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CONTROLLERS FOR NONLINEAR SYSTEMS USING NORMAL FORMS

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We describe a method for the design of feedback stabilization control laws for nonlinear systems using the theory of normal forms and the results from optimal control theory. We show that the resulting controllers can provide a larger region of stability than local linear controllers designed to perform the same task.

Keywords: Normal forms; feedback control; optimal control; nonlinear systems.

1. Introduction

We describe a method of designing local nonlinear controllers for the purposes of stabilizing unstable fixed points (UFP's) of nonlinear systems by using the methods of normal form theory [Wiggins, 1990; Guckenheimer & Holmes, 1983] to determine a coordinate transformation which makes the system "simpler". In the transformed system we design a local linear controller to stabilize the desired UFP. We then "invert" the coordinate transformation to obtain a system of equations which can be solved for the desired applied nonlinear control.

The use of normal forms for control purposes has been proposed by Krener [1988, 1990]. Our work differs slightly in that we use a simplified normal form transformation to enable us to use standard linear control techniques to design the control law. We also only consider hyperbolic fixed points. These simplifications make the calculations much more straightforward, yet retain advantages of Krener's approach, such as a greatly enlarged stability region; our approach also differs in some details from Krener's.

Since the seminal work of Ott, Grebogi and Yorke [1990] presented a method of using small perturbations to stabilize UFP's of chaotic systems using the control law now named OGY, much work has been done on the topic, e.g. [Grebogi & Lai, 1997; Feudal & Grebogi, 1997; Chen & Dong, 1998; Judd *et al.*, 1997; Hill, 2000, 2001]. There are now many variants of and extensions to the original OGY controller but almost all rely on the same principle: *small perturbations of accessible parameters applied locally about the UFP can bring about stability.* In addition many of the resulting control laws can be classified as local linear feedback controllers. Examples of local linear feedback controllers other than OGY include the work of Vincent [1997] and Vincent and Mees [2000] where

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results from optimal controls' linear quadratic regulator (LQR) problem [Jacobs, 1993] are used to design an appropriate feedback control.

Due to the nature of nonlinear systems local linear feedback controllers are often only successful in a small region surrounding the UFP. The work of Paskota [1995] investigates how the size of such a region can be determined. See also Vincent [1997] for alternative calculations. Bounded controls are often required, either because it is necessary in the physical system to keep the perturbations small, or because the nonlinearity means that the effects of large controls are undesirable. Even for linear systems, bounded feedback control is only guaranteed to work in a (typically) small region. For example, consider a simple dynamical system of the form $x_{n+1} = f(x_n) + u_n$ with a fixed point at $\bar{\mathbf{x}}$ when $u = 0$, and suppose the control u is bounded by $\delta > 0$, i.e. $|u| < \delta$. For a linear feedback control $u = -k^T(\mathbf{x} - \bar{\mathbf{x}})$ about the fixed point, $\bar{x}$ we see the region of stability is determined by

$$\|\mathbf{x} - \bar{\mathbf{x}}\| \leq \frac{\delta}{\|k^T\|}\,. \tag{1}$$

As a result, the problem of targeting the region of stability has become an active topic hand-in-hand with the design of control laws [Shinbrot *et al.*, 1990; Kostelich *et al.*, 1993].

In this work we do not concern ourselves with the problem of targeting. Our present interest is more in the design of controllers of nonlinear systems more robust than local linear controls. Since local linear controllers can have very small regions of stability, sometimes too small to be of much use, we regard controllers with a larger region of stability to be more robust. We will give numerical evidence which suggests that the stability region where the control will be successful is larger for our nonlinear controller than local linear controllers.

2. Methods

Consider the nonlinear control system

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n,\, u_n)\,. \tag{2}$$

The system has a fixed point $\bar{\mathbf{x}} = \mathbf{f}(\bar{\mathbf{x}}, 0)$. We can reduce this system to a normal form by making successive changes of state variables and expanding the system to first order in the control. This is one place where our approach differs from Krener [1990]: we disregard all high order control terms *including* cross terms with the state. The normal forms of Krener can contain $\mathbf{x}u$ terms whereas ours do not. The reason for doing this is our wish to use linear feedback control techniques in the normal form. If we retained the cross terms a feedback control law would introduce nonlinear terms into the controlled normal form, which would make the control more difficult.

We have found that for significant improvement in (control) neighborhood size we only need to consider normal form coordinate transformations to second order. This observation is consistent with Krener's remarks [Krener, 1990]. We will, therefore, only describe the normal form method to second order although the extension to higher order transformations is, in principle, straightforward.

We make a first order change of coordinates by translating the fixed point to the origin and transforming the state Jacobian to Jordan canonical form $\mathbf{J}$, i.e.

$$\mathbf{v}_n = \mathbf{T}^{-1}(\mathbf{x}_n - \bar{\mathbf{x}}) \tag{3}$$

where the matrix $\mathbf{T}$ is such that $\mathbf{J} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ and $\mathbf{A} = D_{\mathbf{x}}\mathbf{f}(\bar{\mathbf{x}}, 0)$. The inverse of (3) is

$$\mathbf{x}_n = \mathbf{T}\mathbf{v}_n + \bar{\mathbf{x}}$$

so that the system now takes the form

$$\begin{aligned}\mathbf{v}_{n+1} &= \mathbf{T}^{-1}\mathbf{f}(\mathbf{T}\mathbf{v}_n + \bar{\mathbf{x}},\, u_n) - \mathbf{T}^{-1}\bar{\mathbf{x}} \\ &= \tilde{\mathbf{f}}(\mathbf{v}_n,\, u_n)\,.\end{aligned} \tag{4}$$

If we now Taylor expand (4) about $(\mathbf{0}, 0)$ to first order we will obtain the first order normal form.

In the theory of normal forms, successive nonlinear changes of coordinates of higher order are performed with the purpose of reducing the number of nonlinear terms in the system [Wiggins, 1990]. For example, consider a second order change of coordinates

$$\mathbf{w}_n = \mathbf{v}_n + \mathbf{h}_2(\mathbf{v}_n) \tag{5}$$

with $\mathbf{h}_2$ consisting of homogeneous second order polynomials whose coefficients are to be determined. To the desired second order accuracy, the inverse transformation of (5) is

$$\mathbf{v}_n = \mathbf{w}_n - \mathbf{h}_2(\mathbf{w}_n)\,.$$

Substituting the above into (4) and discarding terms higher than second order in the state and higher than first order in the control or in cross terms gives

$$\mathbf{w}_{n+1} = \mathbf{J}\mathbf{w}_n + \mathbf{C}u_n + \mathbf{h}_2(\mathbf{J}\mathbf{v}_n) - \mathbf{J}\mathbf{h}_2(\mathbf{v}_n) + \mathbf{f}_2(\mathbf{v}_n)$$

where $\mathbf{C} = \mathbf{T}^{-1}\mathbf{B}$ and $\mathbf{B} = D_u\mathbf{f}(\bar{\mathbf{x}}, 0)$. ($\mathbf{f}_2$ consists of the second order contributions of (4).) The coefficients of the nonlinear change of variable $\mathbf{h}_2$ are determined by solving

$$\mathbf{h}_2(\mathbf{J}\mathbf{v}_n) - \mathbf{J}\mathbf{h}_2(\mathbf{v}_n) + \mathbf{f}_2(\mathbf{v}_n) = 0\,. \tag{6}$$

If the matrix $\mathbf{J}$ has no eigenvalues on the unit circle, i.e. the UFP is hyperbolic, then (6) can be solved exactly. If there are eigenvalues on the unit circle then not all second order terms can be removed and more general results need to be considered [Kang, 1998a, 1998b].

The second order normal form transformation is thus

$$\begin{aligned}\mathbf{w}_n &= \mathbf{h}(\mathbf{x}_n)\\ &= \mathbf{T}^{-1}(\mathbf{x}_n - \bar{\mathbf{x}}) + \mathbf{h}_2(\mathbf{T}^{-1}(\mathbf{x}_n - \bar{\mathbf{x}}))\,.\end{aligned} \tag{7}$$

Additional higher order changes of variables can be made in an attempt to knock out cubic, quartic and quintic terms and so on, to produce a normal form to the desired accuracy. These higher order transformations have no effect on the tranformations of lower order, but we do not consider them both because they complicate the calculations and because they are less likely to be useful in practice: their use requires an extremely accurate model, something that is unlikely to be available in many practical applications.

We aim to design a stabilizing controller for the UFP of the system in normal form which can then be transformed back to the original system. Consider the system $\mathbf{f}$ and its normal form $\hat{\mathbf{f}}$ to order r after successive coordinate transformations defined by $\mathbf{h}$. To the desired accuracy we have

$$\mathbf{h}(\mathbf{f}(\mathbf{x}, u)) = \hat{\mathbf{f}}(\mathbf{h}(\mathbf{x}), \hat{u}) \tag{8}$$

where $\hat{u}$ is the (feedback) control applied to the system in normal form. For cases where $(D\mathbf{f})$ has no eigenvalues on the unit circle we find

$$\hat{\mathbf{f}}(\mathbf{w}, \hat{u}) = \mathbf{J}\mathbf{w} + \mathbf{C}\hat{u} + O(r)\,.$$

We design a feedback control $\hat{u} = -\mathbf{K}\mathbf{w}$ such that $(\mathbf{J} - \mathbf{C}\mathbf{K})$ is stable. In order to satisfy (8) we find the control u such that

$$\mathbf{h}(\mathbf{f}(\mathbf{x}, u)) = (\mathbf{J} - \mathbf{C}\mathbf{K})\mathbf{h}(\mathbf{x}).$$

That is, letting $\mathbf{N}(u) = \mathbf{h}(\mathbf{f}(\mathbf{x}, u)) - (\mathbf{J} - \mathbf{C}\mathbf{K})\mathbf{h}(\mathbf{x})$ we see that the required (nonlinear) control is the solution to

$$\mathbf{N}(u) = \mathbf{0}\,. \tag{9}$$

We then use bounded control with this u.

3. Example

We present an example to exhibit the general usefulness of the normal form methods. We choose the Ikeda map, because the stable manifold of the hyperbolic UFP has significant curvature. The motivation for using normal forms will become apparent when we see how the stable manifold is transformed.

It is useful at this stage to be able to approximate the stable and unstable manifolds of a hyperbolic UFP of a map $\mathbf{f}(\mathbf{x}, u)$ in two dimensions. This can be done quite easily [Hill, 2001]. The calculation is not necessary for the normal form method but is useful for demonstrative purposes later. A parametric polynomial approximation to the stable manifold is given by

$$\phi(t) = \bar{\mathbf{x}} + \mathbf{e}_s t + \sum_{i=2}^{N} \phi_i t^i \tag{10}$$

where $\mathbf{e}_s$ is the eigenvector corresponding to the stable eigenvalue λ_s. We know that points on the stable manifold will stay on the stable manifold under the dynamics $\mathbf{f}$ and, furthermore, will be closer to the UFP by a factor λ_s. Thus, points on the stable manifold satisfy

$$\mathbf{f}(\phi(t), u = 0) = \phi(\lambda_s t)\,. \tag{11}$$

We can approximate the left-hand side of (11) using a Taylor series. This enables us to determine the unknown ϕ_i in (10) by equating coefficients of powers of t. (We note that a parametric approximation to the unstable manifold is determined in the same way by replacing $\mathbf{e}_s$ and λ_s with the eigenvector $\mathbf{e}_u$ and its corresponding unstable eigenvalue λ_u.)

The Ikeda map describes the dynamics of a laser pulse in an optical cavity. The state $\mathbf{x}$ of the system can be represented by a complex number $z_n = x_n + iy_n$ related to the amplitude and phase of the nth laser pulse in the cavity. The dynamics

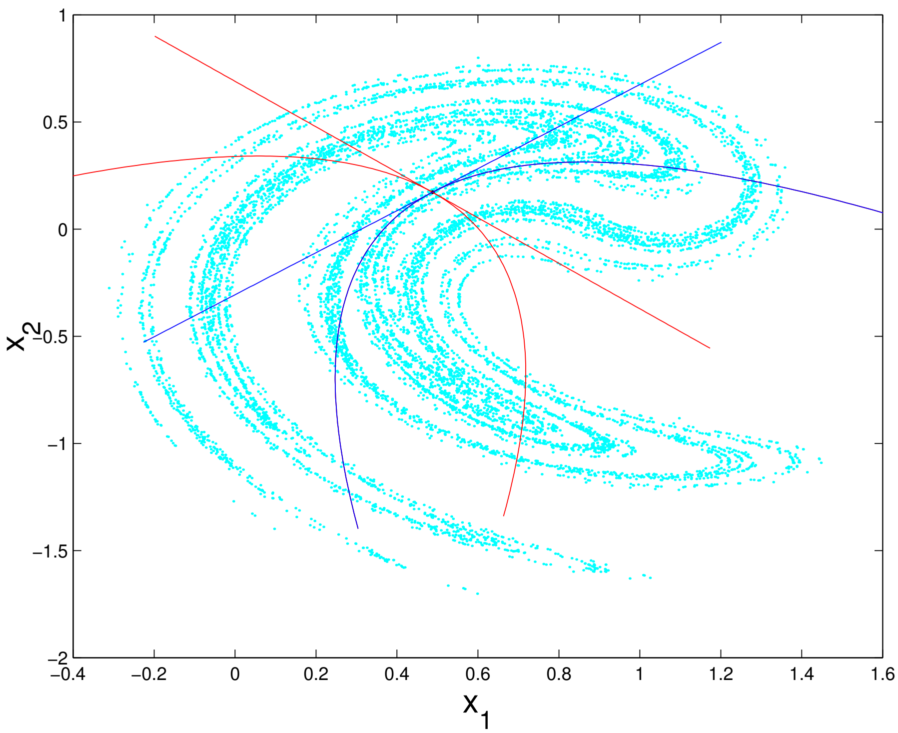

Fig. 1. A plot of a typical trajectory on the attractor. The first and second order approximations to the stable (red) and unstable (blue) manifolds are shown at the UFP.

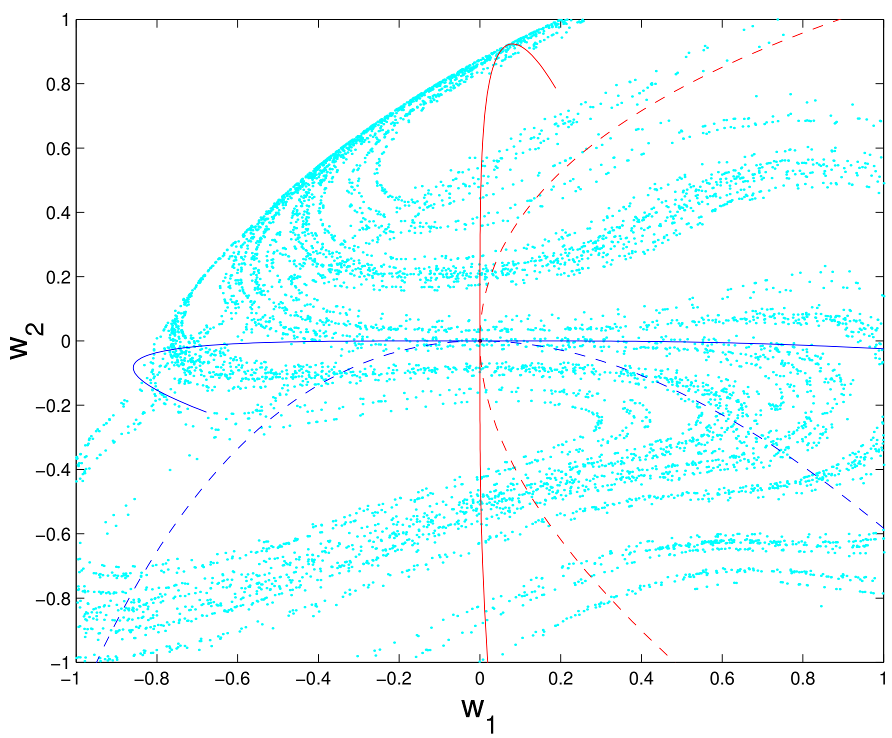

Fig. 2. A plot of how the elements of Fig. 1 transform under a second order normal form transformation. We see the UFP is mapped to the origin and the second order approximation (solid) to the stable (red) and unstable (blue) manifolds are straightened out along the new coordinate axes. The first order approximation (dashed) to the stable (red) and unstable (blue) manifolds are also mapped to the coordinate axes but deviate from them much sooner than the higher order approximation.

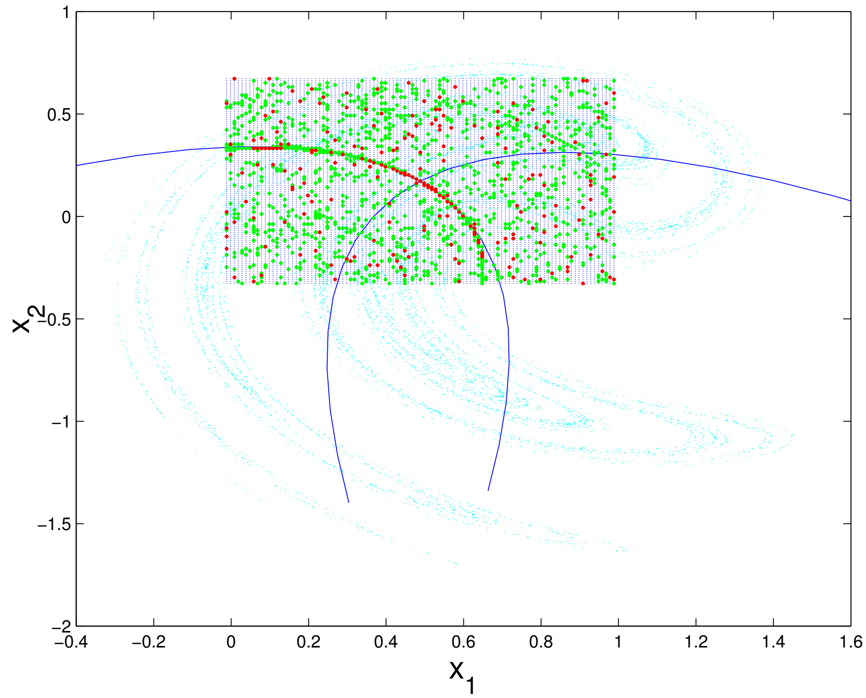

Fig. 3. A plot showing a coarse numerical approximation to the control stability region as described in the text. The blue slab shows 10, 201 initial conditions. Points which are successfully controlled using local LQR control after 50 iterations are marked in red whereas those points successfully controlled using the normal form control method are indicated in green. Those points which are not successfully controlled by either method remain marked in blue. We note that there are far more points marked green than red suggesting the normal form control method is superior.

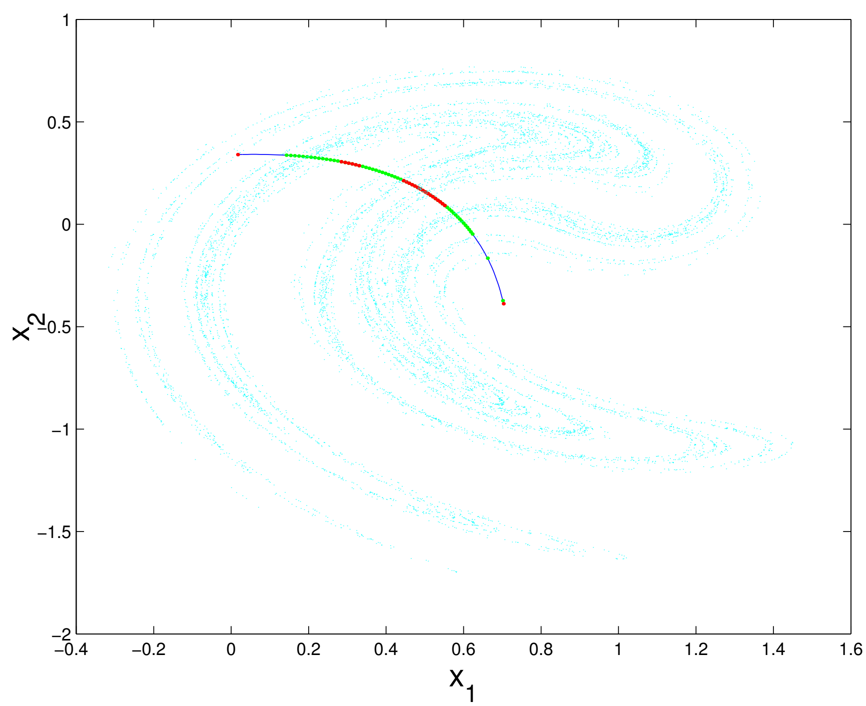

Fig. 4. A plot similar to Fig. 3 where 101 points along the second order approximation to the stable manifold are tested. Once again the red points are those points successfully controlled to the fixed point using local LQR control and the green points are those successfully controlled using the normal form method. Those points not successfully controlled remain marked in blue. We see clearly the advantage of using the normal form method over the usual control method with the stability region being much larger.

written in complex form is given by

$$z_{n+1} = (\gamma + u_n) + \alpha z_n \exp(i\theta_n) \qquad (12)$$

with $\theta_n = 0.4 - \kappa/(1+|z_n|^2)$. The parameter α represents the damping related to reflection properties of the mirror, κ measures the cavity detuning due to a nonlinear dielectric medium. The parameter γ is related to the laser input amplitude and since it is like a forcing parameter it is natural to choose it as an accessible control parameter [Feudal & Grebogi, 1997]. The nominal parameter values we use in this example are ($\gamma = 0.92$, $\alpha = 0.9$, $\kappa = 5.4$). We allow the control perturbations u_n to have a maximum deviation of 1% of the value of γ. Using the parameters we find the Ikeda map has a UFP at (0.4876, 0.1723) with $\lambda_s = -0.4012$ and $\lambda_u = -2.0191$.

We can linearize the system about the UFP and design a local linear feedback control law $u = -\mathbf{K}\mathbf{x}$. The gain $\mathbf{K}$ can be found using, for example, the pole-placement OGY method [Ott *et al.*, 1990], or LQR results [Jacobs, 1993]. We will use LQR techniques and the Matlab routine `dlqr` can be used to return the required gain. This approach works very well, however, the control must be applied when the system state is very close to the UFP to be successful. (We will see what close means shortly.)

A plot of a typical trajectory is shown in Fig. 1 together with the location of the UFP, as well as the stable and unstable eigenspaces, and a second order approximation to the stable and unstable manifolds, i.e. $N = 2$ in (10). The second order approximation to the stable manifold deviates quite sharply from the linear approximation and this deviation occurs quite close to the UFP, so controllers designed using the linear approximation are unlikely to perform well when the stable manifold begins to deviate from the linear eigenspaces.

Making a coordinate transformation to normal form makes the system "more" linear about the UFP. In other words, we can think of the normal form procedure straightening out the stable and unstable manifolds. If this is possible then it is reasonable to expect that a linear controller designed in the normal form coordinate system will be more successful than an ordinary local (LQR) controller as the system will appear linear at greater distances from the UFP.

In Fig. 2 we show how the elements of Fig. 1 map under a second order normal form transformation calculated following the description given in the Methods section. We see that the UFP is transferred to the origin and the second order approximation to the stable and unstable manifolds are straightened out to become close to the orthogonal coordinate axes. We also see that the eigenspaces are transformed to the new coordinate axes but deviate from them much sooner. Thus a control law designed in the transformed system and then "inverted" by solving $N(u) = 0$ [Eq. (9)] to find the corresponding control perturbation to the original system should be much more successful than before.

We design a LQR feedback control to stabilize the second order normal form system to the origin. We then solve (9) and discover that we do indeed find more points further from the UFP can be controlled than earlier.

To see how much larger is the stability region for the normal form control over the usual local LQR control, we choose 10,201 initial points evenly spread over the slab centred on the UFP indicated by the blue points in Fig. 3. For each initial point we apply local LQR control and normal form control for 50 iterations. If the final point is within a squared distance of 10^{-6} of the UFP we mark it as being controlled to the fixed point. We mark those points successfully controlled by local LQR in red and those by normal form control in green on Fig. 3. Those points which are not close after 50 iterations remain blue.

Figure 3 is rather a busy plot but it is apparent that there are more green points than red. We also note that most points which are successfully controlled lie along, or are close to, the approximation of the stable manifold. We can better see the advantages of using the normal form controller by concentrating on points on the stable manifold approximation. Figure 4 is a plot similar to Fig. 3 but this time we have only chosen 101 initial points equally spaced along the second order approximation to the stable manifold. Once again those points successfully controlled to the UFP after 50 iterations using local LQR control are shown in red while those successfully controlled using the normal form controller are shown in green. The improvement is stark and clearly shows the normal form controller to be superior.

An observation arising from Figs. 3 and 4 with respect to targeting is worth making: rather than target a small ball about the UFP, as is typical, it would seem more sensible to target a tube about the stable manifold. This region would be far larger and would therefore improve an overall control implementation of a given system.

4. Conclusions

We have introduced a technique of designing a controller for nonlinear systems using the method of normal forms. We have given numerical evidence supporting our assertion that the new controller provides a much larger region of stability than local linear controls. This fact will have useful implications for the problem of targeting. We have demonstrated the method using an artificial system but there are no restrictions in principle in applying the method to reconstructed models from time series data.

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