R}^3 \to \mathbb{R}$ be defined by $f_1(x_1, x_2, x_3) = x_1^{2r} - x_2$, $f_2(x_1, x_2, x_3) = x_2^{2r} - x_3$ and $f_3(x_1, x_2, x_3) = x_3^{2r}$. It is clear that $f_1, f_2, f_3 \in \Gamma_d(\mathbb{R}^m)$ with $m = 3$ and $d = 2r$. Thus $\kappa(m,d) = (d-1)^m + 1 = (2r-1)^3 + 1$. Define $f := \max\{f_1, f_2, f_3\}$. Then $[f \leq 0] = \{(0,0,0)\}$. Consider $x_n := (1/n, 1/n^{2r}, 1/n^{4r^2})$. Then one has $f(x_n) = 1/n^{8r^3}$ and hence*

$$\frac{d(x_n, [f \leq 0])}{[f(x_n)]_+ + [f(x_n)]_+^{\kappa(m,d)^{-1}}} = \frac{\sqrt{(1/n)^2 + (1/n^{2r})^2 + (1/n^{4r^2})^2}}{1/n^{8r^3} + (1/n^{8r^3})^{\left((2r-1)^3+1\right)^{-1}}} = \frac{O(1/n)}{O(1/n^{\frac{8r^3}{(2r-1)^3+1}})} \to +\infty.$$

Thus (4.18) fails.

5 Conclusions and remarks

In this paper, we present some new and tractable sufficient conditions for convex asymptotic well behaved (AWB) functions. As applications, under the Slater condition, we establish that f has a Lipschitz type global error bound if it satisfies either one of the following 3 conditions: (1) f is a convex separable function; (2) f is a convex polynomial; (3) f can be expressed as maximum of finitely many bounded below convex polynomials. We also establish Hölder type global error bound results for single convex polynomial and for function which can be expressed as maximum of finitely many nonnegative convex polynomials, where the corresponding Hölder exponent can be explicitly determined. In the case where the degree of the convex polynomial is restricted to 2, our Hölder type global error bound result for single convex polynomial coincides with the existing result for single convex quadratic function (see [21]). It should be emphasized that [21] established more general results in the quadratic case. More explicitly, [21] achieved Hölder type global error bound result for convex piecewise quadratic function.

It is also worthy noting that, for a convex quadratic system $\{f_1, \ldots, f_r\}$ (i.e., each f_i is a convex quadratic function), [25, 30] showed that $f := \max\limits_{1 \le i \le r} f_i$ has a Hölder type global error bound with Hölder exponent 2^{-r}. However, unlike the convex quadratic system cases, Example 4.1 suggests that Hölder type global error bound result may fail for a general convex polynomial system $\{f_1, \ldots, f_r\}$ (since the associated multifunction $M : \mathbb{R} \to 2^{\mathbb{R}^m}$, which defined as $M(\lambda) = \{x : \max\limits_{1 \le i \le r} f_i(x) \le \lambda\}$, is discontinuous in the Hausdorff sense). Thus, it would be interesting to investigate what is the appropriate subclass of convex polynomial functions which we can make a full development on error bound theory as in the convex quadratic case. It has been recently shown by V. M. Shronin (see [29] and also [16]) that the associated multifunction M is continuous (in the Hausdorff sense) if each f_i is a simple convex polynomial ¶. Noting that every convex quadratic function is a simple convex polynomial, this might shed the lights on answering this question. Another possible research direction is to investigate whether we can extend the results in this paper to obtain global error bound results for special nonconvex polynomial systems or for some special convex composite functions (Interestingly, some progresses have been made in [8] along this line by imposing coerciveness. However, as the Hölder type global error bound result in [8] relies on the Łojasiewicz inequality, it is not clear that whether the corresponding Hölder exponent in the result in [8] can be determined explicitly or not). These would be further research topics and will be examined in a forthcoming paper.

Acknowledgement: The author would like to express his sincere thanks to the referees for their constructive comments and valuable suggestions which have contributed to the preparation of this paper. In particular, one of the referee pointed out a gap in the original proof of Proposition 3.1 and the other referee suggested Example 4.1. Moreover, the author also would like to thank Professor K.F. Ng and Professor Jeyakumar for valuable suggestions and stimulated discussions.

References

[1] A. Auslender, M. Teboulle, *Asymptotic cones and functions in optimization and variational inequalities.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[2] E. G. Belousov, On types of Hausdorff discontinuity from above for convex closed mappings, Optimization, 49(2001), 303-325.

¶A convex polynomial f is said to be simple if for any two affine subspaces A_1 and A_2 with $H(A_1, A_2) < +\infty$ and $A_i \cap \text{intepi} f \neq \emptyset$, $i = 1, 2$, we have $H(A_1 \cap \text{epi} f, A_2 \cap \text{epi} f) < +\infty$ (where H is the Hausdorff distance). Examples for simple convex polynomial include convex quadratic functions, convex polynomials of 3 variables up to degree 14.

[3] B. Bank, R. Mandel, Parametric integer optimization. Mathematical Research, 39. Akademie-Verlag, Berlin, 1988.

[4] E. G. Belousov and D. Klatte, A Frank-Wolfe type theorem for convex polynomial programs, Computational Optimization and Applications, (22)2002, pp. 37-48.

[5] J.V. Burke and S. Deng, Weak sharp minima revisited Part I: basic theory, Control and Cybernetics, 31(2002), 439-469.

[6] J.V. Burke and S. Deng, Weak sharp minima revisited. II. Application to linear regularity and error bounds. Math. Program. 104(2005), no. 2-3, Ser. B, 235-261.

[7] J.V. Burke and S. Deng, Weak sharp minima revisited. III. Error bounds for differentiable convex inclusions. Math. Program. 116(2009), no. 1-2, Ser. B, 37-56.

[8] S. Deng, Computable error bounds for convex inequality systems in reflexive Banach spaces, SIAM. J. Optim., 7(1997), 274-279.

[9] S. Deng, Pertubation analysis of a condition number for convex inequality systems and global error bounds for analytic systems, Math. Program. 83(1998), 263-276.

[10] S. Deng, Well-posed problems and error bounds in optimization. Reformulation: nonsmooth, piecewise smooth, semismooth and smoothing methods (Lausanne, 1997), Appl. Optim., (22) 1999 Kluwer Acad. Publ., Dordrecht, 117-126.

[11] F. Facchinei and J.S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems Volume I, Springer publication, 2003.

[12] J. Gwoździewicz , The Łojasiewicz exponent of an analytic function at an isolated zero, Commentarii Mathematici Helvetici, (74)1999, No. 3, 364-375.

[13] Y. R. He and J. Sun, Second-order sufficient conditions for error bounds in Banach spaces. SIAM J. Optim. 17 (2006), no. 3, 795-805.

[14] J. B. Hiriart-Urruty and C. Lemarachal, Convex analysis and minimization algorithms. II. Advanced theory and bundle methods. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 306. Springer-Verlag, Berlin, 1993. xviii+346 pp.

[15] H. Hu, Characterizations of local and global error bounds for convex inequalities in Banach spaces. SIAM J. Optim. 18 (2007), no. 1, 309–321.

[16] D. Klatte, Hoffman's error bound for systems of convex inequalities. Mathematical programming with data perturbations, 185–199, Lecture Notes in Pure and Appl. Math., 195, Dekker, New York, 1998.

[17] D. Klatte and W. Li, Asymptotic constraint qualifications and global error bounds for convex inequalities, Math. Program., 84(1999) 137-160.

[18] A. S. Lewis and J. S. Pang, Error bounds for convex inequality systems Generalized Convexity, Generalized Monotonicity J.P. Crouzeix, J.E. Martinez-Legaz and M. Volle (eds) (1998) 75-110.

[19] G. Li and K.F. Ng, Error bounds of generalized D-gap functions for nonsmooth and nonmonotone variational inequality problems , SIAM Journal on Optimization, 20(2009), no. 2, 667-690.

[20] G. Li, C.M. Tang and Z.X. Wei, Error bound results for generalized D-gap functions of nonsmooth variational inequality problems, to appear in Journal of Computional Applied Mathematics, doi:10.1016/j.cam.2009.11.025 .

[21] W. Li, Error bounds for piecewise convex quadratic programs and applications, SIAM J. on Control and Optim. 33(1995), No. 5, 1510– 1529.

[22] X. D. Luo and Z.Q. Luo, Extension of Hoffman's error bound to polynomial systems. SIAM J. Optim. 4 (1994), no. 2, 383–392.

[23] Z. Q. Luo, J. S. Pang, Error bounds for analytic systems and their applications. Math. Programm. 67(1994), 1-28.

[24] Z. Q. Luo, J.S. Pang and D. Ralph, Mathematical Programs with Equilibrium Constraints. Cambridge University Press, Cambridge, 1996.

[25] H. V. Ngai and M. Théra, Error bounds for convex differentiable inequality systems in Banach spaces. Math. Program. 104 (2005), no. 2-3, Ser. B, 465-482.

[26] W. T. Obuchowska, On generalizations of the Frank-Wolfe theorem to convex and quasi-convex programmes, Computational Optimization and Applications, 33(2006), 349–364.

[27] J. S. Pang, Error bounds in mathematical programming. Math Programming, 79(1997), No 2, 299C-332.

[28] R. T. Rockafellar, *Convex Analysis*, Princeton Univ. Press, Princeton, N. J, 1970.

[29] V. M. Shironin, On Hausdorff continuity of convex and convex polynomial mappings (In Russian). In: E.G. Belousov and B. Bank, editors, Mathematical Optimization: Questions of Solvability and Stability, Chapter V, Moscow Univeristy Publ., Moscow (1986).

[30] T. Wang and J.S. Pang, Global error bounds for convex quadratic inequality systems. Optimization, 31(1994), no. 1, 1-12.

[31] Z. L. Wu and J. J. Ye, On error bounds for lower semicontinuous functions. Math. Program. 92 (2002), no. 2, Ser. A, 301-314.

[32] W.H. Yang, Error bounds for convex polynomials. SIAM J. Optim. 19 (2008), no. 4, 1633–1647.

[33] C. Zalinescu, *Convex Analysis in General Vector Spaces*, World Scientific, 2002.

Appendix

In this appendix, we provide the proof of Lemma 2.3.

Proof of Lemma 2.3

Proof. Let f_i, $i = 1, \ldots, r$, be convex polynomials on $\mathbb{R}^m$ with $\inf f_i > -\infty$ and let $f = \max_{1 \le i \le r} f_i$. Let $v \in \mathbb{R}^m$ be such that $f^\infty(v) = 0$. Fix an arbitrary $x \in \mathbb{R}^m$. For each $i \in \{1, 2, \ldots, r\}$, define $g_i(t) = f_i(x + tv)$. Note that

$$v \in \{d : f^\infty(d) = 0\} \subseteq \bigcap_{i=1}^{r} \{d : f_i^\infty(d) \le 0\}.$$

It follows from the definition of recession function that

$$\sup_{t>0} \frac{g_i(t) - g_i(0)}{t} = \sup_{t>0} \frac{f_i(x + tv) - f_i(x)}{t} = f_i^{\infty}(v) \le 0.$$

Since each g_i is a one dimensional convex polynomial (hence is a linear function or a polynomial with even degree q), the preceding relation gives that $g_i(t) = g_i(0) + \alpha_i t$ ($t \ge 0$) for some $\alpha_i \le 0$ ($i = 1, \ldots, r$). That is to say,

$$f_i(x + tv) = f_i(x) + \alpha_i t \text{ for all } t \ge 0. \tag{5.1}$$

From $\inf f_i > -\infty$ and (2.5), it follows that

$$0 \le f_i^{\infty}(v) = \lim_{t \to +\infty} \frac{f_i(x + tv) - f_i(x)}{t} = \alpha_i \le 0.$$

Thus, we see that $\alpha_i = 0$ and hence $f_i(x+tv) = f_i(x)$ for all $t \ge 0$ and for all $i \in \{1, \ldots, r\}$. This implies that $f(x + tv) = f(x)$ for all $t \ge 0$. Since x is arbitrary, replacing x by $x - tv$, we see that the conclusion is true.

$\square$