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# On the isolated spectrum of the Perron–Frobenius operator

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**Abstract.** We discuss the existence of large isolated (non-unit) eigenvalues of the Perron–Frobenius operator for expanding interval maps. Corresponding to these eigenvalues (or 'resonances') are distributions which approach the invariant density (or equilibrium distribution) at a rate *slower* than that prescribed by the minimal expansion rate. We consider the transitional behaviour of the eigenfunctions as the eigenvalues cross this 'minimal expansion rate' threshold, and suggest dynamical implications of the existence and form of these eigenfunctions. A systematic means of constructing maps which possess such isolated eigenvalues is presented.

AMS classification scheme numbers: 37A30 (primary), 37E05, 37D20, 47A10, 47A15 (secondary)

## 1. Introduction

Dynamical systems often exhibit very complicated temporal behaviour. In such cases it is less useful, and perhaps misleading, to compute one single solution of the system for a long period of time; rather, it is more reasonable to approximate the statistics of the underlying dynamics. This information is encoded in the so-called *natural* (or *physically relevant*) *invariant measures*. These specify the probability to observe a typical trajectory within a certain region of state space. Throughout this paper we will be concerned with natural invariant measures that are absolutely continuous with respect to the Lebesgue measure and their corresponding *invariant densities*.

One may consider a density as an ensemble of initial conditions. The action of the dynamical system on this ensemble is described by the *Perron–Frobenius operator* $\mathcal{P}$. Invariant densities are those ensembles fixed under the linear operator $\mathcal{P}$; in other words, they are eigenfunctions with eigenvalue 1.

It is well known [7] that within a certain functional analytical set-up, the Perron–Frobenius operator is Markov, the entire spectrum lies inside the unit disc and $\mathcal{P}$ is *quasi-compact*. Further, it is known that the *essential spectrum* of the operator lies inside a disc of radius $\vartheta$, where $\vartheta$ depends on the weakest ('long-term' or 'sustained') expansion rate of the underlying system.

The spectrum of $\mathcal{P}$ is connected with the *exponential rate of mixing* or the *exponential rate of decay of correlations* of the system. Spectral points inside the disc $\{|z| \leqslant \vartheta\} \subset \mathbb{C}$ indicate exponential mixing rates consistent with the mixing caused by the expansion in the system (or the exponential instability of nearby trajectories). However, the presence of spectral

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points outside the disc $\{|z| \leqslant \vartheta\}$ may not be simply explained by mixing due to expansion or exponential instability in the system. The eigenfunctions of such eigenvalues correspond to ensembles which approach the equilibrium ensemble (or invariant density) at very slow rates—rates which are too slow to be explained by the expansion alone. Therefore, we conclude that the existence of isolated eigenvalues outside the disc $\{|z| \leqslant \vartheta\}$ heralds the existence of macroscopic structures within the state space that contribute directly to the slow mixing rate, via a slow exchange of 'mass' that overrides the expected more rapid decay rates via chaotic expansion and instability.

This discussion indicates that to identify and study certain macroscopic behaviours of the underlying system it is relevant to determine whether there are any isolated eigenvalues outside $\{|z| \leqslant \vartheta\}$. Besides recent analytic considerations [3] this approach has also been used for the numerical identification of *almost-invariant sets* [5] and the approximation of so-called *conformations* of molecules [11]. However, to the authors' knowledge, there are no characterizations in the literature for maps possessing isolated eigenvalues that are not related to the transient behaviour of the system. We expect that isolated eigenvalues exist for many systems arising in applications and, therefore, it is important to understand their existence and related phenomena.

In this paper, we will not just explicitly construct maps which have isolated eigenvalues besides one, but we will also be able to relate the shape of the corresponding eigenfunctions to the underlying dynamical behaviour. To this end, we will present a map depending on a real parameter $t$; varying $t$ will allow us to move an eigenvalue away from the boundary of the essential spectrum to become isolated.

A more detailed outline of the paper is as follows. We begin by formally describing the mathematical objects we will work with and briefly outline pertinent known results. A case study is presented in section 3 in which we derive detailed information about the existence and location of an isolated eigenvalue. In section 4 we begin with some connections between the position of isolated eigenvalues and the behaviour of their eigenfunctions for a class of maps containing all those in the present study. We then present a general framework characterizing a class of maps which possess isolated eigenvalues. This general formulation is particularized to 'odd' interval maps to provide a 'recipe' for drawing graphs of interval maps with isolated eigenvalues. We conclude with a result connecting the behaviour of the second eigenfunction with the location of the second eigenvalue. This leads to a conjecture giving very general conditions (with a physical interpretation) for the existence of isolated eigenvalues.

## 2. Theoretical background

We consider expanding piecewise $C^2$ transformations $T : [0, 1] \circlearrowleft$. In this section we briefly summarize well known facts about the corresponding Perron–Frobenius operator $\mathcal{P} : BV \circlearrowleft$, where

$$(\mathcal{P} f)(x) = \sum_{y \in T^{-1}x} \frac{f(y)}{|T'(y)|}$$

and $BV \subset L^1$ is the space of functions of bounded variation on $[0, 1]$. For details on the Perron–Frobenius operator and its properties on the space $BV$, see [4, 8].

It can be shown that $\mathcal{P}$ is a *Markov operator*†, and so in particular, $\mathcal{P}$ always has the eigenvalue one. Denote by $\sigma(\mathcal{P})$ and $r_\sigma(\mathcal{P})$ the spectrum of $\mathcal{P}$ and the spectral radius of $\mathcal{P}$, respectively. We will frequently use the following facts; see [7] for example.

† A Markov operator is a linear operator that maps non-negative functions to non-negative functions and preserves the norm of non-negative functions.

**Theorem 2.1.** *The Perron–Frobenius operator $\mathcal{P}$ is quasi-compact on $BV$ equipped with the norm $\|f\| = \max\{\|f\|_{L^1}, \mathrm{var}(f)\}$. In particular, $\sigma_{\mathrm{ess}}(\mathcal{P})$ (the essential spectrum of $\mathcal{P}$) is contained inside a disc of radius $r_{\mathrm{ess}} = \sup\{|\lambda| : \lambda \in \sigma_{\mathrm{ess}}(\mathcal{P})\} = \vartheta$, where*

$$\vartheta = \lim_{k\to\infty}\left(\sup_x(1/|(T^k)'(x)|)\right)^{1/k}. \tag{2.1}$$

*Spectral points of $\mathcal{P}$ outside the disc $\{z \in \mathbb{C} : |z| \leqslant \vartheta\}$ are isolated eigenvalues of $\mathcal{P}$ of finite multiplicity.*

In subsequent sections we will restrict ourselves to Markov† maps $T$ to obtain existence results for isolated eigenvalues outside the region $\{|z| \leqslant \vartheta\}$. (In the following, the term *isolated eigenvalue* refers to isolated eigenvalues of modulus strictly less than 1.) Thus, we collect further results concerning piecewise-linear expanding Markov maps. Let $\mathfrak{P}$ be a Markov partition for $T$ with cardinality $q$. Denote by $F$ the $q$-dimensional vector space with basis $\{\chi_I : I \in \mathfrak{P}\}$; it is straightforward to show that $F$ is $\mathcal{P}$-invariant. Define $BV/F$ to be the quotient space where one factors out all test functions in $F$.

**Lemma 2.2.**

*(a) $\sigma(\mathcal{P}) \subset \sigma(\mathcal{P}|_F) \cup \sigma(\mathcal{P}|_{BV/F})$;*
*(b) $r_\sigma(\mathcal{P}|_{BV/F}) \leqslant \vartheta$;*
*(c) $\{z \in \sigma(\mathcal{P}) : |z| > \vartheta\} = \{z \in \sigma(\mathcal{P}|_F) : |z| > \vartheta\}$.*

The proofs of (a) and (b) may be found in [6] and [1], respectively. See [2] for further details on these constructions.

**Proof of (c).** Clearly, $\{z \in \sigma(\mathcal{P}|_F) : |z| > \vartheta\} \subset \{z \in \sigma(\mathcal{P}) : |z| > \vartheta\}$. Containment in the other direction follows immediately from (a) and (b). □

The above lemma tells us that one can find all spectral points outside $\{|z| \leqslant \vartheta\}$ by finding eigenvalues of the matrix representing $\mathcal{P}|_F$.

**Remarks 2.3.** Every eigenvalue of $\mathcal{P}|_F$ is an eigenvalue of $\mathcal{P}$; even eigenvalues inside $\{|z| \leqslant \vartheta\}$. If an eigenvalue of $\mathcal{P}|_F$ is outside $\{|z| \leqslant \vartheta\}$, we know that it is isolated; otherwise, we just know it is an eigenvalue. There may be isolated eigenvalues *inside* $\{|z| \leqslant \vartheta\}$ which are not eigenvalues of $\mathcal{P}|_F$.

## 3. A case study: the four-legs map

In this section we perform, for motivational purposes, a brief study of a particular family of maps where we will show that the corresponding Perron–Frobenius operator has isolated eigenvalues outside the region $\{|z| \leqslant \vartheta\}$.

We consider the following family of one-dimensional maps $T_t : [0, 1] \circlearrowleft$:

$$T_t x = \begin{cases} 2x & 0 \leqslant x < \frac{1}{4} \\ t(x - \frac{1}{4}) & \frac{1}{4} \leqslant x < \frac{1}{2} \\ t(x - \frac{3}{4}) + 1 & \frac{1}{2} \leqslant x < \frac{3}{4} \\ 2(x-1) + 1 & \frac{3}{4} \leqslant x \leqslant 1. \end{cases} \tag{3.1}$$

$T_t$ is piecewise linear and has four branches; two of these have slope 2 and two have slope $t$. The graph of a typical $T$ is shown in figure 1.

† A map $T$ is called *Markov* if there exists a partition $\mathfrak{P} = \{A_1, \ldots, A_n\}$ of [0, 1] such that for each $i = 1, \ldots, n$, $T(A_i) = \bigcup_{j\in\mathcal{J}_i} A_j$, for some index set $\mathcal{J}_i \subset \{1, \ldots, n\}$. Such a partition is known as a *Markov partition*.

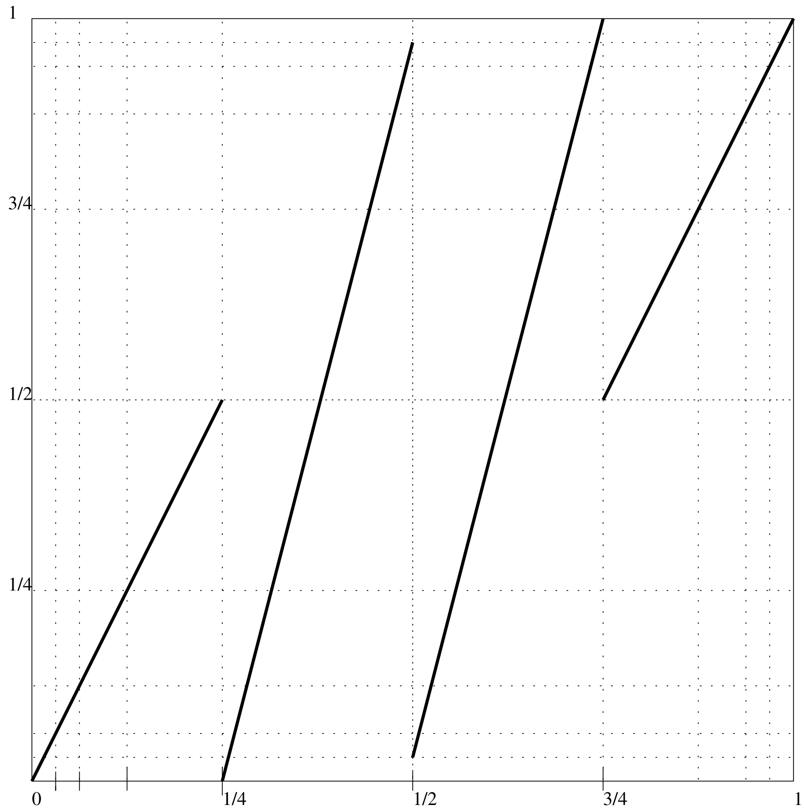


**Figure 1.** Graph of $T_t$ for $t = 4 - \frac{1}{8}$.

We wish to study the spectral properties of the family $T_t$ for $2 \leqslant t \leqslant 4$; in particular, for $t$ close to 4. Since the fixed point 0 lies on a branch with slope 2 we know from proposition 2.1 that for all values of $t$ between 2 and 4, the essential spectrum of the Perron–Frobenius operator acting on $BV$ functions is contained in the circle $|z| \leqslant \frac{1}{2}$.

### *3.1. The case $t = 4$*

The map $T_4$ is Markov with partition $\mathfrak{P} = \{[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}], [\frac{1}{2}, \frac{3}{4}], [\frac{3}{4}, 1]\}$. By restricting the Perron–Frobenius operator to the four-dimensional space consisting of piecewise-constant functions on $\mathfrak{P}$, we obtain the matrix representation (under left multiplication):

$$P_4 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

This matrix has eigenvalues $1, \frac{1}{2}, 0, 0$, the invariant density is $\phi_4 \equiv 1$ and the eigenfunction corresponding to $\frac{1}{2}$ is,

$$\psi_4(x) = \begin{cases} 1 & 0 \leqslant x < \frac{1}{2} \\ -1 & \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

By lemma 2.2 (c), we know that $\mathcal{P}_4$ has no spectral points outside $|z| \leqslant \frac{1}{2}$ (except for the eigenvalue 1).

**Lemma 3.1.** *The eigenvalue $\frac{1}{2}$ is in the essential spectrum of $\mathcal{P}_4$.*

**Proof.** We show that, in fact, any real number $0 \leqslant \lambda \leqslant \frac{1}{2}$ is an eigenvalue of $\mathcal{P}_4$. Thus, the eigenvalue $\frac{1}{2}$ is not isolated and therefore is in the essential spectrum by definition. Choose $0 \leqslant \lambda \leqslant \frac{1}{2}$. We will construct an eigenfunction in $BV$ for this eigenvalue. We restrict ourselves to functions of the form

$$\psi = \sum_{n=1}^{\infty} \alpha_n \left(\chi_{[1/2^{n+1},1/2^n]} - \chi_{[1-1/2^n,1-1/2^{n+1}]}\right).$$

Solving the equation $\mathcal{P}_4\psi = \lambda\psi$ on the interval $[\frac{1}{4}, \frac{1}{2}]$ gives

$$1/2\alpha_2 + 1/4\alpha_1 - 1/4\alpha_1 = \lambda\alpha_1 \qquad \Longrightarrow \qquad \alpha_2 = 2\lambda\alpha_1.$$

Repeating this argument on the intervals $[1/2^n, 1/2^{n-1}]$, we have in general that $\alpha_n = (2\lambda)^{n-1}\alpha_1$. By this construction both var $\psi$ and $\|\psi\|_1$ are clearly finite, so that $\psi \in BV$ is an eigenfunction. □

*3.2. The case $t < 4$*

For $t < 4$, we may construct a Markov partition by choosing $t$ so that $T_t(\frac{1}{2})$ (from the third branch) is mapped onto $x = \frac{1}{2}$ after a finite number of iterations with $T_t^k(\frac{1}{2}) \in [0, \frac{1}{4}]$ for all $1 \leqslant k \leqslant n$. That is, we wish $T_t(\frac{1}{2}) = t(\frac{1}{2} - \frac{3}{4}) + 1 = 1/2^{n+1}$ for any $n \geqslant 1$ of our choosing; solutions for $t$ are $t_n = 4 - 1/2^{n-1}$, giving Markov partitions of cardinality $2n+2$, respectively. We denote the restriction of $\mathcal{P}_{t_n}$ to piecewise-constant functions on the sets

$$[0, 1/2^{n+1}], [1/2^{n+1}, 1/2^n], [1/2^n, 1/2^{n-1}], \ldots, [\tfrac{1}{4}, \tfrac{1}{2}]$$

$$[\tfrac{1}{2}, \tfrac{3}{4}], \ldots, [1 - 1/2^n, 1 - 1/2^{n+1}], [1 - 1/2^{n+1}, 1]$$

by $P_n$. Eigenvalues may be computed numerically for these matrices, and it is observed that there is a single real eigenvalue that moves from near 1, towards $\frac{1}{2}$ along the positive real axis as $n$ is increased (that is, as $t_n$ tends to 4). For example, at $n = 6$, one obtains the spectrum shown in figure 2. In fact, elementary, but somewhat tedious, computations using the characteristic polynomial of $P_n$ yield

**Lemma 3.2.** *The matrix $P_n$ has exactly one real eigenvalue $\lambda_n$ in the interval*

$$\left[\frac{1}{2} + \frac{1}{5n}, \frac{1}{2} + \frac{1}{n^{2/3}}\right]$$

*and no other real eigenvalues in $[\frac{1}{2}, 1)$.*

We will return to this example later to elucidate the connection between the form of the second eigenfunction and the position of the second eigenvalue relative to $\vartheta$.

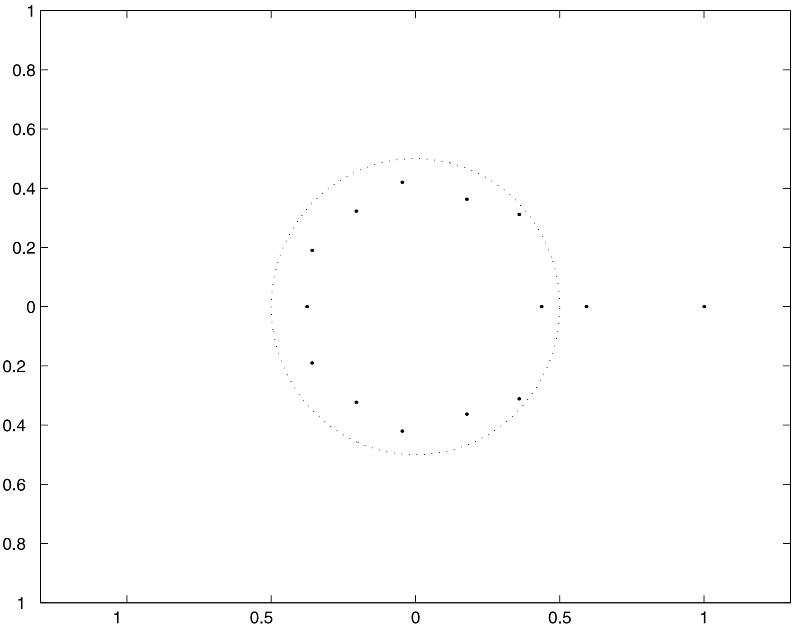


**Figure 2.** Spectral values of the matrix $P_6$. The circle $\{|z| = \frac{1}{2}\}$ is shown as dotted. By lemma 2.2 (c), $\{z \in \sigma(\mathcal{P}_{t_6}) : |z| > \frac{1}{2}\} = \{1, \lambda_{t_6}\}$, where $\lambda_{t_6} \approx 0.5917$. As $n$ is increased, $\lambda_{t_n} \to \frac{1}{2}$ (and additional complex eigenvalues appear outside $\{|z| = \frac{1}{2}\}$ in the vicinity of $z = \frac{1}{2}$).

## 4. General results

We begin this section with a weak general result that links the positions of eigenvalues with the structure of the corresponding eigenmeasure. This indicates a way to progress to achieve our goal of designing maps with isolated eigenvalues. We then construct a class of maps whose Perron–Frobenius operator possesses eigenfunctions with the necessary properties, thus forcing the existence of isolated eigenvalues. This class of maps is later particularized to a special case, allowing one to follow a set of rules for drawing the graph of an interval map with isolated eigenvalues. Finally, we conclude with a conjecture presenting general conditions for the existence of isolated eigenvalues.

### *4.1. Existence of large eigenvalues*

For maps which have a simple structure around one of their fixed points, we show via straightforward arguments that there is a relationship between eigenvalue/eigenmeasure pairs, provided that the eigenmeasure has a specified 'constant' structure around the selected fixed point.

**Lemma 4.1.** *Let $T : [0, 1] \circlearrowleft$ be a (Borel measurable) map that is linear and expanding on an interval $I^*$ with $|T'(x)| = s$ for $x \in I^*$. Suppose, in addition, that $I^*$ contains a fixed point $p$. Let $\mathfrak{S}([0, 1])$ denote the space of Borel signed measures on $[0, 1]$, and let $T^* : \mathfrak{S}([0, 1]) \circlearrowleft$ denote the natural action of $T$ on $\mathfrak{S}([0, 1])$ defined by $T^*\nu = \nu \circ T^{-1}$. Furthermore, suppose that there exists an eigenvalue/eigenmeasure pair $(\lambda, \nu)$ with the eigenmeasure $\nu$*

*being absolutely continuous on the interval $I^*$ with a non-zero constant density. Then*

$$\lambda = \frac{1}{s} + \frac{\nu(T^{-1}(I^*) \cap I^{*c})}{\nu(I^*)}. \tag{4.1}$$

**Proof.** Using the eigenmeasure equation $\lambda\nu(I^*) = T^*\nu(I^*)$, one obtains

$$\lambda\nu(I^*) = \nu\left(T^{-1}(I^*)\right) = \nu(T^{-1}(I^*) \cap I^*) + \nu(T^{-1}(I^*) \cap I^{*c})$$

or

$$\lambda = \frac{\nu(T^{-1}(I^*) \cap I^*)}{\nu(I^*)} + \frac{\nu(T^{-1}(I^*) \cap I^{*c})}{\nu(I^*)}.$$

Since $T$ is linear and expanding on $I^*$ and $p \in I^*$ it follows that $m(T^{-1}I^* \cap I^*) = m(I^*)/s$, where $m$ is the Lebesgue measure. Using the fact that the density of $\nu$ is constant on $I^*$ we conclude that

$$\frac{\nu(T^{-1}(I^*) \cap I^*)}{\nu(I^*)} = \frac{1}{s}$$

which leads to the desired result. □

Particularizing to piecewise-$C^1$ expanding maps, and supposing $s = \inf_x |T'(x)|$, we see that (4.1) is equivalent to

$$\lambda = \vartheta + \frac{\nu(T^{-1}(I^*) \cap I^{*c})}{\nu(I^*)} \tag{4.2}$$

where $\vartheta$ is defined in (2.1).

Now, for example, if it were known that $\nu$ was positive in pre-images of a neighbourhood of $p$, equation (4.2) immediately tells one that the corresponding eigenvalue $\lambda$ is strictly larger than $\vartheta$. Negative and complex eigenvalues may also in principle be produced by finding appropriate eigenmeasures $\nu$. We now take up this question of finding suitable eigenmeasures.

Numerical studies have suggested that a general template for such maps are those where a fixed point coincides with a minimum for the derivative, combined with a 'non-onto' condition; this will be discussed further in a later section. For now, our aim is to describe a class of simple interval maps with non-trivial chaotic behaviour which exhibit a (real) eigenvalue larger than the magnitude of the inverse of the minimum slope of the map. These maps should be non-trivial in the sense that they are mixing on the entire interval with respect to some everywhere-positive density, and the slow rate of decay is not due to a slow escape of mass from some transient region.

Roughly speaking, we construct mappings such that the eigenfunction of the Perron–Frobenius operator corresponding to the second largest eigenvalue is positive (say) on one 'side' of this distinguished fixed point, and negative on the other side. The existence of such an eigenfunction will follow from properties of the Perron–Frobenius operator, which in turn gives us a description of suitable maps. Once the existence of the eigenfunction is established, it is a simple matter to use the above arguments to show that the corresponding eigenvalue must be larger than the magnitude of the inverse of the minimal slope of the map.

For clarity, our constructions will be based upon piecewise-linear Markov maps with properties to be made precise in the following theorem. We introduce the notion of an interval exchanging map.

**Definition 4.2.** Let $\mathfrak{P}$ be a Markov partition of [0, 1] of cardinality $2q$ for a piecewise-linear Markov map $T : [0,1]\circlearrowleft$. Form any two complementary subsets of $\mathfrak{P}$, say $\mathfrak{P}_A$ and $\mathfrak{P}_A^c$, with $\mathfrak{P}_A \cap \mathfrak{P}_A^c = \emptyset$ and $\mathfrak{P}_A \cup \mathfrak{P}_A^c = \mathfrak{P}$. We define an *interval exchanging map* $\mathcal{S} : [0,1]\circlearrowleft$ by pairing each interval $I \in \mathfrak{P}_A$ with a corresponding interval $I' \in \mathfrak{P}_A^c$, and setting $\mathcal{S}(I) = I'$ and $\mathcal{S}(I') = I$, with $\mathcal{S}$ mapping the intervals $I$ and $I'$ onto each other in a linear manner. Further, we insist that

(a) $\mathcal{S}$ commutes with the action of $T$, so that $T \circ \mathcal{S} = \mathcal{S} \circ T$; and
(b) $|S'| = 1$.

Define $F = \mathrm{sp}\{\chi_I : I \in \mathfrak{P}\}$, and set $P = \mathcal{P}|_F$. Further, define the $q$-dimensional subspace $\mathcal{A} = \mathrm{sp}\{\chi_I - \chi_I \circ \mathcal{S} : I \in \mathfrak{P}\}$ and set $A = \mathcal{P}|_{\mathcal{A}}$.

**Lemma 4.3.** *The subspace $\mathcal{A}$ is $\mathcal{P}$-invariant.*

**Proof.** We claim that

$$(\mathcal{P}f) \circ \mathcal{S} = \mathcal{P}(f \circ \mathcal{S}) \qquad \text{for any } f \in L^1$$

and the result will then follow since $\mathcal{P}(f - f \circ \mathcal{S}) = \mathcal{P}f - \mathcal{P}(f \circ \mathcal{S}) = \mathcal{P}f - (\mathcal{P}f) \circ \mathcal{S}$, and $\mathcal{P}\chi_I \in F$ for $I \in \mathfrak{P}$.

Now, let $B \subset [0, 1]$. Then

$$\begin{aligned}\int_B (\mathcal{P}f) \circ \mathcal{S}\, \mathrm{d}m &= \int_{\mathcal{S}B} \frac{1}{|\mathcal{S}' \circ \mathcal{S}^{-1}|} \mathcal{P}f\, \mathrm{d}m = \int_{T^{-1}\circ\mathcal{S}B} f\, \mathrm{d}m = \int_{\mathcal{S}\circ T^{-1}B} f\, \mathrm{d}m \\ &= \int_{T^{-1}B} f \circ \mathcal{S} \cdot |\mathcal{S}'|\, \mathrm{d}m = \int_B \mathcal{P}(f \circ \mathcal{S})\, \mathrm{d}m.\end{aligned}$$

and the claim follows. □

**Remarks 4.4.** The graph of $T$ is invariant under the action of $\mathcal{S}$, since $\mathcal{S}(x, Tx) = (\mathcal{S}x, \mathcal{S}Tx) = (\mathcal{S}x, T(\mathcal{S}x))$. Clearly $\mathcal{S}^2 = \mathrm{Id}$, and the only possible fixed points of $\mathcal{S}$ are the boundary points of the intervals in $\mathfrak{P}$.

**Example 4.5.** Setting $\mathfrak{P}_A = \{[0, \frac{1}{4}], [\frac{1}{4}, \frac{1}{2}]\}$ and $\mathfrak{P}_A^c = \{[\frac{1}{2}, \frac{3}{4}], [\frac{3}{4}, 1]\}$ the map $T_4$ of section 3.1 commutes with the interval exchange map $\mathcal{S}x = 1 - x$.

**Definition 4.6.** We say that a map $T : [0,1]\circlearrowleft$ is *of class* $\mathfrak{Y}$ if $T$ is a piecewise-linear expanding Markov map with Markov partition $\mathfrak{P} = \{I_1, \ldots, I_{2q}\}$, such that there is a fixed point $p$ satisfying $|T'(p)| = \min_{x\in I} |T'(x)| =: s$, and the matrix $P$ of definition 4.2 is eventually positive†.

**Theorem 4.7.** *Let $T : [0,1]\circlearrowleft$ be of class $\mathfrak{Y}$. In the notation of definition 4.2, if A is a non-negative eventually positive matrix, and there exists a $\delta > 0$ such that $T^{-1}([p, p+\delta]) \subset \bigcup_{I\in\mathfrak{P}_A} I$, where either*

† That is, there exists $k > 0$ such that $(P^k)_{ij} > 0$ for all $i, j = 1, \ldots, 2q$. It is known that if $T$ is mixing, then $P = \mathcal{P}|_F$ is eventually positive. For non-negative matrices, eventual positivity is equivalent to irreducibility and aperiodicity (theorem 4.5.8 in [9], for example.)

*(a) p is a left boundary point of some* $I^* \in \mathfrak{P}_A$, $[p, p+\delta] \subset I^*$, *and* $T'(p+\delta) > 0$,
*(b) p lies in the interior of some* $I^* \in \mathfrak{P}_A$ *and* $[p-\delta/s, p+\delta] \subset I^*$

*then* $\mathcal{P} : BV \circlearrowleft$ *has a real positive eigenvalue* $\lambda$ *with* $1/s < \lambda < 1$. *The corresponding eigenfunction is positive on* $\mathfrak{P}_A$ *and negative on* $\mathfrak{P}_A^c$.

**Remark 4.8.** One may replace $\delta > 0$ with $\delta < 0$, and $[p, p+\delta]$ with $[p-\delta, p]$ throughout, changing 'left boundary point' to 'right boundary point' in the statement of theorem 4.7. A similar formulation may be used to create negative isolated eigenvalues.

**Proof of theorem.** Using eventual positivity of $P$, the Perron–Frobenius theorem (see theorem 9.10 [10], for example), implies that $P$ has a unique maximal positive eigenvalue (in this case the eigenvalue 1) and a corresponding unique positive left eigenvector $p$. This (suitably normalized) eigenvector $p$ defines the unique invariant density $\phi$ by setting $\phi(x) \equiv p_i$ for $x \in I_i$, $i = 1, \ldots, 2q$. By the Perron–Frobenius theorem, all other eigenvalues of $P$ are strictly less than 1. Any eigenvalues of $A$ must be less than one, since functions in $\mathcal{A}$ have both positive and negative values, and these cannot be eigenfunctions for the eigenvalue 1, which is everywhere positive. We now show that under the above assumptions, one must have $\lambda > 1/s$.

For notational purposes, set $\mathfrak{P}_A = \{I_1', \ldots, I_q'\}$. By eventual positivity of $A$, the Perron–Frobenius theorem tells us that $A$ has a positive eigenvalue $\lambda$ with corresponding left eigenvector $v$ with all positive entries. This (again suitably normalized) eigenvector $v$ defines a signed density $\psi$ on [0, 1] by setting $\psi(x) \equiv v_j$ and $\psi(\mathcal{S}x) = -v_j$ for $x \in I_j'$, $j = 1, \ldots, q$. The signed density $\psi$ satisfies the equation $\mathcal{P}\psi = \lambda\psi$ (since $\psi \in \mathcal{A}$). Note that $\psi$ is positive on $\mathfrak{P}_A$ and negative on $\mathfrak{P}_A^c$; $\psi$ has been normalized so that $\int_{[0,1]} |\psi| \, \mathrm{d}m = 1$. Let $\nu$ denote the signed measure corresponding to the signed density $\psi$.

In (a), $T'(p+\delta) > 0$, and therefore $T^{-1}([p, p+\delta]) \cap [p, p+\delta] = [p, p+\delta/s]$. Clearly, $\nu(T^{-1}([p, p+\delta]) \cap [p, p+\delta]) = s\nu([p, p+\delta])$. In (b), we have that $[p-\delta/s, p+\delta/s] \subset I^*$ and therefore again, $\nu(T^{-1}([p, p+\delta]) \cap [p, p+\delta]) = s\nu([p, p+\delta])$. We evaluate the eigenequation $\nu \circ T^{-1} = \lambda\nu$ on the set $[p, p+\delta]$ to obtain (WLOG we assume that $T^{-1}([p, p+\delta]) \cap [p, p+\delta] = [p, p+\delta/s]$)

$$\nu([p, p+\delta/s]) + \nu(B) = \lambda\nu([p, p+\delta]) \tag{4.3}$$

where $B$ is a collection of intervals in $\mathfrak{P}_A$ by assumption. We may replace $\nu([p, p+\delta/s])$ with $\nu([p, p+\delta])/s$ since $\nu$ has a constant density on $[p, p+\delta]$ as $[p, p+\delta] \subset I^* \in \mathfrak{P}_A$ to obtain

$$\nu(B) = (\lambda - 1/s)\nu([p, p+\delta]). \tag{4.4}$$

Since $B \subset \mathfrak{P}_A$ and $\nu$ is positive on $\mathfrak{P}_A$, $\nu(B) > 0$, so that we must have $\lambda > 1/s$. □

## *4.2. When A is eventually positive: odd maps*

Theorem 4.7 gave conditions on the matrix $A$ and the map $T$ for the existence of eigenvalues outside the disc $\{|z| \leqslant 1/s\}$. The purpose of this subsection is to interpret non-negativity and eventual positivity of $A$ in terms of the map $T$, and to arrive at a 'recipe' for constructing maps with large isolated eigenvalues. For simplicity, we choose $\mathcal{S}x = 1 - x$, a reflection about $x = \frac{1}{2}$; maps which commute with this interval exchange map are odd maps about the symmetry point $x = \frac{1}{2}$. We begin by considering an example map and discussing the salient features of the graph which will be used in a rigorous theorem to follow.

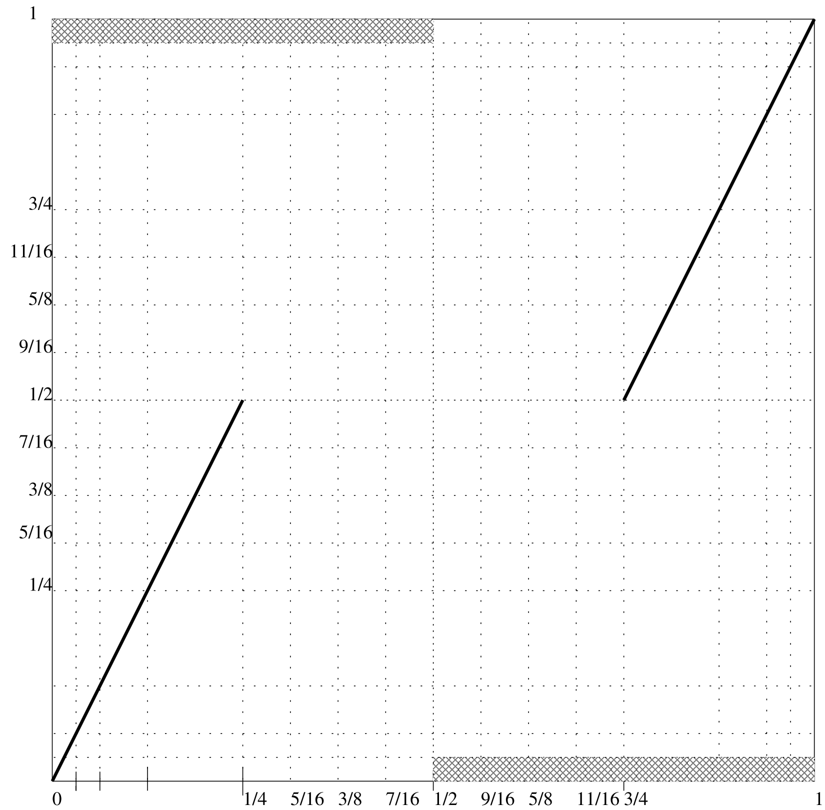


**Figure 3.** The branch with fixed point $p = 0$ having minimal slope.

**Example 4.9.** Our map will be an odd piecewise-linear expanding Markov map; we discuss the construction on $[0, \frac{1}{2}]$ and the remaining half is obtained by symmetry.

The first branch shown in figure 3 maps $[0, b_1]$ ($b_1 = \frac{1}{4}$ here) onto $[0, \frac{1}{2}]$ and has a fixed point at the origin ($p = 0$). This branch has slope $s = 2$, and this will be the minimal slope of $T$. The boundary points of the Markov partition $\mathfrak{P}$ of $T$ in $[0, \frac{1}{2}]$ consist only of break points of $T$ and inverse images of $b_1$ contained in $[0, b_1]$, namely the points $x = s^{-l}/2, l = 1 \ldots, 4$ ($x = \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$). By the oddness of $T$, the Markov partition in $[\frac{1}{2}, 1]$ is given by the images of the partition sets in $[0, \frac{1}{2}]$ under $\mathcal{S}$. Denote the partition sets between $b_1$ and $\frac{1}{2}$ as

$$I_1 = \left[\tfrac{1}{4}, \tfrac{5}{16}\right] \qquad I_2 = \left[\tfrac{5}{16}, \tfrac{3}{8}\right] \qquad I_3 = \left[\tfrac{3}{8}, \tfrac{7}{16}\right] \qquad \text{and} \qquad I_4 = \left[\tfrac{7}{16}, \tfrac{1}{2}\right].$$

In figure 4 we add the branches on each interval $I_i$ that has $1 - s^{-4}/2 = 1 - \frac{1}{32} \in \partial(TI_i)$. There must be at least one branch which has $x = 1 - \frac{1}{32}$ as a boundary point of its image; otherwise there is no reason to include this point as a boundary point in the Markov partition. To ensure non-negativity of $A$, a further restriction to all additional branches is that the left endpoint of $T(I_i)$ is strictly† less than $\mathcal{S}$ (the right endpoint of $T(I_i)$); roughly speaking, on the graph of $T$, all branches must be 'more below $y = \frac{1}{2}$ than above $y = \frac{1}{2}$'. This forces this second branch in figure 4 to map onto $x = 0$, and also disallows any branches mapping into the shaded region of the graph.

† 'Strictly' is needed later to show eventual positivity of $A$.

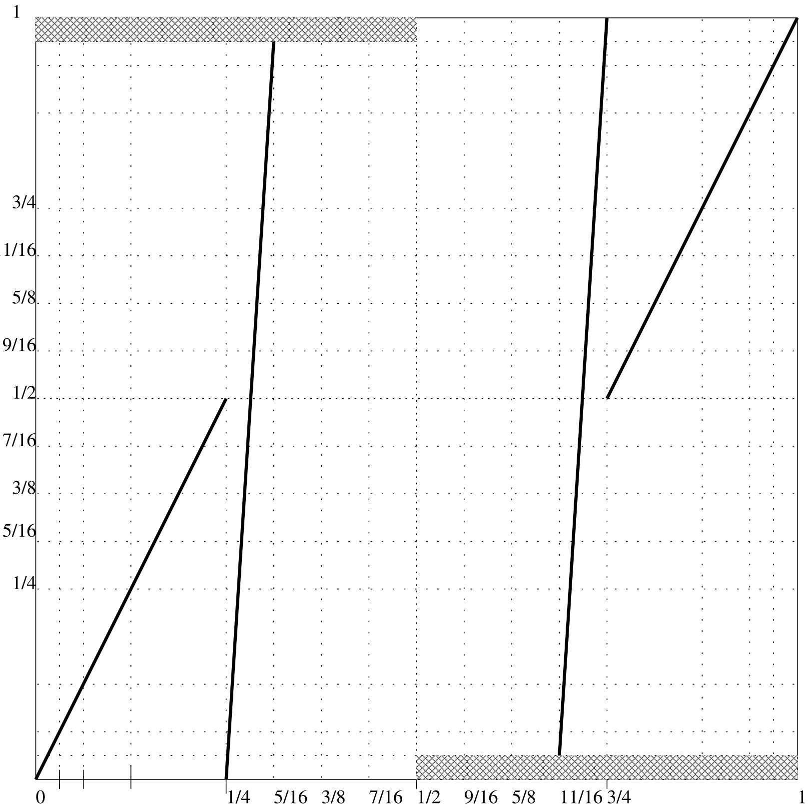


**Figure 4.** Additional branches on the intervals $I_i$ which satisfy $1 - \frac{1}{32} \in \partial(TI_i)$.

In figure 5 we now insert all other branches on intervals $I_i$ for which $[0, \frac{1}{32}] \subset T(I_i)$. Denote the indices of such intervals by $\mathcal{K}$; in this example, $\mathcal{K} = \{1, 3\}$, since $T(I_1)$, $T(I_3)$ both intersect $[0, \frac{1}{32}]$. It is important that $\mathcal{K} \neq \emptyset$.

Finally, in figure 6 we insert the remaining branches (for which $[0, \frac{1}{32}] \cap T(I_i) = \emptyset$). The indices of these branches will belong to $\mathcal{K}^c$; here $\mathcal{K}^c = \{2, 4\}$. To ensure eventual positivity of $A$, it is sufficient that the leftmost interval of $\mathfrak{P}$ contained in $T(I_j)$, $j \in \mathcal{K}^c$, intersects, the union $[0, b_1] \cup \bigcup_{k\in\mathcal{K}} I_k$. This is true for this map since $T(I_2) = \mathbf{[\frac{1}{32}, \frac{1}{16}]} \cup [\frac{1}{16}, \frac{1}{8}] \cup [\frac{1}{8}, \frac{1}{4}] \cup I_1 \cup I_2 \cup I_3$, and $T(I_4) = \boldsymbol{I_3} \cup I_4 \cup \mathcal{S}(I_4)$, where the bold intervals belong to $[0, b_1] \cup \bigcup_{k\in\mathcal{K}} I_k$. These properties will be made precise in the next theorem, and the reader should refer back to this illustrative example.

**Definition 4.10.** Let $T : [0, 1] \circlearrowleft$ be a piecewise-linear expanding Markov map satisfying the oddness condition $T(x) = 1 - T(1 - x)$. The class of maps we now construct will be called *class* $\mathfrak{C}$.

We describe $T$ on $[0, \frac{1}{2}]$ and construct the remainder of $T$ by symmetry. For $x \in [0, b_1]$, $b_1 < \frac{1}{2}$, $T(x) = x/2b_1$; we assume that $s := 1/2b_1 = \min_{x\in I} |T'(x)|$. In the region $[b_1, \frac{1}{2}]$, $T$ consists of finitely many linear branches with intervals of monotonicity $I_1, \ldots, I_q$. The boundary points of the Markov partition $\mathfrak{P}$ of $T$ consist only of the break points of $T$ and inverse images of $\frac{1}{2}$ contained in $[0, b_1]$; that is $\partial\mathfrak{P} \cap [0, \frac{1}{2}] = \{0\} \cup \bigcup_{i=1}^{q} \partial I_i \cup \bigcup_{i=1}^{M} s^{-i}/2$ and $\partial\mathfrak{P} \cap [\frac{1}{2}, 1]$ is constructed by reflection about $x = \frac{1}{2}$ $(M \geqslant 1)$. Let $\mathcal{K}$ be defined by

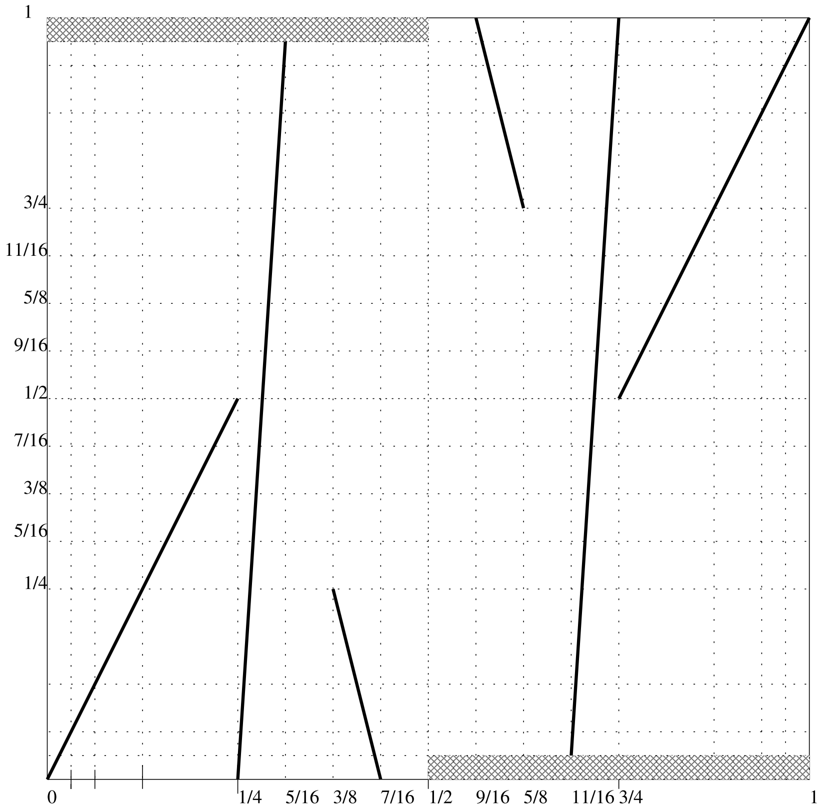


**Figure 5.** Additional branches on the intervals $I_i$ which satisfy $[0, \frac{1}{32}] \subset T(I_i)$.

$[0, s^{-M}/2] \subset T(I_k) \implies k \in \mathcal{K}$, and assume that $\mathcal{K} \neq \emptyset$. Let $\partial I^-$, $\partial I^+$ denote the lower and upper boundaries of an interval $I$, respectively. Further assume that $\partial(TI_j)^- < 1 - \partial(TI_j)^+$ for all $j = 1 \ldots, q$. Finally, for every $j \in \mathcal{K}^c := \{1, \ldots, q\} \setminus \mathcal{K}$, suppose that the leftmost interval contained in $T(I_j)$ is a subset of $[0, b_1] \cup \bigcup_{k\in\mathcal{K}} I_k$.

**Theorem 4.11.** *Let $T : [0, 1] \circlearrowleft$ be of class $\mathfrak{C}$. Then $T$ has a unique positive invariant density $\phi$, and $T$ is exact with respect to $\phi$. The essential spectrum of $\mathcal{P}|_{BV}$ is contained in the disc $\{|z| \leqslant 1/s\}$, while $\mathcal{P}|_{BV}$ has a real positive eigenvalue $1/s < \lambda < 1$. The corresponding eigenfunction is positive on $[0, \frac{1}{2}]$ and negative on $[\frac{1}{2}, 1]$.*

**Remarks 4.12.**

(a) Note that the four-legs map and the map of example 4.9 are of class $\mathfrak{C}$.

(b) The fact that $M \geqslant 1$ automatically implies that $T(0, \frac{1}{2}) \cap (\frac{1}{2}, 1) \neq \emptyset$. The theorem would also hold if we were to allow $M \geqslant 0$, while additionally insisting that $T(0, \frac{1}{2}) \cap (\frac{1}{2}, 1) \neq \emptyset$. We require the two intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ to communicate so that it is possible for $P$ to be mixing.

(c) The fact that $\mathcal{K} \neq \emptyset$ means that there is at least one other branch (besides the branch through 0) which maps onto the Markov partition set containing 0. The purpose of this is to allow returns near the fixed point, so that (a) $T$ will be mixing and (b) the matrix $A$ has a chance of eventually being positive.

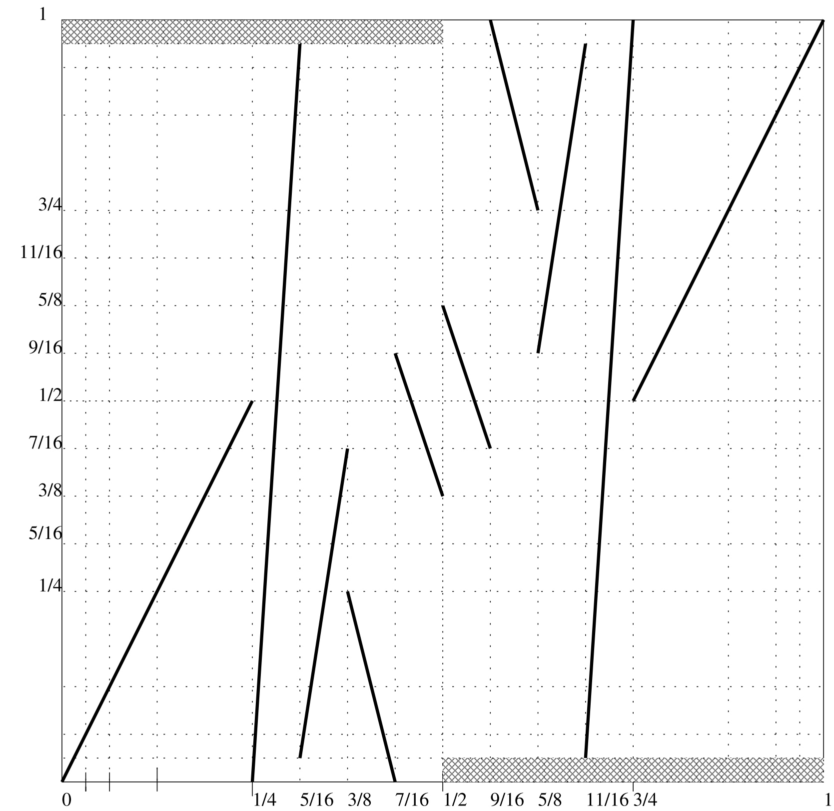


**Figure 6.** Additional branches on intervals $I_i$ which have the leftmost interval in $T(I_i)$ contained in $[0, b_1] \cup \bigcup_{k\in\mathcal{K}} I_k$.

(d) The property that $\partial(TI_j)^- < 1 - \partial(TI_j)^+$ for all $j = 1 \ldots, q$ is sufficient to ensure that the 'antisymmetric' matrix $A$ is non-negative. It also automatically precludes any images too close to 1; that is, it forces $T([0, \frac{1}{2}]) \subset [0, 1 - s^{-M}/2]$, and produces a 'dip' in $\mathcal{P}\mathbf{1}$ (cf conjecture 4.16).

(e) The property concerning those intervals $I_j$ with $j \in \mathcal{K}^c$ forces the intervals which do not map back to near 0 to at least map onto intervals which *do* map back near to 0 (that is, onto those intervals $I_k$ with $k \in \mathcal{K}$). In fact, this property is even stronger, requiring that the *leftmost interval in the image* maps onto a branch that maps to near 0. This extra property is sufficient to ensure eventual positivity of the 'antisymmetric' matrix $A$. Without the 'leftmost' property, we still have eventual positivity of the full matrix $P$.

**Proof of theorem 4.11.** By linearity of $T$, we may restrict $\mathcal{P}$ to the basis of piecewise-constant functions $F$ as in theorem 4.7. Set $P = \mathcal{P}|_F$ and $A = \mathcal{P}|_{\mathcal{A}}$, where $\mathcal{S}(x) = 1 - x$. Consider first the matrix $P$; we show that it is eventually positive. The cyclic structure of $P$ emanating from the first state means that beginning at the first state (corresponding to the interval $[0, s^{-M}/2]$) it is possible to cyclically move through all states in $[0, \frac{1}{2}]$; similarly, starting from the final state (corresponding to $[1 - s^{-M}/2, 1]$), one can move through $[\frac{1}{2}, 1]$ from right to left. Since $\mathcal{K} \neq \emptyset$, we are re-injected into this first (and last) state by at least one of the intervals in $[b_1, \frac{1}{2}]$. Thus, starting anywhere in $[0, b_1] \cup \bigcup_{k\in\mathcal{K}} I_k$, we can reach anywhere else in $[0, b_1] \cup \bigcup_{k\in\mathcal{K}} I_k$, and starting in $[1 - b_1, 1] \cup \bigcup_{k\in\mathcal{K}}(1 - I_k)$ we can reach anywhere else in $[1 - b_1, 1] \cup \bigcup_{k\in\mathcal{K}}(1 - I_k)$. By the property concerning $I_j$, $j \in \mathcal{K}^c$, intervals in $[b_1, \frac{1}{2}]$ which do not map into $[0, s^{-M}/2]$ do, however, map into $[0, b_1] \cup \bigcup_{k\in\mathcal{K}} I_k$, and thus we can move back and forth from this collection of intervals to those with indices $\mathcal{K}^c$. Thus we can move from anywhere in $[0, \frac{1}{2}]$ to anywhere else in $[0, \frac{1}{2}]$. By remark 4.12 (b), the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ communicate and therefore we can move from anywhere in $[0, 1]$ to anywhere else in $[0, 1]$ and so given any two states $i$, $j$ there is an $N$ such that $(P^N)_{ij} > 0$. This does *not*, however, show that $P$ is eventually positive. For this, *all* entries $(i, j)$ of $P^n$ must be positive for *every* $n \geqslant N$, some finite $N$. However, this is also true, because we can 'wait' in the state corresponding to $[0, s^{-M}/2]$ (or $[1 - s^{-M}/2, 1]$) as long as we like. Suppose we can move from a state $i$ to a state $j$ in $n$ steps ($n$ is bigger than the dimension of $P$); then we can go from $i$ to $j$ in any number of steps greater than $n$ by waiting in the first (or last) state as long as is necessary. Now $P$ is eventually positive, and by a standard result $T$ is exact with respect to $\phi$.

Consider now the $(M + q) \times (M + q)$ matrix $A$. Firstly, by the property that $\partial(TI_j)^- < 1 - \partial(TI_j)^+$ for all $j = 1 \ldots, q$, we see that $A$ must be non-negative. By a similar argument to that used for $P$ above, we cyclically move from the (first) state corresponding to the basis function $\chi_{[0,s^{-M}/2]} - \chi_{[1-s^{-M}/2,1]}$ through all states to the (final) state corresponding to $\chi_{I_q} - \chi_{1-I_q}$. We are again re-injected into the first state by the fact that $\mathcal{K} \neq \emptyset$ and that $\partial(TI_j)^- < 1 - \partial(TI_j)^+$ for all $j$. Thus there is no problem with $A$ being eventually positive on the states corresponding to $\chi_{[0,b_1]\cup\bigcup_{k\in\mathcal{K}} I_k} - \chi_{1-([0,b_1]\cup\bigcup_{k\in\mathcal{K}} I_k)}$. For the remaining states (corresponding to basis functions $\chi_{I_j} - \chi_{1-I_j}$ with $j \in \mathcal{K}^c$), we argue as follows. The fact that $\partial(TI_j)^- < 1 - \partial(TI_j)^+$ for all $j = 1 \ldots, q$ ensures that each row of $A$ has at least one positive entry, namely $A_{jl}$, where $I_l$ is the leftmost interval in $T(I_j)$. By the property that for every $j \in \mathcal{K}^c := \{1, \ldots, q\} \setminus \mathcal{K}$, the leftmost interval contained in $T(I_j)$ is a subset of $[0, b_1] \cup \bigcup_{k\in\mathcal{K}} I_k$, we know that at the next iteration, this leftmost interval will be sent to a state with index not in $\mathcal{K}^c$, and therefore we have communication between states with indices in $\mathcal{K}^c$ and the remaining states as before. Thus, $A$ is also eventually positive (since the first state may remain fixed forever so we can verify aperiodicity as for $P$), and there exists a strictly positive eigenvector $v$, with eigenvalue $\lambda$. This defines an eigenmeasure $\nu$ as in the proof of theorem 4.7, and since our map $T$ satisfies the conditions of this theorem, we again conclude that $1/s < \lambda < 1$. This completes the proof of theorem 4.11. □

**Remark 4.13.** Theorem 4.11 describes the situation where the fixed point $p = 0$ and the low-slope branch of $T$ containing $p$ has positive slope. We could also produce versions where $p = \frac{1}{2}$ (say), and the low-slope branch containing $P$ could have either a positive or a negative slope. We would require the negative-slope case if we wished to find a negative isolated eigenvalue.

### *4.3. Possible discontinuities in the second eigenfunction*

In this subsection we study possible 'spiking' behaviour in the second eigenfunctions that can occur when isolated eigenvalues approach $1/s$, where $s$ is the slope of the branch of a map containing the fixed point $p$.

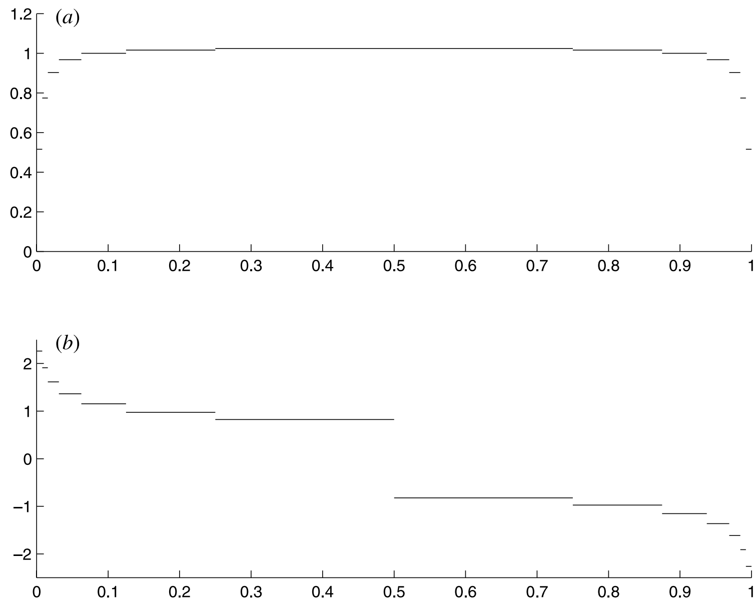

**Figure 7.** (*a*) Graph of the invariant density $\phi_{t_6}$ ($\mathcal{P}_{t_6}\phi_{t_6} = \phi_{t_6}$). (*b*) Graph of the eigenfunction $\psi_{t_6}$ corresponding to the second largest eigenvalue $\lambda_{t_6} \approx 0.5917$.

**Corollary 4.14.** *Under the hypotheses of theorem 4.7, one has (letting* $\psi$ *denote the eigenfunction corresponding to the eigenvalue* $\lambda$*)*

$$\lambda - 1/s = (1/\psi|_{[p,p+\delta]}) \sum_{\substack{I \in \mathfrak{P}_A \\ [p,p+\delta] \subset T(I)}} \frac{\psi|_I}{|T'|_I|}. \tag{4.5}$$

**Proof.** This follows immediately by evaluating the eigenvalue equation $\mathcal{P}\psi = \lambda\psi$ on the interval $[p, p+\delta]$. □

Considering corollary 4.14, we know from theorem 4.7 that $\psi$ is positive on $\mathfrak{P}_A$; thus both sides of (4.5) are positive. Now consider a family of mappings $\{T_n\}$ satisfying the conditions of theorem 4.7 which share the fixed point $p$ (and the local slope $s$ about $p$) and for which $\lambda_n \to 1/s$. If one assumes that $\psi_n|_I$ is uniformly bounded below (uniformly in $n$), for all $I \in \mathfrak{P}_{n,A}$ with $[p, p+\delta] \subset T_n(I)$, then equation (4.5) says that $\psi_n|_{[p,p+\delta]}$ *must become unbounded as* $n \to \infty$. If $\psi_n|_I$ is not uniformly bounded below, then $\psi_n|_{[p,p+\delta]}$ must still become unbounded from normalization considerations of $\psi$. Either way, a 'spike' occurs in the eigenfunctions $\psi_n$ in a neighbourhood of the fixed point $p$, as $\lambda_n$ approaches $1/s$. We will now briefly show that indeed this spike is observed in the second eigenfunction of the four-legs map in section 3 (see figure 7).

**Lemma 4.15.** *Let* $T_t$ *denote the map of section 3, and let* $\nu_n$ *be a signed measure satisfying* $\nu_n \circ T_{t_n}^{-1} = \lambda_n \nu_n$, *with* $\nu_n([0, 1]) = 0$ *and* $\nu_n([0, \frac{1}{2}]) = \frac{1}{2}$, *and let* $\psi_{t_n}$ *denote the density of* $\nu_n$. *Then*

$$\frac{n^{2/3}}{t_n} \leqslant \frac{\sup_{x\in[0,1/2]} \psi_{t_n}(x)}{\inf_{x\in[0,1/2]} \psi_{t_n}(x)} \leqslant \frac{5n}{t_n} \tag{4.6}$$

*with the supremum occurring in the interval* $[0, 1/2^{n+1}]$ *and the infimum occurring in* $[\frac{1}{4}, \frac{1}{2}]$ *(clearly* $\psi_{t_n}$ *is constant on these intervals).*

The dips in the invariant density at $x = 0, 1$ are related to the fact that for $t < 4$, small neighbourhoods of the fixed points have fewer pre-images than for $t = 4$ (and therefore less 'mass' is injected into these regions). The spikes in the second eigenfunction may also informally be thought of as a consequence of the missing pre-images; the lack of pre-image deprives 'cancellation' of the spike when the eigenfunction is acted on by the Perron–Frobenius operator (see also the discussion in section 4.3). These considerations, backed by our numerical experience, lead us naturally to the following conjecture.

### *4.4. A conjecture*

To conclude this section, we present a conjecture to suggest very general conditions under which piecewise-linear expanding interval maps may possess eigenvalues outside the disc $\{|z| \leqslant \vartheta\}$. The hypotheses of the conjecture were formulated after numerical studies of a large number of interval maps. We note that the hypotheses of theorem 4.7 and those of the following conjecture overlap, but one set of hypotheses is not a subset of the other.

**Conjecture 4.16.** Let $T : [0, 1]\circlearrowleft$ be a piecewise-linear expanding map with a unique everywhere-positive invariant density $\phi$. Assume that $T$ has a fixed point $p$ on the interior of the branch of minimal slope; that is, $|T'(p)| = \min_{x\in[0,1]} |T'(x)|$ and $T$ is $C^2$ in an open neighbourhood† about $p$. Further, suppose that the function $g := \mathcal{P}\mathbf{1}$ satisfies

(a) $g(x) = g(p)$ for $p < x \leqslant p + \epsilon_1^+$ and $g(x) > g(p)$ for $p + \epsilon_1^+ < x$ for some $\epsilon_1^+ \geqslant 0$,
(b) $g(x) = g(p)$ for $p - \epsilon_1^- < x < p$ and $g(x) > g(p)$ for $x < p - \epsilon_1^-$ for some $\epsilon_1^- \geqslant 0$.

where at least one of $\epsilon_1^+$ and $\epsilon_1^-$ is strictly greater than 0. Then for all sufficiently small $\epsilon_1^{\pm}$, $\mathcal{P}$ has an eigenfunction of bounded variation with eigenvalue outside the disc of radius $\vartheta = 1/\inf_{x\in I} |T'(x)|$.

The reasoning behind this conjecture is as follows. The existence of large isolated eigenvalues is equivalent to the existence of initial densities $\phi'$ for which $\mathcal{P}^k\phi' \to \phi$ very slowly (at a rate slower than $O(\vartheta^k)$). A possible mechanism for the existence of such densities is the existence of a fixed point $p$ on a branch of low slope, coupled with a 'less than average' injection of mass into a neighbourhood of the fixed point.

It is often observed numerically that maps which possess large isolated eigenvalues have corresponding eigenfunctions $\psi'$ which have 'spikes' in a neighbourhood of the fixed point $p$; these spikes often change sign when crossing over the fixed point. If $(\lambda', \psi')$ is a genuine eigenvalue/eigenfunction pair for $\mathcal{P}$, we may set $\phi' = \phi + \alpha\psi'$, where $\alpha$ is sufficiently small to ensure that $\phi' \geqslant 0$, and have that $\|\mathcal{P}^k\phi' - \phi\|_{BV} = \lambda'^k\|\psi'\|_{BV}$.

A large negative 'spike' in $\psi'$ translates into a 'dip' in $\phi + \alpha\psi'$ near the fixed point. Considering only the slope at the fixed point, this 'dip' will be 'washed away' at a rate given by the slope. However, our 'dip' conditions (a) and (b) slow down this washing away just a little, pushing the eigenvalue outside the disc $\{|z| \leqslant \vartheta\}$.

## 5. Discussion and conclusions

The current work is preliminary. We do not pretend to give a systematic treatment of the theory concerning the isolated spectra of Perron–Frobenius operators arising from expanding interval

† If $p = 0$ or 1, then $T$ should be $C^2$ when $T : [0, 1]\circlearrowleft$ is regarded as a map of the circle $T : S^1\circlearrowleft$ by identifying the boundary points 0 and 1.

maps. Rather, we have scratched the surface of this interesting, important and seemingly largely unstudied subject.

**Remark 5.1.** At this point, we would like to describe an alternative method of producing an isolated eigenvalue as pointed out by one of the referees. Take a mixing piecewise-linear Markov map $T$ with $k$ linear branches whose Perron–Frobenius operator $\mathcal{P}$ has an eigenvalue $0 < \lambda < 1$; suppose that $|\lambda| < \vartheta$. For each $i = 1, \dots, k$, replace the $i$th branch of $T$ (denoted $T_i$) by $l$ smaller branches $\tilde{T}_{i_1}, \dots, \tilde{T}_{i_l}$ with slope $l|T_i'|$ such that the image of each $\tilde{T}_{i_j}$, $j = 1, \dots, l$, is equal to the image of $T_i$. Denoting by $\tilde{\mathcal{P}}$ and $\tilde{\vartheta}$ the Perron–Frobenius operator and essential spectral radius bound for $\tilde{T}$, it is clear that $\tilde{\vartheta} = \vartheta/l$. It is also clear that the eigenvalue $\lambda$ is still an eigenvalue for $\tilde{\mathcal{P}}$ and so by making $l$ sufficiently large, we may shrink the essential spectrum so that $\lambda$ eventually lies outside it.

Future work will include enlarging the class of maps for which we can rigorously show the existence of isolated eigenvalues. It is also desirable for the conditions on the maps to show more transparently the mechanism by which the isolated eigenvalues appear, and the connections with the slow mixing rates and corresponding macroscopic structures. (Perhaps conjecture 4.16 or some variant can be proven.)

An understanding in greater generality of the transition of eigenvalues across the circle $\{|z| = \vartheta\}$ is also of interest. For example, understanding how the map $T$ must vary, and how the eigenfunctions $\psi$ must behave during the transition.

Also of importance is the connection between numerical techniques and mathematical rigour. Finite-dimensional (matrix) approximations of the Perron–Frobenius operator and the corresponding spectra may be calculated numerically. Such approximations are often produced for very general (higher-dimensional, chaotic) maps, and it is of great interest to understand what of the dynamics can be extracted from the numerically calculated spectral points and their eigenvectors. For example, whether outlying eigenvalues give information about large slowly mixing structures embedded in the overall (on average more rapidly mixing) chaotic dynamics.

## Acknowledgments

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