Finite Approximation of Sinai-Bowen-Ruelle Measures for Anosov Systems in Two Dimensions

Gary Froyland
Department of Mathematics
The University of Western Australia
Nedlands WA 6907, AUSTRALIA

Abstract

We describe a computational method of approximating the "physical" or Sinai-Bowen-Ruelle measure of an Anosov system in two dimensions. The approximation may either be viewed as a fixed point of an approximate Perron-Frobenius operator or as an invariant measure of a randomly perturbed system.

Keywords: invariant measure, Perron-Frobenius operator, small random perturbation.

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1 Introduction

The existence and computation of important invariant measures of deterministic dynamical systems are still major concerns in ergodic theory. In this note, we do not address the problem of existence, but provide a small step in the computation of important measures when they are known to exist. In one dimension, absolutely continuous invariant measures are considered to be important from a computational point of view because it is absolutely continuous measures that show up on computer simulations for most starting points. If μ is an ergodic absolutely continuous invariant measure for any Borel measurable $f : [0,1] \circlearrowleft$, then by the Birkhoff Theorem [13, 19] we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^i x) = \int_{[0,1]} g \, d\mu, \tag{1}$$

for μ almost all $x \in [0,1]$ and any given continuous $g : [0,1] \to \mathbb{R}$. Since $\mu \ll \text{Leb}$, we may as well say that (1) holds for Lebesgue almost all $x \in [0,1]$; μ is clearly the unique measure with this property. In this sense, it seems natural to call absolutely continuous measures "physical" measures, and ignoring round-off errors, it is not so surprising that such measures commonly show up in computer simulations. There is an analogue of absolutely continuous measures for well-behaved higher dimensional systems, such as Anosov and Axiom A maps. This measure is known as a Sinai-Bowen-Ruelle (SBR) measure [2, 3, 15], denoted μ_{SBR} and has the property that it is exhibited by Lebesgue almost all initial points in the Anosov case and Lebesgue almost all points in a fundamental neighbourhood for Axiom A maps. More formally, we have the following theorem

Theorem 1.1 ([2, 3, 15]): *For $f : M \circlearrowleft$ a C^2 Axiom A diffeomorphism with fundamental neighbourhood $U \subset M$ satisfying $\overline{fU} \subset U$, and invariant attracting set $\Lambda = \bigcap_{i=0}^{\infty} f^i U$, we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} g(f^i x) = \int_M g \, d\mu_{\text{SBR}} \tag{2}$$

for Lebesgue *almost all $x \in U$ and any given continuous $g : M \to \mathbb{R}$. In the case of Anosov diffeomorphisms, $U = \Lambda = M$.*

Throughout the paper, f satisfies the conditions of this theorem. SBR measures have clear physical significance as they are exhibited by a large (in the Lebesgue measure sense) set of points. Approximations of the SBR measure are generally defined by experimentalists to be the LHS of (2) with $g(x) = \delta_x$, evaluated up to some large n for a randomly chosen x. Many statistical indicators such as Lyapunov exponents are estimated using this time-average. A discussion of the shortcomings of time-averaging and an application of our invariant measure approximation to the evaluation of Lyapunov exponents may be found in [6].

In 1976, Tien-Yien Li [12] resolved a conjecture of Ulam [17] by showing that a unique absolutely continuous invariant measure of a one-dimensional system could be estimated using a finite approximation of the Perron-Frobenius operator. The unit interval was partitioned into a finite number of subintervals $\{I_i\}_{i=0}^{m-1}$ and the length of overlap of inverse images of the subintervals produced the matrix approximation

$$P_{ij} = \frac{\ell\left(f^{-1}(I_j) \cap I_i\right)}{\ell(I_i)} \tag{3}$$

to the infinite-dimensional Perron-Frobenius operator; ℓ is one-dimensional Lebesgue measure. The invariant density of the Markov chain governed by P defined a piecewise constant approximation of the absolutely continuous invariant measure. As the maximum length of the subintervals went to zero by refining the partition, strong limit points of the invariant densities gave the unique absolutely continuous invariant measure. Our goal was to extend this construction to dimensions ≥ 1 to provide us with a finite approximation of the SBR measure of a given system, when it exists. The natural thing to do is to partition the space M into r sets $\{\Omega_i\}_{i=0}^{r-1}$ with $\bigcup_{i=0}^{m-1} \Omega_i = M$ and $\operatorname{Int}\Omega_i \cap \operatorname{Int}\Omega_j = \emptyset$ for $i \neq j$ and use the obvious extension of (3), namely

$$P_{ij} = \frac{m\left(f^{-1}(\Omega_j) \cap \Omega_i\right)}{m(\Omega_i)},$$

where m is normalised Riemannian volume on M. By refining our partition so that the maximum diameter of the partition sets goes to zero, we extract an invariant measure as a weak limit point of the invariant densities of the sequence of Markov chains governed by P. Our result is that if at each stage of refinement, the partition is Markov, then the limiting measure is the SBR measure. More formally, the main result is

Theorem 1.2 (Main Result): *Let $f : M \circlearrowleft$ be a C^2 Anosov diffeomorphism (or expanding C^2 map) of a smooth compact 2-dimensional (d-dimensional) Riemannian manifold M, and denote by m normalised Riemannian volume. Let $\mathfrak{P}^n = \{\Omega_1^n, \ldots, \Omega_{r(n)}^n\}$, $n = 0, 1, \ldots$ be a sequence of Markov partitions of M with $\max_{1 \leq i \leq r(n)} \operatorname{diam}\Omega_i^n < \epsilon(n)$ for all $n \geq 0$ and $\epsilon(n) \to 0$ as $n \to \infty$. Define*

$$P_{ij}^n = \frac{m\left(f^{-1}(\Omega_j^n) \cap \Omega_i^n\right)}{m(\Omega_i^n)}. \tag{4}$$

Let p^n be the normalised left eigenvector of P^n of eigenvalue 1, that is, $p^n P^n = p^n$, and define

$$\mu^n(E) = \sum_{i=0}^{r(n)} \frac{m(E \cap \Omega_i^n)}{m(\Omega_i^n)} \cdot p_i^n. \tag{5}$$

The sequence $\{\mu^n\}$ has a unique weak limit point, namely the SBR measure of f.

An outline of our approach follows. To show that the limiting measure is f-invariant, we first cast the deterministic system (f, M) as a randomly perturbed system governed by the finite state Markov chain P^n. We show that as the diameters of the partition sets decrease, so do the random perturbations, and it is an easy matter to show f-invariance of the limiting measure. We then introduce equilibrium states (special invariant measures) of f with respect to weight functions ϕ. It is known [2, 3, 15] that for a special weight function, namely $-\log$(local expansion in unstable directions), the equilibrium state of f is the SBR measure. We use the relative areas of intersection of the partition sets with their inverse images to provide us with an approximation of the special weight function. It then turns out that the appropriate matrix equation to solve to obtain an approximate equilibrium state is none other than (4).

2 Invariance of limit measures

We show in this section that weak limit points of our construction are f-invariant. The rest of the paper will be devoted to showing that this invariant measure is the SBR measure and the uniqueness follows. The stochastic matrix P^n may be thought of as a transition matrix of a finite state Markov chain. From this transition matrix, we may define a transition *function* $P_n : M \times \mathfrak{B}(M) \to \mathbb{R}$ by

$$P_n(x,\Gamma) = \sum_{j=1}^{r(n)} \frac{m(\Gamma \cap \Omega_j^n)}{m(\Omega_j^n)} \cdot \frac{m(f^{-1}\Omega_j^n \cap \Omega_x^n)}{m(\Omega_x^n)}, \tag{6}$$

where Ω_x^n is the unique partition set containing $x \in M$. The transition function P_n has μ^n as its invariant density. P_{ij}^n may be thought of as the probability that a point in Ω_i^n moves into Ω_j^n after one iteration of f. Inspection of (4) shows that this is not an unreasonable interpretation. The construction of P_n allows us to discuss a concrete random perturbation of the deterministic map f, with $P_n(x,\Gamma)$ to be thought of as the probability that the random image of x lies in the set $\Gamma \subset M$. The following definition is taken from [11].

Definition 2.1: The Markov chains governed by a family of transition functions P_n are called *small random perturbations* of $f : M \circlearrowleft$ if for every continuous function $g : M \to \mathbb{R}$,

$$\lim_{n\to\infty} \sup_{x\in M} \left| \int_M g(y) P_n(x,dy) - g(f(x)) \right| = 0. \tag{7}$$

The LHS of the difference in (7) represents the effect of random "noise" which is added after applying the function f to the point $x \in M$. The integral averages the value of g over the allowed noise neighbourhood. The requirement for the Markov process to be a small random perturbation of f is roughly that the noise neighbourhood applied after a transition from x coalesces about the deterministic image $f(x)$.

Lemma 2.2: *Our family of transition functions $P_n(\cdot,\cdot)$ is a small random perturbation of f.*

PROOF: Define

$$\begin{aligned}
\omega_g(\epsilon) &= \sup_{x,y\in M} \{|g(x)-g(y)| : \|x-y\| < \epsilon\}, \\
\omega'_f(\epsilon) &= \sup_{x,y\in M} \{\|f(x)-f(y)\| : \|x-y\| < \epsilon\} \quad \text{and} \\
\omega''_{f^{-1}}(\epsilon) &= \sup_{x,y\in M} \{\|f^{-1}(x)-f^{-1}(y)\| : \|x-y\| < \epsilon\}.
\end{aligned}$$

$$\begin{aligned}
\left| \int_M P_n(x,dy) g(y) - g(f(x)) \right| &= \left| \int_M \sum_{j=1}^{r(n)} \frac{\chi_{\Omega_j^n}(y)}{m(\Omega_j^n)} \frac{m(f^{-1}\Omega_j^n \cap \Omega_x^n)}{m(\Omega_x^n)} g(y)\, dm(y) - g(f(x)) \right| \\
&= \left| \sum_{j=1}^{r(n)} \int_{\Omega_j^n} \frac{m(f^{-1}\Omega_j^n \cap \Omega_x^n)}{m(\Omega_j^n) m(\Omega_x)} g(y)\, dm(y) - g(f(x)) \right| \\
&\leq \sum_{j=1}^{r(n)} \frac{m(f^{-1}\Omega_j^n \cap \Omega_x^n)}{m(\Omega_x^n)} \left(\frac{1}{m(\Omega_j^n)} \int_{\Omega_j^n} |g(y) - g(f(x))|\, dm(y) \right) \\
&\leq \sum_{j=1}^{r(n)} \frac{m(f^{-1}\Omega_j^n \cap \Omega_x^n)}{m(\Omega_x^n)} \cdot \omega_g\left(\omega'_f \left(\epsilon(n) + \omega''_{f^{-1}}(\epsilon(n)) \right) \right) \\
&= \omega_g\left(\omega'_f \left(\epsilon(n) + \omega''_{f^{-1}}(\epsilon(n)) \right) \right) \to 0 \quad \text{as } n \to \infty
\end{aligned}$$

If $m(f^{-1}\Omega_j^n \cap \Omega_x^n) > 0$, then Ω_x^n and $f^{-1}\Omega_j^n$ intersect and the greatest distance a point in such an Ω_j^n can be from $f(x)$ is less than $\omega'_f\left(\text{diameter of } \Omega_x^n \text{ plus the diameter of } f^{-1}\Omega_j^n\right)$ which is less than $\omega'_f\left(\epsilon(n) + \omega'_{f^{-1}}(\epsilon)\right)$. □

We reproduce here the simple proof of [9] that weak limit points of invariant densities of small random perturbations are invariant measures of the unperturbed map.

Proposition 2.3: *Let $\{\mu^n\}$ be a sequence of invariant densities obtained from the sequence of transition functions $\{P_n\}$, and suppose that $\mu^n \to \tilde{\mu}$ weakly. If the family $\{P_n\}$ is a small random perturbation of f, then $\tilde{\mu}$ is f-invariant.*

Proof: Let $g \in C(M)$.

$$\begin{aligned}\left|\int_M g(f(x))\, d\tilde{\mu}(x) - \int_M g(x)\, d\tilde{\mu}(x)\right| &\leq \left|\int_M g(f(x))\, d\tilde{\mu}(x)) - \int_M g(f(x))\, d\mu^n(x)\right| \\ &+ \left|\int_M \left(g(f(x)) - \int_M P_n(x,dy)g(y)\right)\, d\mu_n(x)\right| \\ &+ \left|\int_M g(x)\, d\mu_n(x) - \int_M g(x)\, d\tilde{\mu}(x)\right|.\end{aligned}$$

The first and third terms go to zero by weak convergence and the middle term goes to zero as $\{P_n\}$ is a small random perturbation. □

So now we know that the measure that we extract from repeated refinements of our partition is f-invariant. The rest of the paper is devoted to demonstrating it is the SBR measure.

3 Equilibrium states

We consider the *pressure* of f with respect to a weight function $\phi : M \to \mathbb{R}$ defined by

$$\rho_f(\phi) = \sup_{\mu \in \mathcal{M}_f(M)} \left(h_\mu(f) + \int_M \phi\, d\mu\right) \tag{8}$$

where $h_\mu(f)$ is the measure-theoretic entropy of f, and $\mathcal{M}_f(M)$ is the space of all f-invariant Borel probability measures. For the weight function $\phi^{(u)}(x) = -\log\left|\det Df(x)|_{E_x^u}\right|$, where E_x^u is the unstable subspace at x, the measure which maximises the quantity $h_\mu(f) + \int_M \phi^{(u)}\, d\mu$ is the SBR measure [2, 3]. We will use symbolic dynamics to compute an approximation to this maximal measure, known as the *equilibrium state* for the weight function $\phi^{(u)}$. For brevity of notation, we drop the n dependence of the variables and consider a fixed Markov partition $\mathfrak{P} = \{\Omega_1, \dots, \Omega_r\}$. Define

$$A_{ij} = \begin{cases} 0, & \text{if } \operatorname{Int}\Omega_i \cap f^{-1}\operatorname{Int}\Omega_j = \emptyset \\ 1, & \text{otherwise.}\end{cases}$$

A defines a subset $\Sigma_A = \{\underset{\sim}{x} \in \Sigma_n : A_{x_i x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\} \subset \{1, 2, \dots, r\}^{\mathbb{Z}} = \Sigma_n$ of allowable sequences. Given a bi-infinite sequence $(\dots, x_{-1}, x_0, x_1, \dots) \in \Sigma_A$, there is a unique point in $x \in M$ with the property that $f^i(x) \in \Omega_{x_i}$ for all $i \in \mathbb{Z}$. We denote the mapping from a sequence $\underset{\sim}{x} \in \Sigma_A$ to its corresponding point $x \in M$ by π. We have the following commutative diagram; $\pi : \Sigma_A \to M$ is continuous and surjective.

$$\begin{array}{ccc} \Sigma_A & \xrightarrow{\sigma} & \Sigma_A \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & M \end{array}$$

If f is transitive (resp. mixing) then σ is transitive (resp. mixing); see Proposition 3.19 [2]. If we remove the boundary points of the partition, along with all of their foward and inverse images, π is bijective on the remaining (residual) set; see Theorem 3.18 [2]. That is, π is bijective on $M \setminus \bigcup_{j\in\mathbb{Z}} f^j \left(\bigcup_{i=1}^n \partial\Omega_i\right)$.

For $\phi : M \to \mathbb{R}$ Hölder, there is a unique equilibrium state (Theorem 4.1 [2]) and the equilibrium state for ϕ is the projection of the equilibrium state for $\varphi = \phi \circ \pi : \Sigma_A \to \mathbb{R}$ under π; that is, $\mu_\phi = \pi^* \mu_{\underset{\sim}{\varphi}}$. Thus, we may find the equilibrium state for the smooth dynamical system (M, f) with Hölder weight function ϕ, by projecting down the equilibrium state from a suitably chosen subshift of finite type (the one derived from a Markov partition) and weight function (the one defined by $\varphi = \phi \circ \pi$). In fact what we are going to do is use the relative areas of intersection of the partition sets with their inverse images to provide us with an estimate of the weight function $\phi_n : M \to \mathbb{R}$ which will approximate $\phi^{(u)}$. We then lift this ϕ_n to a $\varphi_n : \Sigma_A \to \mathbb{R}$, calculate the corresponding equilibrium state μ_{φ_n}, and map this down to the equilibrium state μ_{ϕ_n} for ϕ_n. In the next section, we see how to approximate the weight function $\phi^{(u)}$ using only the Markov partition and its inverse images.

4 Approximation of the weight function

For starters, we deal with the much simpler case of expanding maps to give an idea of the direction we'll take for the Anosov case.

Lemma 4.1 (The Expanding Case):

$$\phi_n = \log\left(\frac{m(f^{-1}(\Omega^n_{x_1}) \cap \Omega^n_{x_0})}{m(\Omega^n_{x_1})}\right) \circ \pi^{-1} \to \phi^{(u)}$$

uniformly as $n \to \infty$.

Proof: In the expanding case, $\phi^{(u)}(x) = -\log|\det Df(x)|$ as all directions are expanding. We want to estimate the Jacobian of f over the set $f^{-1}(\Omega_j) \cap \Omega_i$. Let $m(E)$ denote the Riemannian volume of a Borel subset E of our Riemannian manifold M. Clearly,

$$\inf_{x \in f^{-1}(\Omega^n_j) \cap \Omega^n_i} |\det Df(x)| \leq \frac{m(\Omega^n_j)}{m(f^{-1}(\Omega^n_j) \cap \Omega^n_i)} \leq \sup_{x \in f^{-1}(\Omega^n_j) \cap \Omega^n_i} |\det Df(x)|, \qquad (9)$$

as either $f^{-1}(\Omega^n_j) \subset \Omega^n_i$ or $f^{-1}(\mathrm{Int}(\Omega^n_j)) \cap \mathrm{Int}(\Omega^n_i) = \emptyset$ for all $1 \leq i, j \leq n$, since $\mathfrak{P} = \{\Omega^n_1, \ldots, \Omega^n_n\}$ is a Markov partition. From (9) we have

$$\inf_{x \in f^{-1}(\Omega^n_j) \cap \Omega^n_i} \frac{1}{|\det Df(x)|} \leq \frac{m(f^{-1}(\Omega^n_j) \cap \Omega^n_i)}{m(\Omega^n_j)} \leq \sup_{x \in f^{-1}(\Omega^n_j) \cap \Omega^n_i} \frac{1}{|\det Df(x)|},$$

and since f is C^2, as the diameter of the partition sets goes to zero,

$$\frac{m(f^{-1}(\Omega^n_j) \cap \Omega^n_i)}{m(\Omega^n_j)} \to \frac{1}{|\det Df(x)|_{f^{-1}(\Omega^n_j) \cap \Omega^n_i}|}$$

uniformly. If we put

$$\varphi_n(\underset{\sim}{x}) = -\log\left(\frac{m(f^{-1}(\Omega^n_{x_1}) \cap \Omega^n_{x_0})}{m(\Omega^n_{x_0})}\right),$$

then $\phi_n = \varphi_n \circ \pi^{-1} : M \to \mathbb{R}$ is an approximation of $\phi^{(u)}$ with $\phi_n \to \phi^{(u)}$ uniformly as $n \to \infty$ (since f is C^2 and π^{-1} is uniformly continuous on its domain of definition). Note that $\phi_n(x)$ is a piecewise constant approximation of $|\det Df(x)|$; constant on each of the sets in $\mathfrak{P} \vee f^{-1}\mathfrak{P}$. □

Remark 4.2: Theorem 1.2 holds if

$$P^n_{ij} = \frac{\nu(f^{-1}\Omega^n_j \cap \Omega^n_i)}{\nu(\Omega^n_i)},$$

where ν is any probability measure equivalent to m ($\nu \ll m$ and $m \ll \nu$).

In the proofs section we find a similar estimate for two-dimensional Anosov systems.

5 Computing the approximate equilibrium state

For n large enough, ϕ_n is a good approximation of $\phi^{(u)}$. Recall that the equilibrium state for ϕ_n is equal to the projection of the equilibrium state for φ_n under π. So let's find the equilibrium state for φ_n; it's a locally constant function depending only on the two symbols to the right of centre, and we know how to do that.

Theorem 5.1 ([8]): *Let (Σ_A, σ) be mixing so that A is irreducible and aperiodic. Suppose that we have a weight function $\varphi : \Sigma_A \to \mathbb{R}$ which depends only the on the two symbols to the right of centre, that is, $\varphi(\underset{\sim}{x}) = \varphi(x_0, x_1)$. Define*

$$G_{ij} = \begin{cases} 0, & \text{if } A_{ij} = 0 \\ e^{\varphi(x)}, & \text{if } A_{ij} = 1 \text{ and } x_0 = i, x_1 = j. \end{cases} \tag{10}$$

Let u, v be the left and right eigenvectors respectively, corresponding to the maximal non-negative eigenvalue λ, so that $uG = \lambda u$ and $Gv = \lambda v$. Further, define

$$P_{ij} = \frac{G_{ij} v_j}{\lambda v_i}. \tag{11}$$

The Markov measure μ_φ generated by P is the equilibrium state for φ. (If $pP = p$, then $\mu_\varphi(\{\underset{\sim}{x} \in \Sigma_A : x_0 = a_0, \ldots, x_{m-1} = a_{m-1}\}) = \mu_\varphi([a_0, a_1, \ldots, a_{r-1}]) = p_{a_0} P_{a_0 a_1} \ldots P_{a_{r-2} a_{r-1}}$.)

Lemma 5.2: *The stochastic matrix that generates the Markov measure that is the equilibrium state for φ_n is given by*

$$P_{ij}^{(n)} = \frac{m(f^{-1}\Omega_j^n \cap \Omega_i^n)}{m(\Omega_i^n)}.$$

PROOF: By Theorem 5.1,

$$P_{ij}^{(n)} = \frac{G_{ij}^{(n)} v_j^{(n)}}{\lambda^{(n)} v_i^{(n)}}, \tag{12}$$

where

$$G_{ij}^{(n)} = e^{\varphi_n(i,j)} = \frac{m(f^{-1}(\Omega_j^n) \cap \Omega_i^n)}{m(\Omega_j^n)} \tag{13}$$

Now note that $G_{ij}^{(n)} = L^{(n)} Q^{(n)} (L^{(n)})^{-1}$, where $L_{ij}^{(n)} = \delta_{ij} m(\Omega_i^n)$ and $Q_{ij}^{(n)} = m(f^{-1}\Omega_j^n \cap \Omega_i^n)/m(\Omega_i^n)$. Since $Q^{(n)}$ is stochastic and irreducible, we know that the unique (up to scalar multiples) right eigenvector with nonnegative eigenvalue is the vector $(1, 1, \ldots, 1)$ with eigenvalue 1, and so $v^{(n)} = (m(\Omega_1^n), m(\Omega_2^n), \ldots, m(\Omega_{r(n)}^n))$, and $\lambda^{(n)} = 1$. Thus

$$\begin{aligned} P_{ij}^{(n)} &= \frac{m(f^{-1}\Omega_j^n \cap \Omega_i^n)}{m(\Omega_j^n)} \cdot \frac{m(\Omega_j^n)}{1 \cdot m(\Omega_i^n)} \\ &= \frac{m(f^{-1}(\Omega_j^n) \cap \Omega_i^n)}{m(\Omega_i^n)}, \end{aligned} \tag{14}$$

by (12) and (13). □

This means that the equilibrium state μ_{φ_n} is given by the Markov measure

$$\mu_{\varphi_n}([a_0, \ldots, a_{m-1}]) = p_{a_0}^{(n)} P_{a_0 a_1}^{(n)} \cdots P_{a_{m-2} a_{m-1}}^{(n)},$$

where $p^{(n)}$ is the invariant density of $P^{(n)}$. To obtain the equilibrium state μ_{ϕ_n} for ϕ_n, we merely project down via π; that is, $\mu_{\phi_n} = \pi^* \mu_{\varphi_n}$.

Remark 5.3: The same argument applies to Anosov maps. This is done in the proofs section.

Remark 5.4: In practice, for $E \subset \mathfrak{B}(M)$ we will most likely take

$$\hat{\mu}_{\phi_n}(E) := \sum_{i=1}^{n} \frac{m(E \cap \Omega_i)}{m(\Omega_i)} \cdot p_i^{(n)}. \tag{15}$$

$\hat{\mu}_{\phi_n}$ has a uniform density on each of the partition sets, and $\hat{\mu}_{\phi_n}(\Omega_i) = \mu_{\phi_n}(\Omega_i)$ for every $1 \leq i \leq n$. What we are doing is taking the weight given to each partition set by μ_{ϕ_n} and distributing it uniformly (according to the natural volume measure) over the entire partition set to obtain the measure $\hat{\mu}_{\phi_n}$. This extra error goes away as the size of the sets decreases since the distribution of mass within a partition set does not affect weak limits.

Equation (14) is a generalisation (to higher dimensions) of the approximation of the Perron-Frobenius operator commonly used to estimate invariant densities of one-dimensional systems with unique absolutely continuous invariant measures (see Li [12] for example). What we have shown (well, we're nearly there) is that this generalisation also provides us with an approximation of the absolutely continous invariant measure of an d-dimensional expanding map and the "physical" invariant measure of an Anosov map (later) in two dimensions when the partition is Markov. To show this, we used the relative volumes of inverse images of partition sets to estimate the Jacobian of the map in the expanding directions. The resulting piecewise constant approximation generated a stochastic matrix which in turn defined our estimate of the invariant measure. This stochastic matrix was nothing other than the higher dimensional generalisation of Li's approximation. So we can find an approximate equilibrium state of the Anosov diffeomorphism, simply by computing $P^{(n)}$ and its left eigenvector $p^{(n)}$. To finish, we see that we do indeed have an approximation, in the sense that if ϕ_n is close to $\phi^{(u)}$ in the uniform topology (C^0 close), then μ_{ϕ_n} is close to $\mu_{\phi^{(u)}}$ in the weak topology.

Lemma 5.5: *$\mu_{\phi_n} \to \mu_{\mathrm{SBR}}$ weakly as $\phi_n \to \phi^{(u)}$ uniformly.*

Proof: Let ϕ_n be a sequence of approximations of $\phi^{(u)}$ obtained from shift systems so that $\|\phi_n - \phi^{(u)}\|_\infty \to 0$. The following argument is from the proof of Lemma 4.9 [18]. Choose some convergent subsequence $\mu_{\phi_{n_i}} \to \tilde{\mu}$, with $\tilde{\mu}$ f-invariant. We know that $\rho_f(\phi)$ is Lipschitz with respect to the C^0 topology for ϕ, provided $\rho_f(\cdot)$ is finite; (see [18] Theorem 2.1(v) or [19] Theorem 9.7(iv)).

$$\begin{aligned}
\rho_f(\phi^{(u)}) &= \lim_{n\to\infty} \rho_f(\phi_n) \qquad \text{by the Lipschitz property} \\
&= \lim_{n\to\infty} \left(h_{\mu_{\phi_n}}(f) + \mu_n(\phi_n) \right) \\
&\leq \lim_{n\to\infty} \left(h_{\mu_{\phi_n}}(f) + \mu_n(\phi^{(u)}) + \|\phi_n - \phi^{(u)}\|_\infty \right) \\
&= \lim_{n\to\infty} h_{\mu_{\phi_n}}(f) + \tilde{\mu}(\phi^{(u)}) \qquad \text{by weak convergence} \\
&\leq h_{\tilde{\mu}}(f) + \tilde{\mu}(\phi^{(u)}),
\end{aligned}$$

where the final inequality follows from the upper semicontinuity of the entropy map $\mu \mapsto h_\mu(f)$ for expansive f; see [18] Lemma 4.5 or [19] Theorem 8.2. Now, by (8), we see that $\rho_{\phi^{(u)}} = h_{\tilde{\mu}}(f) + \tilde{\mu}(\phi^{(u)})$. In other words, $\tilde{\mu}$ is the unique equilibrium state for $\phi^{(u)}$, and so we must have $\tilde{\mu} = \mu_{\phi^{(u)}} = \mu_{\mathrm{SBR}}$.

□

So in our setup, as we refine the partition, the weight function ϕ_n converges to $\phi^{(u)}$, and we know from perturbation arguments that the approximations μ_{ϕ_n} converge to an invariant measure $\tilde{\mu}$. What Lemma 5.5 tells us is that $\tilde{\mu}$ is in fact the equilibrium state for $\phi^{(u)}$. The measure that we obtain as a weak limit of our approximations is the unique invariant measure that is exhibited by Lebesgue almost all starting points in phase space. All that is used to estimate this measure are the areas of intersections of Markov partitions with their inverse images.

Using Lemmata 4.1, 5.2, 5.5 and Remark 5.4, Theorem 1.2 is proven in the expanding case.

Remark 5.6: The preceding theory may be extended to two-dimensional Axiom A attractors by suitably defining a "Markovian" partition of M.

6 Discussion

In this note we have shown that Li's formula may be used to approximate the SBR measure in two-dimensional Anosov systems and expanding maps in n-dimensions, provided the partitions are Markov. The resulting estimate may be viewed as either the fixed point of an approximate (finite dimensional) Perron-Frobenius operator or as an invariant measure of a small random perturbation of the original map. Work on the former interpretation was done by Li for one-dimensional piecewise C^2 expanding maps. A Markov partition was not necessary; using bounded variation arguments, Li's work [12] showed any finite partition would do. Boyarsky and Lou [4] extended Li's result to Jablonski transformations (a special class of piecewise expanding transformations) in n-dimensions. Jablonski transformations satisfy a type of separability condition, and Boyarsky and Lou used Li's bounded variations arguments to obtain the result. For more general transformations in higher dimensions the problem becomes more difficult as the bounded variation arguments are not able to be used. Baladi and Young [1] have shown for some model expanding maps and types of random perturbations with continuous densities, that the Perron-Frobenius operators of the random systems approximate the Perron-Frobenius operator of the unperturbed system sufficiently well to give convergence of densities to the smooth invariant measure as the noise goes to zero.

Interpreting the approximation as an invariant measure of a randomly perturbed system, results have been obtained by Kifer [10] and Young [20]. Kifer has set out a number of technical hypotheses on the random perturbations that guarantee convergence of the invariant measures of the perturbed systems to the SBR measure of the unperturbed system. Young has looked at random compositions of perturbed maps and shown that as the perturbations go to zero the invariant measures of the random process approach the SBR measure of the original map. These two approaches are of little computational use since the problem of computing the invariant measure of these random processes is just as difficult as the original problem of computing the SBR measure of the deterministic system. This is also the case for the work of [1] where the invariant densities of a continuously perturbed Perron-Frobenius operator are impossible to compute except in the simplest of cases. For numerical purposes one really needs a finite dimensional approximation of the Perron-Frobenius operator or a discontinuous random perturbation.

Our method allows a simple calculation of the invariant measure of the perturbed system as a left eigenvector of a stochastic matrix. While this in principle provides a rigorous method of approximating the SBR measure by computer, the fact that Markov partitions need to be constructed is a significant restriction for the practical application of the technique. A matter for future work will be to consider the validity of the result if the Markov condition is relaxed. Numerical results using non-Markov partitions are encouraging [7].

7 Proofs

The Anosov case in two dimensions

We restrict ourselves to two dimensions as Markov partitions of Anosov diffeomorphisms in dimension three or above are difficult to deal with (see [13], pages 182–184). Each set in our partition has two sides which are segments of stable foliations and two sides that are segments of unstable foliations; see Appendix 2 [14]. For any point on one of the stable boundaries, we have an unstable foliation running through the point, and across to an "opposing" point on the other stable boundary. We have a similar situation for the unstable boundaries. We introduce some notation, leaving out the n dependence of the partitions for convenience.

Let ℓ_i^u be the "average" unstable length of the partition set Ω_i, calculated by averaging the unstable lengths integrating along one of the stable boundaries of Ω_i. More precisely, let $\gamma : [0,1] \to \Omega^s_{i,\text{left}}$ be a parametrisation of the "left" stable boundary of Ω_i, and let $\ell^u_{i,x} : \Omega^s_{i,\text{left}} \to \mathbb{R}^+$ denote the length of the unstable foliation passing through $x \in \Omega^s_{i,\text{left}}$. Then

$$\ell_i^u := \int_0^1 \ell^u_{i,\gamma(t)}\, dt.$$

Here, length means the length of the curve representing a segment of the unstable foliation from one stable boundary to the other. Similarly define ℓ_i^s.

Define $\ell_i^{u,\max}$ to be the maximal length of an unstable foliation traversing the partition set Ω_i (from one stable boundary to the other), and put $\ell_i^{u,\min}$ as the minimal such length. Similarly define $\ell_i^{s,\max}$ and $\ell_i^{s,\min}$.

Throughout this proof we will be finding upper and lower bounds of lengths and areas. Without loss of generality, we calculate everything in local coordinates, as the upper and lower bounds will still be such for the average values when everything is transferred back to the manifold.

At each point $x \in \Omega^i$, we may calculate the sine of the angle between E_x^u and E_x^s. Put $\sin\theta_i^{\max}$ as the supremum of these values taken over all of Ω_i, and similarly define $\sin\theta_i^{\min}$.

Finally define

$$\mathcal{A}_i^{\max} := \ell_i^{u,\max}\ell_i^{s,\max}\sin\theta_i^{\max},$$

$$\mathcal{A}_i^{\min} := \ell_i^{u,\min}\ell_i^{s,\min}\sin\theta_i^{\min},$$

and denote by $\mathcal{A}_i$, the area of Ω_i.

Clearly,

$$\ell_i^{u,\min} \leq \ell_i^u \leq \ell_i^{u,\max}$$

$$\ell_i^{s,\min} \leq \ell_i^s \leq \ell_i^{s,\max}$$

$$\mathcal{A}_i^{\min} \leq \mathcal{A}_i \leq \mathcal{A}_i^{\max}$$

The important thing is that as the diameter of the partition sets decreases to zero, we have convergence of the maximum and minimum values. That is,

$$\ell_i^{u,\min} \to \ell_i^u \leftarrow \ell_i^{u,\max}$$

$$\ell_i^{s,\min} \to \ell_i^s \leftarrow \ell_i^{s,\max}$$

$$\mathcal{A}_i^{\min} \to \mathcal{A}_i \leftarrow \mathcal{A}_i^{\max}$$

as the partition is refined. This is because the boundaries of our partition sets are C^2, and the angle function is Hölder. (For a C^2 diffeomorphism of a C^2 manifold with a closed hyperbolic invariant set Λ, the transition functions $x \mapsto E_x^u$ and $x \mapsto E_x^s$ are Hölder continuous; see [16], p.48.)

Lemma 7.1: *With the previous notation,*

$$\phi_n = -\log\left(\frac{m\left(f^{-1}(\Omega_{x_1}^n)\cap\Omega_{x_0}^n\right)}{m(\Omega_{x_0}^n)}\cdot\frac{\ell_{x_0}^{u,n}}{\ell_{x_1}^{u,n}}\right)\circ\pi^{-1} \to \phi^{(u)} \tag{16}$$

uniformly as $n \to \infty$.

PROOF: We want to put some bounds on the value of the Jacobian of f restricted to the unstable subspace. Define $\kappa : M \to \mathbb{R}^+$ by $\kappa(x) = |1/Df(x)|E_x^u|$ We denote by κ_i, an "average" value of the restriction of κ to Ω_i. This average is calculated by integrating $\kappa(x)$ over Ω_i, using suitably normalised Riemannian volume. We wish to estimate the value of $\kappa_{f^{-1}j\cap i}$; this is the average value of $\kappa(x)$ restricted to $f^{-1}\Omega_j \cap \Omega_i$. Upper bounds first. We have that

$$\kappa_{f^{-1}j\cap i} \leq \frac{\ell_{f^{-1}j}^{u,\max}}{\ell_j^{u,\min}}, \tag{17}$$

as the right hand side is greater than or equal to the average value of $\kappa(x)$ restricted to any of the unstable foliations passing through Ω_j. We claim that

$$\frac{\ell_{f^{-1}j\cap i}^{u,\max}}{\ell_j^{u,\min}} \leq \frac{\mathcal{A}_{f^{-1}j\cap i}^{\max}}{\mathcal{A}_i^{\min}}\cdot\frac{\ell_i^{u,\min}}{\ell_j^{u,\min}} \leq \frac{\mathcal{A}_{f^{-1}j\cap i}^{\max}}{\mathcal{A}_i^{\min}}\cdot\frac{\ell_i^{u,\max}}{\ell_j^{u,\min}}. \tag{18}$$

The second inequality is obvious; the first is true if

$$\begin{aligned} \ell^{u,\max}_{f^{-1}j\cap i} &\le \frac{\mathcal{A}^{\max}_{f^{-1}j\cap i}}{\mathcal{A}^{\min}_i}\cdot \ell^{u,\min}_i \\ \iff \quad \ell^{u,\max}_{f^{-1}j\cap i} &\le \frac{\ell^{u,\max}_{f^{-1}j\cap i}\ell^{s,\max}_{f^{-1}j\cap i}\sin\theta^{\max}_{f^{-1}\cap i}}{\ell^{u,\min}_i\ell^{s,\min}_i\sin\theta^{\min}_i}\cdot \ell^{u,\min}_i \\ \iff \ell^{s,\min}_i\sin\theta^{\min}_i &\le \ell^{s,\max}_{f^{-1}j\cap i}\sin\theta^{\max}_{f^{-1}j\cap i}, \end{aligned}$$

and the final inequality always holds. Thus we have proved our claim. For the lower bound, we have that

$$\frac{\ell^{u,\min}_{f^{-1}j}}{\ell^{u,\max}_j} \le \kappa_{f^{-1}j\cap i}, \tag{19}$$

and we wish to show that

$$\frac{\mathcal{A}^{\min}_{f^{-1}j\cap i}}{\mathcal{A}^{\max}_i}\cdot\frac{\ell^{u,\min}_i}{\ell^{u,\max}_j} \le \frac{\mathcal{A}^{\min}_{f^{-1}j\cap i}}{\mathcal{A}^{\max}_i}\cdot\frac{\ell^{u,\max}_i}{\ell^{u,\max}_j} \le \frac{\ell^{u,\min}_{f^{-1}j\cap i}}{\ell^{u,\max}_j}. \tag{20}$$

Again, the first inequality is obvious; the second is true if

$$\begin{aligned} \frac{\mathcal{A}^{\min}_{f^{-1}j\cap i}}{\mathcal{A}^{\max}_i}\cdot \ell^{u,\max}_i &\le \ell^{u,\min}_{f^{-1}j\cap i} \\ \iff \frac{\ell^{u,\min}_{f^{-1}j\cap i}\ell^{s,\min}_{f^{-1}j\cap i}\sin\theta^{\min}_{f^{-1}j\cap i}}{\ell^{u,\max}_i\ell^{s,\max}_i\sin\theta^{\max}_i}\cdot \ell^{u,\max}_i &\le \ell^{u,\min}_{f^{-1}j\cap i} \\ \iff \quad \ell^{s,\min}_{f^{-1}j\cap i}\sin\theta^{\min}_{f^{-1}j\cap i} &\le \ell^{s,\max}_i\sin\theta^{\max}_i. \end{aligned}$$

The final inequality is always true, so (20) holds.

By (17) and (18), and (19) and (20) we have

$$\frac{\mathcal{A}^{\min}_{f^{-1}j\cap i}}{\mathcal{A}^{\max}_i}\cdot\frac{\ell^{u,\min}_i}{\ell^{u,\max}_j} \le \kappa_{f^{-1}j\cap i} \le \frac{\mathcal{A}^{\max}_{f^{-1}j\cap i}}{\mathcal{A}^{\min}_i}\cdot\frac{\ell^{u,\max}_i}{\ell^{u,\min}_j} \tag{21}$$

and clearly also,

$$\frac{\mathcal{A}^{\min}_{f^{-1}j\cap i}}{\mathcal{A}^{\max}_i}\cdot\frac{\ell^{u,\min}_i}{\ell^{u,\max}_j} \le \frac{\mathcal{A}_{f^{-1}j\cap i}}{\mathcal{A}_i}\cdot\frac{\ell^{u}_i}{\ell^{u}_j} \le \frac{\mathcal{A}^{\max}_{f^{-1}j\cap i}}{\mathcal{A}^{\min}_i}\cdot\frac{\ell^{u,\max}_i}{\ell^{u,\min}_j}. \tag{22}$$

In fact, as the diameters of the partition sets go to zero, the left and right hand sides of the inequality (22) converge to the average (centre) value. Thus, as the diameters of the partition sets go to zero,

$$\left|\kappa_{f^{-1}j\cap i} - \frac{\mathcal{A}_{f^{-1}j\cap i}}{\mathcal{A}_i}\cdot\frac{\ell^u_i}{\ell^u_j}\right| \to 0. \tag{23}$$

By putting

$$\varphi_n(\underset{\sim}{x}) = -\log\left(\frac{\mathcal{A}_{f^{-1}x_1\cap x_0}}{\mathcal{A}_{x_0}}\cdot\frac{\ell^u_{x_0}}{\ell^u_{x_1}}\right) \tag{24}$$

and

$$\phi_n = \varphi_n\circ\pi^{-1}, \tag{25}$$

we see that $\phi_n \to \phi^{(u)}$ uniformly (f is C^2, and π^{-1} is uniformly continuous on $M\setminus\bigcup_{j\in\mathbb{Z}} f^j\left(\bigcup_{i=1}^n \partial\Omega_i\right)$, the set on which π is invertible) as $n\to\infty$. Again, ϕ_n is just a piecewise constant approximation of $\phi^{(u)}$, constant on the sets $\mathfrak{P}\vee f^{-1}\mathfrak{P}$. □

Remarks 7.2: There is a simple interpretation of the above estimate for $\kappa_{f^{-1}j\cap i}$. We are essentially approximating $\ell^u_{f^{-1}j\cap i}$ by $\left(\mathcal{A}_{f^{-1}j\cap i}/\mathcal{A}_i\right)\cdot \ell^u_i$. Since the sets in the Markov partition are almost parallelograms (if they are small enough), and because of the way inverse images of partition sets overlap, it is simple to see how this works (draw a picture!).

Also note the similarity between the expressions for the estimates of $\kappa_{f^{-1}j\cap i}$ for the expanding and hyperbolic cases. Recall that for f expanding we had

$$\kappa_{f^{-1}j\cap i} \approx \frac{\mathcal{A}_{f^{-1}j\cap i}}{\mathcal{A}_j}. \tag{26}$$

Our estimate of $\kappa_{f^{-1}j\cap i}$ in the Anosov case reduces to the above estimate for expanding maps as here all directions are expanding, and so ℓ^u_i coincides with $\mathcal{A}_i$.

Lemma 7.3: *The stochastic matrix that generates the Markov measure for our equilibrium state of φ_n is*

$$P^{(n)}_{ij} = \frac{m(f^{-1}\Omega^n_j \cap \Omega^n_i)}{m(\Omega^n_i)}.$$

Proof: Set

$$G^{(n)}_{ij} = e^{\varphi_n(i,j)} = \frac{\mathcal{A}^n_{f^{-1}j\cap i}}{\mathcal{A}^n_i}\cdot\frac{\ell^{u,n}_i}{\ell^{u,n}_j}. \tag{27}$$

As in the expanding case, note that $G^{(n)}_{ij} = L^{(n)}Q^{(n)}(L^{(n)})^{-1}$, where $L^{(n)}_{ij} = \delta_{ij}\ell^{u,n}_i$ and $Q^{(n)}_{ij} = \mathcal{A}^n_{f^{-1}j\cap i}/\mathcal{A}^n_i$. Since $Q^{(n)}$ is stochastic and irreducible, we know that the unique (up to scalar multiples) right eigenvector with nonnegative eigenvalue is the vector $(1,1,\ldots,1)$ with eigenvalue 1, and so $v^{(n)} = (\ell^{u,n}_1, \ell^{u,n}_2, \ldots, \ell^{u,n}_n)$, and $\lambda^{(n)} = 1$. Thus

$$\begin{aligned} P^{(n)}_{ij} &= \frac{\mathcal{A}^n_{f^{-1}j\cap i}\ell^{u,n}_i}{\mathcal{A}^n_i\ell^{u,n}_j}\cdot\frac{\ell^{u,n}_j}{1\cdot\ell^{u,n}_i} \\ &= \frac{\mathcal{A}^n_{f^{-1}j\cap i}}{\mathcal{A}^n_i}, \end{aligned} \tag{28}$$

by (12) and (27). □

By Lemmata 7.1, 7.3, 5.5 and Remark 5.4 we have proven Theorem 1.2 in the Anosov case.

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Note Added in Proof: A similar convergence result has been proven by Ding and Zhou [5] for higher dimensional expanding maps using piecewise linear and piecewise quadratic approximations on general partitions. The author is grateful to a referee for bringing this work to his attention.

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