DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **14**, Number **2**, September **2010**

doi:10.3934/dcdsb.2010.14.457

pp. **457–472**

ESCAPE RATES AND PERRON-FROBENIUS OPERATORS: OPEN AND CLOSED DYNAMICAL SYSTEMS

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Abstract. We study the Perron-Frobenius operator $\mathcal{P}$ of closed dynamical systems and certain open dynamical systems. We prove that the presence of a large positive eigenvalue ρ of $\mathcal{P}$ guarantees the existence of a 2-partition of the phase space for which the escape rates of the open systems defined on the two partition sets are both slower than $-\log\rho$. The open systems with slow escape rates are easily identified from the Perron-Frobenius operators of the closed systems. Numerical results are presented for expanding maps of the unit interval. We also apply our technique to shifts of finite type to show that if the adjacency matrix for the shift has a large positive second eigenvalue, then the shift may be decomposed into two disjoint subshifts, both of which have high topological entropies.

1. **Introduction.** Our aim is to explore the relationship between closed dynamical systems and certain associated open dynamical systems formed by the introduction of a hole. Let $T : (X, \mathcal{B}, m) \circlearrowleft$ be a dynamical system, with m a finite reference measure on X. We will call this system *closed*. We may construct an open system from a closed system by introducing a $\mathcal{B}$-measurable hole $H \subset X$. Let $A := X \setminus H$, $T_A := T|_A$ be the restriction of T to the set A, and $m_A := m|_A$ be the restriction of m to $\mathcal{B} \cap A$. The system $T_A : (A, \mathcal{B} \cap A, m_A) \to (X, \mathcal{B}, m)$ is called *open* as trajectories may leave A, never to return.

Pianigiani and Yorke [32] contains early work on open dynamical systems. A series of papers by Collet *et al.* [11, 12, 13] followed, studying Markov systems with Markov holes. Anosov systems with non-Markov holes [8] and open billiards [31] have also been studied. More recently, Collet *et al.* [9, 10] obtained results for a wide class of systems with holes. Lasota-Yorke maps with small holes have been extensively studied; [4, 17, 30, 35]. Bunimovich and Yurchenko [6] carried out a case study of the effect of hole position for the doubling map. Open systems as perturbations of closed systems have been considered in recent work [27, 34]. Applications of Ulam's method to open systems include [1, 2, 20]. For a survey paper with more references and discussion, see [18].

2000 *Mathematics Subject Classification.* Primary: 37A30, 37M25; Secondary: 37C40, 37E05, 37B10, 37B40.

Key words and phrases. Perron-Frobenius operator, open dynamical system, escape rate, almost-invariant set, topological entropy.

The research of GF is supported in part by an ARC Discovery Project and the ARC Centre of Excellence for Mathematics and Statistics of Complex Systems (MASCOS). OS is supported by an Australian Postgraduate Award and MASCOS.

Let the *time of escape* of a point $x \in A$ be the smallest positive integer $\xi(x)$ such that $T^{\xi(x)}(x) \in H$. Define A^n to be the set of all points that stay in A up to the n^{th} iterate of T; that is, all points that have not escaped by time n

$$A^n := \{x \in A \ : \ \xi(x) > n\} = T^{-n}(A) \cap T^{-n+1}(A) \cap \cdots \cap A.$$

A natural question concerning open systems is the rate of decrease of the measure of A^n.

Definition 1.1. Let m be a finite reference measure and $A \subset X$ a measurable set. Define the upper and lower escape rates as follows:

$$\overline{E}_m(A) := -\liminf_{n\to\infty} \frac{1}{n} \log m(A^n);$$

$$\underline{E}_m(A) := -\limsup_{n\to\infty} \frac{1}{n} \log m(A^n).$$

If $\overline{E}_m(A) = \underline{E}_m(A)$, then we say that the *escape rate* of the measure m from A exists and is

$$E_m(A) := -\lim_{n\to\infty} \frac{1}{n} \log m(A^n) \in [0, \infty].$$

In fact, if the escape rate exists and $m(\{x : \xi(x) > 0\}) = 1$, then [6]:

$$E_m(A) = -\lim_{n\to\infty} \frac{1}{n} \log m\,\{x \ : \ \xi(x) = n\}\,.$$

Remark 1. Note that if ν and m are equivalent measures, and the density $d\nu/dm$ is bounded away from zero and infinity, then one has $E_\nu(A) = E_m(A)$.

If T is nonsingular with respect to m we may form the Perron-Frobenius operator $\mathcal{P}$ of T.

Definition 1.2. Let $T : (X, \mathcal{B}, m) \circlearrowleft$ be a closed dynamical system where (T, m) is nonsingular. The *Perron-Frobenius operator* is the unique operator $\mathcal{P} : L^1(X, \mathcal{B}, m) \circlearrowleft$ that satisfies

$$\int_B \mathcal{P}f \ dm = \int_{T^{-1}(B)} f \ dm, \quad \forall B \in \mathcal{B}, \quad \forall f \in L^1(X, \mathcal{B}, m).$$

The adjoint of the Perron-Frobenius operator is the *Koopman operator*, defined on $L^\infty(X, \mathcal{B}, m)$ by $f \mapsto f \circ T$. We will throughout assume that there is a nonnegative density $\tilde{f} \in L^1$ fixed by $\mathcal{P}$. If $\mathcal{P}f = \rho f$ for some $0 < \rho < 1$ and $f \in L^1$, then $\int f \ dm = \int \mathcal{P}f \ dm = \int \rho f \ dm$ implies that $\int f \ dm = 0$. By setting $A_+ := \{f > 0\}$ and $A_\ominus := \{f \le 0\}$, we may form two open systems T_{A_+} and $T_{A_\ominus}$. Our main result is control of the escape rate for these two open systems.

Main Theorem. *Let $T : (X, \mathcal{B}, m) \circlearrowleft$ be nonsingular. Suppose that $0 < \rho < 1$ is a real positive eigenvalue of $\mathcal{P} : L^1(X, \mathcal{B}, m) \circlearrowleft$ and that the corresponding eigenfunction is bounded. Then setting $A_+ = \{f > 0\}$ and $A_\ominus = \{f \le 0\}$, one has*

$$\overline{E}_m(A_+) \le -\log \rho, \qquad \overline{E}_m(A_\ominus) \le -\log \rho.$$

In words, if the Perron-Frobenius operator for our *closed system* T has a real eigenvalue ρ, then we may break the phase space X into two pieces, forming *two open systems*, both of which have escape rates slower than $-\log \rho$. When ρ is an eigenvalue close to 1, the escape rate of each A_+ and $A_\ominus$ is low. As A_+ and $A_\ominus$ partition X, escape from A_+ corresponds to entry into $A_\ominus$ and vice-versa. In the closed system, for large ρ, this exchange may lead to rates of mixing slower than rates of local separation of trajectories. The use of eigenfunctions corresponding to large ρ to determine *almost-invariant sets* for the closed system has been considered in [16, 21, 23].

An outline of this paper is as follows. Section 2 introduces conditional Perron-Frobenius operators and conditionally invariant measures, proves Theorem 2.4, and begins to investigate its consequences. Section 3 considers the implications of Theorem 2.4 for Lasota-Yorke maps, discusses an example where the escape rate of both A_+ and $A_\ominus$ is slower than the rate of local separation of trajectories, compares the notions of escape rate and almost-invariance, and discusses related work. In Section 4 we develop a version of Theorem 2.4 for shifts of finite type and illustrate with an example.

2. **Closed and open systems.**

2.1. **Conditionally invariant measures and conditional Perron-Frobenius operators.** The notion of conditionally invariant probability measures is central to the study of open systems.

Definition 2.1. A measure μ on A is called a *conditionally invariant probability measure* of the open system $T_A : A \to X$ if for every measurable $B \subset A$

$$\mu(T_A^{-1}(B)) = \mu(A^1)\mu(B).$$

If μ is conditionally invariant then $E_\mu(A) = -\log \mu(A^1)$ as $-\frac{1}{n}\log \mu(A^n) = -\log \mu(A^1)$ for all $n \geq 1$.

Definition 2.2. For $T_A : A \to X$, where $A \subset X$, the *conditional Perron-Frobenius operator* $\mathcal{P}_A : L^1(X, \mathcal{B}, m) \circlearrowleft$ is defined by:

$$\int_B \mathcal{P}_A f \; dm = \int_{T_A^{-1}(B)} f \; dm, \quad \forall B \in \mathcal{B}, \quad \forall f \in L^1(X, \mathcal{B}, m).$$

Remark 2. We can write $\mathcal{P}_A f = \mathcal{P}(f\chi_A)$, where χ_A is the indicator function of A and more generally $\mathcal{P}_A^n f = \mathcal{P}(f\chi_{A^{n-1}})$. Thus $\|\mathcal{P}_A^n 1\|_1 = \|\mathcal{P}(\chi_{A^{n-1}})\|_1 = m(A^{n-1})$ and so

$$E_m(A) = -\lim_{n\to\infty} \frac{1}{n} \log \|\mathcal{P}_A^n 1\|_1,$$

if this limit exists.

Proposition 1 ([32])**.** *For $\lambda > 0$ and f nonnegative, $\mathcal{P}_A f = \lambda f$ if and only if the absolutely continuous measure μ, with density $f = d\mu/dm$, is conditionally invariant. It follows that $E_\mu(A) = -\log \lambda$.*

Define $\lambda_m(A) := \exp(-E_m(A))$. Following [4, 17], we call $\lambda_m(A)$ the *eigenvalue of the measure m* with respect to the set A.

From now on, we will assume Lebesgue measure ℓ to be the reference measure. All of our results will hold if a general finite reference measure m is used. For notational simplicity we will write $E(A)$ for $E_\ell(A)$ and $\lambda(A)$ for $\lambda_\ell(A)$. Unless otherwise stated, *escape rate* will refer to Lebesgue escape rate.

Existence of an absolutely continuous (w.r.t. Lebesgue) conditionally invariant probability measure (ACCIPM) whose density is bounded away from zero and infinity implies existence of Lebesgue escape rate. Existence and uniqueness of such ACCIPM for Markov maps satisfying a suitable transitivity condition was proved in [32]. Subsequent work has largely been aimed at relaxing the Markov condition, unfortunately at the expense of limiting the size of the hole. Collet *et al.* [9, 10]

proved the existence of ACCIPMs for non-Markov maps under some technical assumptions. Chernov *et al.* [8] developed some results for existence of ACCIPMs for Anosov diffeomorphisms with small holes. Liverani and Maume-Deschamps [30] proved existence of ACCIPMs for Lasota-Yorke maps with small holes using a perturbation result based on [26], as well as for maps with larger holes under the assumption that the map has enough full branches outside of the hole.

Remark 3. Generally, there will exist multiple, even uncountably many [18] ACCIPMs, with different escape rates. Of course not all are of physical relevance. In an ideal case (see [18]) suppose that $\mathcal{P}_A$ is quasi-compact in an appropriate Banach space, $(\mathbb{B}, \|\cdot\|) \subset L^1(X)$, and $\mathcal{P}_A f = \lambda f$ where λ is of multiplicity 1 and of maximal modulus. Then we may write $\mathbb{B} = \{f\} \oplus \mathbb{H}$ where $\{f\}$ is the space spanned by f and $\mathcal{P}_A(\mathbb{H}) \subset \mathbb{H}$. If there exists a $g \in \mathbb{B} \setminus \mathbb{H}$ such that $C^{-1} \le g \le C$ for some $C > 0$, then Lebesgue escape rate exists and $E(A) = -\log \lambda$. All other absolutely continuous measures have higher escape rates.

2.2. **Connecting closed and open systems.** We now restate and prove our first main result which, roughly speaking, relates eigenvalues of the operator $\mathcal{P}$ to the largest eigenvalue of a particular conditional operator $\mathcal{P}_A$.

Definition 2.3. For a function $f \in L^1(X)$ we denote by $\mathrm{supp}(f) := \{x \in X : f(x) \neq 0\}$ the *support* of f. Also define $f^+, f^- \in L^1(X)$ by $f^+ := \max(f, 0)$ and $f^- := \max(-f, 0)$.

Theorem 2.4 (Main Theorem)**.** *Let $T : X \circlearrowleft$ be a nonsingular transformation on the finite measure space $(X, \mathcal{B}, \ell)$ and let $\mathcal{P} : L^1(X, \mathcal{B}, \ell) \circlearrowleft$ be the corresponding Perron-Frobenius operator. Suppose that $\mathcal{P}$ has a real positive eigenvalue $0 < \rho < 1$, with corresponding bounded eigenfunction $-\infty < f < \infty$. Define the measurable sets $A_+, A_- \subset X$ by*

$$A_+ := \mathrm{supp}(f^+) \quad \text{and} \quad A_- := \mathrm{supp}(f^-).$$

Then one has $\overline{E}(A_+) \le -\log \rho$ and $\overline{E}(A_-) \le -\log \rho$.

For the proof we will need the following lemma.

Lemma 2.5. *For a finite measure ν, let $A \subset X$ be measurable and $0 < \gamma \le 1$.*

(i) If $\nu(A^{n+1}) \ge \gamma \nu(A^n)$ for all $n \ge 0$, then $\overline{E}_\nu(A) \le -\log \gamma$.
(ii) If $\nu(A^{n+1}) \le \gamma \nu(A^n)$ for all $n \ge 0$, then $\underline{E}_\nu(A) \ge -\log \gamma$;

Proof.
(i) By induction it follows that $\nu(A^n) \ge \gamma^n \nu(A)$.

$$\overline{E}_\nu(A) \quad = \quad -\liminf_{n\to\infty} \frac{1}{n} \log \nu(A^n) \le -\liminf_{n\to\infty} \frac{1}{n} \log(\gamma^n \nu(A)) = -\log \gamma.$$

(ii) is analogous to *(i)*. ◻

Proof of Theorem 2.4. Define a finite measure ν on X by

$$\nu(B) := \int_B |f| \, d\ell, \quad B \in \mathcal{B}.$$

Now note that for all $n \geq 0$ we have $f > 0$ on A_+^n. Also $A_+^{n+1} = T^{-1}(A_+^n) \cap A_+$, therefore

$$\begin{aligned}
\rho\nu(A_+^n) &= \rho \int_{A_+^n} f \, d\ell \\
&= \int_{A_+^n} \mathcal{P} f \, d\ell \\
&= \int_{T^{-1}(A_+^n)} f \, d\ell \\
&= \int_{T^{-1}(A_+^n) \cap A_+} f \, d\ell + \int_{T^{-1}(A_+^n) \cap (X \setminus A_+)} f \, d\ell \\
&\leq \int_{T^{-1}(A_+^n) \cap A_+} f \, d\ell \\
&= \nu(A_+^{n+1}),
\end{aligned}$$

where the inequality above is due to $f \leq 0$ on $X \setminus A_+$. Since $\nu(A_+^{n+1}) \geq \rho\nu(A_+^n)$, by (i) of Lemma 2.5 we have $\overline{E}_\nu(A_+) \leq -\log \rho$. It remains to show that $\overline{E}(A_+) \leq \overline{E}_\nu(A_+)$. Since $f \leq C$ for some constant $C > 0$, we have $\nu(A_+^n) \leq C\ell(A_+^n)$ for all $n \geq 0$. This gives $\overline{E}_\nu(A_+) = -\liminf_{n\to\infty} \frac{1}{n} \log(\nu(A_+^n)) \geq -\liminf_{n\to\infty} \frac{1}{n} \log(C\ell(A_+^n)) = \overline{E}(A_+)$. Thus $\overline{E}(A_+) \leq \overline{E}_\nu(A_+) \leq -\log \rho$. The inequality for A_- is obtained by considering $-f$ in place of f. □

Remark 4. If one wishes to create a 2-partition of X such that each element of the partition has upper escape rate lower than $-\log \rho$, then the set $\{f = 0\}$ may be absorbed into either A_+ or A_-. Enlarging A_+ does not increase $\overline{E}(A_+)$ so Theorem 2.4 also holds for $A_\oplus := X \setminus A_-$ and $A_\ominus := X \setminus A_+$. The desired 2-partition is then $\{A_+, A_\ominus\}$ or $\{A_\oplus, A_-\}$ (or any other redistribution of $\{f = 0\}$ among the two sets).

Lemma 2.6. *$E(A) = E(A^N)$ for any measurable set $A \subset X$ and integer $N \geq 0$.*

Proof. It is a simple exercise to show that $(A^N)^n = A^{N+n}$ for all $n, N \geq 0$. The result follows:

$$\begin{aligned}
E(A^N) &= -\lim_{n\to\infty} \frac{1}{n} \log \ell((A^N)^n) = -\lim_{n\to\infty} \frac{1}{n} \log \ell(A^{N+n}) \\
&= -\lim_{n\to\infty} \frac{1}{n-N} \log \ell(A^n) = E(A).
\end{aligned}$$

□

Remark 5. By Lemma 2.6, we may replace T_A with T_{A^1} and obtain an open system with an identical escape rate. We may think of T_{A^1} as an open system on A with hole $A \setminus T^{-1}A$. Consider now our partition $\{A_+, A_\ominus\}$ of X formed from the positive and nonpositive parts of some $f \in L^1$ satisfying $\mathcal{P}f = \rho f$, $0 < \rho < 1$. By the above remarks, the open system T_{A_+} has the same escape rate as the open system $T_{A_+^1}$, where the hole for the latter system is $A_+ \setminus T^{-1}A_+ = A_+ \cap T^{-1}A_\ominus \subset A_+$. Thus, while the hole $H = A_\ominus$ for the open system T_{A_+} is very large in measure, we may easily construct another system $T_{A_+^1}$ with the same escape rate, but a hole $H = A_+ \cap T^{-1}A_\ominus$ that is likely to be much smaller in terms of ℓ. Similarly, we may define an open system $T_{A^1_\ominus}$, with hole $A_\ominus \setminus T^{-1}A_\ominus = A_\ominus \cap T^{-1}A_+ \subset A_\ominus$; this open system has the same escape rate as $T_{A_\ominus}$.

2.3. **Spectrum of $\mathcal{P}$ in L^1.** Let $\sigma(\mathcal{P})$ denote the L^1 spectrum of $\mathcal{P}$. Ding *et al.* [19] (Corollary 3.2) state that for $(X, \mathcal{B}, \ell)$ a σ-finite measure space, T nonsingular and $\mathcal{P} : L^1(X, \ell) \circlearrowleft$, with a positive fixed density, if $0 \in \sigma(\mathcal{P})$, then $\sigma(\mathcal{P}) = \{z \in \mathbb{C} : |z| \le 1\}$.

Consider $T : [0,1] \circlearrowleft$, piecewise monotonic and C^2 on each monotone branch. Then we may represent $\mathcal{P} : L^1([0,1], \ell) \circlearrowleft$ as

$$\mathcal{P}f(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{|T'(y)|}, \tag{1}$$

where $T'(y)$ is defined by continuity along an inverse branch if the derivative of T fails to exist at y. The following Lemma shows that if 0 is an eigenvalue, then every point in the open unit disk is also an eigenvalue.

Lemma 2.7. *Let $\mathcal{P}$ be as above. Suppose there is a nonzero $\hat{f} \in L^1([0,1], \ell)$ satisfying $\mathcal{P}\hat{f} = 0$. Every $\rho \in \mathbb{C}$ such that $|\rho| < 1$ is an eigenvalue of $\mathcal{P}$.*

Proof. This proof appears in a slightly different context in the proof of Theorem 1.5 (7) [3]. If $\rho = 0$ we are done. Let $\rho \ne 0$ and let $h > 0$ be the fixed density such that $\mathcal{P}h = h$. Then $f := \sum_{n=0}^{\infty} \rho^n (\hat{f}/h) \circ T^n \cdot h \in L^1$ is an eigenfunction with eigenvalue ρ. To see this, we note that $f \in L^1$ and compute

$$\begin{aligned}
\mathcal{P}f(x) &= \sum_{y \in T^{-1}x} \sum_{n=0}^{\infty} \rho^n (\hat{f}/h) \circ T^n(y) \cdot h(y)/|T'(y)| \\
&= \sum_{y \in T^{-1}x} \hat{f}(y)/|T'(y)| + \sum_{y \in T^{-1}x} \sum_{n=1}^{\infty} \rho^n (\hat{f}/h) \circ T^n(y) \cdot h(y)/|T'(y)| \\
&= 0 + \rho \sum_{y \in T^{-1}x} \sum_{n=0}^{\infty} \rho^n (\hat{f}/h) \circ T^n(x) \cdot h(y)/|T'(y)| \\
&= \rho \sum_{n=0}^{\infty} \rho^n (\hat{f}/h) \circ T^n(x) \sum_{y \in T^{-1}x} h(y)/|T'(y)| = \rho f(x).
\end{aligned}$$

□

Remark 6. A related result of Collet and Isola [7] shows that if T is a piecewise C^∞ expanding Markov map with bounded first and second derivatives, then the spectrum of $\mathcal{P}$, acting on C^0 functions, is the entire unit disk and every spectral point is an eigenvalue of infinite multiplicity.

Example 2.8. Figure 1 shows three eigenfunctions for the doubling map $x \mapsto 2x$ on S^1. We may apply Theorem 2.4 to any one of these eigenfunctions to obtain two open systems, both of which have escape rates slower than $-\log \rho$. Each eigenfunction produces a very large hole, and Theorem 2.4 says that one may set ρ as close to unity as one wishes, to obtain very slow escape rates. The penalty that one pays for producing escape rates less than $\log 2$ are sets A_+ that may be very complicated. We discuss this further in the next section.

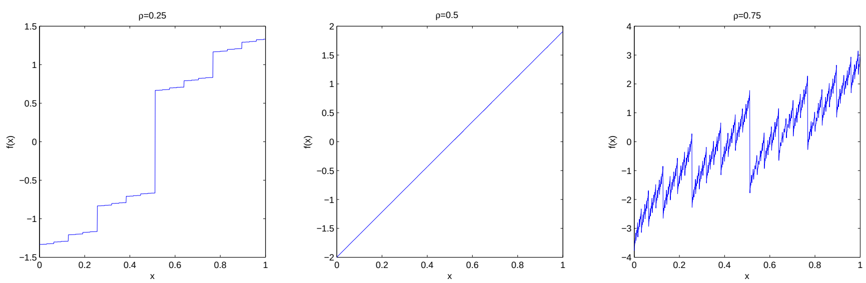

FIGURE 1. Graphs of L^1 eigenfunctions for the circle map $x \mapsto 2x$ for $\rho = 0.25, 0.5$, and 0.75.

3. **Application to Lasota-Yorke maps.** Let us focus on the case where $I := [0,1]$ and $T : I \circlearrowleft$ is piecewise monotone and C^2; that is, there is a finite partition $\{a_0, a_1, \ldots, a_n\}$ with $a_0 = 0, a_n = 1$ so that T is monotone and C^2 on the interior of each interval (a_{i-1}, a_i), $i = 1, \ldots, n$. Furthermore we assume that T is expanding, that is $\tau := \inf |T'| > 1$ where the infimum is taken over all points in $[0,1]$ for which the derivative exists. Such maps are known as *Lasota-Yorke maps*.

We begin this section by stating that we can expect the L^1 spectrum to be the entire unit disk for interesting Lasota-Yorke maps.

Lemma 3.1. *Let T be a Lasota-Yorke map and suppose that there are two monotone branches $T_i := T|_{(a_{i-1},a_i)}, T_j := T|_{(a_{j-1},a_j)}$, $i \neq j$, for which $T_i((a_{i-1}, a_i)) \cap T_j((a_{j-1}, a_j)) \neq \emptyset$. Then there exists a nonzero $f \in L^1$ such that $\mathcal{P}f = 0$.*

Proof. We construct a nonzero $f \in L^1$ with $\mathcal{P}f = 0$. As T_i, T_j are monotonic and expanding, $T_i((a_{i-1}, a_i)) \cap T_j((a_{j-1}, a_j))$ is an open interval, which we denote (x_1, x_2). Let $f(x) = 0$ for $x \in [0,1] \setminus (T_i^{-1}(x_1, x_2) \cup T_j^{-1}(x_1, x_2))$, and $f(x) = 1$ for $x \in T_i^{-1}(x_1, x_2)$. We now determine the value of $f(x)$ for $x \in T_j^{-1}(x_1, x_2)$.

By (1), for $x \in (x_1, x_2)$ we have

$$\mathcal{P}f(x) = 1/|T_i'(T_i^{-1}(x))| + f(T_j^{-1}(x))/|T_j'(T_j^{-1}(x))|.$$

Equating the RHS with zero and rearranging, we obtain $f(T_j^{-1}(x)) = -|T_j'(T_j^{-1}(x))|/|T_i'(T_i^{-1}(x))|$, defining $f(x)$ for $x \in T_j^{-1}(x_1, x_2)$. For $x \notin (x_1, x_2)$ $\mathcal{P}f(x)$ is clearly also zero by the definition of f. □

3.1. **Spectrum of $\mathcal{P}$ on $BV(I)$.** By replacing $(L^1(X), \|\cdot\|_1)$ with $(\mathrm{BV}(I), \|\cdot\|_{\mathrm{BV}})$, the space of functions of bounded variation where $\|\cdot\|_{\mathrm{BV}} = \max\{\mathrm{var}_I(\cdot), \|\cdot\|_1\}$, the operator $\mathcal{P} : (\mathrm{BV}, \|\cdot\|_{\mathrm{BV}}) \circlearrowleft$ becomes *quasi-compact* (see eg. [3]). Eigenfunctions of $\mathcal{P}$ that lie in BV give rise to sets A_+ with a reasonably simple structure.

Definition 3.2. Let $\mathcal{I}$ be the family of sets $A \subset I$ where A can be written as a countable union of intervals (where singleton sets $\{x\} = [x, x]$ are included).

Proposition 2 ([28])**.** *If $f \in BV$ then* $\mathrm{supp}(f) \in \mathcal{I}$.

Corollary 1. *If $f \in BV$ then $f^+, f^- \in BV$; thus the sets A_+ and A_- from Theorem 2.4 belong to $\mathcal{I}$.*

Example 3.3. Returning to the doubling map $x \mapsto 2x$, considered as a map on I, it is well known that the spectrum of $\mathcal{P} : \mathrm{BV}(I) \circlearrowleft$ is contained in $\{|z| \leq 1/2\} \cup \{1\}$.

Thus, all BV eigenfunctions corresponding to eigenvalues $0 < \rho < 1$ must in fact have $\rho \leq 1/2 = 1/\tau$. In particular, this excludes the third, more irregular L^1 eigenfunction in Figure 1.

Thus, for the doubling map in the BV setting, Theorem 2.4 guarantees the existence of open subsystems defined on reasonably regular domains (in the sense of Definition 3.2) with escape rates less than $\log C$ where $C \geq \tau$; the theorem does not, however, guarantee the existence of open systems on regular domains with escape rates *less than* $\log 2 = \log \tau$. The following section investigates a map for which Theorem 2.4 does predict open systems on regular domains with escape rates slower than $\log \tau$.

3.2. **A map with escape rate slower than** $\log \tau$**.** In this section we exhibit a map for which we may identify two disjoint open subsystems, both of which have an escape rate slower than $\log \tau$. The sets A_+ and A_- constructed in Theorem 2.4 are one good way to define such open systems. Via numerical exploration, we investigate whether there are other decompositions into open systems with even slower escape rates than the decomposition identified by Theorem 2.4.

As an objective means of comparison, given a closed system, we propose to maximise the following quantity

$$\psi(A) := \min(\lambda(A), \lambda(I \setminus A)), \quad A \in \mathcal{I}.$$

Example 3.4. Consider the following piecewise affine map $T : I \circlearrowleft$ [22].

$$T(x) = \begin{cases} 4x, & x \in [0, 1/8); \\ 4x - 1/2, & x \in [1/8, 2/8); \\ 4x - 1, & x \in [2/8, 4/8); \\ 4x - 2, & x \in [4/8, 6/8); \\ 4x - 5/2, & x \in [6/8, 7/8); \\ 4x - 3, & x \in [7/8, 1]. \end{cases}$$

The graph of T is shown in Figure 2. The Perron-Frobenius operator of T has an isolated second largest eigenvalue $\rho_2 = 1/2$ with the corresponding eigenfunction $f_2 \in \mathrm{BV}$, shown in Figure 3.

By considering where f_2 is positive and where it is negative, we can partition the domain of T into two sets, $A_- = [0, 1/2)$ and $A_+ = [1/2, 1]$. The escape rate of both of these sets is much lower than $\log \tau$: $E(A_-) = E(A_+) = -\log 3/4 = \log 4/3$, compared to $\log \tau = \log 4$, and both satisfy the inequality of Theorem 2.4. Sets with even lower escape rates do exist (for example, if we take $A = [0, 1-\epsilon]$ for small enough ϵ, then we can make $E(A)$ as close as we like to zero). However it is not immediately obvious that there exists a set $A \in \mathcal{I}$ with $\psi(A) > 3/4$; that is, the escape rate from both A and the complement of A is lower than $-\log 3/4$ (note the escape rate of $X \setminus A = X \setminus [0, 1-\epsilon] = (1-\epsilon, 1]$ is $-\log \frac{1}{4}$).

Intervals of length 1/2. First, we will maximise $\psi(A)$ over the class of all intervals of length 1/2. Let $I_{\alpha,1/2}$ be an interval of length 1/2 centered at $x = \alpha$. Figure 4 suggests that $\psi(I_{\alpha,1/2})$ is maximised when $\alpha = 1/4$, that is $I_{\alpha,1/2} = [0, 1/2]$, coinciding with the set A_- identified by Theorem 2.4.

Intervals of varying length. We also considered intervals $I_{\alpha,l}$ with centres and lengths $\alpha, l \in \{i/512\}_{i=0,\ldots,255}$. Again, we found that $\psi(I_{\alpha,l}) \leq 3/4$ for all α, l considered, with the maximum achieved by $I_{1/4,1/2}$.

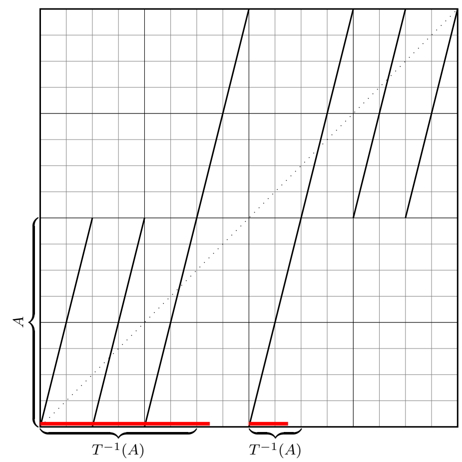

FIGURE 2. Graph of T in Example 3.4. The set $A = [0, \frac{1}{2}]$ and its pre-image are shown.

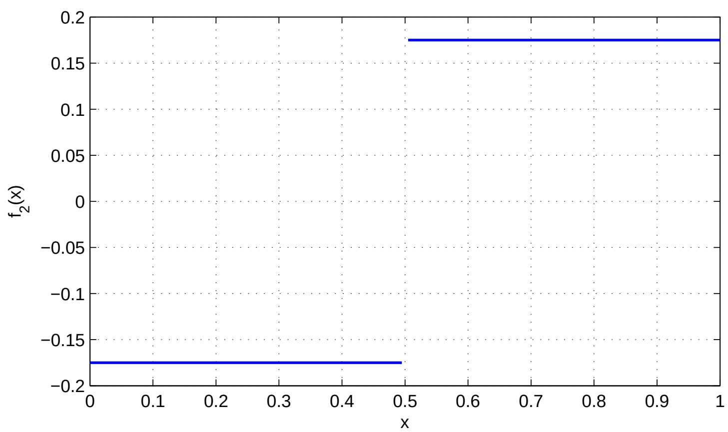

FIGURE 3. Graph of second eigenfunction f_2 of $\mathcal{P}$.

Finite unions of intervals. We may also consider A to be a finite union of elements from an interval partition of I. We maximise $\psi(A)$ over all unions of intervals in the partition $\mathcal{I}_{16} := \{[i/16, (i+1)/16) : i = 0, \ldots, 15\}$ and find $\psi(A) \approx 0.799$ for $A = [0, \frac{7}{16}) \cup [\frac{1}{2}, \frac{9}{16})$. If we repeat on the finer partition $\mathcal{I}_{32} := \{[i/32, (i+1)/32) : i = 0, \ldots, 31\}$ we obtain maximal $\psi(A) \approx 0.8198$ for $A = [0, \frac{13}{32}) \cup [\frac{16}{32}, \frac{19}{32})$. This set is coloured in red in Figure 2.

If we allow more complicated sets than those in $\mathcal{I}$, then combining Theorem 2.4, Lemma 3.1, and Lemma 2.7 we see that $\sup\{\psi(A) : A \subset X\} = 1$ as per the discussion in Section 2.3 for the doubling map.

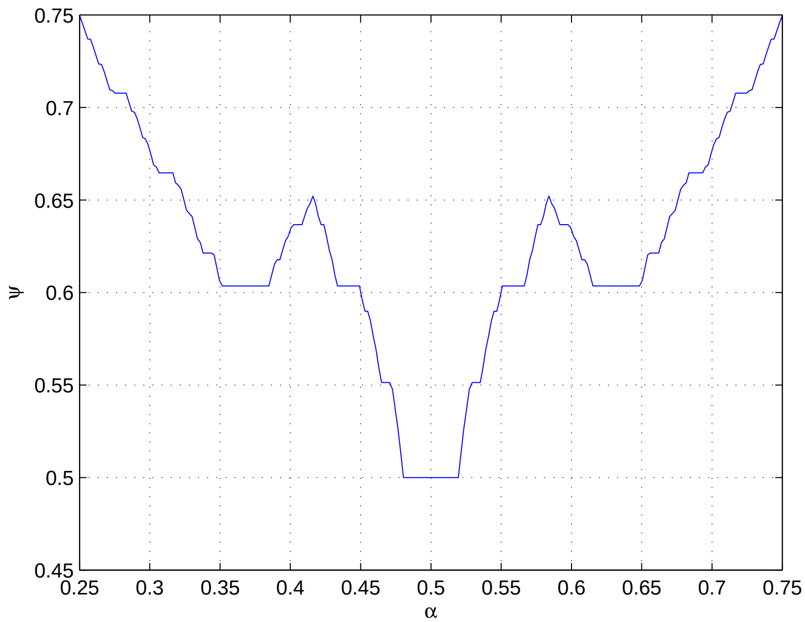

FIGURE 4. Graph of $\psi(I_{\alpha,1/2})$ where $I_{\alpha,1/2}$ is an interval of length 1/2 with varying center point α.

3.3. **Escape rate and almost-invariant sets.** *Almost-invariant sets* [16, 21, 23] are sets for which the invariance ratio

$$\varrho(A) := \frac{\ell(T^{-1}(A) \cap A)}{\ell(A)}$$

is close to 1. Dynamical systems that are close to nonergodic typically have a decomposition into nontrivial sets, each of which has a high invariance ratio. The identification of such almost-invariant sets is often very difficult; see [24] for a recent computational study. Application areas include molecular dynamics [33] and ocean dynamics [15, 25].

The construction of A_+ and A_- in Theorem 2.4 is based on an algorithm in [16] for determining almost-invariant sets. In the Lasota-Yorke map setting, with $\mathcal{P} : BV \circlearrowleft$, almost-invariant sets have formally been associated with isolated spectral points of $\mathcal{P}$ [14]. In such a setting, if the map is additionally Markov and one restricts oneself to searching for almost-invariant sets that are unions of Markov partition sets, then lower and upper bounds for the largest possible almost-invariance ratio are given by the second largest eigenvalue of an associated Markov chain [21].

Thus, there is a strong connection between almost-invariant sets and the construction we have used to define our slow escape sets A_+ and A_-. One might therefore naively expect that sets with low escape rate should have a high invariance ratio and vice-versa. However, escape rate is an asymptotic quantity, while almost-invariance measures escape over just one iteration of a map. We give examples below to demonstrate that a set may simultaneously have (i) high almost-invariance and high escape rate and (ii) low almost-invariance and low escape rate.

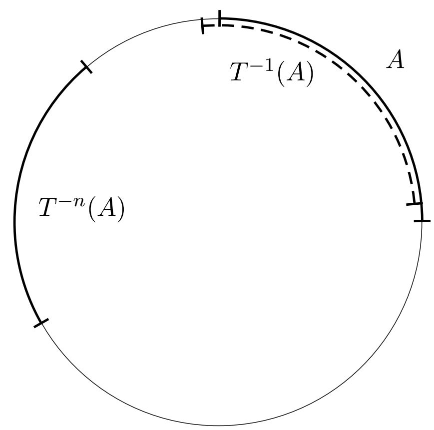

FIGURE 5. Illustration of Example 3.5.

Example 3.5 (High almost-invariance, infinite escape rate)**.** Let $T : S^1 \circlearrowleft$ be the irrational rotation of the circle, $T(x) := x + 2\pi\alpha$ where α is small. Let $A = [0, \frac{\pi}{2}]$. The pre-image of A is given by $T^{-1}(A) = [2\pi\alpha, \frac{\pi}{2} + 2\pi\alpha]$. Thus the invariance ratio of A with respect to Haar measure on the circle is $(\frac{\pi}{2} - 2\pi\alpha)/(\frac{\pi}{2}) \approx 1$. However, for $\frac{1}{4\alpha} < n < \frac{3}{4\alpha}$ we have $T^{-n}(A) \cap A = \emptyset$, therefore escape rate from A with respect to any measure is infinite. See Figure 5.

Example 3.6 (Low almost-invariance, arbitrarily low escape rate)**.** Let $T : [0, 1) \circlearrowleft$ be defined as follows:

$$T(x) = \begin{cases} (1+\epsilon)x, & 0 \le x \le 1/4; \\ 2x \pmod 1, & 1/4 < x < 1. \end{cases}$$

Let $A = [0, 1/2]$. The invariance ratio of A with respect to Lebesgue measure equals to $1/2$. However its escape rate is $\log(1 + \epsilon) \approx 0$. See Figure 6.

3.4. **Related work.** Bunimovich and Yurchenko [6] study the doubling map $x \mapsto 2x$ on the circle with reference measure Lebesgue and consider Markov holes. They show that the escape rate is related to the first return time of a positive measure subset of the hole: longer return time to the hole implies faster escape rate into the hole. More precisely, for times longer than the return time, longer return time to the hole implies smaller survivor sets. Unfortunately, the proofs rely heavily on combinatorial arguments based upon the full 2-shift (or k-shift) structure, and thus are specific to the doubling map and systems metrically conjugate to the doubling map. Even for reasonably simple systems such as piecewise affine expanding Markov maps, similar results are not known. Numerical investigations such as Figure 4 clearly display the dependence of escape rate on the position of the hole, and support our observation that the holes identified by Theorem 2.4 are positioned so as to form open systems with very low escape rates.

Keller and Liverani [27] study the escape rates of systems with very small holes. They consider Lasota-Yorke maps with possibly countably many branches and a family of compact interval holes I_ϵ shrinking to a point z as $\epsilon \to 0$. To each ϵ is associated a conditional Perron-Frobenius operator with leading eigenvalue λ_ϵ.

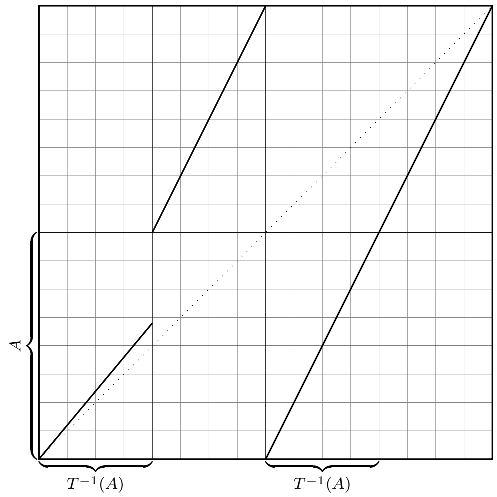

FIGURE 6. Graph of T in Example 3.6.

Formulae are provided for the ratio of $1 - \lambda_\epsilon$ to the size of the hole I_ϵ for periodic and non-periodic z. Holes shrinking to a fixed interval are also discussed. Results on exchange rates, similar to that of [16], are obtained when T has two mixing ergodic components that are joined into a single ergodic component by the addition of smooth noise.

Tokman *et al.* [34] study Lasota-Yorke maps that possess two invariant subsets of positive Lebesgue measure and exactly two ergodic absolutely continuous invariant probability measures (ACIPMs). They perturb such maps slightly to destroy the two invariant subsets and show that the (now unique) ACIPM may be approximated by a convex combination of the two initial ergodic ACIPMs. The holes considered in [34] are the holes $A_+ \cap T^{-1}A_\ominus \subset A_+$ and $A_\ominus \cap T^{-1}A_+ \subset A_\ominus$ discussed in Remark 5. Our results may be viewed as generalised converses to [34], who study the particular setting of Lasota-Yorke maps and require very precise knowledge on the initial closed dynamical system. In contrast, we begin with a closed system about which we know very little, apart from the existence of eigenvalues for its Perron-Frobenius operator. From the eigenvalue and eigenfunction information, we are able to *determine* two holes and form two open systems from which the rate of escape is guaranteed to be slower than the rate given by the eigenvalue. In general, the identification of such open systems is far from obvious. Our approach may handle very general settings (only non-singularity is required to define the Perron-Frobenius operator), and provides useful information even for macroscopic holes when the closed system may be far from nonergodic.

4. **Shifts of finite type.** Let us introduce some common notation and well known results (see e.g. [29]). Let Z_K be a finite *alphabet* of length K and let $X = Z_K^{\mathbb{Z}}$ be the space of all bi-infinite sequences of elements of Z_K. We denote an element of X by $x = (x_i)_{i\in\mathbb{Z}} = \dots x_{-2}x_{-1}.x_0x_1x_2\dots$. Define the *left shift map* $\sigma : X \circlearrowleft$ by $(\sigma x)_i = x_{i+1}$. A *block* of length k is a finite sequence of k elements from Z_K. A *shift of finite type*, denoted $X_\mathcal{F}$, is a σ-invariant subspace of X where $\mathcal{F}$ is a finite collection of *forbidden blocks*. A shift $X_\mathcal{F}$ is said to be of memory 1 if all forbidden

blocks in $\mathcal{F}$ are of length 2. It is always possible to *recode* a shift of finite type into a conjugate shift of memory 1, so without loss of generality we may assume that this has already been done. We call $X_{\mathcal{G}}$ a *subshift* of $X_{\mathcal{F}}$ if $\mathcal{F} \subset \mathcal{G}$. Define the *adjacency matrix* of a memory 1 shift $X_{\mathcal{F}}$ to be the 0–1 matrix M such that $M_{ij} = 0$ if and only if ij is a forbidden block. If M is the adjacency matrix of the shift $X_{\mathcal{F}}$ and M' is the adjacency matrix of its subshift $X_{\mathcal{G}}$, then $M'_{ij} = 1 \Rightarrow M_{ij} = 1$. By the Perron-Frobenius theorem for nonnegative matrices, M has a real eigenvalue equal to the spectral radius $r(M)$. The *topological entropy* $h(X_{\mathcal{F}})$ of a shift $(X_{\mathcal{F}}, \sigma)$ is given by $h(X_{\mathcal{F}}) = \log r(M)$.

We now state a partitioning theorem, similar in theme to Theorem 2.4, for shifts of finite type. The aim is to identify two disjoint subshifts of $X_{\mathcal{F}}$, both of which have high entropy.

Theorem 4.1. *Let $(X_{\mathcal{F}}, \sigma)$ be a memory 1 shift of finite type, with corresponding $K \times K$ adjacency matrix M. Let $0 < \rho < r(M)$ be another real eigenvalue of M with eigenvector $v \in \mathbb{R}^K$. Define A_+ and A_- to be the two sets of indices for which v is positive and negative, respectively:*

$$A_+ := \{i \in Z_K \ : \ v_i > 0\}, \quad A_- := \{i \in Z_K \ : \ v_i < 0\}.$$

Let M_{A_+} and M_{A_-} be the restrictions of M to indices in A_+ and A_- respectively. These adjacency matrices define two disjoint memory 1 subshifts of $X_{\mathcal{F}}$ on disjoint symbol sets, denoted by $X_{\mathcal{G}}$ and $X_{\mathcal{H}}$.

One has $h(X_{\mathcal{G}}) \geq \log \rho$ and $h(X_{\mathcal{H}}) \geq \log \rho$.

Proof. It is sufficient to show that $r(M_{A_+}) \geq \rho$, where $r(M_{A_+})$ is the spectral radius of M_{A_+}. For every $i \in A_+$

$$\begin{aligned} \rho v_i &= \sum_{j \in Z_K} M_{ij} v_j \\ &= \sum_{j \in A_+} M_{ij} v_j + \sum_{j \notin A_+} M_{ij} v_j \\ &\leq \sum_{j \in A_+} (M_{A_+})_{ij} v_j. \end{aligned}$$

It follows that $(\rho^n v^+)_i \leq (M^n_{A_+} v^+)_i$ for all $n \geq 1$ and $i \in A_+$, where v^+ is the restriction of v to A_+, thus $\rho^n \|v^+\| \leq \|M^n_{A_+} v^+\|$. By Gelfand's spectral radius formula (see e.g. [5]) we obtain $\rho \leq r(M_{A_+})$, therefore $h(X_{\mathcal{G}}) = \log r(M_{A_+}) \geq \log \rho$. By considering $-v$ in place of v we obtain $h(X_{\mathcal{H}}) \geq \log \rho$. ☐

Example 4.2. Let $X_{\mathcal{F}} \subset Z_9^{\mathbb{Z}}$ be the memory 1 shift whose allowed transition graph is shown in Figure 7. The adjacency matrix M of $X_{\mathcal{F}}$ is given by

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

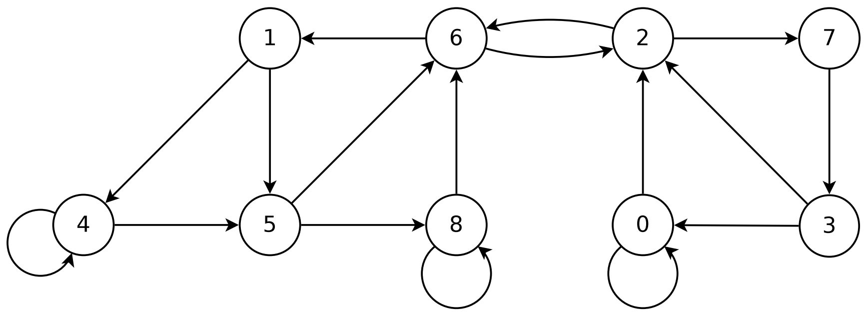

FIGURE 7. Transition graph of $X_{\mathcal{F}}$.

We have deliberately constructed the shift $X_{\mathcal{F}}$ so that its graph of allowed transitions consists of two weakly linked subgraphs, each of which is highly internally linked. The dynamics restricted to each of the two subgraphs generate almost as much entropy as the dynamics on the whole graph. We expect that the adjacency matrix M has a real positive eigenvalue ρ close to $r(M)$. If so, we may use Theorem 4.1 to identify two disjoint subshifts of $X_{\mathcal{F}}$, namely $X_{\mathcal{G}}$ and $X_{\mathcal{H}}$ with entropy of each larger than $\log \rho$.

We find that M has largest eigenvalue $r(M) \approx 1.92$, and second largest eigenvalue $\rho \approx 1.42$. The eigenvector v corresponding to ρ is shown in Figure 8; thus we define $A_+ = \{0, 2, 3, 7\}$ and $A_- = \{1, 4, 5, 6, 8\}$. This corresponds to breaking

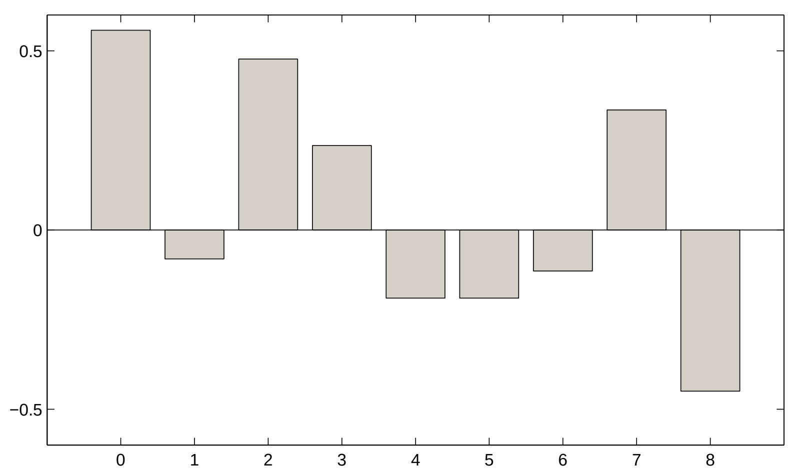

FIGURE 8. Second eigenvector of M.

the connections between vertices 6 and 2 in Figure 7, and taking each of the two connected components to be the allowed transition graphs of $X_{\mathcal{G}}$ and $X_{\mathcal{H}}$. We calculate the topological entropy of each of the newly obtained subshifts: $h(X_{\mathcal{G}}) \approx \log 1.47$ and $h(X_{\mathcal{H}}) \approx \log 1.76$. Both entropies are greater than $\log \rho$, as guaranteed by Theorem 4.1. In this example, we guessed a good partitioning of the set of states and this guess coincided with the conclusion of Theorem 4.1. For larger, more complicated examples, Theorem 4.1 can be used to discover good partitions.

Example 4.3. We may also partition memory 2 or higher shifts by first recoding them to memory 1 shifts. In Example 4.2, $X_{\mathcal{F}}$ is a recoding of a memory 2 shift $Y_{\mathcal{F}'} \subset Z_3^{\mathbb{Z}}$ where $\mathcal{F}' = \{001, 010, 022, 101, 110, 121, 200, 211, 212, 221\}$. The sliding block code $\phi : Y_{\mathcal{F}'} \to X_{\mathcal{F}}$ given by $x = \phi(y)$, where $x_i = 3y_{i-1} + y_i$, $i \in \mathbb{Z}$, conjugates the two shifts. As in Example 4.2 we partition $X_{\mathcal{F}}$ into two disjoint subshifts $X_{\mathcal{G}}$ and $X_{\mathcal{H}}$ of high entropy, relative to the entropy of $X_{\mathcal{F}}$. By applying ϕ^{-1} and using the fact that ϕ is a conjugacy, we create a partition of $Y_{\mathcal{F}'}$ into two disjoint subshifts of high entropy: $Y_{\mathcal{G}'} = \phi^{-1}(X_{\mathcal{G}})$ and $Y_{\mathcal{H}'} = \phi^{-1}(X_{\mathcal{H}})$. From the calculations in Example 4.2 we have $h(Y_{\mathcal{F}'}) \approx \log 1.92$, $h(Y_{\mathcal{G}'}) \approx \log 1.47$ and $h(Y_{\mathcal{H}'}) \approx \log 1.76$. Moreover, $\mathcal{G}' = \mathcal{F}' \cup \{011, 012, 111, 112, 120, 122, 201, 220, 222\}$ and $\mathcal{H}' = \mathcal{F}' \cup \{000, 002, 020, 021, 100, 102, 202, 210\}$; thus $\mathcal{G}' \cap \mathcal{H}' = \mathcal{F}'$ and $\mathcal{G}' \cup \mathcal{H}'$ contains all words of length three on 3 symbols.

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Received August 2009; revised January 2010.

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