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Computer-Assisted Bounds for the Rate of Decay of Correlations

Gary Froyland[*]

Department of Mathematics, The University of Western Australia, Nedlands WA 6907, Australia.
E-mail: gary@maths.uwa.edu.au

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Abstract: The rate of decay of correlations quantitatively describes the rate at which a chaotic system "mixes" the state space. We present a new rigorous method to estimate a bound for this rate of mixing. The technique may be implemented on a computer and is applicable to both multidimensional expanding and hyperbolic systems. The bounds produced are significantly less conservative than current rigorous bounds. In some situations it is possible to approximate resonant eigenfunctions and to strengthen our bound to an estimate of the decay rate. Order of convergence results are stated.

1. Introduction and Motivation

One of the major concerns in dynamical systems and ergodic theory is the rate at which systems settle down into their regular (statistically speaking) behaviour. Let $T : M \circlearrowleft$ govern a discrete dynamical system on a compact Riemannian manifold M. We assume that our system (M, T) has a unique asymptotic distribution for a "large" set of initial points, that is, $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \delta_{T^i x} \to \mu$ weakly for Lebesgue almost all x. The measure μ is called the "physical" or "natural" invariant measure of the system (M, T) [2, 18]. We are interested in the rate at which an initial concentration of mass in phase space approaches the distribution given by μ under the evolution of T.

Definition 1.1. *The system* (T, μ) *is called* **mixing** *if for each pair of Borel sets* $A, B \in \mathfrak{B}(M)$*,*

$$\lim_{k\to\infty} \mu(A \cap T^{-k}B) \to \mu(A)\mu(B). \tag{1}$$

We may rewrite this Eq. as $\lim_{k\to\infty} \mu(A \cap T^{-k}B)/\mu(A) = \mu(B)$, and interpret it as saying that the probability (according to μ) of a point x moving from a set A to a set B

[*] URL:http://maths.uwa.edu.au/∼gary/

in k iterations, given that $x \in A$, is (for k large) roughly the probability of x being in B, or the "size" of B (according to μ). As this Eq. holds for any A, we see that $T^{-k}B$ is being evenly dispersed throughout phase space according to the physical measure μ. It is the rate at which this dispersion occurs that we are concerned with.

In this paper we shall assume that T is at least piecewise C^1 with Hölder continuous derivative. As we are after rigorous results, T is assumed to be either (i) expanding or (ii) transitive Anosov, and possess an absolutely continuous invariant measure. For manifolds M of dimension $d \geq 2$, we call a map T *expanding* if $\|D_xT(v)\| > \|v\|$ for all $x \in M$, $v \in T_xM$. It is known (Mañé [15] and Bowen [2], for example), that μ is the *unique* absolutely continuous invariant measure, that (T, μ) is mixing, and that the density $h : M \to \mathbb{R}^+$ of μ is positive and at least Hölder continuous.

Definition 1.2. *Let $\varphi, \psi : M \to \mathbb{R}$ be C^1 test functions. We define the* **correlation function** *of φ and ψ as*

$$\mathcal{C}_{\varphi,\psi}(k) = \left| \int_M \varphi \circ T^k \cdot \psi \, d\mu - \int_M \varphi \, d\mu \cdot \int_M \psi \, d\mu \right|. \tag{2}$$

By putting $\varphi = \chi_B$ and $\psi = \chi_A$, we see that (2) reduces to (1), and that $\mathcal{C}_{\chi_A,\chi_B}(k) \to 0$ as $k \to \infty$. In fact, by approximating φ and ψ by step functions it may be shown that $\mathcal{C}_{\varphi,\psi}(k) \to 0$ as $k \to \infty$ for all $\varphi, \psi \in L^2(M, \mu)$ is equivalent to (T, μ) mixing. We shall, however, restrict φ and ψ to be at least piecewise continuously differentiable, as we are then guaranteed that $\mathcal{C}_{\varphi,\psi}(k)$ approaches zero *exponentially* fast with n; (Bowen [2], Ruelle [18]). The test functions φ and ψ may be thought of as physical observables of our dynamical system. The special case of $\varphi = \psi$ gives what is known as the *autocorrelation function*:

$$\mathcal{C}_{\varphi,\varphi}(k) = \left| \int_M \varphi \circ T^k \cdot \varphi \, d\mu - \left(\int_M \varphi \, d\mu \right)^2 \right|. \tag{3}$$

$\mathcal{C}_{\varphi,\varphi}(k)$ gives an indication of how much the observable φ at time $t = 0$ is correlated with itself at time $t = k$.

Remark 1.3. It is reasonable to suppose that our physical observables vary smoothly in phase space. If we choose our observables φ and ψ from the larger spaces $L^2(M, \mu)$ or $C^0(M, \mathbb{R})$, pathological examples may be found for which the rate of decay of correlations is very slow (subexponential). For instance, in the case where $M = \mathbb{T}^2$ the 2-torus, $T : \mathbb{T}^2 \circlearrowleft$ is defined by the linear map $T(x_1, x_2) = (2x_1 + x_2, x_1 + x_2) \pmod 1$, and μ is normalised Lebesgue measure, Crawford and Cary [5] construct a C^0 test function φ for which $\mathcal{C}_{\varphi,\varphi}(k) = 2/k^2$.

Since (2) approaches zero at an exponential rate for $\varphi, \psi \in C^1(M, \mathbb{R})$, one may find constants $C = C(\varphi, \psi)$ and $0 < r_0 < 1$ such that $\mathcal{C}_{\varphi,\psi}(k) \leq C(\varphi, \psi) r^k$ for all $r > r_0$. We shall call the minimal such r_0 the *rate of decay of correlations* for the system (T, μ). Only relatively recently has work on rigorous bounds for the rate of decay of correlations been done; Rychlik [21], Liverani [13, 14]. Unfortunately, the estimates of [21] and [13, 14] are extremely conservative. For example, one may easily show that the rate of decay of correlations for the tent map with respect to C^1 test functions is 0.5. The estimates provided by Rychlik's and Liverani's papers are 0.9986 and 0.5 respectively. While the estimate of [13] is optimal in this case, we shall demonstrate later that the bounds quickly worsen as small nonlinearities are introduced into T.

One could consider a naive direct approach to the estimation of r_0 by scattering a very large number of test points $x \in M$ throughout M, distributed according to the measure μ. Once specific functions φ and ψ are chosen, the integrals in (2) could then be approximately evaluated as a finite sum over the scattered points. A linear fit of $\log \mathcal{C}_{\varphi,\psi}(k)$ vs. k would then estimate r_0. This approach is infeasible, however, because of the exponentially stretching nature of the map T. If the points are scattered so that they are roughly a distance ϵ apart, then nearby scatter points may be stretched apart to a distance of order the diameter of M after only $k = \log(\mathrm{diam}\, M/\epsilon)/\log(1/\lambda)$, where λ is the expansion constant ($\lambda = 1/\left(\inf_{x\in M} \inf_{v\in T_xM} \|D_xT(v)\|/\|v\|\right)$). Thus the number of scatter points required grows exponentially with k, making such a calculation infeasible, or at the very least, unreliable for low values of k. One can also not be sure that the particular functions φ and ψ used in this calculation yield an uncharacteristically low value for r_0.

Other methods of bounding the rate of decay are discussed in the final section. These methods compute rates of decay for test functions that are either real-analytic [3, 20] or of bounded variation [1] (one-dimensional systems). We argue that such function spaces are not good models of physical observables. We instead have concentrated on piecewise C^γ or C^1 test functions as many physical measurements naturally fall into this class.

To reiterate, our aim is to provide a rigorous computational method of bounding the value of r_0. We expect our method to provide much better bounds than those of [21] or [13, 14]. In addition, we believe our bounds to be of greater physical significance than those of [1] and [3]. In later sections we discuss the order of convergence of our bounds. In some instances we are able to compute resonant eigenfunctions of our system (T, μ), thus strengthening our bound to an *estimate* of the rate of decay. Particular features of our result are that it may be applied to multidimensional and hyperbolic systems, and that an estimate of the physical invariant measure is obtained "for free" with no extra work. The order of convergence of our bounds are also discussed.

2. Outline of Method

Without loss of generality, we henceforth assume that $\int_M \psi \, d\mu = 0$. Define the Perron-Frobenius operator $\mathcal{P}$ for expanding T by

$$\mathcal{P}\psi(x) = \sum_{y\in T^{-1}x} \frac{\psi(y)}{|\det D_yT|}. \tag{4}$$

We shall consider $\mathcal{P} : C^\gamma(M,\mathbb{C})\circlearrowleft$, where the Banach space $C^\gamma(M,\mathbb{C})$, $0 < \gamma \leq 1$ has norm

$$\|\psi\|_\gamma = \sup_{x\in M} |\psi(x)| + \sup_{x,y\in M} \frac{|\psi(x) - \psi(y)|}{\|x-y\|^\gamma} := |\psi|_\infty + |\psi|_\gamma. \tag{5}$$

We produce an upper bound for $\mathcal{C}_{\varphi,\psi}(k)$ by rewriting (2) in terms of the Perron-Frobenius operator:

$$\begin{aligned} \mathcal{C}_{\varphi,\psi}(k) &= \left| \int_M \varphi \circ T^k \cdot \psi \cdot h \, dm \right| \\ &= \left| \int_M \varphi \cdot \mathcal{P}^k(\psi \cdot h) \, dm \right| \end{aligned} \tag{6}$$

$$\leq \|\varphi\|_\gamma \|\mathcal{P}^k(\psi \cdot h)\|_\gamma$$
$$\leq \|\varphi\|_\gamma \|\psi \cdot h\|_\gamma H r^k. \quad (7)$$

Here $r > r_0$, the spectral radius of $\mathcal{P}$ restricted to $\mathbb{C}_\gamma^\perp = \{\psi \in C^\gamma(M, \mathbb{C}) : \int_M \psi \, d\mu = 0\}$. Thus the problem boils down to one of estimating the spectral radius of the Perron-Frobenius operator acting on C^γ test functions of zero μ-integral.

Definition 2.1. *We define the* **essential spectral radius** r_{ess} *to be the smallest nonnegative number for which elements of* $\sigma(\mathcal{P})$ *outside the disk* $\{z \in \mathbb{C} : |z| \leq r_{\text{ess}}\}$ *are isolated eigenvalues of finite multiplicity.*

The spectrum of $\mathcal{P}$ may now be divided into the part that lies inside the disk $|z| \leq r_{\text{ess}}$ and that which lies outside this disk.

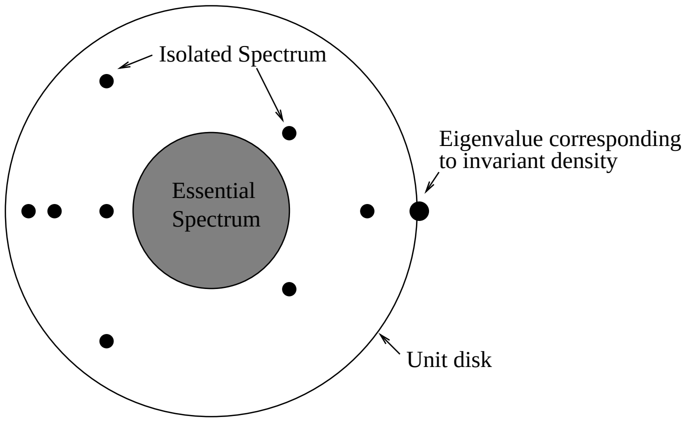

Fig. 1. Schematic representation of the spectrum (as a subset of the complex plane) of the Perron-Frobenius operator acting on the space of C^γ test functions

We are not concerned with the composition of the part of $\sigma(\mathcal{P})$ inside $|z| \leq \lambda^\gamma$. It is known (Ruelle, [19]) that $r_{\text{ess}} \leq \lambda^\gamma$, where $\lambda = 1/\left(\inf_{x\in M} \inf_{v \in T_x M} \|D_x T(v)\|/\|v\|\right)$. Collet & Isola [4] have better estimates for one-dimensional Markov maps, but these are difficult to compute. We can thus obtain reasonable bounds for the radius of the essential spectrum, and now need only worry about a bound for the radius of the remainder of the non-unit spectrum, that which consists of isolated eigenvalues of finite multiplicity. It is this remaining spectrum of $\mathcal{P}$ in the region $|z| > \lambda^\gamma$ that we wish to approximate.

We shall partition the state space M into a finite number of connected sets $A_1, \ldots, A_n$ and form the $n \times n$ transition matrix

$$P_{ij} = \frac{m(A_i \cap T^{-1} A_j)}{m(A_i)}, \quad (8)$$

where m is the Riemannian volume measure on M. One may think of the entry P_{ij} as representing the probability of a point in the region A_i moving into the region A_j in one step. Ulam [23] originally proposed this matrix as a finite-dimensional approximation to the Perron-Frobenius operator in the case where T was an expanding interval map. In [9] the author proved that provided that the regions $A_1, \ldots, A_n$ are carefully chosen, the matrix P is a very good approximation of the Perron-Frobenius operator for both multidimensional expanding maps and uniformly hyperbolic maps. The main result of [9] is that the left eigenvector of the stochastic matrix P defines a good approximation of the physical measure μ. In the present paper, we go on to show that the same matrix in fact gives us more information, namely a bound on the rate of decay of correlations for T with respect to the physical measure μ.

Theorem 2.2 (Main Result). *Let $T : M \circlearrowleft$ be a $C^{1+\gamma}$, $(0 < \gamma \leq 1)$, expanding (resp. Anosov) map of a compact d-dimensional (resp. 2-dimensional) Riemannian manifold M. Denote by $\{\mathfrak{P}_n\}_{n=n_0}^{\infty}$ a sequence of Markov partitions for T on M, with the property that* $\max_{A\in\mathfrak{P}_n} \operatorname{diam} A \to 0$ *as $n \to \infty$. Construct the matrix*

$$P_{n,ij} = \frac{m(A_{n,i} \cap T^{-1}A_{n,j})}{m(A_{n,j})}, \qquad A_{n,i}, A_{n,j} \in \mathfrak{P}_n, \quad 1 \leq i, j \leq \operatorname{card} \mathfrak{P}_n. \tag{9}$$

Denote by $\sigma'(P_n)$ the spectral values of P_n that lie in the region $|z| > \lambda^{\gamma'}$, and by $\sigma(\mathcal{P})$ the spectrum of $\mathcal{P} : C^{\gamma'}(M, \mathbb{C})\circlearrowleft$, $(0 < \gamma' < \gamma)$. Then given $\epsilon > 0$, there is $n_\epsilon \in \mathbb{Z}^+$ such that

$$\sigma(\mathcal{P}) \setminus \{|z| \leq \lambda^{\gamma'}\} \subset B_\epsilon(\sigma'(P_n)) \qquad \textit{for all } n > n_\epsilon.$$

Simply put, the above theorem says that the spectral values of the matrices P_n lying outside the disk $\{|z| \leq \lambda^{\gamma'}\}$ converge to a set containing the isolated spectrum of $\mathcal{P} : C^{\gamma'}(M, \mathbb{C})\circlearrowleft$ outside this disk. The matrices P_n are relatively simple to compute, and provide us with a means of approximating the important portion of the isolated spectrum of the Perron-Frobenius operator. In the case of an example computation, if it so happens that all of $\sigma(P_n) \setminus \{1\}$ is contained in the disk $|z| \leq \lambda^{\gamma'}$, then we take $\lambda^{\gamma'}$ as our bound for the rate of decay of correlations.

Remarks 2.3. (i) Finite Markov partitions exist for the class of maps that we are considering [2]. If T is Anosov, the behaviour of the boundaries of the Markov partition sets become difficult to control in dimensions $d \geq 3$ (see Mañé, p.184 for a discussion).

(ii) For Anosov T, the operator $\mathcal{P}$ is not the Perron-Frobenius operator for the full map T. A standard construction is followed, where the dynamics of T is projected onto the unstable boundaries of the Markov partition. This projected dynamics induces an expanding map on the unstable boundaries, and we *define* $\mathcal{P}$ for T to be the Perron-Frobenius operator for this induced expanding map.

(iii) The matrix defined in (9) is slightly different from that of (8). The two, however, are related via a similarity transformation using the matrix $Q_{ij} = \delta_{ij} m(A_i)$ and thus have the same eigenvalues.

(iv) We have enlarged our space of test functions from $C^\gamma(M, \mathbb{C})$ to $C^{\gamma'}(M, \mathbb{C})$, $\gamma' < \gamma$ for technical reasons. This slight change makes no practicable difference to computations.

Corollary 2.4. *Let $M, T, \mathcal{P}$, and P_n be defined as in Theorem 2.2. An upper bound for the rate of decay of correlations of T with respect to $C^{\gamma'}$ test functions $(0 < \gamma' < \gamma)$ is:*

Expanding T:

$$\max\left\{\lambda^{\gamma'}, \lim_{n\to\infty} \max_{z\in\sigma(P_n)\setminus\{1\}} |z|\right\}. \tag{10}$$

Anosov T:

$$\left(\max\left\{\lambda^{\gamma'}, \lim_{n\to\infty} \max_{z\in\sigma(P_n)\setminus\{1\}} |z|\right\}\right)^{1/3}. \tag{11}$$

Proof of Corollary. The expanding case follows from Theorem 2.2 and the inequality (7). The Anosov situation is described in Sect. 6. □

Thus the Ulam matrices provide us with estimates for the isolated spectrum of the Perron-Frobenius operator; it is these isolated spectral values that cause the estimates of [21] and [13] to be so poor. If there are no isolated eigenvalues for $\mathcal{P}$ (besides unity) then we take the much more reasonable value $\lambda^{\gamma'}$ as a bound for the decay rate.

3. Coding

We deal with the expanding situation first; the Anosov case will follow from this by extracting its expanding part as described in Sect. 6. It is convenient for us to code the dynamics of our map T using the symbolic dynamics provided by our Markov partition $\mathfrak{P}$. We shall determine a bound for the rate of decay of our induced system and use this as a bound for our original smooth system. Define

$$A_{ij} = \begin{cases} 0, & \text{if Int } A_i \cap T^{-1} \text{ Int } A_j = \emptyset \\ 1, & \text{otherwise.} \end{cases}$$

The matrix A defines a subset $\Sigma_A^+ = \{\xi \in \Sigma^+ : A_{\xi_i\xi_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}^+\} \subset \{1,2,\ldots,s\}^{\mathbb{Z}^+} = \Sigma^+$ of "allowable" sequences. This subset Σ_A^+ is invariant under the one-sided left-shift $\sigma : \Sigma^+ \circlearrowleft$ defined by $[\sigma(\xi)]_i = \xi_{i+1}$. Denote by $[\xi_0,\ldots,\xi_{N-1}]$ the cylinder set $\{\eta \in \Sigma_A^+ : \eta_0 = \xi_0, \ldots, \eta_{N-1} = \xi_{N-1}\}$. We define $\pi : \Sigma^+ \to M$ to be the semiconjugacy between Σ^+ and M that maps the sequence $(\xi_0, \xi_1, \ldots)$ onto the unique point x satisfying $T^i x \in A_{\xi_i}$ for all $i \geq 0$. We have the following commutative diagram; $\pi : \Sigma_A^+ \to M$ is continuous and surjective; see Bowen [2] for details:

$$\begin{array}{ccc} \Sigma_A^+ & \xrightarrow{\sigma} & \Sigma_A^+ \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{T} & M \end{array}$$

The metric on Σ_A^+ will be defined by $d_\theta(\xi,\eta) = \theta^N$, where N is the maximal integer for which $\xi_i = \eta_i$, $i = 0,\ldots,N-1$, and $0 < \theta < 1$ is fixed. Define a norm on the space of complex-valued functions on Σ_A^+ by

$$\|\phi\|_\theta = |\phi|_\infty + \sup_{N\geq 0} \sup_{\substack{\xi,\eta\in\Sigma_A^+ \\ \xi_0=\eta_0,\ldots,\xi_{N-1}=\eta_{N-1}}} \frac{|\phi(\xi)-\phi(\eta)|}{\theta^N} := |\phi|_\infty + |\phi|_\theta. \tag{12}$$

Lemma 3.1. *If φ is γ-Hölder on M ($0 < \gamma \leq 1$), the function $\varphi \circ \pi$ is Lipschitz on Σ_A^+ with respect to the d_{λ^γ} metric.*

Proof. Let $\varphi \in C^\gamma(M, \mathbb{R})$; that is, there is a constant $C_2 < \infty$ such that $|\varphi(x) - \varphi(y)| \leq C_2 \|x - y\|^\gamma$ for all $x, y \in M$. Let $|\varphi|_\gamma$ denote the minimal such constant. We note that if $1/\lambda := \inf_{x \in M} \inf_{v \in T_x M} \|D_x T(v)\| / \|v\|$ we may find a universal constant $C_1 < \infty$ such that if $\xi_i = \eta_i$ for $i < N$, then $\pi([\xi_0, \xi_1, \ldots, \xi_{N-1}]) \subset M$ has diameter less than $C_1 \lambda^{-N}$ (Ruelle [18]). Also suppose that $d_\theta(\xi, \eta) = \theta^N$. Then

$$\begin{aligned} |\varphi \circ \pi(\xi) - \varphi \circ \pi(\eta)| &\leq |\varphi|_\gamma \|\pi(\xi) - \pi(\eta)\|^\gamma \\ &\leq |\varphi|_\gamma \cdot \left(C_1 \lambda^N\right)^\gamma \\ &= |\varphi|_\gamma C_1^\gamma \left(\lambda^\gamma\right)^N \\ &= |\varphi|_\gamma C_1^\gamma d_{\lambda^\gamma}(\xi, \eta). \qquad \Box \end{aligned} \tag{13}$$

We denote this space of Lipschitz functions on Σ_A^+ by

$$\mathfrak{F}_{\lambda^\gamma} = \{\phi : \Sigma_A^+ \to \mathbb{C} : |\phi|_{\lambda^\gamma} < \infty\}.$$

One has

$$\begin{aligned} \|\varphi \circ \pi\|_{\lambda^\gamma} &= |\varphi \circ \pi|_\infty + |\varphi \circ \pi|_{\lambda^\gamma} \\ &= |\varphi|_\infty + \sup_{\xi, \eta \in \Sigma_A^+} \frac{|\varphi \circ \pi(\xi) - \varphi \circ \pi(\eta)|}{d_{\lambda^\gamma}(\xi, \eta)} \\ &\leq |\varphi|_\infty + |\varphi|_\gamma C_1^\gamma \qquad \text{by (13)} \\ &\leq \max\{1, C_1^\gamma\} \|\varphi\|_\gamma. \end{aligned} \tag{14}$$

We are now in a position to define a rate of decay for our symbolic system. Given a T-invariant probability measure μ on M, there is a σ-invariant probability measure ν on Σ_A^+ such that $\pi^* \nu = \mu$ ([2] p.91); thus $\psi \circ \pi$ has ν integral zero. The measure ν plays the role of the "physical" measure μ for our symbolic system. Denote by $\mathfrak{F}_{\lambda^\gamma}^\perp$ the subspace of $\mathfrak{F}_{\lambda^\gamma}$ with zero ν integral, and define the Perron-Frobenius operator (or transfer operator) $\mathcal{L} : (\mathfrak{F}_{\lambda^\gamma}, \|\cdot\|_{\lambda^\gamma}) \circlearrowleft$ by

$$(\mathcal{L}\phi)(\xi) = \sum_{\eta \in \sigma^{-1}\xi} \frac{\phi(\eta)}{|\det D_{\pi(\eta)} T|}. \tag{15}$$

$\mathfrak{F}_{\lambda^\gamma}^\perp$ is the space of test functions on Σ_A^+ that corresponds to the space $\mathbb{C}_\gamma^\perp$ of test functions on the smooth space M. $\mathcal{L}$ is the operator corresponding to $\mathcal{P}$. Recall that if $x \mapsto |\det D_x T|$ is C^γ then $\xi \mapsto |\det D_{\pi(\xi)} T|$ is in $\mathfrak{F}_{\lambda^\gamma}$.

We use the symbolic dynamics to rewrite (6) as follows: ($h_\nu \in \mathfrak{F}_{\lambda^\gamma}$ satisfies $\mathcal{L}(h_\nu) = h_\nu$ and m_Σ is the probability measure on Σ_A^+ that assigns equal weight to cylinder sets of equal length. $\nu = h_\nu \cdot m_\Sigma$ in the same way that $\mu = h \cdot m$.)

$$\begin{aligned} \left|\int_M \varphi \circ T^k \cdot \psi \, d\mu\right| &= \left|\int_{\Sigma_A^+} (\varphi \circ T^k) \circ \pi \cdot \psi \circ \pi \, d\nu\right| \\ &= \left|\int_{\Sigma_A^+} (\varphi \circ T^k) \circ \pi \cdot (\psi \cdot h_\nu) \circ \pi \, dm_\Sigma\right| \\ &= \left|\int_{\Sigma_A^+} (\varphi \circ \pi) \cdot \mathcal{L}^k((\psi \cdot h_\nu) \circ \pi) \, dm_\Sigma\right| \end{aligned}$$

$$\leq \|\varphi \circ \pi\| \|\mathcal{L}^k((\psi \cdot h_\nu) \circ \pi)\|_{\lambda^\gamma}$$
$$\leq H_2 \|\varphi \circ \pi\|_{\lambda^\gamma} \|((\psi \cdot h_\nu) \circ \pi\|_{\lambda^\gamma} R^k$$
$$\leq H_3 \|\varphi\|_\gamma \|\psi \cdot h_\nu\|_\gamma R^k. \tag{16}$$

Here R is any positive number greater than R_0, the spectral radius of $\mathcal{L}|_{\mathfrak{F}^\perp_{\lambda^\gamma}}$.

Remark 3.2. The class of functions $\mathfrak{F}^\perp_{\lambda^\gamma}$ is larger than the space $\mathbb{C}^\perp_\gamma \circ \pi = \{\varphi \circ \pi : \varphi \in \mathbb{C}^\perp_\gamma\}$. Functions $\varphi \circ \pi$ with φ discontinuous on boundaries of partition sets in M (and inverse images of the boundaries) are allowed in $\mathfrak{F}^\perp_{\lambda^\gamma}$ because $\pi(\xi)$ and $\pi(\eta)$ may be very close, but will be assigned a distance of 1 if $\xi_0 \neq \eta_0$. Thus R_0, the spectral radius of $\mathcal{L}|_{\mathfrak{F}^\perp_{\lambda^\gamma}}$, will in general be larger than r_0, the spectral radius of $\mathcal{P}|_{\mathbb{C}^\perp_\gamma}$.

4. Approximating $\mathcal{L}$

There are two main ingredients to the approximation of the spectrum of the operator $\mathcal{L}$. The first thing we do is define a simpler operator $\mathcal{L}_N$ that is close in norm to $\mathcal{L}$. Because $\mathcal{L}_N$ is close to $\mathcal{L}$ in the operator norm topology, standard perturbation theory tells us that their spectra are also close. The second step is to restrict the simpler operator $\mathcal{L}_N$ to a small (finite-dimensional) invariant subspace. The operator restricted to this finite-dimensional space now has a matrix representation, and its spectrum is easily computed as the eigenvalues of this matrix.

4.1. A Simpler Operator. To make finding the spectrum of $\mathcal{L}$ tractable, we construct a simpler operator

$$(\mathcal{L}_N \phi)(\xi) = \sum_{\eta \in \sigma^{-1}\xi} g_N \phi(\eta). \tag{17}$$

We shall think of $g_N : \Sigma_A^+ \to \mathbb{R}^+$ as an approximation to $g(\xi) := 1/|\det D_{\pi(\xi)} T|$ that is constant on N-cylinders $[\xi_0, \xi_1, \ldots, \xi_{N-1}]$. Formally, define

$$g_N = \frac{m(\pi([\xi_0, \ldots, \xi_{N-2}]) \cap T^{-1}\pi([\xi_1, \ldots, \xi_{N-1}]))}{m(\pi([\xi_1, \ldots, \xi_{N-1}]))}; \tag{18}$$

see Fig. 2.

Lemma 4.1. $|g - g_N|_\infty \leq |g|_\theta \theta^N$.

Proof. Clearly,

$$\inf_{\xi \in [\xi_0, \ldots, \xi_{N-1}]} |\det D_{\pi(\xi)} T| \leq \frac{m(\pi([\xi_1, \ldots, \xi_{N-1}]))}{m(\pi([\xi_0, \xi_1, \ldots, \xi_{N-1}]))} \leq \sup_{\xi \in [\xi_0, \ldots, \xi_{N-1}]} |\det D_{\pi(\xi)} T|,$$

so that

$$\inf_{\xi \in [\xi_0, \ldots, \xi_{N-1}]} \frac{1}{|\det D_{\pi(\xi)} T|} \leq \frac{m(\pi([\xi_0, \xi_1, \ldots, \xi_{N-2}]) \cap T^{-1}\pi([\xi_1, \ldots, \xi_{N-1}]))}{m(\pi([\xi_1, \ldots, \xi_{N-1}]))}$$
$$\leq \sup_{\xi \in [\xi_0, \ldots, \xi_{N-1}]} \frac{1}{|\det D_{\pi(\xi)} T|}.$$

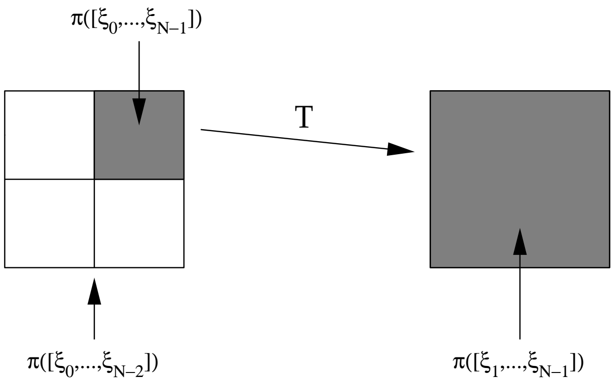

Fig. 2. Schematic representation of the sets in M involved in defining g_N. We identify $\pi([\xi_0, \ldots, \xi_{N-2}])$ with $A_{n,i}$ and $\pi([\xi_1, \ldots, \xi_{N-1}])$ with $A_{n,j}$ in Eq. (9)

Hence,

$$\begin{aligned} &|g - g_N|_\infty \\ &\leq \max_{\xi_0,\ldots,\xi_{N-1}\in\{1,\ldots,r\}} \left| \sup_{\xi\in[\xi_0,\ldots,\xi_{N-1}]} \frac{1}{|\det D_{\pi(\xi)}T|} - \inf_{\xi\in[\xi_0,\ldots,\xi_{N-1}]} \frac{1}{|\det D_{\pi(\xi)}T|} \right| \\ &= \sup_{\substack{\xi,\eta\in\Sigma_A^+ \\ \xi_0=\eta_0,\ldots,\xi_{N-1}=\eta_{N-1}}} |g(\xi) - g(\eta)| \\ &\leq |g|_\theta \theta^N, \end{aligned}$$

where we have used the definition of $|g|_\theta$ in Eq. (12). □

By standard perturbation theory (e.g. Kato [12] or Dunford & Schwartz [6] Lemma 6.3) if the operators $\mathcal{L}$ and $\mathcal{L}_N$ are close in operator norm, then their spectra are also close in the sense of Hausdorff distance. We may simply bound the norm of the difference $\mathcal{L} - \mathcal{L}_N$ as follows.

$$\begin{aligned} \|(\mathcal{L} - \mathcal{L}_N)\phi\|_\theta &\leq \sum_{\eta\in\sigma^{-1}\xi} \|\phi\|_\theta \left\| \frac{1}{|\det D_{\pi(\eta)}T|} - g_N(\eta) \right\|_\theta \\ &\leq \Phi\|\phi\|_\theta \left\| \frac{1}{|\det D_{\pi(\cdot)}T|} - g_N(\cdot) \right\|_\theta . \end{aligned} \tag{19}$$

where Φ is the maximum number of inverse branches of T. It is shown in Froyland [9], Lemma 4.1 that $g_N \circ \pi^{-1}(\cdot) \to 1/|\det D.T|$ uniformly in the C^0 topology on M. However, this corresponds to convergence of $g_N(\cdot) \to 1/|\det D_{\pi(\cdot)}T|$ in $\|\cdot\|_\theta$ with $\theta = 1$ and this is not enough for convergence of $\mathcal{L}_N$ to $\mathcal{L}$ in the $\|\cdot\|_\theta$ operator norm, where $0 \leq \theta < 1$. To obtain the required convergence we use a small trick, namely slightly relaxing our norm and enlarging our space of test functions.

Proposition 4.2. *Let $g \in \mathfrak{F}_\theta^+$ and suppose g_N is constructed as above. For any $0 < \theta < \theta' < 1$ we have that*

$$|g - g_N|_{\theta'} \leq |g|_\theta \left(\frac{\theta}{\theta'}\right)^N, \qquad N \geq 0. \tag{20}$$

Proof. The only property required of the g_N's is that listed in Lemma 4.1. We refer the reader to Proposition 1.3, Parry & Pollicott [16] for a proof. □

If T is $C^{1+\gamma}$, then $|\det DT|$ is C^γ. We shall consider $\mathcal{L}$ to be acting on the Banach space $(\mathfrak{F}^{\perp}_{\lambda^{\gamma'}}, \|\cdot\|_{\lambda^{\gamma'}})$, where $0 < \gamma' < \gamma$. For brevity we shall sometimes write $\|\cdot\|_{\theta'}$ to mean $\|\cdot\|_{\lambda^{\gamma'}}$. By virtue of Proposition 4.2 and Lemma 4.1, $g_N \to 1/|\det D_{\pi(\cdot)}T|$ in the $\|\cdot\|_{\theta'}$ norm (see Eq. (22)), and so $\|(\mathcal{L} - \mathcal{L}_N)\|_{\theta'} \to 0$. As we now have norm convergence of $\mathcal{L}_N$ to $\mathcal{L}$ we may apply standard perturbation results to the spectrum of $\mathcal{L}$. The following lemma is a suitably modified version of Lemma 6.3 [6].

Lemma 4.3. *If*

$$\|\mathcal{L} - \mathcal{L}_N\|_{\theta'} \le \|R(z, \mathcal{L}_N)\|^{-1}_{\theta'} := \|(\mathcal{L}_N - zI)^{-1}\|^{-1}_{\theta'}, \tag{21}$$

for all $z \in \mathbb{C} \setminus B_\epsilon(\sigma(\mathcal{L}_N))$*, then*

$$\sigma(\mathcal{L}) \subset B_\epsilon(\sigma(\mathcal{L}_N)),$$

where $B_\epsilon(\sigma(\mathcal{L}_N))$ *denotes an* ϵ*-neighbourhood of the spectrum of* $\mathcal{L}_N$*.*

Proof. Recall that if $\mathcal{L}_N - zI$ is invertible, then so too are all operators in a ball centred at $\mathcal{L}_N - zI$ of radius $\|(\mathcal{L}_N - zI)^{-1}\|^{-1}_{\theta'}$. Thus if $\|(\mathcal{L}_N - zI) - (\mathcal{L} - zI)\|_{\theta'} \le \|(\mathcal{L}_N - zI)^{-1}\|^{-1}_{\theta'}$ for $z \in \mathbb{C} \setminus B_\epsilon(\sigma(\mathcal{L}_N))$ then $\mathcal{L} - zI$ is invertible on $\mathbb{C} \setminus B_\epsilon(\sigma(\mathcal{L}_N))$ and so $\sigma(\mathcal{L}) \subset B_\epsilon(\sigma(\mathcal{L}_N))$. □

We summarise our findings so far.

Proposition 4.4. $\mathcal{H}(\sigma(\mathcal{L}_N), \sigma(\mathcal{L})) \to 0$ *as* $N \to \infty$*, where* $\mathcal{H}$ *denotes the Hausdorff metric on* $\mathbb{C}$ *defined by* $\mathcal{H}(E, F) = \max\{\sup_{x\in E} \operatorname{dist}(x, F), \sup_{y\in F} \operatorname{dist}(E, y)\}$ *for* $E, F \subset \mathbb{C}$*.*

Proof. Putting together (19), Lemma 4.1, and Proposition 4.2, we see that

$$\begin{aligned} \|\mathcal{L} - \mathcal{L}_N\|_{\theta'} &\le \Phi\left(|g - g_N|_\infty + |g - g_N|_{\theta'}\right) \\ &\le \Phi\left(|g|_{\theta'}\left(\theta'\right)^N + |g|_\theta \left(\frac{\theta}{\theta'}\right)^N\right) \\ &\to 0 \quad \text{as } N \to \infty. \end{aligned} \tag{22}$$

By applying Lemma 4.3 twice (reversing the roles of $\mathcal{L}$ and $\mathcal{L}_N$) we are done. □

We shall now show how to extract the isolated part of the spectrum of $\mathcal{L}_N$.

4.2. A Smaller Space. The operator $\mathcal{L}_N$ has a finite dimensional invariant subspace, namely those functions that are constant on N-cylinders $[\xi_0, \xi_1, \ldots, \xi_{N-1}]$ in Σ^+_A. Recall that $\pi([\xi_0, \xi_1, \ldots, \xi_{N-1}]) = A_{\xi_0} \cap T^{-1}A_{\xi_1} \cap \cdots \cap T^{-(N-1)}A_{\xi_{N-1}}$, so that functions $\phi : \Sigma^+_A \to \mathbb{R}$ piecewise constant on N-cylinders correspond to functions $\varphi : M \to \mathbb{R}$ piecewise constant on t he refined Markov partition $\bigvee_{i=0}^{N-1} T^{-i}\mathfrak{P}$. Let $V_N = \operatorname{sp}\{\chi_{[\xi_0,\xi_1,\ldots,\xi_{N-1}]} : \xi_i \in \{1, \ldots, r\}\} \subset \mathfrak{F}_{\theta'}$ be the span of the test functions constant on N-cylinders. We may now define two operators, namely $\mathcal{L}_N|_{V_N} : V_N \circlearrowleft$ (the restriction of $\mathcal{L}_N$ to V_N), and $\mathcal{L}_N/V_N : \mathfrak{F}_{\theta'}/V_N \circlearrowleft$ (the quotient operator on the quotient space $\mathfrak{F}_{\theta'}/V_N$). There are three very important points regarding these induced operators:

(i) The operator $\mathcal{L}_N|_{V_N}$ has matrix representation

$$[P_n]_{ij} = \frac{m(A_{n,i} \cap T^{-1}A_{n,j})}{m(A_{n,j})},$$

where the partition $\mathfrak{P}_n = \{A_{n,1}, \dots, A_{n,n}\}$ in M is formed from the images (under π) of the N-cylinder sets. That is, $\mathfrak{P}_n = \{\pi([\xi_0, \dots, \xi_{N-1}]) : [\xi_0, \dots, \xi_{N-1}] \subset \Sigma_A^+\}$. This is clear from the construction of the weight function g_N in (18).

(ii) The spectrum of $\mathcal{L}_N$ is contained in the union of the spectra of $\mathcal{L}_N|_{V_N}$ and $\mathcal{L}_N/V_N$; symbolically,

$$\sigma(\mathcal{L}_N) \subset \sigma(\mathcal{L}_N|_{V_N}) \cup \sigma(\mathcal{L}_N/V_N).$$

See, for example, Erdelyi and Lange [7]. Note that the spectrum of $\mathcal{L}_N|_{V_N}$ is simply the set of eigenvalues of the matrix representing $\mathcal{L}_N|_{V_N}$, that is, the eigenvalues of the matrix P_n defined above.

(iii) The spectral radius of $\mathcal{L}_N/V_N$ in $\mathfrak{F}_{\theta'}/V_N$ is bounded above by θ. This is proven in Pollicott [17].

By putting these three facts together, we see that the part of $\sigma(\mathcal{L}_N)$ contained outside the disk $\{|z| > \theta'\}$ must be wholly contained in $\sigma(\mathcal{L}_N|_{V_N})$. Further, $\sigma(\mathcal{L}_N|_{V_N})$ may be simply computed as the eigenvalues of the matrix P_n. Thus

$$R_{0,N} := r_\sigma(\mathcal{L}_N|_{\mathfrak{F}_{\theta'}^\perp}) \le \max\left\{\theta', \max_{z \in \sigma(P_n)\setminus\{1\}} |z|\right\}. \tag{23}$$

We now have an upper bound for the spectral radius of the simpler operator $\mathcal{L}_N$ acting on the space $\mathfrak{F}_{\theta'}$, and by our perturbation theorems, this will be close to an upper bound for $\mathcal{L}$ acting on $\mathfrak{F}_{\theta'}$.

Proof (of Main Result). We first note that if φ is an eigenfunction of $\mathcal{P}$, then $\varphi \circ \pi$ is an eigenfunction of $\mathcal{L}$. Thus $\sigma(\mathcal{P}) \setminus \{|z| \le \lambda^{\gamma'}\} \subset \sigma(\mathcal{L}) \setminus \{|z| \le \theta'\}$. By Proposition 4.4 we have in particular that $\mathcal{H}(\sigma(\mathcal{L}) \setminus \{|z| \le \theta'\}, \sigma(\mathcal{L}_N) \setminus \{|z| \le \theta'\}) \to 0$ as $N \to \infty$. Finally, we have just shown that $\sigma(\mathcal{L}_N) \setminus \{|z| \le \theta'\} \subset \sigma(P_n)$. By putting these three observations together we are done. □

Remark 4.5. Corollary 3.3 of [19] states that the part of the discrete spectrum of our transfer operator $\mathcal{L}$ that we are considering does not vary as we vary the norm $\|\cdot\|_\theta$ and function space $\mathfrak{F}_\theta$ with θ. Formally, the spectra of $\mathcal{L} : (\mathfrak{F}_\theta, \|\cdot\|_\theta) \circlearrowleft$ and $\mathcal{L} : (\mathfrak{F}_{\theta'}, \|\cdot\|_{\theta'}) \circlearrowleft$ coincide in the region $\{z : |z| > \theta'\}$. In particular, this tells us that the choice of θ' is not overly important in practice.

5. When T is Anosov

In the case where T is Anosov (all of M is uniformly hyperbolic), one must not deal directly with the full Perron-Frobenius operator of T, as the action of $\mathcal{P}$ would be to stretch in unstable directions (decrease the slope of a test function) and to compress in stable directions (increase the slope of a test function). Thus under the action of $\mathcal{P}$, the C^γ norm of test functions would most likely *increase* with each application of $\mathcal{P}$. We instead induce an expanding map from T on the unstable boundaries of a Markov partition as described in Remark 2.3 (ii). This induced expanding map will have its own Perron-Frobenius operator, and we may calculate its rate of mixing as before. We then

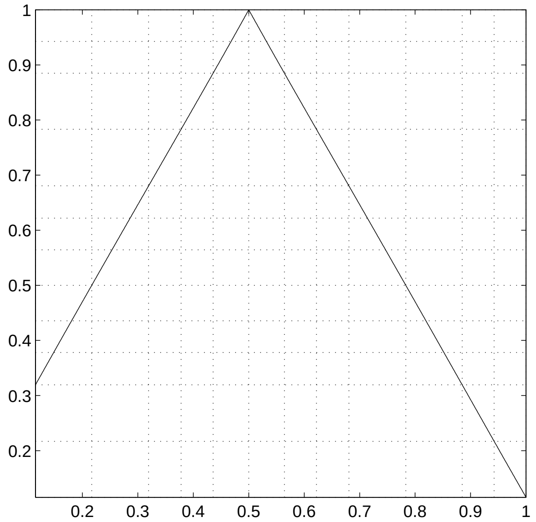

Fig. 3. Graph of the piecewise quartic interval map T as defined by (24)

appeal to a standard result to show that if R_0 is a bound for the rate of mixing of the induced expanding map, then $R_0^{1/3}$ is a bound for the rate of mixing of the full Anosov map. The advantage of our matrix technique is that the induced expanding map *need never be constructed*. We compute the matrices P_n as before using the full map T, and simply take the cube root of the maximal non-unit eigenvalue to obtain a bound.

It is shown in [9] Lemmas 7.1 and 7.3 that the matrix P_n as defined in (9) may be taken via a similarity transformation into the matrix representing $\mathcal{L}_N|_{V_N}$ (here $\mathcal{L}_N$ is the Perron-Frobenius operator for the induced expanding map acting on test functions on symbol space). Thus as in the expanding case, the eigenvalues of P_n outside the disk $\{|z| \leq \lambda^{\gamma'}\}$ coincide with the spectrum of $\mathcal{L}_N$ outside $\{|z| \leq \lambda^{\gamma'}\}$. We calculate P_n rather than the true matrix representation of $\mathcal{L}_N|_{V_N}$ as it is much easier to compute and does not require the construction of the induced expanding map.

The fact that we may take $R_0^{1/3}$ as a bound for the rate of decay of correlations for C^γ test functions, when R_0 is a bound for the induced expanding map, can be found in [16], Proposition 2.4, for example.

6. An Example

For ease of presentation, we illustrate our technique for a one-dimensional system. The map that we choose is a piecewise quartic interval map with two inverse branches, defined by[1]

$$Tx = \begin{cases} 0.9982x^4 - 0.8104x^3 + 0.1390x^2 + 1.7821x + 0.1131, & 0.1154 \leq x \leq 1/2 \\ 1.1572x^4 - 3.5886x^3 + 4.0826x^2 - 3.7830x + 2.2471, & 1/2 \leq x \leq 1 \end{cases} \tag{24}$$

[1] All numerical values in this section are approximate only.

This rather complicated-looking piecewise polynomial is just a slight nonlinear perturbation of a piecewise linear Markov map; see Fig. 3. The perturbation is so slight, that the graph of T appears linear. That T is not linear is evident from the graph of the unique invariant density h of T as shown in Fig. 4. If T were linear, this invariant

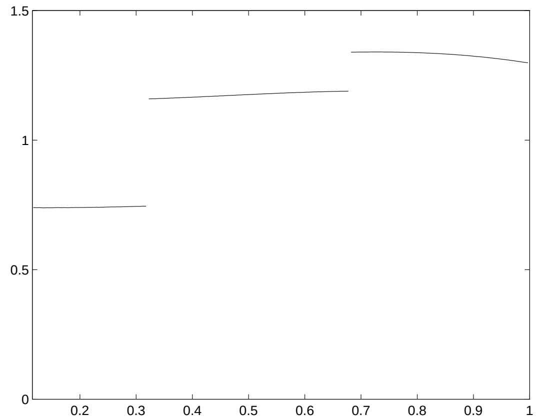

Fig. 4. Graph of the unique invariant density h of T

density would be piecewise constant. The density h defines a probability measure μ, and as h is everywhere positive, the Birkhoff theorem tells us that *Lebesgue* almost all $x \in [0.1154, 1]$ exhibit μ. Clearly, μ is a natural candidate for the physical measure of T. We are interested in how quickly initial blobs of mass in state space distribute themselves according to the measure μ under the action of T. That is, we want an estimate of the rate of mixing, or the rate of decay of correlations for T with respect to the measure μ.

The value of $\lambda = 1/\left(\min_{x\in[0.1154,1]} |T'(x)|\right)$ is approximately 0.5706. This shall be an initial bound for our rate of decay. As T is piecewise C^2 we choose $\mathcal{P}$ to act on the space of piecewise C^γ real-valued test functions, $\gamma < 1$. Typically, if there are kinks in the smoothness of T on a set $K \subset M$, we allow breaks in the smoothness of the test functions at $B = \bigcup_{i>0} T^i K$. In our example, $K = \{1/2\}$ and $B = \{T^3(1/2), T^4(1/2)\} = \{0.3194, 0.6806\}$ (as $T^4(1/2)$ is fixed under T; we have discarded the endpoints of $M = [0.1154, 1]$.) Thus $\mathcal{P}$ will act on test functions in $C^\gamma(M, \mathbb{R})$, $\gamma < 1$, with breaks allowed at 0.3194 and 0.6806. The norm for this space is $\|\cdot\|_\gamma$ as defined in (5) where each pair $x, y \in M$ are both elements of one of the three subintervals $[0.1154, 0.3194], [0.3194, 0.6806], [0.6806, 1]$. Our strategy is as follows. Choose $\gamma < 1$. If the Perron-Frobenius operator for T has no isolated spectrum, then we take 0.5706^γ as our upper bound. If, however, there are some isolated eigenvalues of the Perron-Frobenius operator lying outside the disk $\{|z| \leq 0.5706^\gamma\}$, we will approximate these as eigenvalues of our matrices P_n, and use these values as our upper bounds instead. The map T has been chosen for our example because it is close to a map $\tilde{T}$ whose Perron-Frobenius operator $\tilde{P}$ is known to have some eigenvalues lying outside the region $\{|z| \leq \lambda\}$. The author is grateful to Viviane Baladi for communicating the piecewise linear map $\tilde{T}x = -1.7525|x - 1/2| + 1$ as an example of a map with this property. Our nonlinear map T is a small perturbation of $\tilde{T}$ that demonstrates a nontrivial application of our technique.

An initial Markov partition of $\mathfrak{P} = \{[0.1154, 0.3194], [0.3194, 0.5], [0.5, 0.6806], [0.6806, 1]\}$ is formed. Refined Markov partitions are constructed by simply taking the join of $\mathfrak{P}$ with its inverse images, that is, define $\mathfrak{P}^{(N)} = \bigvee_{i=0}^{N-1} T^{-i}\mathfrak{P}$, where $\mathfrak{A} \vee \mathfrak{B} := \{A \cap B : A \in \mathfrak{A}, B \in \mathfrak{B}\}$. The 12 set partition $\mathfrak{P}^{(2)}$ is illustrated in Fig. 3. At each stage of refinement, a transition matrix P_n is constructed (note that P_n is an $n \times n$ matrix, whereas N refers to the number of inverse iterates required to construct the partition; thus P_{12} is the matrix for corresponding to $\mathfrak{P}^{(2)}$). The absolute values of the maximum non-unit eigenvalues of the sequence of matrices for $N = 0, \ldots, 9$ are given in Table 1.

Table 1. Estimates for the magnitude of the largest non-unit eigenvalue of the transfer operator of $\mathcal{L}$

Inverse Iterates N	Bound for rate of decay $\max_{z\in\sigma(P_n)\setminus\{1\}} \lvert z\rvert$
0	0.60037
1	0.59687
2	0.59649
3	0.59846
4	0.59815
5	0.59812
6	0.59823
7	0.59812
8	0.59819
9	0.59816

Figure 5 shows the position of the eigenvalues of P_{212} (corresponding to $N = 7$) relative to the region bounding the essential spectrum. From the above results, we can

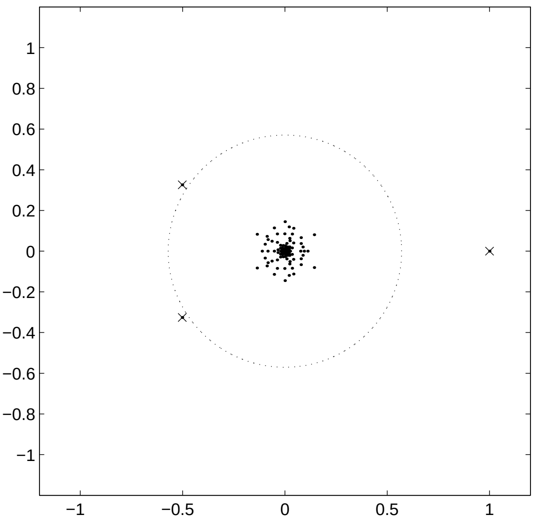

Fig. 5. A plot of the spectrum of P_{212} (the transition matrix for T constructed from a Markov partition refined from 7 inverse iterations). The dotted circle is given by $\{|z| = 0.5706\}$; it is known that the essential spectrum is contained in this region. The crosses represent the eigenvalues of P_{212} that are estimating the isolated spectrum of $\mathcal{P}$

fairly confidently state that $\mathcal{P}$ *does* have a non-trivial isolated spectrum, and that therefore, it is this isolated spectrum that will control the rate of decay. Corollary 2.4 says that a bound for the rate of decay is $\max\left\{\lambda^\gamma, \lim_{n\to\infty} \max_{z\in\sigma(P_n)\setminus\{1\}} |z|\right\}$. We thus take 0.599 as a safe upper bound for the mixing rate. By Remark 4.5, 0.599 is also the rate of decay with respect to piecewise C^1 test functions. In Sect. 8 we shall see that not only is 0.599 a bound, but that 0.5982 is an *estimate* of the rate of decay. This result will follow from an analysis of the eigenvectors of P_n.

As mentioned earlier, if it so happened that all of the eigenvalues of P_{212} were contained in $\{|z| \le 0.5706^\gamma\}$, we would simply take 0.5706^γ as our upper bound instead.

7. Order of Convergence of the Spectrum

In this section we consider the situation where $\mathcal{L}$ has nontrivial isolated spectrum. We wish to know the rate at which the isolated spectrum of $\mathcal{L}_N$ approaches that of $\mathcal{L}$. Recall that any isolated spectral values of $\mathcal{L}_N$ appear as eigenvalues of the matrix P_n, and that the spectral radius of $\mathcal{L}_N|_{\mathfrak{F}_{\theta'}^\perp}$ coincides with the magnitude of the second largest eigenvalue of P_n.

Denote by $\hat{z}$ one of the eigenvalues of $\mathcal{L}$ satisfying $|\hat{z}| = \max_{z\in\sigma(\mathcal{L})\setminus\{1\}} |z|$. Whether or not this maximum is attained by more than one eigenvalue of $\mathcal{L}$ (in the case of a complex conjugate pair, for example) we do not care, as we are only concerned with the magnitude of the eigenvalue. We decompose the spectrum as $\sigma(\mathcal{L}) = \Sigma_0 \cup \Sigma_1$, where $\Sigma_0 = \{\hat{z}\}$, and $\Sigma_1 = \sigma(\mathcal{L}) \setminus \{\hat{z}\}$. One has the $\mathcal{L}$-invariant subspace decomposition $\mathfrak{F}_{\theta'} = X_0 \oplus X_1$, satisfying $\sigma(\mathcal{L}|_{X_i}) = \Sigma_i$, $i = 0, 1$ ([12] Theorem III-6.17). Denote by $\Gamma \subset \mathbb{C}$ a simple closed curve containing an open neighbourhood of $\hat{z}$, but no elements of Σ_1. The operator $\Pi_0 : \mathfrak{F}_{\theta'} \to X_0$ defined by

$$\Pi_0 = -\frac{1}{2\pi i}\int_\Gamma (\mathcal{L} - zI)^{-1}\, dz \tag{25}$$

is a projection onto X_0 along the direction X_1. The following theorem is a modified version of Theorem IV-3.16 [12].

Theorem 7.1. *If $\|\mathcal{L} - \mathcal{L}_N\|_{\theta'}$ is sufficiently small, the spectrum of $\mathcal{L}_N$ is separated by Γ into two parts $\Sigma_{0,N}$ and $\Sigma_{1,N}$. There is an associated $\mathcal{L}_N$-invariant decomposition $\mathfrak{F}_{\theta'} = X_{0,N} \oplus X_{1,N}$ with* $\dim(X_{i,N}) = \dim(X_i)$, $i = 0, 1$. *Furthermore the projection onto $X_{0,N}$ along $X_{1,N}$ given by*

$$\Pi_{0,N} = -\frac{1}{2\pi i}\int_\Gamma (\mathcal{L}_N - zI)^{-1}\, dz$$

approaches Π_0 in norm as $\|\mathcal{L} - \mathcal{L}_N\|_{\theta'} \to 0$.

Proof. See [12] p.213. □

For large enough N we have that a portion of $\sigma(\mathcal{L}_N)$ is isolated inside the closed curve Γ, and that the dimension of the corresponding invariant subspace is equal to the dimension of the eigenspace associated with $\hat{z}$. We now go on to bound the distance between the elements of $\Sigma_{0,N}$ and $\hat{z}$. The following two results are borrowed from Chatelin [8] p.278.

Lemma 7.2. *$\Pi_{0,N}|_{X_0} : X_0 \to X_{0,N}$ is a bijection for sufficiently large N.*

Proof. Let $\phi_0 \in X_0$. One has

$$|\|\phi_0\|_{\theta'} - \|\Pi_{0,N}\phi_0\|_{\theta'}| = |\|\Pi_0\phi_0\|_{\theta'} - \|\Pi_{0,N}\phi_0\|_{\theta'}| \leq \|(\Pi_0 - \Pi_{0,N})\phi_0\|_{\theta'} \leq \|\phi\|_{\theta'}/2,$$

for sufficiently large N by Theorem 7.1. Thus $\|\Pi_{0,N}\phi_0\|_{\theta'} \geq \|\phi_0\|_{\theta'}/2$, and so $\|\Pi_{0,N}|_{X_0}\|_{\theta'} \geq 1/2 \implies \|\left(\Pi_{0,N}|_{X_0}\right)^{-1}\|_{\theta'} \leq 2$. □

Proposition 7.3. *Let N be large enough to make the conclusions of Theorem 7.1 and Lemma 7.2 true. Define $L, L_N : X_0 \circlearrowleft$ by $L = \mathcal{L}|_{X_0}$ and $L_N = \left(\Pi_{0,N}|_{X_0}\right)^{-1} \circ \mathcal{L}_N \circ \Pi_{0,N}|_{X_0}$. Then*

$$\|L - L_N\|_{\theta'} \leq 2\|\mathcal{L} - \mathcal{L}_N\|_{\theta'}), \qquad \textit{and}$$
$$\mathcal{H}(\Sigma_{0,N}, \{\hat{z}\}) = O(\|\mathcal{L} - \mathcal{L}_N\|_{\theta'}^{1/\dim(X_0)}).$$

Proof. Clearly $\sigma(L) = \{\hat{z}\}$ and $\sigma(L_N) = \Sigma_{0,N}$. Now,

$$\begin{aligned}
&\left\|\left(\mathcal{L} - \left(\Pi_{0,N}|_{X_0}\right)^{-1} \circ \mathcal{L}_N \circ \Pi_{0,N}\right)|_{X_0}\right\|_{\theta'} \\
&= \left\|\left(\mathcal{L} - \left(\Pi_{0,N}|_{X_0}\right)^{-1} \circ \Pi_{0,N} \circ \mathcal{L}_N\right)|_{X_0}\right\|_{\theta'} \\
&= \left\|\left(\Pi_{0,N}|_{X_0}\right)^{-1} \circ \Pi_{0,N} \circ (\mathcal{L} - \mathcal{L}_N)\,|_{X_0}\right\|_{\theta'} \\
&\leq \left\|\left(\Pi_{0,N}|_{X_0}\right)^{-1} \circ \Pi_{0,N}\right\|_{\theta'} \|\mathcal{L} - \mathcal{L}_N\|_{\theta'} \\
&\leq 2\|\mathcal{L} - \mathcal{L}_N\|_{\theta'} \qquad \text{for } N \text{ as in Lemma 7.2.}
\end{aligned}$$

For the second part of the proposition we apply a standard result of Elsner (see Stewart & Sun [22], p.168, for example). Let $[L], [L_N]$ denote matrix representations of L and L_N with respect to some basis of X_0.

Sublemma 7.4.

$$\mathcal{H}(L, L_N) \leq \left(\|[L]\|_2 + \|[L_N]\|_2\right)^{1-1/\dim(X_0)} \|[L] - [L_N]\|_2^{1/\dim(X_0)},$$

where $\|\cdot\|_2$ is the standard L^2 matrix norm.

Using the fact that all norms are equivalent on finite dimensional spaces, by Sublemma 7.4 we are done. □

Eq. (22) and Proposition 7.3 imply that the order of convergence of $\max_{z\in\sigma(P_n)\setminus\{1\}}|z|$ to $\hat{z}$ is at least exponential in N with rate $\max\{\theta', \theta/\theta'\}$. Such behaviour is corroborated by Fig. 6. In our example, $\theta = \lambda = 0.5706$, and we may take $\theta' = \lambda^{\gamma'} = 0.597$, say. The exact figure of θ' doesn't matter, so long as it lies between θ and where we think our discrete eigenvalues are. In this case, the rate of convergence is at least $(0.5706/0.597)^N = 0.9558^N$.

If we wish to consider how quickly our estimates converge when compared to an increase in the size of P_n (in other words, the size of the partition $\mathfrak{P}_n$.), we may proceed as follows. Define $\Theta := \Lambda := 1/\sup_{x\in M}|\det D_xT|$ (T expanding). There exists a constant C such that the number of sets n in a Markov partition constructed from N inverse iterates of the original partition satisfies $n \leq C/\Theta^N$. Choose $\kappa > 0$ such that $\Theta^\kappa = \max\{\theta', \theta/\theta'\}$; in other words, put $\kappa = \log\left(\max\{\theta', \theta/\theta'\}\right)/\log\Theta$. Then

$$\|\mathcal{L} - \mathcal{L}_N\|_{\theta'} \leq \Phi|g|_\theta\left((\Theta^\kappa)^N\right) \leq \Phi|g|_\theta\frac{C^\kappa}{n^\kappa} = O\left(\frac{1}{n^\kappa}\right).$$

In our example, $\Theta = 0.5515$, and $\kappa = \log(0.5706/0.597)/\log(0.5515) = 0.0760$.

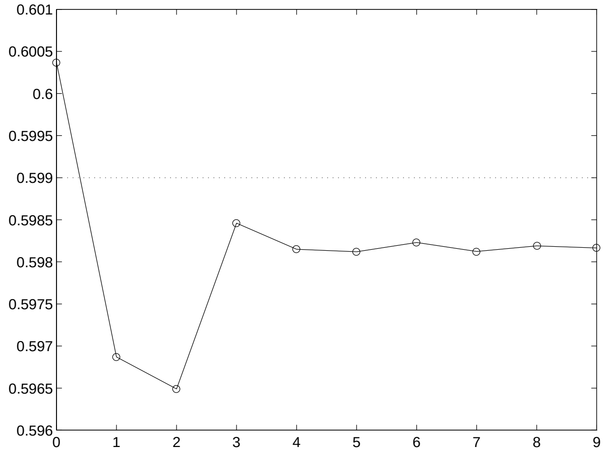

Fig. 6. A plot of $\max_{z \in \sigma(P_n)\setminus\{1\}} |z|$ versus N. Note that the convergence to a single value appears to be occurring exponentially as expected. The safe bound of 0.599 is shown dotted

8. Estimating the Rate of Decay Using Eigenvectors

By computing the eigenvectors of the matrix P_n in addition to its eigenvalues, one may further refine the analysis of the rate of decay. We assume that we are in the situation where $\mathcal{L}$ has nontrivial isolated spectrum, and N is large enough to satisfy the conditions of Proposition 7.3. We denote by $\hat{z}_N$ one of the eigenvalues of P_n satisfying $|\hat{z}_N| = \max_{z \in \sigma(P_n)\setminus=\{1\}} |z|$. Norm convergence of $\mathcal{L}_N$ to $\mathcal{L}$ not only guarantees that the isolated spectrum of $\mathcal{L}$ is approached by that of $\mathcal{L}_N$ (that is, $\hat{z}_N \to \hat{z}$), but that the corresponding eigenvectors (eigenfunctions) of $\mathcal{L}$ are also approximated in $\|\cdot\|_{\theta'}$ norm by those of $\mathcal{L}_N$. Formally, one has the following result; see [8], Theorem 6.7.

Theorem 8.1. *Under the hypotheses of Proposition 7.3, let $\hat{\phi}_N$ be an eigenvector of P_n corresponding to the eigenvalue $\hat{z}_N \in \Sigma_{0,N}$. Consider $\hat{\phi}_N$ as a test function in $X_{0,N}$. Then*

$$\mathrm{dist}(\hat{\phi}_N, X_0) := \min_{\phi \in X_0} \|\hat{\phi}_N - \phi\|_{\theta'} = O\left(\|\mathcal{L} - \mathcal{L}_N\|_{\theta'}^{1/\dim(X_0)}\right). \tag{26}$$

Such convergence results sometimes allow us to refine our bounds into *estimates* of the rate of decay. If the maximal non-unit eigenvalue $\hat{z}_N$ of P_n satisfies $|\hat{z}_N| > \theta'$, then $|\hat{z}_N|$ actually gives us an *estimate*, rather than just a bound, for the spectral radius of $\mathcal{L}|_{\mathfrak{F}_{\theta'}^{\perp}}$. As mentioned in Remark 3.2, the space $\mathfrak{F}_{\theta'}^{\perp}$ is a good deal larger than $C^{\gamma'}(M, \mathbb{R})$ and $C^{\gamma'}(M, \mathbb{C})$. $\mathfrak{F}_{\theta'}^{\perp}$ contains a lot of functions that do not concern us, namely those with more discontinuities than we are allowing. Recall that if T is piecewise smooth, we allow breaks in the smoothness of test functions at all forward images of the breaks in T. However, functions in $\mathfrak{F}_{\theta'}^{\perp}$ are allowed breaks at infinitely many places. We can identify these unwanted eigenfunctions by numerically examining the eigenvectors of P_n. If any of these eigenvectors have more breaks in smoothness than is allowed, we disregard their corresponding eigenvalues. By removing the offending eigenfunctions and corresponding eigenvalues, we sharpen our bounds on the rate of decay. If, after removal of all irrelevant eigenvalues, there still remains an eigenvalue that is clearly outside the disk $\{|z| \leq \lambda^{\gamma'}\}$, then the magnitude of this eigenvalue is an *estimate* of the rate of decay of correlations, rather than merely a bound. This is because we have found an explicit approximation to an eigenfunction, namely the corresponding eigenvector, for which (6) displays a decay rate given by the magnitude of the eigenvalue.

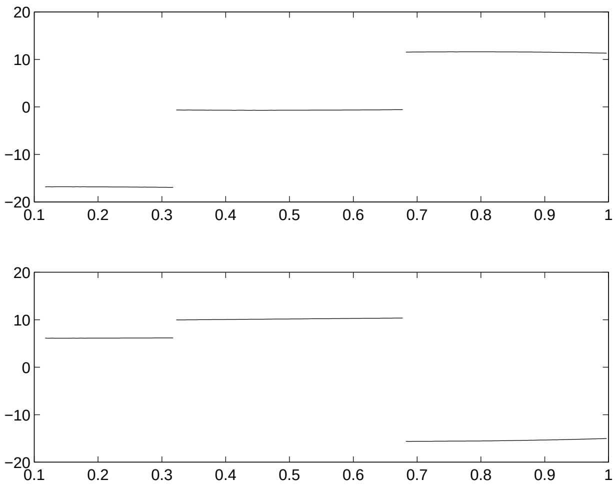

Fig. 7. The real and imaginary part of the (approximate) eigenfunction corresponding to the eigenvalue $-0.5014 + 0.3261i$ calculated using the matrix P_{212} ($N = 7$). These two functions (considered separately as real-valued) also span a two-dimensional real invariant subspace associated with the complex eigenvalues $-0.5014 \pm 0.3261i$

In our example, both of our eigenvalues of P_n lying outside $\{|z| \leq 0.5706\}$ are complex. We compute the eigenvectors corresponding to this complex conjugate pair. From Fig. 7 it is apparent that the corresponding eigenfunction is piecewise smooth with breaks only at the allowed positions of $x = 0.3194$ and $x = 0.6806$. Thus 0.59816 is an *estimate* of the rate of decay of (T, μ) with respect to complex-valued C^1 test functions, and the function shown in Fig. 7 is an example of a function for which (6) decays at this rate. This value is also an estimate for the rate of decay of (T, μ) with respect to real-valued C^1 test functions. The complex conjugate eigenvalue pair have an associated two-dimensional invariant subspace of real-valued functions (spanned by the two functions shown in Fig. 7). This two-dimensional invariant subspace contains no eigenfunctions, but all functions in the subspace decay in norm under the action of $\mathcal{P}$ at the same rate as the complex eigenfunctions.

9. Comparisons and Discussion

We have presented a method of computing a bound for the rate of decay of correlations for $C^{1+\gamma}$ expanding and hyperbolic maps acting on C^γ, $0 < \gamma \leq 1$, test functions. Our technique used the relative volumes of intersection of Markov partition sets with their inverse images to provide us with a matrix approximation of the Perron-Frobenius operator for the map. This approximation was close to $\mathcal{P}$ in the operator norm topology, and so their spectra were close also. To compute the spectrum of our matrix approximation was a simple matter. Our method also yields a numerical approximation of the physical invariant measure of T "for free"; see [9] for details.

In the introduction, we mentioned a simple-minded method of calculating the correlation function (1.2) by choosing specific real-valued functions φ and ψ, sprinkling a large number of points in M distributed according to μ, and iterating them forward k steps. The integral with respect to μ is then approximated by averaging the contributions from the sprinkled points; in this way $\mathcal{C}_{\varphi,\psi}(k)$ is "computed". Once one has values for $\mathcal{C}_{\varphi,\psi}(k)$ for $k = 1, \ldots, K$, one may subject this sequence to a number of analyses to determine a rate of decay. The simplest is to perform a sum-of-exponentials fit to the data. One may also perform Fourier analysis (see Isola [11], for example) on the sequence to try to extract the frequencies corresponding to decay rates. Either way, the number of sprinkled points required to maintain accuracy becomes prohibitively large very quickly because of the very exponential stretching properties one is trying to measure. Usually, this distribution of points approximating μ is obtained by running out a single long orbit of T. This procedure itself is an unreliable method for approximating μ because of computer roundoff and the possibility that the orbit chosen represents atypical behaviour (for example, it may be an orbit falling into a weak periodic sink, while we only observe its initial transient effects and do not detect the eventual periodicity). Our construction is a single-step method, thus avoiding problems such as compounding computer roundoff and long-term transient effects. Finally, the particular functions φ and ψ chosen may not be representative of the slowest possible decay rate. Results obtained from such an analysis must be treated with caution.

Rugh [20] has shown that one may compute the spectrum of the Perron-Frobenius operator for real analytic hyperbolic maps acting on the space of analytic test functions, using the periodic points of the map. This method of using the periodic orbits of a map to build up a picture of the dynamics has been popular with physicists for some time; see Christiansen *et al.* [3] for numerical examples of the determination of correlation spectra. However, the technique is very restrictive, requiring extremely smooth (analytic) maps and observables φ, ψ. From a physical point of view, one would expect to have to deal with both systems and physical observables that are not infinitely differentiable. In this paper, we have made the choice of observables that are C^1 or at least C^γ, $0 < \gamma < 1$, and believe that such a model of the world is much more realistic.

Our main result is of a similar vein to that of Baladi *et al.* [1]. In their paper they consider expanding Markov maps of the interval, with the Perron-Frobenius operator acting on test functions of bounded variation, and use the variational norm to compute the spectra. Again, we consider physical observables that are C^1 or Hölder continuous to be more realistic than observables of bounded variation. Their result is also restrictive in the sense that it is only applicable to one-dimensional expanding systems. Although the matrix construction of [1] turns out to be identical to ours, their method of proof is entirely different.

Our goal was to improve the known theoretical bounds for the rate of mixing with respect to C^1 test functions. In the table below, we compare the estimates of [21] and [13, 14] with our computer-assisted bounds.

It is clear that just a small amount of nonlinearity greatly affects the known theoretical bounds. It is interesting that although Ruelle's [19] bound of λ for the essential spectral radius increases as we add nonlinearities to $\tilde{T}x$ (because the minimum slope $1/\lambda$ decreases), our bound for the mixing rate actually improves. That is, our bound for the isolated spectrum moves in from a magnitude of 0.6009 to somewhere around 0.598–0.599. Thus in this case, adding nonlinearities may actually speed up the mixing of phase space. Such an observation would not be possible using the bounds of [21] or [13, 14] as added nonlinearities always worsen their estimates.

Table 2. Our computed-assisted bound for the rate of decay with respect to piecewise C^1 test functions compared to known theoretical bounds[1]

Map	Ruelle bound (essential spectrum)	Our bound	Liverani bound	Rychlik bound
$\tilde{T}x = -1.7525\|x - 1/2\| + 1$	0.5652	0.6009	0.6009	0.9986
T as in (24)	0.5706	0.599	0.9162	0.99999915

[1] Liverani's results strictly only apply to interval maps for which each branch is onto. As the left branches of $\tilde{T}$ and T are not onto, we generously make this estimate the best possible for $\tilde{T}$, assuming that his bound actually finds the isolated eigenvalues. When the map is piecewise linear and all branches are onto, his algorithm does return the best estimate, however, for such maps there are *no* isolated eigenvalues to find. For the map T, we compute his bound again assuming that both branches of this nonlinear map are onto.

We have established an order of convergence of our bounds to the "optimal" bounds in terms of (i) the number of inverse iterates N of T used to construct the Markov partition and (ii) the number of partition sets n used to construct our matrix approximation P_n. A computation of the eigenvectors of P_n may allow a refinement of our bound, in some cases producing an *estimate* of the rate of decay rather than merely a bound. The order of convergence of such estimates is identical to the order of convergence of our bounds.

Finally, we acknowledge that although the construction of Markov partitions for one-dimensional maps is relatively easy, it may be time consuming for some higher dimensional maps. If one is in a hurry, one may construct matrices defined by (9) using any finite partition of M into connected subsets (a triangulation, for example), and compute its eigenvalues. Because one no longer has the coding machinery available, rigorous results are not as easy to come by. In the author's experience, "most" randomly chosen partitions tend to give pessimistic estimates of the rate of mixing. That is, for systems with known rates of mixing, matrices constructed from arbitrary partitions tend to have eigenvalues lying outside the disk containing all the true eigenvalues of the Perron-Frobenius operator. A discussion of these results and which mixing rate is actually being approximated (be it with respect to C^1 test functions, L^1 test functions, or test functions of bounded variation) is contained in [10].

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