MR2276432 (2008b:37143) 37M25 (37C30 37D20)

Froyland, Gary (5-NSW-SM)

On Ulam approximation of the isolated spectrum and eigenfunctions of hyperbolic maps. (English summary)

Discrete Contin. Dyn. Syst. **17** (2007), *no. 3,* 671–689 (*electronic*).

Ulam's discretization scheme, which is easily implemented on a computer, has proven to be successful in approximating the isolated spectrum of the Perron-Frobenius operator associated with a one-dimensional map. However, for two-dimensional hyperbolic maps, counterexamples to spectral stability of Ulam's scheme were provided by M. Blank, G. Keller and C. Liverani in [Nonlinearity **15** (2002), no. 6, 1905–1973; MR1938476 (2003m:37033)]. Instead, by introducing additional smooth random perturbations, they constructed a 'modified' Ulam scheme that well approximates the Perron-Frobenius operator in a proper sense.

In this note, the author addresses the following question: When does the 'original' Ulam scheme (without smoothing) well approximate the isolated spectrum of the Perron-Frobenius operator of a given two-dimensional hyperbolic map?

For this purpose, he constructs a hyperbolic map and a suitable space of distributions on the unit square. He shows that when acting on this space the Perron-Frobenius operator associated with the map is quasi-compact. In particular, he shows that 1/2 and 1 are isolated eigenvalues. For this particular map the author proves that Ulam's method well approximates the isolated eigenvalues and their corresponding eigendistributions. The proof relies upon the ability to separate the unstable and stable dynamics into two one-dimensional problems.

The note ends with desirable lines of generalizations and an open question.

Reviewed by *Wael Bahsoun*

References

1. Viviane Baladi, Unpublished, 1989.
2. Viviane Baladi and Masato Tsujii, *Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms*, Preprint, 2005. cf. MR 2008d:37034
3. Viviane Baladi and Lai-Sang Young, *On the spectra of randomly perturbed expanding maps*, Communications in Mathematical Physics, **156** (1993), 355–385. MR1233850 (94g:58172)
4. Michael Blank and Gerhard Keller, *Random perturbations of chaotic dynamical systems: stability of the spectrum*, Nonlinearity, **11** (1998), 1351–1365. MR1644405 (99h:58145)
5. Michael Blank, Gerhard Keller, and Carlangelo Liverani, *Ruelle-Perron-Frobenius spectrum for Anosov maps*, Nonlinearity, **15** (2002), 1905–1973. MR1938476 (2003m:37033)
6. F. Brini, S. Siboni, G. Turchetti, and S. Vaienti, *Decay of correlations for the automorphism of the torus* $\mathbb{T}^2$, Nonlinearity, **10** (1997), 1257–1268. MR1473382 (98h:58133)
7. P. Collet and J.-P. Eckmann, *Liapunov multipliers and decay of correlations in dynamical systems*, Journal of Statistical Physics, **115** (2004), 217–253. MR2070095 (2006f:37008)

8. Michael Dellnitz, Gary Froyland, and Oliver Junge, "The Algorithms behind GAIO-Set Oriented Numerical Methods for Dynamical Systems. Ergodic theory, analysis, and efficient simulation of dynamical systems," pages 145–174. Springer, Berlin, 2001. MR1850305 (2002k:65217)
9. Michael Dellnitz, Gary Froyland, and Stefan Sertl, *On the isolated spectrum of the Perron-Frobenius operator*, Nonlinearity, **13** (2000), 1171–1188. MR1767953 (2001e:37033)
10. Michael Dellnitz and Oliver Junge, *On the approximation of complicated dynamical behaviour*, SIAM Journal for Numerical Analysis, **36** (1999), 491–515. MR1668207 (2000c:37026)
11. Jiu Ding and Aihui Zhou, *Finite approximations of Frobenius-Perron operators. a solution of Ulam's conjecture to multi-dimensional transformations*, Physica D, **92** (1996), 61–68. MR1384679 (97a:58097)
12. Gary Froyland, *Finite approximation of Sinai-Bowen-Ruelle measures for Anosov systems in two dimensions*, Random and Computational Dynamics, **3** (1995), 251–263. MR1362773 (96k:58133)
13. Gary Froyland, *Computer-assisted bounds for the rate of decay of correlations*, Communications in Mathematical Physics, **189** (1997), 237–257. MR1478538 (98i:58138)
14. Gary Froyland and Michael Dellnitz, *Detecting and locating near-optimal almos-invariant sets and cycles*, SIAM J. Sci. Comput., **24** (2003), 1839–1863. MR2005610 (2004e:37132)
15. Sébastien Gouëzel and Carlangelo Liverani, *Banach spaces adapted to Anosov systems*, Ergodic Theory and Dynamical Systems, **26** (2006), 189–217. MR2201945 (2007h:37037)
16. Hiroshi Hasegawa and William Saphir, *Unitarity and irreversibility in chaotic systems*, Physical Review A, **46** (1992), 7401–7423. MR1197304 (93k:58179)
17. Franz Hofbauer and Gerhard Keller, *Ergodic properties of invariant measures for piecewise monotonic transformations*, Mathematische Zeitschrift, **180** (1982), 119–140. MR0656227 (83h:28028)
18. Gerhard Keller and Hans Henrik Rugh, *Eigenfunctions for smooth expanding circle maps*, Nonlinearity, **17** (2004), 1723–1730. MR2086147 (2005d:37081)
19. Andrzej Lasota and Michael C. Mackey, "Chaos, Fractals, and Noise: Stochastic Aspects of Dynamics," volume 97 of Applied Mathematical Sciences, Springer-Verlag, New York, 2nd edition, 1994. MR1244104 (94j:58102)
20. Tien-Yien Li, *Finite approximation for the Frobenius-Perron operator. a solution to Ulam's conjecture*, Journal of Approximation Theory, **17** (1976), 177–186. MR0412689 (54 #811)
21. Prashant Mehta, Personal communication, 2005.
22. Rua Murray, *Approximation error for invariant density calculations*, Discrete and Continuous Dynamical Systems, **4** (1998), 535–557. MR1612772 (99e:28033)
23. A. Pikovsky and O. Popovych, *Persistent patterns in deterministic mixing flows*, Europhys. Lett., **61** (2003), 625–631.
24. David Ruelle, *The thermodynamic formalism for expanding maps*, Commun. Math. Phys., **125** (1989), 239–262. MR1016871 (91a:58149)
25. Marek Rychlik, *Bounded variation and invariant measures*, Studia Mathematica, **76** (1983), 69–80. MR0728198 (85h:28019)
26. Christof Schütte, Wilhelm Huisinga, and Peter Deuflhard, *Transfer operator approach to con-*

formational dynamics in biomolecular systems, in "Ergodic Theory, Analysis, and Efficient Simulation of Dynamical Systems" (eds. Bernold Fiedler), pages 191–223. Springer, Berlin, 2001. MR1850307 (2002g:92013)

27. S. Ulam, "Problems in Modern Mathematics," Interscience, 1964. MR0280310 (43 #6031)
28. Joachim Weber, Fritz Haake, Petr A Braun, Christopher Manderfeld, and Petr Seba, *Resonances of the Frobenius-Perron operator for a Hamiltonian map with a mixed phase space*, Journal of Physics A, **34** (2001), 7195–7211. MR1862763 (2002i:82055)

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.