Optimal algorithms for doubly weighted approximation of univariate functions

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Abstract

We consider a ϱ-weighted L_q approximation in the space of univariate functions $f : \mathbb{R}_+ \to \mathbb{R}$ with finite $\|f^{(r)}\psi\|_{L_p}$. Let $\alpha = r - 1/p + 1/q$ and $\omega = \varrho/\psi$. Assuming that ψ and ω are non-increasing and the quasi-norm $\|\omega\|_{L_{1/\alpha}}$ is finite, we construct algorithms using function/derivatives evaluations at n points with the worst case errors proportional to $\|\omega\|_{L_{1/\alpha}} n^{-r+(1/p-1/q)_+}$. In addition we show that this bound is sharp; in particular, if $\|\omega\|_{L_{1/\alpha}} = \infty$ then the rate $n^{-r+(1/p-1/q)_+}$ cannot be achieved. Our results generalize known results for bounded domains such as $[0,1]$ and $\varrho = \psi \equiv 1$. We also provide a numerical illustration.

1 Introduction

We study in this paper the approximation of univariate real-valued functions $f : D \to \mathbb{R}$ where the domain is $D = \mathbb{R}_+ = [0, \infty)$ and the error of approximation is measured in a ϱ-weighted L_q (semi-)norm,

$$\|f\varrho\|_{L_q} = \left(\int_D |f(x)\,\varrho(x)|^q \,\mathrm{d}x\right)^{1/q}, \qquad 1 \le q \le \infty.$$

Here $\varrho : D \to \mathbb{R}_+$ is a nonnegative and measurable weight function. The restriction to $D = [0, \infty)$ is to simplify the notation only, since all the results can be easily extended to D being an arbitrary interval including $D = \mathbb{R}$.

We assume that approximation algorithms use function and/or derivatives values at n points, and we study the worst case errors of such algorithms with respect to the unit balls of the following spaces $F = F(r, p, \psi)$. For given positive integer r, $1 \le p \le \infty$, and a positive and measurable weight function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, the space F consists of functions with (locally) absolutely continuous derivative $f^{(r-1)}$ and

$$\|f^{(r)}\psi\|_{L_p} \,<\, \infty.$$

A special case of such a problem is the unweighted approximation on a compact interval $[0, T]$, which corresponds to $\psi \equiv 1$ and $\varrho(x) = 1$ for $x \in [0, T]$ and $\varrho(x) = 0$ otherwise. It follows from, e.g., [2, 8, 13, 14] that the nth minimal worst case errors are then proportional to

$$T^{\,r-1/p+1/q}\, n^{-r+(1/p-1/q)_+} \quad \text{where} \quad x_+ \,=\, \max(0, x). \tag{1}$$

Note that for $p = r = 1$ and $q = \infty$ the errors do not converge to zero; therefore this case is excluded from our considerations.

Doubly weighted approximation problems were first investigated in [16], see also [9–12]. Moreover, the spaces $F(r, p, \psi)$ and the corresponding results were used to construct weighted tensor product spaces of multivariate and ∞-variate functions, see, e.g., [3–6, 15, 17].

The results of [16] were obtained under rather complicated assumptions. In the current paper, we obtain more accurate results using simpler assumptions and deliver different algorithms from those in [16]. Assumptions and results of this paper are rather comparable to those in [11], where the weighted integration problem with $r = 1$, $p = \infty$, $\psi \equiv 1$, and the weight $\varrho(x) = \exp(-x)$ was considered; see also [1, 10, 12]. We believe that our new algorithms are more suitable for constructing Smolyak's (often called Sparse Grid) algorithms for multivariate approximation problems.

We now discuss the main results of the paper. Define

$$\alpha \,=\, r - \frac{1}{p} + \frac{1}{q} \qquad \text{and} \qquad \omega \,=\, \frac{\varrho}{\;}.$$

We assume that ψ and ω are monotonically non-increasing, and that

$$\|\omega^{1/\alpha}\|_{L_1} \,=\, \int_{\mathbb{R}_+} \omega^{1/\alpha}(x)\,\mathrm{d}x \,<\, \infty.$$

Clearly, $\|\omega^{1/\alpha}\|_{L_1}^{\alpha} \,=\, \|\omega\|_{L_{1/\alpha}}$ is the $L_{1/\alpha}$ *quasi*-norm of ω.

For given $n \geq 1$, let the points $x_{n,i}$ for $i = 1, \dots, n$ be given by

$$\int_0^{x_{n,i}} \omega^{1/\alpha}(x)\,\mathrm{d}x \,=\, \frac{i-1}{n}\,\|\omega^{1/\alpha}\|_{L_1}.$$

That is, $x_{n,i}$ are chosen so that the integrals of $\omega^{1/\alpha}$ between successive points are equal. We prove that the algorithm based on piecewise Taylor polynomials of degree $r-1$ at the points

$x_{n,i}$ has the worst case error bounded from above by (see Theorem 1)

$$\frac{\|\omega\|_{L_{1/\alpha}}}{(r-1)!\,((r-1)\,p^*+1)^{1/p^*}}\, n^{-r+(1/p-1/q)_+}, \tag{2}$$

where $x_+ = \max(x,0)$ for any $x \in \mathbb{R}$. Recall that $n^{-r+(1/p-1/q)_+}$ is the best possible convergence rate for the interval $D = [0,T]$ and $\psi = \varrho \equiv 1$. Moreover, in the latter case $\|\omega\|_{L_{1/\alpha}} = T^\alpha$, cf. (1).

Since Taylor's algorithm uses derivatives of f at $x_{n,i}$ which can be too demanding, we also consider an algorithm based on piecewise Lagrange extrapolation that requires function values only. It turns out that the latter algorithm enjoys an error bound similar to (2) (see Theorem 2 and Remark 3). We stress that in general the use of extrapolation instead of interpolation is crucial. While interpolation gives smaller worst case error than extrapolation for $\psi \equiv 1$ and arbitrary ϱ, it completely fails (even for large values of n) when the weight ψ rapidly decreases and this decrease is not compensated with much faster decreasing ϱ. In such cases the Lagrange interpolation can achieve the rate $n^{-r+(1/p-1/q)_+}$ only asymptotically as $n \to \infty$ (with best asymptotic constant among the three algorithms), see the comments at the end of Section 3.

In addition we show that the Taylor and Lagrange extrapolation algorithms are optimal not only with respect to the rate, but also with respect to the weights ψ and ϱ, and that the dependence of the error on the weights via $\|\omega\|_{L_{1/\alpha}}$ is crucial. Specifically, if $e^*(n)$ denotes the minimal worst case error that can be achieved using n points, then (see Theorem 3)

$$\liminf_{n\to\infty}\; e^*(n)\, n^{r-(1/p+1/q)_+} \;\ge\; c\,\|\omega\|_{L_{1/\alpha}}$$

with $c > 0$ independent of ψ and ϱ. Hence, if $\|\omega\|_{L_{1/\alpha}} = \infty$ then the rate $n^{-r+(1/p-1/q)_+}$ cannot be achieved.

The content of the paper is as follows. The problem is precisely formulated in Section 2. The particular algorithms are analyzed in Section 3. Section 4 is devoted to the lower bound of Theorem 3, and numerical results are presented in Section 5.

2 Problem formulation

We consider the following approximation problem. Let

$$\psi : \mathbb{R}_+ \to \mathbb{R}_+$$

be a given positive and measurable function. For a positive integer r and $p \in [1,\infty]$, let $F = F(r,p,\psi)$ be the linear space of functions $f : \mathbb{R}_+ \to \mathbb{R}$ with (locally) absolutely continuous derivative $f^{(r-1)}$ and

$$\left\|f^{(r)}\psi\right\|_{L_p} \;=\; \left(\int_{\mathbb{R}_+} \left|f^{(r)}(x)\,\psi(x)\right|^p \,\mathrm{d}x\right)^{1/p} \;<\infty.$$

In particular, if $p=\infty$ then

$$\left\|f^{(r)}\psi\right\|_{L_\infty} \;=\; \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} \left|f^{(r)}(x)\,\psi(x)\right| \;<\; \infty.$$

The function spaces $F=F(r,p,\psi)$ were introduced in [16], see also [3–6, 9–12, 15, 17].

Our goal is to construct efficient algorithms for a weighted L_q approximation of functions $f\in F$. More precisely, we allow algorithms A_n such that, for every $f\in F$, $A_nf:\mathbb{R}_+\to\mathbb{R}$ is a measurable function φ depending on f only via values of f and its derivatives at n points. That is,

$$A_nf \;=\; \varphi\left(f^{(s)}(x_i):\ 1\le i\le n,\, 0\le s\le k\right),$$

where $0\le k\le r-1$ and $x_i\in\mathbb{R}_+$ for all i. Then the error of A_n at f is given as

$$e(A_n;f) \;=\; \|(f-A_nf)\varrho\|_{L_q} \;=\; \left(\int_{\mathbb{R}_+} |(f(x)-(A_nf)(x))\,\varrho(x)|^q \,\mathrm{d}x\right)^{1/q}.$$

Here $1\le q\le\infty$ and

$$\varrho:\mathbb{R}_+\to\mathbb{R}_+$$

is another weight function, which is measurable, non-negative, and positive on a set of positive Lebesgue measure. Clearly, for $q=\infty$ we have

$$e(A_n;f) \;=\; \operatorname*{ess\,sup}_{x\in\mathbb{R}_+} |f(x)-(A_nf)(x)|\;\varrho(x).$$

We define the worst case error of A_n as

$$e^{\mathrm{wor}}(A_n) \;=\; \sup_{\|f^{(r)}\psi\|_{L_p}\le 1} e(A_n;f).$$

Then for algorithms A_n that are linear in f we have

$$e(A_n;f) \;\le\; e^{\mathrm{wor}}(A_n)\,\left\|f^{(r)}\psi\right\|_{L_p} \qquad \text{for all} \quad f\in F.$$

Observe that our approximation problem is defined by five parameters: r, p, ψ define the space F, and q, ϱ define the error.

Such approximation problems have been investigated in [16]; however, under relatively complicated assumptions concerning the five parameters. In this paper we derive efficient algorithms under the following assumptions on weights ψ, ϱ and the parameters p, q, r.

Let

$$\alpha \,=\, \alpha(r,p,q) \,=\, r - \frac{1}{p} + \frac{1}{q}$$

(where $q < \infty$ if $r = p = 1$) and

$$\omega = \frac{\varrho}{\psi} : \mathbb{R}_+ \to \mathbb{R}_+.$$

We assume that

$$\|\omega^{1/\alpha}\|_{L_1} \,=\, \int_{\mathbb{R}_+} \omega^{1/\alpha}(x)\,\mathrm{d}x \,<\, \infty, \tag{3}$$

and that ψ and ω are non-increasing,

$$\psi(x_1) \geq \psi(x_2) \quad \text{and} \quad \omega(x_1) \geq \omega(x_2) \qquad \text{for all} \quad 0 \leq x_1 \leq x_2. \tag{4}$$

Clearly, (4) implies that ϱ is also non-increasing.

Remark 1 *We could relax (4) and assume that ψ and ω are only asymptotically non-increasing, i.e., the inequalities in (4) hold for $T \leq x_1 \leq x_2$ for some (possibly large) $T \geq 0$. Then the results of this paper remain true (in the cases of Proposition 1, Theorem 3) or can be generalized (in the cases of Theorems 1 and 2) provided ψ and ϱ are in addition (locally) Riemann integrable, see Remark 5. An advantage of working with the relaxed assumption is that it covers approximation on a compact interval where the weights need not be monotonic.*

We end this section with the following observation. In general, $\|f\varrho\|_{L_q}$ need not be finite for all polynomials of degree smaller than r. However, since the algorithms to be presented are based on interpolation/extrapolation by piecewise polynomials of degree $r-1$, the errors of such algorithms are finite for all $f \in F(r,p,\psi)$. We have the following proposition.

Proposition 1 *Let* (3) *and* (4) *hold. Let $v_r(x) = x^{r-1}$. Then $\|v_r\varrho\|_{L_q} < \infty$ is a necessary and sufficient condition for $\|f\varrho\|_{L_q} < \infty$ for all $f \in F(r,p,\psi)$.*

Proof. We present the proof for $q < \infty$ since the proof for $q = \infty$ is even easier. Since polynomials of degree at most $r-1$ belong to $F(r,p,\psi)$ and the condition $\|v_r\varrho\|_{L_q} < \infty$ implies that $\|v\varrho\|_{L_q} < \infty$ for all polynomials v of degree $\leq r-1$, we conclude from Taylor's theorem that it is sufficient to show that $\|f\varrho\|_{L_q}$ is finite for any f of the form $f(x) = \int_{\mathbb{R}_+} f^{(r)}(t)\,(x-t)_+^{r-1}\,\mathrm{d}t$. Using $\varrho = \omega\,\psi$, we have

$$
\begin{aligned}
\|f\varrho\|_{L_q} &= \left(\int_{\mathbb{R}_+} \varrho^q(x) \left|\int_{\mathbb{R}_+} f^{(r)}(t)\,(x-t)_+^{r-1}\,\mathrm{d}t\right|^q \mathrm{d}x\right)^{1/q} \\
&= \left(\int_{\mathbb{R}_+} \omega^q(x) \left|\int_{\mathbb{R}_+} f^{(r)}(t)\,\psi(t)\,\frac{\psi(x)}{\psi(t)}\,(x-t)_+^{r-1}\,\mathrm{d}t\right|^q \mathrm{d}x\right)^{1/q} \\
&\le \left\|f^{(r)}\psi\right\|_{L_p} \left(\int_{\mathbb{R}_+} \omega^q(x) \left(\int_0^x (x-t)^{(r-1)p^*}\,\mathrm{d}t\right)^{q/p^*} \mathrm{d}x\right)^{1/q} \\
&= \frac{\left\|f^{(r)}\psi\right\|_{L_p}}{((r-1)\,p^*+1)^{1/p^*}} \left(\int_{\mathbb{R}_+} \omega^q(x)\,x^{(r-1/p)q}\,\mathrm{d}x\right)^{1/q},
\end{aligned}
$$

where we used the fact that ψ is monotonic, i.e.,

$$
\frac{\psi(x)}{\psi(t)} \le 1 \quad \text{for} \quad x \ge t \ge 0, \tag{5}
$$

and next applied Hölder's inequality with conjugate exponents $1/p + 1/p^* = 1$.

The remaining integral can be estimated as follows. Since $\alpha = r - 1/p + 1/q$, we have $\omega = \omega^{\alpha/\alpha} = (\omega^{1/\alpha})^{1/q}\,(\omega^{1/\alpha})^{r-1/p}$ and

$$
\omega(x)\,(x-u)^{r-1/p} = \left(\omega^{1/\alpha}(x)\right)^{1/q} \left(\omega^{1/\alpha}(x)\,(x-u)\right)^{r-1/p} \quad \text{for} \quad x \ge u \ge 0. \tag{6}
$$

Since ω is non-increasing,

$$
\omega^{1/\alpha}(x)\,(x-u) \le \int_u^x \omega^{1/\alpha}(t)\,\mathrm{d}t \quad \text{for} \quad x \ge u \ge 0. \tag{7}
$$

Taking $u = 0$ in (6) and (7), we obtain

$$
\begin{aligned}
\int_{\mathbb{R}_+} \omega^q(x)\,x^{(r-1/p)q}\,\mathrm{d}x &\le \int_{\mathbb{R}_+} \omega^{1/\alpha}(x) \left(\int_0^x \omega^{1/\alpha}(z)\,\mathrm{d}z\right)^{(r-1/p)q} \mathrm{d}x \\
&\le \left\|\omega^{1/\alpha}\right\|_{L_1} \left\|\omega^{1/\alpha}\right\|_{L_1}^{(r-1/p)q} = \|\omega\|_{L_{1/\alpha}}^q < \infty.
\end{aligned}
$$

The proof is complete. □

3 Algorithms

The algorithms that we now present and analyze use the points $x_i = x_{n,i}$, $1 \le i \le n$, defined by the equations

$$\int_0^{x_i} \omega^{1/\alpha}(x)\,\mathrm{d}x \;=\; \frac{i-1}{n}\,\|\omega^{1/\alpha}\|_{L_1}, \tag{8}$$

so that the integral of $\omega^{1/\alpha}(x)$ between successive points is exactly $\|\omega^{1/\alpha}\|_{L_1}/n$. We have $x_1 = 0$. For convenience, we also set $x_{n+1} = +\infty$.

3.1 Taylor-based approximation

Consider first a piecewise Taylor approximation. For $x_i \le x < x_{i+1}$ we approximate $f(x)$ by a Taylor polynomial $\mathcal{T}_i^r f$ of degree at most $(r-1)$ using the values $f(x_i), f'(x_i), \ldots, f^{(r-1)}(x_i)$. That is, for $x_i \le x < x_{i+1}$ we define

$$\left(A_n^{\mathcal{T}} f\right)(x) \;=\; \left(\mathcal{T}_i^r f\right)(x), \qquad \text{where} \qquad \left(\mathcal{T}_i^r f\right)(x) \;=\; \sum_{s=0}^{r-1} f^{(s)}(x_i)\,\frac{(x-x_i)^s}{s!}.$$

Let

$$c_1 \;=\; c_1(r,p) \;=\; \frac{1}{(r-1)!\,((r-1)p^*+1)^{1/p^*}}, \tag{9}$$

where p^* is the conjugate of p, i.e., $1/p + 1/p^* = 1$. In particular, $c_1(r,1) = 1/(r-1)!$ and $c_1(r,\infty) = 1/r!$.

Theorem 1 *For the Taylor-based approximation we have*

$$e\left(A_n^{\mathcal{T}}; f\right) \;\le\; c_1\,\|\omega\|_{L_{1/\alpha}}\,\left\|f^{(r)}\,\psi\right\|_{L_p}\,n^{-r+(1/p-1/q)_+}.$$

Proof. We provide the proof for $1 < p < \infty$ and $1 \le q < \infty$ only, as the proof in the other cases is similar. Using the integral form of the Taylor remainder formula

$$f(x) - (\mathcal{T}_i^r f)(x) \;=\; \int_{x_i}^{x} f^{(r)}(t)\,K^{\mathcal{T}}(x,t)\,\mathrm{d}t \qquad \text{with} \qquad K^{\mathcal{T}}(x,t) = \frac{(x-t)^{r-1}}{(r-1)!}, \tag{10}$$

for $x_i \le x < x_{i+1}$, together with (5) and Hölder's inequality, we obtain

$$\begin{aligned}
\varrho(x)\left|f(x)-(A_n^{\mathcal{T}}f)(x)\right| &= \varrho(x)\left|\int_{x_i}^{x} f^{(r)}(t)\,K^{\mathcal{T}}(x,t)\,\mathrm{d}t\right| \\
&= \omega(x)\left|\int_{x_i}^{x} f^{(r)}(t)\,\psi(t)\left(\frac{\psi(x)}{\psi(t)}\,K^{\mathcal{T}}(x,t)\right)\,\mathrm{d}t\right| \\
&\le \omega(x)\left(\int_{x_i}^{x}\left|f^{(r)}(t)\,\psi(t)\right|^{p}\,\mathrm{d}t\right)^{1/p}\left(\int_{x_i}^{x}\left|K^{\mathcal{T}}(x,t)\right|^{p^*}\,\mathrm{d}t\right)^{1/p^*}.
\end{aligned}$$

The formula for $K^{\mathcal{T}}$ in (10) yields

$$\left(\int_{x_i}^{x}\left|K^{\mathcal{T}}(x,t)\right|^{p^*}\,\mathrm{d}t\right)^{1/p^*} = c_1\,(x-x_i)^{r-1/p},$$

where c_1 is defined in (9). Taking $u = x_i$ in (6) and (7), and using (8), we obtain

$$\begin{aligned}
\omega(x)\,(x-x_i)^{r-1/p} &\le \left(\omega^{1/\alpha}(x)\right)^{1/q}\left(\int_{x_i}^{x}\omega^{1/\alpha}(t)\,\mathrm{d}t\right)^{r-1/p} \\
&\le \left(\omega^{1/\alpha}(x)\right)^{1/q}\left(\frac{\|\omega^{1/\alpha}\|_{L_1}}{n}\right)^{r-1/p}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\left(\int_{x_i}^{x_{i+1}}\varrho^q(x)\,\left|f(x)-(A_n^{\mathcal{T}}f)(x)\right|^q\,\mathrm{d}x\right)^{1/q} \\
&\le c_1\left(\frac{\|\omega^{1/\alpha}\|_{L_1}}{n}\right)^{r-1/p}\left(\int_{x_i}^{x_{i+1}}\omega^{1/\alpha}(x)\left(\int_{x_i}^{x}\left|f^{(r)}(t)\,\psi(t)\right|^p\,\mathrm{d}t\right)^{q/p}\mathrm{d}x\right)^{1/q} \\
&\le c_1\left(\frac{\|\omega^{1/\alpha}\|_{L_1}}{n}\right)^{\alpha}\left(\int_{x_i}^{x_{i+1}}\left|f^{(r)}(t)\,\psi(t)\right|^p\,\mathrm{d}t\right)^{1/p},
\end{aligned}$$

and the total error satisfies

$$\begin{aligned}
\|(f-A_n^{\mathcal{T}}f)\varrho\|_{L_q} &= \left(\sum_{i=1}^{n}\int_{x_i}^{x_{i+1}}\varrho^q(x)\,\left|f(x)-(A_n^{\mathcal{T}}f)(x)\right|^q\,\mathrm{d}x\right)^{1/q} \\
&\le c_1\left(\frac{\|\omega^{1/\alpha}\|_{L_1}}{n}\right)^{\alpha}\left(\sum_{i=1}^{n}\left(\int_{x_i}^{x_{i+1}}\left|f^{(r)}(t)\,\psi(t)\right|^p\,\mathrm{d}t\right)^{q/p}\right)^{1/q}.
\end{aligned}$$

If $q \geq p$ then Jensen's inequality yields

$$\sum_{i=1}^{n} \left(\int_{x_i}^{x_{i+1}} \left| f^{(r)}(t)\, \psi(t) \right|^p \, \mathrm{d}t \right)^{q/p} \leq \left\| f^{(r)}\, \psi \right\|_{L_p}^q ,$$

and if $p > q$ then Hölder's inequality yields

$$\sum_{i=1}^{n} \left(\int_{x_i}^{x_{i+1}} \left| f^{(r)}(t)\, \psi(t) \right|^p \, \mathrm{d}t \right)^{q/p} \leq \left\| f^{(r)}\, \psi \right\|_{L_p}^q \, n^{1-q/p}.$$

The proof is complete. □

3.2 Lagrange-based approximation

In practice, we often do not have access to derivatives of f and have to rely on function values only. We now show that in such cases a piecewise Lagrange approximation can be successfully used. Since in the case $r = 1$ the algorithm $A_n^{\mathcal{T}}$ does not use any derivatives, throughout this subsection we assume that $r \geq 2$.

We first need the following auxiliary result. Denote by $L_{\boldsymbol{z}}^r f$ the Lagrange polynomial of degree $(r-1)$ that interpolates f at the points $\boldsymbol{z} = (z_1, z_2, \ldots, z_r)$ with $z_1 < z_2 < \ldots < z_r$,

$$(L_{\boldsymbol{z}}^r f)(x) = \sum_{s=1}^{r} f(z_s)\, \ell_{\boldsymbol{z},s}(x), \qquad \text{where} \qquad \ell_{\boldsymbol{z},s}(x) = \prod_{\substack{j=1 \\ j \neq s}}^{r} \frac{x - z_j}{z_s - z_j}.$$

Since for any fixed x the interpolation error $f \mapsto f(x) - (L_{\boldsymbol{z}}^r f)(x)$ is a linear functional that depends only on $f(x)$ and $f(z_j)$ for $1 \leq j \leq r$, it can be expressed as

$$f(x) - (L_{\boldsymbol{z}}^r f)(x) = \int_{\min(x,z_1)}^{\max(x,z_r)} f^{(r)}(t)\, K_{\boldsymbol{z}}(x,t)\, \mathrm{d}t,$$

where $K_{\boldsymbol{z}}$ is the corresponding Peano kernel. Recall that $K_{\boldsymbol{z}}(x,t)$ is the interpolation error for the 'test' function

$$\phi_r(x,t) = \frac{(x-t)_+^{r-1}}{(r-1)!}, \tag{11}$$

i.e.,

$$K_{\boldsymbol{z}}(x,t) = \phi_r(x,t) - (L_{\boldsymbol{z}}^r \phi_r(\cdot,t))(x), \tag{12}$$

and therefore $K_{\boldsymbol{z}}(x,t) = 0$ for $t \notin [\min(x,z_1), \max(x,z_r)]$.

Let $c_1 = c_1(r,p)$ be as in (9) and

$$c_2 = c_2(r,p) \;=\; \sup_{0=u_1<\cdots<u_r=1} \; \sup_{0\le x\le 1} \left(\int_0^1 |K_{\mathbf{u}}(x,t)|^{p^*}\,\mathrm{d}t\right)^{1/p^*}, \tag{13}$$

where $\mathbf{u} = (u_1,\ldots,u_r)$.

Lemma 1 *Let $x \ge z_1$ and $\bar{x} = \max(x, z_r)$. Then*

$$\left(\int_{z_1}^{\bar{x}} |K_{\boldsymbol{z}}(x,t)|^{p^*}\,\mathrm{d}t\right)^{1/p^*} \;\le\; \bar{c}\,(\bar{x}-z_1)^{r-1/p}, \tag{14}$$

where $\bar{c} = c_1$ for $x \ge z_r$ and $\bar{c} = c_2$ for $z_1 \le x < z_r$.

Proof. Suppose first that $x \ge z_r$. Denote $p_r(x,t) = L^r_{\boldsymbol{z}}(\phi_r(\cdot,t))(x)$ with $\phi_r(x,t)$ given by (11). We show that for $z_1 \le t \le x$,

$$0 \;\le\; p_r(x,t) \;\le\; \phi_r(x,t). \tag{15}$$

We use induction on r. The inequalities (15) are obvious for $r = 1$ since then $p_r \equiv 0$. Suppose that $r \ge 2$. Since $\phi_r(\cdot,t) - p_r(\cdot,t)$ has r zeros $z_1,\ldots,z_r$, for fixed t the derivative (with respect to the first variable)

$$(\phi_r(\cdot,t) - p_r(\cdot,t))' \;=\; \phi_{r-1}(\cdot,t) - p_r'(\cdot,t)$$

has $r-1$ zeros $\hat{z}_i$ with $z_i \le \hat{z}_i \le z_{i+1}$, $1 \le i \le r-1$. Hence $p_r'(\cdot,t)$ is the polynomial of degree $r-2$ interpolating $\phi_{r-1}(\cdot,t)$ at $\hat{z}_i$. By the induction hypothesis we then have that

$$0 \le p_r'(x,t) \le \phi_{r-1}(x,t).$$

This implies $p_r(x,t) \ge p_r(z_r,t) = \phi_r(z_r,t) \ge 0$ and

$$p_r(x,t) \;=\; p_r(z_r,t) + \int_{z_r}^x p_r'(u,t)\,\mathrm{d}u \;\le\; \phi_r(z_r,t) + \int_{z_r}^x \phi_{r-1}(u,t)\,\mathrm{d}u \;=\; \phi_r(x,t),$$

as claimed in (15).

Now, the inequality (14) in the case $x \ge z_r$ immediately follows from the observation that (15) is equivalent to the bounds

$$0 \;\le\; K_{\boldsymbol{z}}(x,t) \le \phi_r(x,t).$$

In the case $z_1 \le x < z_r$ we apply the change of variables $y = (x - z_1)/(z_r - z_1)$, $u = (t - z_1)/(z_r - z_1)$, and $w_i = (z_i - z_1)/(z_r - z_1)$ for $1 \le i \le r$. Then the formula (12) implies that

$$K_{\boldsymbol{z}}(x,t) \;=\; (z_r - z_1)^{r-1} K_{\mathbf{w}}(y,u),$$

which together with the definition (13) of c_2 completes the proof.

□

We now define our piecewise Lagrange approximation $A_n^{\mathcal{E}}$ and prove an error bound. Let

$$\mathcal{L}_i^r \;=\; L^r_{(x_{i-r+1},\dots,x_i)}$$

be the Lagrange interpolation of degree at most $(r-1)$ using the points $x_{i-r+1},\dots,x_i$ given by (8). Then we define

$$(A_n^{\mathcal{E}} f)(x) \;=\; \begin{cases} (\mathcal{L}_r^r f)(x) & \text{for} \quad 0 \le x < x_r, \\ (\mathcal{L}_i^r f)(x) & \text{for} \quad x_i \le x < x_{i+1} \quad \text{with} \quad i \ge r. \end{cases}$$

That is, $A_n^{\mathcal{E}}$ applies extrapolation rather than interpolation to approximate f at $x \ge x_r$. (Therefore we use the superscript $\mathcal{E}$ to denote this algorithm.) The importance of using extrapolation will be explained later.

Theorem 2 *For the Lagrange-based approximation we have*

$$\begin{aligned} e(A_n^{\mathcal{E}}; f) \;&\le\; c_1\, r^r\, \|\omega\|_{L_{1/\alpha}}\, \left\|f^{(r)}\psi\right\|_{L_p}\, n^{-r+(1/p-1/q)_+} \\ &\quad + c_2\,(r-1)^\alpha\, \frac{\psi(0^+)}{\psi(x_r)} \left(\frac{\omega(0^+)}{\omega(x_r)}\right)^{1-1/(q\,\alpha)} \|\omega\|_{L_{1/\alpha}}\, \left\|f^{(r)}\psi\right\|_{L_p([0,x_r])}\, n^{-\alpha}, \end{aligned}$$

where the notation $\|\cdot\|_{L_p([a,b])}$ *indicates that we restrict the integral to the interval* $[a,b]$.

Proof. Consider first $x_r \le x_i \le x < x_{i+1} \le \infty$. Proceeding as in the proof of Theorem 1 we obtain

$$\begin{aligned} \varrho(x)\left|f(x) - (A_n^{\mathcal{E}} f)(x)\right| \;&=\; \varrho(x)\left|\int_{x_{i-r+1}}^{x} f^{(r)}(x)\, K_i^{\mathcal{E}}(x,t)\,\mathrm{d}t\right| \\ &=\; \omega(x)\left|\int_{x_{i-r+1}}^{x} f^{(r)}(x)\,\psi(t)\,\frac{\psi(x)}{\psi(t)}\,K_i^{\mathcal{E}}(x,t)\,\mathrm{d}t\right| \\ &\le\; \omega(x)\left(\int_{x_{i-r+1}}^{x} |f^{(r)}(t)\,\psi(t)|^p\,\mathrm{d}t\right)^{1/p}\left(\int_{x_i-r+1}^{x}\left|K_i^{\mathcal{E}}(x,t)\right|^{p^*}\,\mathrm{d}t\right)^{1/p^*}, \end{aligned} \qquad (16)$$

where $K_i^{\mathcal{E}}(x,t)$ is the kernel (12) corresponding to $\boldsymbol{z} = (x_{i-r+1}, \ldots, x_i)$, and we made use of (5) and Hölder's inequality with conjugate exponents $1/p + 1/p^* = 1$. The property (14) yields

$$\left(\int_{x_i-r+1}^{x} \left|K_i^{\mathcal{E}}(x,t)\right|^{p^*} \, \mathrm{d}t\right)^{1/p^*} \le c_1 (x - x_{i-r+1})^{r-1/p}.$$

Taking $u = x_{i-r+1}$ in (6) and (7), and using (8), we obtain

$$\begin{aligned}
\omega(x)\ (x - x_{i-r+1})^{r-1/p} &= \left(\omega^{1/\alpha}(x)\right)^{1/q} \left(\omega^{1/\alpha}(x)\,(x - x_{i-r+1})\right)^{r-1/p} \\
&\le \left(\omega^{1/\alpha}(x)\right)^{1/q} \left(\int_{x_{i-r+1}}^{x} \omega^{1/\alpha}(t)\,\mathrm{d}t\right)^{r-1/p} \\
&\le \left(\omega^{1/\alpha}(x)\right)^{1/q} \left(\int_{x_{i-r+1}}^{x_{i+1}} \omega^{1/\alpha}(t)\,\mathrm{d}t\right)^{r-1/p} \\
&= \left(\omega^{1/\alpha}(x)\right)^{1/q} \left(\frac{r\,\|\omega^{1/\alpha}\|_{L_1}}{n}\right)^{r-1/p}.
\end{aligned}$$

Hence

$$\begin{aligned}
\left(\int_{x_r}^{\infty} \varrho^q(x)\,|f(x) - (A_n^{\mathcal{E}} f)(x)|^q \,\mathrm{d}x\right)^{1/q} &= \left(\sum_{i=r}^{n} \int_{x_i}^{x_{i+1}} \varrho^q(x)\,|f(x) - (A_n^{\mathcal{E}} f)(x)|^q \,\mathrm{d}x\right)^{1/q} \\
\le c_1\, r^{r-1/p} \left(\frac{\|\omega^{1/\alpha}\|_{L_1}}{n}\right)^{r-1/p} &\left(\sum_{i=r}^{n} \int_{x_i}^{x_{i+1}} \omega^{1/\alpha}(x) \left(\int_{x_{i-r+1}}^{x} \left|f^{(r)}(t)\,\psi(t)\right|^p \,\mathrm{d}t\right)^{q/p} \mathrm{d}x\right)^{1/q} \\
\le c_1\, r^{r-1/p} \left(\frac{\|\omega^{1/\alpha}\|_{L_1}}{n}\right)^{\alpha} &\left(\sum_{i=r}^{n} \left(\int_{x_{i-r+1}}^{x_{i+1}} \left|f^{(r)}(t)\,\psi(t)\right|^p \,\mathrm{d}t\right)^{q/p}\right)^{1/q}. \qquad (17)
\end{aligned}$$

If $q \ge p$ then Jensen's inequality yields

$$\sum_{i=r}^{n} \left(\int_{x_{i-r+1}}^{x_{i+1}} \left|f^{(r)}(t)\,\psi(t)\right|^p \,\mathrm{d}t\right)^{q/p} \le \left(\sum_{i=r}^{n} \int_{x_{i-r+1}}^{x_{i+1}} \left|f^{(r)}(t)\,\psi(t)\right|^p \,\mathrm{d}t\right)^{q/p} \le \left(r\,\|f^{(r)}\psi\|_{L_p}^p\right)^{q/p},$$

and thus the last factor in (17) is upper bounded by $r^{1/p}\,\|f^{(r)}\psi\|_{L_p}$. If $p > q$ then Hölder's inequality yields

$$\begin{aligned}
\sum_{i=r}^{n} \left(\int_{x_{i-r+1}}^{x_{i+1}} \left|f^{(r)}(t)\,\psi(t)\right|^p \,\mathrm{d}t\right)^{q/p} &\le \left(\sum_{i=r}^{n} \int_{x_{i-r+1}}^{x_{i+1}} \left|f^{(r)}(t)\,\psi(t)\right|^p \,\mathrm{d}t\right)^{q/p} \left(\sum_{i=r}^{n} 1\right)^{1-q/p} \\
&\le \left(r\,\|f^{(r)}\psi\|_{L_p}^p\right)^{q/p} (n - r + 1)^{1-q/p},
\end{aligned}$$

and thus the last factor in (17) is bounded by $r^{1/p}\,\|f^{(r)}\psi\|_{L_p}(n-r+1)^{1/q-1/p}$.

For $0 \le x < x_r$ we have

$$\begin{aligned}\varrho(x)|f(x)-(A_n^{\mathcal{E}}f)(x)| \;&=\; \omega(x)\left|\int_0^{x_r} f^{(r)}(x)\,\psi(t)\,\frac{\psi(x)}{\psi(t)}\,K_r^{\mathcal{E}}(x,t)\,\mathrm{d}t\right| \\ &\le\; \omega(x)\,\frac{\psi(0^+)}{\psi(x_r)}\left(\int_0^{x_r}|f^{(r)}(t)\,\psi(t)|^p\,\mathrm{d}t\right)^{1/p}\left(\int_0^{x_r}\left|K_r^{\mathcal{E}}(x,t)\right|^{p^*}\,\mathrm{d}t\right)^{1/p^*} \\ &\le\; c_2\,\omega(x)\,x_r^{r-1/p}\,\frac{\psi(0^+)}{\psi(x_r)}\left(\int_0^{x_r}|f^{(r)}(t)\,\psi(t)|^p\,\mathrm{d}t\right)^{1/p},\end{aligned}$$

where we again used (14). Taking $x = x_r$ and $u = 0$ in (6) and (7), and using (8), we obtain

$$\begin{aligned}\omega(x)\,x_r^{r-1/p} \;&=\; \frac{\omega(x)}{\omega(x_r)}\,\omega(x_r)\,x_r^{r-1/p} \;\le\; \frac{\omega(x)}{\omega(x_r)}\left(\omega^{1/\alpha}(x_r)\right)^{1/q}\left(\int_0^{x_r}\omega^{1/\alpha}(t)\,\mathrm{d}t\right)^{r-1/p} \\ &\le\; \frac{\omega(x)}{\left(\omega(x_r)\right)^{1-1/(q\alpha)}}\left(\frac{(r-1)\,\|\omega^{1/\alpha}\|_{L_1}}{n}\right)^{r-1/p} \\ &\le\; \left(\omega^{1/\alpha}(x)\right)^{1/q}\left(\frac{\omega(0^+)}{\omega(x_r)}\right)^{1-1/(q\alpha)}\left(\frac{(r-1)\,\|\omega^{1/\alpha}\|_{L_1}}{n}\right)^{r-1/p},\end{aligned}$$

where we used $\omega(x) = \left(\omega^{1/\alpha}(x)\right)^{1/q}(\omega(x))^{1-1/(q\alpha)} \le \left(\omega^{1/\alpha}(x)\right)^{1/q}(\omega(0^+))^{1-1/(q\alpha)}$. Therefore

$$\begin{aligned}&\left(\int_0^{x_r}\varrho^q(x)\,|f(x)-(A_n^{\mathcal{E}}f)(x)|^q\,\mathrm{d}x\right)^{1/q} \\ &\quad\le\; c_2\,\frac{\psi(0^+)}{\psi(x_r)}\left(\frac{\omega(0^+)}{\omega(x_r)}\right)^{1-1/(q\alpha)}\left(\frac{(r-1)\,\|\omega^{1/\alpha}\|_{L_1}}{n}\right)^{\alpha}\left(\int_0^{x_r}|f^{(r)}(t)\,\psi(t)|^p\,\mathrm{d}t\right)^{1/p}.\end{aligned}$$

Putting everything together we obtain the desired result. □

Remark 2 *We assumed that $r \ge 2$; however, the algorithm $A_n^{\mathcal{E}}$ is well defined also for $r = 1$. In this case $A_n^{\mathcal{E}} = A_n^{\mathcal{T}}$ and Theorem 2 reproduces the bound of Theorem 1.*

Remark 3 *Since $x_r = x_{n,r}$ decreases to zero as $n \to \infty$, we have that*

$$\lim_{n\to\infty}\frac{\psi(0^+)}{\psi(x_r)}\left(\frac{\omega(0^+)}{\omega(x_r)}\right)^{1-1/(q\alpha)} \;=\; 1.$$

This implies that the worst case error of $A_n^{\mathcal{E}}$ *is proportional to* $\|\omega\|_{L_{1/\alpha}} n^{-r+(1/p-1/q)_+}$*. More precisely, the upper limit*

$$\limsup_{n\to\infty} \frac{e^{\mathrm{wor}}(A_n^{\mathcal{E}})\, n^{r-(1/p-1/q)_+}}{\|\omega\|_{L_{1/\alpha}}} \le \begin{cases} c_1\, r^r, & p > q, \\ c_1\, r^r + c_2(r-1)^\alpha, & p \le q, \end{cases} \tag{18}$$

is finite and independent of ψ *and* ϱ*.*

We also note that for $p < \infty$ *we have* $\lim_{n\to\infty} \|f^{(r)}\psi\|_{L_p([0,x_r])} = 0$*, and for* $q < p = \infty$ *we have* $\alpha > r - (1/p - 1/q)_+$*. This means that if either* $p \ne \infty$ *or* $q \ne \infty$*, then for any function* $f \in F(r,p,\psi)$

$$\limsup_{n\to\infty} \frac{\|(f - A_n^{\mathcal{E}} f)\, \varrho\|_{L_q}\, n^{r-(1/p-1/q)_+}}{\|f^{(r)}\psi\|_{L_p}} \le c_1\, r^r\, \|\omega\|_{L_{1/\alpha}}.$$

Remark 4 *Since the algorithm* $A_n^{\mathcal{T}}$ *uses altogether* nr *function and derivatives values, its error should be compared with the error of* $A_{nr}^{\mathcal{E}}$ *which is*

$$\begin{aligned} e(A_{nr}^{\mathcal{E}}; f) \;&\le\; c_1\, r^{(1/p-1/q)_+}\, \|\omega\|_{L_{1/\alpha}} \left\|f^{(r)}\psi\right\|_{L_p} n^{-r+(1/p-1/q)_+} \\ &\quad + c_2 \left(1 - \frac{1}{r}\right)^\alpha \frac{\psi(0^+)}{\psi(x_r)} \left(\frac{\omega(0^+)}{\omega(x_r)}\right)^{1-1/(q\,\alpha)} \|\omega\|_{L_{1/\alpha}} \left\|f^{(r)}\psi\right\|_{L_p([0,x_r])} n^{-\alpha}. \end{aligned}$$

Hence for $q \le p < \infty$ *or* $q < p = \infty$ *the upper bounds for the worst case errors of* $A_n^{\mathcal{T}}$ *and* $A_{nr}^{\mathcal{E}}$ *are asymptotically identical.*

Remark 5 *If instead of* (4) *we assume that* ψ *and* ϱ *are Riemann integrable and asymptotically non-increasing (see Remark 1) then the inequalities of Theorems 1 and 2 have to be replaced by their asymptotic counterparts. In particular we have the asymptotic bound* (18) *for the worst case error of* $A_n^{\mathcal{E}}$*, and the corresponding bound for* $A_n^{\mathcal{T}}$ *is*

$$\limsup_{n\to\infty} e^{\mathrm{wor}}(A_n^{\mathcal{T}})\, n^{r-(1/p-1/q)_+} \le c_1\, \|\omega\|_{L_{1/\alpha}}.$$

We end this section with a comment on the use of interpolation instead of extrapolation. Assume for simplicity that $(n-1)$ is a multiple of $(r-1)$, i.e., $n = k(r-1)+1$. Denote by $A_n^{\mathcal{I}}$ the algorithm that interpolates f with $\mathcal{L}_i^r f$ in the corresponding intervals $[x_{i-r+1}, x_i]$ for $i = j(r-1)+1$, $1 \le j \le k$, and extrapolates f with $\mathcal{L}_n^r f$ in $[x_n, \infty]$.

Consider a particular interval $[x_{i-r+1}, x_i]$. Then the corresponding kernel $K_i^{\mathcal{I}}$ is supported on $[x_{i-r+1}, x_i]$ and (16) now reads

$$\begin{aligned} &\varrho(x)\left|f(x) - (A_n^{\mathcal{I}} f)(x)\right| \\ &\quad \le \omega(x) \left(\int_{x_{i-r+1}}^{x_i} |f^{(r)}(t)\,\psi(t)|^p\,\mathrm{d}t\right)^{1/p} \left(\int_{x_{i-r+1}}^{x_i} \left|\frac{\psi(x)}{\psi(t)} K_i^{\mathcal{I}}(x,t)\right|^{p^*} \mathrm{d}t\right)^{1/p^*}. \end{aligned}$$

If $\psi \equiv 1$ then the factor $\psi(x)/\psi(t)$ above is 1 and we can use previous arguments to obtain

$$\begin{aligned} e(A_n^{\mathcal{I}}; f) \;\le\; & c_2\,(r-1)^r\,\|\varrho\|_{L_{1/\alpha}}\,\left\|f^{(r)}\right\|_{L_p([0,x_n])}\,n^{-r+(1/p-1/q)_+} \\ & + c_1\,r^{r-1/p}\,\|\varrho\|_{L_{1/\alpha}}\,\left\|f^{(r)}\right\|_{L_p([x_{n-r+1},\infty])}\,n^{-\alpha}, \end{aligned}$$

where the first component comes from the interpolation in $[0, x_n)$ and the second component from the extrapolation in $[x_n, \infty)$. However, if $\psi \not\equiv 1$ then $\psi(x)/\psi(t)$ can be arbitrarily large. If ψ rapidly decreases and this is not compensated with much faster decrease of ϱ then the worst case error of interpolation can be quite large, even for large values of n. This phenomenon is illustrated by a numerical example of Section 5.

4 Lower bound

We proved that if $\|\omega\|_{L_{1/\alpha}} < \infty$ then the upper bounds on the worst case errors of the algorithms $A_n^{\mathcal{T}}$ and $A_n^{\mathcal{E}}$ are proportional to $\|\omega\|_{L_{1/\alpha}} n^{-r+(1/p-1/q)_+}$. A comparison with unweighted approximation (1) shows that the convergence rate $n^{-r+(1/p-1/q)_+}$ is best possible. We shall see that also the dependence on the weights ψ and ϱ via $\|\omega\|_{L_{1/\alpha}}$ is crucial and cannot be improved.

Denote by $e^*(n)$ the minimal worst case error that can be achieved by algorithms that use function and/or derivatives evaluations at n points,

$$e^*(n) \;=\; \inf_{A_n}\; e^{\mathrm{wor}}(A_n).$$

Theorem 3 *There exists $c = c(r,p,q) > 0$ with the following property: if $\|\omega\|_{L_{1/\alpha}} < \infty$ then*

$$\liminf_{n\to\infty}\; e^*(n)\,n^{r-(1/p-1/q)_+} \;\ge\; c\,\|\omega\|_{L_{1/\alpha}}.$$

Proof. We consider only $p, q < \infty$, the cases $p = \infty$ or $q = \infty$ are similar and simpler. Since $\|\omega\|_{L_{1/\alpha}} = \lim_{t\to\infty} \|\omega\|_{L_{1/\alpha}([0,t])}$, we can assume without loss of generality that ω is supported on $[0, T]$ for some $T < \infty$.

Observe that there exists $c = c(p,q,r) > 0$ with the following property. For every finite interval $I = [a,b] \subset [0,T]$, any $m \ge 0$, and any points $t_0 = a \le t_1 < \cdots < t_m \le b = t_{m+1}$, there is an r-times continuously differentiable function $g : \mathbb{R}_+ \to \mathbb{R}$ such that

(i) g is supported on $[a,b]$,

(ii) $g^{(j)}(t_\ell) = 0$ for all $0 \le j \le r-1$ and $0 \le \ell \le m+1$,

(iii) $\|g\|_{L_q([a,b])} \ \geq\ c\,(b-a)^\alpha\,(n+1)^{-r+(1/p-1/q)_+}\,\|g^{(r)}\|_{L_p([a,b])}$,

(iv) $\|g^{(r)}\psi\|_{L_p([a,b])} = 1$.

Here g is a 'multi-bump' function. It can be obtained in a standard way that is often used to prove lower bounds for the worst case error of problems like (1), see, e.g., [7]. Moreover, (iv) is a normalizing condition.

For a fixed $k \geq 1$, define the k equally spaced intervals in $[0,T]$

$$I_j = [T(j-1)/k, Tj/k], \qquad 1 \leq j \leq k.$$

Let A_n be an arbitrary algorithm that uses n_j points in each I_j so that $n = \sum_{j=1}^k n_j$. We let f_j be the function satisfying the conditions (i)–(iv) for $[a,b] = I_j$ with $m = n_j$ points in I_j used by A_n, and consider the class $\mathcal{F}$ of all functions f of the form

$$f \ = \sum_{j=1}^k \beta_j\, f_j \qquad \text{where} \qquad \sum_{j=1}^k |\beta_j|^p \leq 1.$$

We obviously have $\mathcal{F} \subseteq F(r,p,\psi)$, and by (iv)

$$\|f^{(r)}\,\psi\|^p_{L_p([0,T])} \ = \sum_{j=1}^k |\beta_j|^p\,\|f_j^{(r)}\,\psi\|^p_{L_p(I_j)} \ \leq\ 1 \quad \text{for all } f \in \mathcal{F}.$$

From (ii) it follows that all $f \in \mathcal{F}$ share the same zero information and therefore are approximated by the same function $f_0 = A_n(0)$. Since, in addition, $\mathcal{F}$ is symmetric about zero, the error of A_n is lower bounded as

$$e^{\rm wor}(A_n) \ \geq\ \sup_{f\in\mathcal{F}} \|(f-f_0)\,\varrho\|_{L_q([0,T])} \ \geq\ \sup_{f\in\mathcal{F}} \|f\,\varrho\|_{L_q([0,T])}.$$

Due to monotonicity of ϱ and ψ and the properties (iii) and (iv) we have that

$$\|f_j\,\varrho\|_{L_q(I_j)} \ \geq\ \varrho(Tj/k)\,\|f_j\|_{L_q(I_j)} \ \geq\ \varrho(Tj/k)\,c\,(T/k)^\alpha\,\frac{\|f_j^{(r)}\|_{L_p(I_j)}}{(n_j+1)^{r-(1/p-1/q)_+}}$$

and

$$\|f_j^{(r)}\|_{L_p(I_j)} \ \geq\ \frac{\|f_j^{(r)}\,\psi\|_{L_p(I_j)}}{\psi(T(j-1)/k)} \ =\ \frac{1}{\psi(T(j-1)/k)}.$$

Thus

$$\|f\,\varrho\|_{L_q([0,T])} \;\geq\; c\,(T/k)^{\alpha}\,\left(\sum_{j=1}^{k}\left(\frac{w_j\,|\beta_j|}{(n_j+1)^{r-(1/p-1/q)_+}}\right)^q\right)^{1/q}, \tag{19}$$

where

$$w_j \;=\; \frac{\varrho(Tj/k)}{\psi(T(j-1)/k)}.$$

We now maximize the right-hand-side of (19) with respect to $(\beta_1,\dots,\beta_k)\in\mathbb{R}^k$ subject to $\sum_{j=1}^k|\beta_j|^p\le 1$. Letting $s=p/q$,

$$U_j=(w_j(n_j+1)^{-r+(1/p-1/q)_+})^q \qquad \text{and} \qquad W_j=\beta_j^q, \qquad 1\le j\le k,$$

we arrive at the following problem: for given $\vec{U}=(U_1,\dots,U_k)\in\mathbb{R}_+^k$

$$\text{maximize} \qquad \langle\vec{U},\vec{W}\rangle_2=\sum_{j=1}^{k}U_jW_j \tag{20}$$

over all $\vec{W}=(W_1,\dots,W_k)\in\mathbb{R}_+^k$ with $\|\vec{W}\|_s=(\sum_{j=1}^k W_j^s)^{1/s}\le 1$.

The solution $\vec{W}^*$ of (20) can be obtained using standard techniques. If $s>1$ then

$$W_j^*=\frac{U_j^{1/(s-1)}}{(\sum_{i=1}^k U_i^{s/(s-1)})^{1/s}}, \qquad 1\le j\le k,$$

and $\langle\vec{U},\vec{W}^*\rangle_2=\|\vec{U}\|_{s/(s-1)}$. If $s\le 1$ then $\vec{W}^*=\vec{e}_\ell$ is the ℓth unit vector of the standard basis of $\mathbb{R}^k$ with ℓ such that $U_\ell=\|\vec{U}\|_\infty$, and $\langle\vec{U},\vec{W}^*\rangle_2=\|\vec{U}\|_\infty$. Translating this back to the worst case error of A_n we have the following.

Suppose first that $p>q$ and so $s>1$. Then

$$e^{\mathrm{wor}}(A_n) \;\geq\; c\,(T/k)^{\alpha}\left(\sum_{j=1}^{k}\left(w_j(n_j+1)^{-r}\right)^{pq/(p-q)}\right)^{(p-q)/(pq)}. \tag{21}$$

Standard minimization of (21) with respect to $\sum_{j=1}^k n_j=n$ gives optimal

$$n_j^*=(n+k)\,\frac{z_j^{1/(t+1)}}{\sum_{i=1}^k z_i^{1/(t+1)}}-1 \qquad \text{with} \quad z_j=w_j^{pq/(p-q)}, \quad t=r\frac{pq}{p-q}.$$

(Note that n_j here can be negative or not an integer.) Substituting n_j with n_j^* in (21) we obtain

$$e^*(n) \;\geq\; c\left(\sum_{j=1}^{k} \frac{T}{k}\, w_j^{1/\alpha}\right)^{\alpha} (n+k)^{-r}.$$

For $p \leq q$ and so $s \leq 1$, we have

$$e^{\mathrm{wor}}(A_n) \;\geq\; c\,(T/k)^{\alpha} \max_{1\leq j\leq k}\, w_j (n_j+1)^{-\alpha}.$$

This is minimized by $n_j^* = (n+k)\, w_j^{1/\alpha}/(\sum_{i=1}^{k} w_i^{1/\alpha}) - 1$ and therefore

$$e^*(n) \;\geq\; c\left(\sum_{j=1}^{k} \frac{T}{k}\, w_j^{1/\alpha}\right)^{\alpha} (n+k)^{-\alpha}.$$

Clearly, $\omega(Tj/k) \leq w_j \leq \omega(T(j-1)/k)$ implies that

$$\sum_{j=1}^{k} \frac{T}{k}\, \omega^{1/\alpha}(Tj/k) \;\leq\; \sum_{j=1}^{k} \frac{T}{k}\, w_j^{1/\alpha} \;\leq\; \sum_{j=1}^{k} \frac{T}{k}\, \omega^{1/\alpha}(T(j-1)/k).$$

Since ω is measurable and non-increasing, it is also Riemann integrable. Hence taking k sufficiently large we can make the sum $(T/k)\sum_{j=1}^{k} w_j^{1/\alpha}$ arbitrarily close to $\|\omega^{1/\alpha}\|_{L_1}$. The proof is complete. $\square$

Theorem 3 implies in particular that if $\|\omega\|_{L_{1/\alpha}} = \infty$ then the convergence rate $n^{-r+(1/p-1/q)_+}$ cannot be achieved.

5 Numerical illustration

We now show results of some numerical tests for a particular set of parameters. We assume that $r = 2$, $p = \infty$, $q = 1$, and

$$\psi(x) = \exp(-\lambda_\psi x), \qquad \varrho(x) = \exp(-\lambda_\varrho x) \qquad \text{with} \quad \lambda_\psi < \lambda_\varrho.$$

That is, we consider L_1-weighted approximation where

$$e(A_n; f) \;=\; \int_0^\infty |f(x) - (A_n f)(x)|\, \exp(-\lambda_\varrho x)\, \mathrm{d}x,$$

in the class $F = F(2,\infty,\lambda_\psi)$ of functions f with

$$\|f^{(r)}\psi\|_{L_\infty} \;=\; \operatorname*{ess\,sup}_{0\le x<\infty} \left|f^{(2)}(x)\right| \,\exp(-\lambda_\psi x) \;<\; \infty.$$

Then $r-(1/p-1/q)_+ = 2$, $\alpha = 3$,

$$\omega(x) = \exp\left(-(\lambda_\varrho - \lambda_\psi)\,x\right), \qquad \|\omega\|_{L_{1/\alpha}} \;=\; \frac{27}{(\lambda_\varrho-\lambda_\psi)^3},$$

and the sample points are

$$x_i \;=\; \frac{3}{\lambda_\varrho-\lambda_\psi}\,\ln\left(\frac{n}{n-i+1}\right), \qquad 1\le i\le n.$$

We present results for the Taylor approximation $A_n^{\mathcal{T}}$, Lagrange extrapolation $A_n^{\mathcal{E}}$, and Lagrange interpolation $A_n^{\mathcal{I}}$. We tested the behavior of the error $e(A_n;f)$ as n increases for the function

$$f(x) = \exp(\lambda x).$$

Obviously $f \in F(2,\infty,\lambda_\psi)$ iff $\lambda \le \lambda_\psi$. The results are presented in logarithmic scales, i.e., $-\log_{10} e(A_n;f)$ versus $\log_{10}(2n)$ for $A_n = A_n^{\mathcal{T}}$, and $-\log_{10} e(A_n;f)$ versus $\log_{10} n$ for $A_n \in \{A_n^{\mathcal{E}}, A_n^{\mathcal{I}}\}$.

For $\lambda \le \lambda_\psi < \lambda_\varrho$ we observed convergence of $A_n^{\mathcal{T}}$ and $A_n^{\mathcal{E}}$ at the theoretical rate n^{-2}, see Figure 1. For given $\lambda < \lambda_\varrho$ the best choice of λ_ψ is $\lambda_\psi = \lambda$ since then the sampling is adjusted to the smallest function class containing f. It is worth mentioning that the extrapolation algorithm has about 37% smaller asymptotic constant than the Taylor algorithm. This should not be surprising since $A_n^{\mathcal{T}}$ approximates f better than $A_{2n}^{\mathcal{E}}$ in $(x_{n,i}, x_{n,i+1/2})$ but worse in $(x_{n,i+1/2}, x_{n,i+1})$. The algorithm $A_n^{\mathcal{I}}$ also (asymptotically) converges at rate n^{-2} and has about 5 times better asymptotic constant than $A_n^{\mathcal{E}}$. However, even for large n its error can be quite large compared to the error of the other algorithms, as illustrated in Figure 2.

Figure 3 shows the importance of the right sampling strategy. If $\lambda_\psi < \lambda$ so that $f \notin F(2,\infty,\lambda_\psi)$ then the convergence rate n^{-2} is completely lost.

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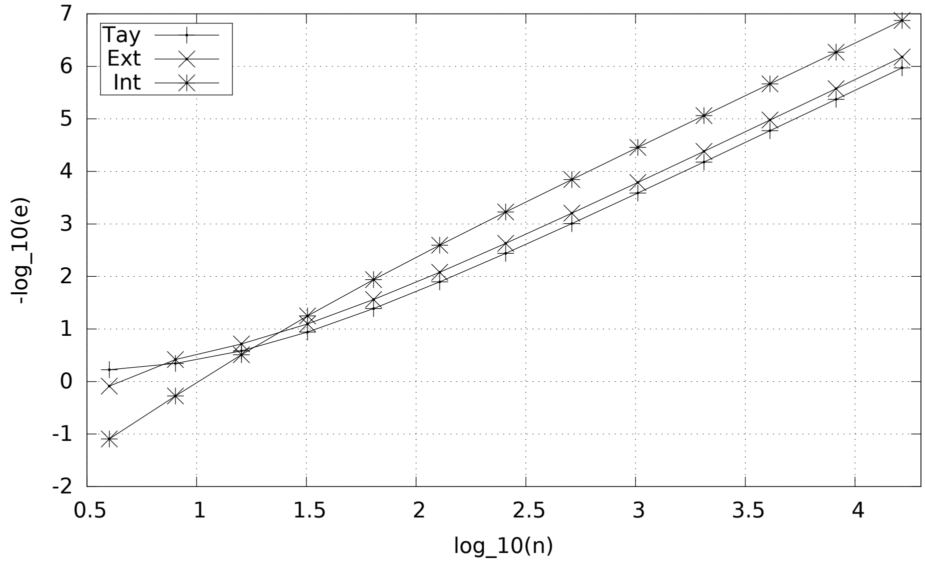

Figure 1: $(\lambda, \lambda_\psi, \lambda_\varrho) = (4, 4, 5)$

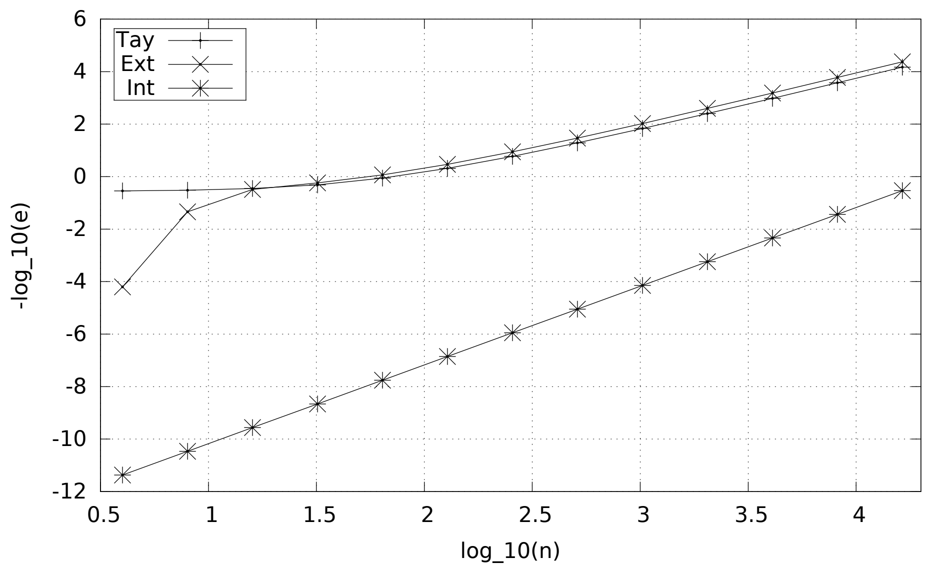

Figure 2: $(\lambda, \lambda_\psi, \lambda_\varrho) = (4, 4, 4.25)$

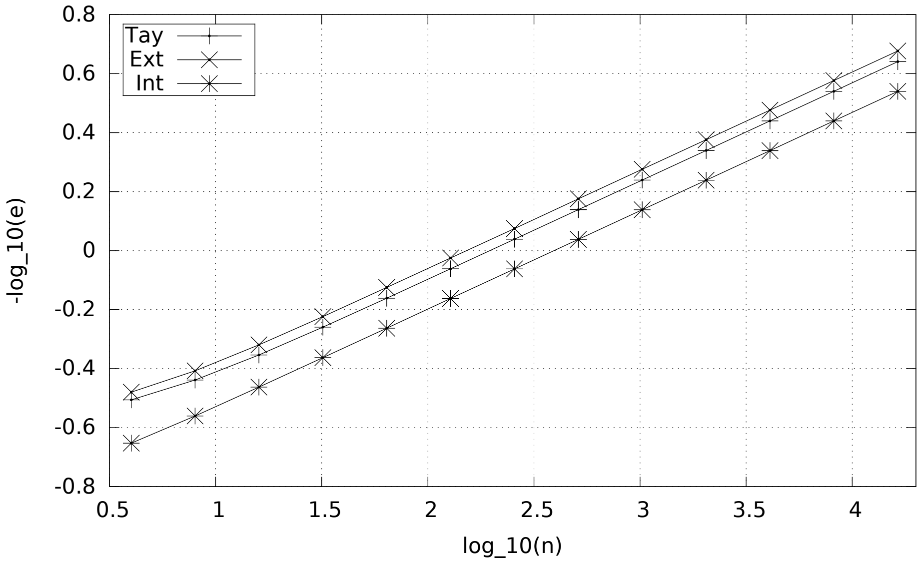

Figure 3: $(\lambda, \lambda_\psi, \lambda_\varrho) = (4, 2, 4.25)$

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