K-THEORY AND THE ADAMS OPERATIONS

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Introduction

Algebraic invariants are often constructed by defining some objects on a topological space and turning the set of these objects into an algebraic object. Often these algebraic invariants give topological information of the space. For example, recall that the fundamental group is constructed from the homotopy classes of loops defined on a topological space X. The topological information that the fundamental group carries are, intuitively, 'holes' in X.

The most important use of algebraic invariants is that by defining these algebraic invariants, alot of topological problems can be reduced to algebraic problems, which are easier most of the time. For example a classical problem in algebraic topology is whether the disk $\mathbb{D}^2$ can be continuously deformed onto its boundary S^1 can be solved using such techniques.

A **division algebra**, intuitively, is an algebra over a field in which division is possible. If we take $\mathbb{R}$ as the field, then the well-known division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$, with corresponding dimensions 1, 2, 4, 8. A natural question to ask is whether there are division algebras of higher dimension. It turn's out that the only finite dimensional division algebras only $\mathbb{R}$ must be isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$. So up to isomorohism, there are only four division algebras only the real numbers. In 1960 Frank Adams and Michael Atiyah gave a very shorter proof of the above theorem using K-theory and the Adams operations. This thesis aims to give an exposition of the proof.

In essence, if a multiplication map exists on $\mathbb{R}^n$, then by restricting to a sphere we get a continuous multiplication map on the sphere. If we want to show $\mathbb{R}^n$ is not a division algebra for $n \neq 1, 2, 4, 8$, it suffices to show that the restricted continuous map does not exist on sphere besides S^0, S^1, S^3, S^7. So to show $\mathbb{R}^n$ is in a division algebra it is enough to show that a certain continuous map does not exist on S^{n-1}.

The idea is similar to showing that the disk cannot be continuously deformed onto the circle. For if that was true that we had a homotopy from the disk to the circle, which is a certain kind continuous map. This continuous map induces an isomorphism between $\pi_1(D)$ and $\pi_1(S^1)$ which is a contradiction as $\pi_1(D) = 0 \neq \mathbb{Z} = \pi_1(S^1)$. The idea is to assoiate a continous map to a homomorphism between some algebraic invariant of a topological space. It is easier to show the nonexistence of certain homomorphism then continuous maps since homomorphism preserves algebraic structures so in general showing nonexistence of homomorphism is easier then showing nonexistence of continuous maps.

In a similar vein, for spheres S^{n-1} we would like to associate the multiplication map with a homomorphism. To do this we need to define an algebraic invariant of S^{n-1}. It turns out that the invariant we need is the ring $K(S^{n-1})$, and this is where K-theory comes into play in the proof of the above theorem.

Author's note

At the start of the year my goal for this thesis was to give an exposition of the proof of the non-existence of division algebras $\mathbb{R}^n$ over the reals besides $n = 1, 2, 4, 8$. During the first semester Daniel found an old paper of Atiyah in which Atiyah defines the Adams operations in terms of the symmetric group. Atiyah proceeds to show that ψ^k is a ring homomorphism using his new definition. My goal made a change in which then I focused most of my time on showing the remaining properties of the Adam's operations using Atiyah's definition.

In the end, after spending many hours in Daniel's office, we were able to recover most properties using Atiyah's new definition. However we were only able to prove $\psi^k\psi^l = \psi^l\psi^k$ for any positive integers k, l which is a weaker property of $\psi^k\psi^l = \psi^{kl}$. This weaker condition is enough to show the non-existence of division algebras so unfortunately we were not able to show everything.

One thing we got around is the splitting principle. It was not needed to prove the properties of the Adams operations.

Contents

Chapter 1

Topological constructions

In this prelimilary chapter we will introduce some topological constructions which will be used in the upcoming chapters. Throughout this chapter we let X and Y be topological spaces with chosen base points $\{x_0\}$ and $\{y_0\}$. For more of these constructions we refer the reader to [5].

1.1 Wedge sum

The **wedge sum** of X and Y, denoted by $X \vee Y$, is defined as

$$X \vee Y = \frac{X \bigsqcup Y}{x_0 \sim y_0},$$

equipped with the quotient topology. Intuitively the wedge sum are two spaces joined at a point. For example, the wedge sum of two circles is an eight shaped figure.

As a set, $X \vee Y = X \times \{y_0\} \cup Y \times \{x_0\}$ which can be thought of as a subspace in $X \times Y$.

1.2 Smash product

The **smash product** $X \wedge Y$ is defined as the quotient

$$X \wedge Y = \frac{X \times Y}{X \vee Y}$$

1.3 Cone

The **cone** is defined as

$$CX = \frac{X \times I}{X \times \{0\}}$$

One remark is that even if X is not contractible, the cone CX is contractible by sending all elements to the point $[X \times \{0\}]$

1.4 Suspension

The **suspension** of X, denoted by SX is defined as

$$SX = \frac{X \times I}{X \times \{0\} \cup X \times \{1\}}$$

Basically we take the cylinder $X \times I$ and collpase the top and bottom ends to two points. Intuitively taking the suspension is the same as attaching two cones on top and bottom of X.

One remark is that

1.5 Reduced suspension

For X with a given base point x_0, we define the **reduced suspension** ΣX as

$$\Sigma X = \frac{X \times I}{X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I}$$

This is equivalent to taking SX and collapsing the subspace $\{x_0\} \times I$ to a point.

1.6 Additional properties

Some properties that will be needed are the following

(i) $S^n \wedge X = \Sigma^n X$, that is the smash product of X with S^n is the reduced suspension on X.

(ii) $S^m S^n = S^{m+n}$.

Chapter 2

Vector bundles

We will first give definition of vector bundle. Intuitively a vector bundle is a topological space which locally looks like a cartesian product. In some sense it is a generalisation of vector spaces, in the senses that whatever we can do to vector spaces, we can do to vector bundles.

2.1 Vector bundles

Definition 2.1.1. Given a compact Hausdorff space X, a **vector bundle** of rank n over X is a topological space E with a continuous map $\pi : E \to X$ such that each $\pi^{-1}(x)$ is a complex n-dimensional vector space. We also demand that for every $x \in X$ there is an open cover $\{U_\alpha\}$ such that there is a homeomorphism $h_\alpha : p^{-1}(U) \to U \times \mathbb{C}^n$ such that $h_\alpha|_x : p^{-1}(x) \to \{x\} \times \mathbb{C}^n$ is a vector space isomorphism for all $x \in U$.

A vector bundle of rank n is sometimes also called an n-dimensional vector bundle. To properly define a homeomorphism $h_\alpha : p^{-1}(U) \to U \times \mathbb{C}^n$ we need to specify a topology on $U \times \mathbb{C}^n$. We take the topology on $\mathbb{C}^n$ to be the normal Euclidean topology and we take the product topology on $U \times \mathbb{C}^n$. Each h_α is then called a **local trivilisation** of E. For each $x \in X$ the vector space $\pi^{-1}(x)$ is called the **fiber** over x, normally denoted by $E|_x$. With this definition, we call X the *base* space and E the *total* space. The notation we write is (E, X, π). For simplicity we normally write E when the context is clear.

Definition 2.1.2. If all the fibers of a vector bundle E are one-dimensional, we call E a **line bundle**.

Example 2.1.3. The product bundle $X \times \mathbb{C}^n$ is a vector bundle. We call such a bundle a **trivial bundle**. □ We will denote a trivial bundle of dimension n to be T^n. **Example 2.1.4.** The tangent bundle of a manifold is a vector bundle. Since manifolds are locally contractible, the tangent bundle is locally trivial. In general a vector bundle over a contractible space is trivial, a fact we will prove later. □

Example 2.1.5. We will now introduce an important line bundle called the **tautological line bundle**, denoted by H. We first introduce the complex projective line $\mathbb{C}P^1$, defined as

$$\mathbb{C}P^1 = \left\{(z_0, z_1) \in \mathbb{C}^2 | (z_0, z_1) \sim (\lambda z_0, \lambda z_1)\right\}$$

Intuitively, $\mathbb{C}P^1$ is the set of all lines in $\mathbb{C}\backslash\{0\}$ through the origin, and is homeomorphic to the sphere. We will briefly show that there is a bijection from $\mathbb{C}P^1$

to the sphere. Let $[z_0, z_1]$ be the equivalence class of (z_0, z_1) in $\mathbb{C}P^1$. Notice that by rescaling λ we can assume $[z_0, z_1] = [z, 1]$ for some $z \in \mathbb{C}$. Then all the points in $\mathbb{C}P^1$ are of the form $[z, 1]$ with an extra point $[1, 0]$. Intuitively this is $\mathbb{C}$ with a point at infinity. Using sterographic projection we see that points on $\mathbb{C}P^1$ have a one to one correspondence to points on the sphere. To define H, notice that every point on $\mathbb{C}P^1$ is a line through the origin in $\mathbb{C}^2$ collapsed into a point. On every point we define the fiber to be the line that is quotiented out. If we view $\mathbb{C}P^1$ as S^2 then H is a line bundle on S^2. □

Similar to vector subspaces, we will now define a subbundle of a vector bundle.

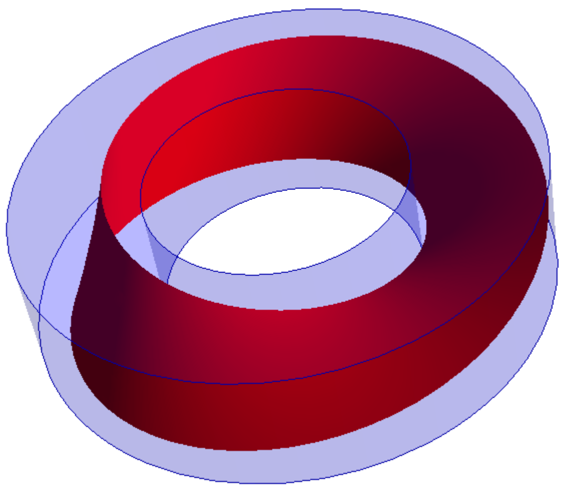
Definition 2.1.6. A **subbundle** (F, X, π') of a vector bundle (E, X, π) is a vector bundle such that $F \subseteq E$ and each ${\pi'}^{-1}(x)$ is a vector subspace of $\pi^{-1}(x)$.

We will now give an example of a subbundle inside a vector bundle. We first define the Mobius bundle M to be the quotient

$$M = \frac{I \times \mathbb{R}}{(0, t) \sim (1, -t)}.$$

Roughly speaking this definition is an 'infinite' Mobius band, an infinite cylinder with a twist. Locally M is homeomoprhic to $U \times \mathbb{R}$ where U is an open subset of S^1.

Example 2.1.7. Here we have a trivial bundle $S^1 \times \mathbb{R}^2$, with the Mobius bundle sitting inside as a subbundle.



□

The above example are vector bundles where the fibers are real vector spaces. Throughout this thesis we will generally be interested in complex vector bundles only.

We will now define a homomorphism between two vector bundles.

Definition 2.1.8. Given two vector bundles $p : E \to X$, $q : F \to X$ over a compact Hausdorff space X, a **homomorphism** is a continuous map $\phi : E \to F$ such that

(i) $q \circ \phi = p$

(ii) for each $x \in X$, $\phi|_{p^{-1}(x)} : p^{-1}(x) \to q^{-1}(x)$ is a linear map between the two vector spaces $p^{-1}(x)$ and $q^{-1}(x)$.

Definition 2.1.9. Given vector bundles E, F over a compact Hausdorff space X, a homomorphism $\phi : E \to F$ is an **isomorphism** if ϕ is bijective and ϕ^{-1} is continuous. If an isomorphism exists between E, F we say E and F are isomorphic and we write $E \approx F$.

Intuitively an isomorphism between two vector bundles is a homeomorphism which also preserves the vector space structure on each fiber. Sometimes we will also use the symbol $\approx$ to denote a homeomorphism between two topological spaces. We leave it to the reader to determine from context which is meant.

For vector spaces we know that an isomorphism is a bijective homomorphism. We have a similar result for vector bundles. In particular, a bijective homomorphism between vector bundles is indeed a vector bundle isomorphism. This is not at all obvious, as this implies that the inverse of the homomorphism is continuous. This result will be useful for what is to come.

Lemma 2.1.10. *A continuous map $\phi : E \to F$ between two vector bundles is an isomorphism if $\phi|_x : E_x \to F_x$ is a linear isomorphism for all $x \in X$.*

We now define the restriction of a vector bundle. Suppose that $Y \subseteq X$ and $p : E \to X$ a vector bundle over X. We define the **restriction of** E **at** Y, denoted by $E|_Y$, as the set $p^{-1}(Y)$. This is essentially restricted a bundle over X to a 'smaller' bundle over a subspace $Y \subset X$.

We will conclude this section which is useful result.

Proposition 2.1.11. *If $E \to X \times I$ is a vector bundle and X is compact Hausdorff, then $E|_{X\times\{0\}} \approx F|_{X\times\{1\}}$.*

2.2 Operations on vector bundles

Given two vector spaces V, W, we can construct new vector spaces $V \oplus W$ and $V \otimes W$. A natural question to ask here is whether we can extend these operations on vector spaces to operations on vector bundles. Given two vector bundles E, F over the same space, suppose that $E = \bigsqcup_{x\in X} E_x$, $F = \bigsqcup_{x\in X} F_x$. We can naively define $E \oplus F$, $E \otimes F$ by taking the direct sums and tensor products of each fibers E_x, F_x over each $x \in X$. One problem we need to do is to put a topology on these sets and to ensure local triviality if we want to realise these sets as vector bundles. Here we will define these operations. For the explicit description of the topologies on these sets we refer the reader to [4],[2].

Let E and F be vector bundles over X. We will now give an example of three operations on vector bundles which will be important for the later chapters.

Example 2.2.1. The **direct sum** $E \oplus F$ is defined as the set

$$E \oplus F = \bigsqcup_{x\in X} E_x \oplus F_x$$

□

We have a similar definition for the tensor product

Example 2.2.2. The **tensor product** $E \otimes F$ is defined as

$$E \otimes F = \bigsqcup_{x \in X} E_x \otimes F_x$$

□

Example 2.2.3. For each $x \in X$, the set of all vector space homomorphisms from E_x to F_x, $\text{Hom}(E_x, F_x)$ has a vector space structure. We define

$$\text{Hom}(E, F) = \bigsqcup_{x \in X} \text{Hom}(E_x, F_x)$$

□

One thing to note here is that the topologies on these vector bundles $E \oplus F$, $E \otimes F$, $\text{Hom}(E, F)$ are natural in the sense that a vector space isomorphisms between the fibers for each $x \in X$ extends to a vector bundle isomorphism. For example, given vector bundles E and F, the fibers of $E \oplus F$ are defined as $E|_x \oplus F|_x$. For vector spaces we know that for each x, $E|_x \oplus F|_x$ is naturally isomorphic to $E|_x \oplus F|_x$. These isomorphisms extend to an vector bundle isomorphism $E \oplus F \approx F \oplus E$. We will here state the following isomorphisms between vector bundles E, F, F':

(i) $E \oplus F \approx F \oplus E$.
(ii) $E \otimes F \cong F \otimes E$.
(iii) $E \otimes (F \oplus F') \approx E \otimes F \oplus E \otimes F'$.

2.3 Vector bundles on compact Hausdorff spaces

Given a vector space V, we can put the standard inner product of V by identifying V with $\mathbb{C}^n$. With this inner product and a given subspace $W \subset V$ we can define the orthogonal complement $W^\perp$. There is a natural isomorphism $V = W \oplus W^\perp$. One question is whether we can do the same thing for vector bundles. That is, given a vector bundle E and a subbundle $F \subset E$, we ask whether there exists a subbundle F' such that $E \approx F \oplus F'$. For vector bundles over an arbitrary topological space X the answer is generally no. However, this is true if X is compact Hausdorff. More precisely, we have

Proposition 2.3.1. *If $E \to X$ is a vector bundle over a compact Hausdorff space X and $F \to X$ is a subbundle of E, then there exists a subbundle $F' \subset E$ such that $E \approx F \oplus F'$.*

Proposition 2.3.2. *For each vector bundle $E \to X$ with X compact Hausdorff there exists a vector bundle $F \to X$ such that $E \oplus F$ is trivial.*

To see why this can be true, consider example 2.1.7. We actually have $S^1 \times \mathbb{R}^2 \approx M \oplus M$. To see this, imagine putting another Mobius bundle in $S^1 \times \mathbb{R}^2$ but rotated by 90° about the base space S^1. Then over each fiber we get the direct sum $\mathbb{R} \oplus \mathbb{R}$ which is just $\mathbb{R}^2$. Hence we just get $S^1 \times \mathbb{R}^2$.

We wil now introduce some a lemma

Lemma 2.3.3. *Let Y be a closed subspace of a compact Hausdorff space X, and let E, F be two vector bundles over X. If $f : E|_Y \to F|_Y$ is an isomorphism, then* *there exists an open set U containing Y and an extension $f : E|_U \to F|_U$ which is an isomorphism.*

2.4 Pullback bundles

We will now introduce the idea of a pullback. Given a continuous map $f : X \to Y$ and a vector bundle $p : E \to Y$, we define a vector bundle on X, denoted by $f^*(E)$ as

$$f^*(E) = (x, v) \in X \times E | p(v) = f(x)$$

Intuitively on every point $x \in X$, we look at the point $f(x) \in Y$ and we 'pullback' the fiber over Y onto X.

Example 2.4.1. Given $Y \subset X$, let $i : Y \to X$ be the inclusion and $E \to X$ a vector bundle over X. Then we have $i^*(E) = E|_Y$, which is just the vector bundle restricted to Y. □

We will now state some properties of pullback

(i) $(fg)^*(E) \approx g^*(f^*(E))$
(ii) $1^*(E) \approx E$
(iii) $f^*(E_1 \oplus E_2) \approx f^*(E_1) \oplus f^*(E_2)$
(iv) $f^*(E_1 \otimes E_2) \approx f^*(E_1) \otimes f^*(E_2)$
(v) $1^*(E) = E$
(vi) If $f : X \to Y$ and $A \subset X$, then $(f|_A)^*(E) = f^*(E)|_A$.
(vii) If f_0, f_1 are homotopic, then $f_0^*(E) \approx f_1^*(E)$.
(viii) The pullback of a trivial bundle is trivial.

We will now denote the set of isomorphism classes of n-dimensional complex vector bundles on a space X by $\mathrm{Vect}^n(X)$.

Recall that two spaces X, Y are **homotopy equivalent** if there exist two continuous maps $f : X \to Y$, $g : Y \to X$ such that gf is homotopic to the identity on X and fg is homotopic to the identity on Y. Intuitively two spaces are homotopic equivalent if one can be continuously deformed into another. A space which is homotopic equivalent to a point is called **contractible**.

Lemma 2.4.2.

(i) If $f : X \to Y$ is a homotopy equivalence, then $f^* : Vect(Y) \to Vect(X)$ is bijective.
(ii) If X is contractible, every vector bundle over X is trivial.

Proof. To see that (i) implies (ii), by definition a contractible space is one which is homotopy equivalenct to a point. Since all vector bundles over a point is trivial, it follows that all vector bundles over X are trivial.

To prove (i), we first check that the map f^* is well defined. It means if E and F are isomorphic vector bundles over Y, then $f^*(E) \cong f^*(F)$. Now suppose that $f : X \to Y$ and $g : Y \to X$ are continuous maps such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Then $f^* g^* = (gf)^* = (\mathrm{id}_X)* = \mathrm{id}_{\mathrm{Vect}(X)}$ and similarly $g^* f^* = \mathrm{id}_{\mathrm{Vect}(Y)}$. So we see that f^*, g^* are inverse of each other which implies that f^* is bijective. □

To conclude the chapter, we will state a two results that will be used later on.

Proposition 2.4.3. *All vector bundles on S^1 is trivial.*

Chapter 3

K-Theory

Given a compact Hausdorff space X, we will now define the ring $K(X)$. Recall that the set $\text{Vect}_{\mathbb{C}}(X)$ has two natural operations: the direct sum and tensor product. With respect to these two operations we see that $\text{Vect}_{\mathbb{C}}(X)$ has an addition and a multiplication structure. The additive identity is the vector bundle $X \times \{0\}$, where the fibers are just the zero vector. The multiplicative identity on the other hand is the trivial bundle $X \times \mathbb{C}$, since for any complex vector space V, we have $V \otimes \mathbb{C} \cong V$ and this isomorphsim extends to vector bundles.

For a general vector bundle E of rank n in $\text{Vect}_{\mathbb{C}}(X)$, E has neither an additive inverse nor a multiplicative inverse. In some sense the set $\text{Vect}_{\mathbb{C}}(X)$ is similar to the natural numbers $\mathbb{N}$. In fact if Y is a compact Hausdorff space in which all the vector bundles on Y are trivial e.g. when Y is just a point, then we can make the identification $\phi : \text{Vect}_{\mathbb{C}}(Y) \to \mathbb{N}$ by $[\epsilon^n] \to n$, where $[\epsilon^n]$ is the isomorphism class of trivial bundles of rank n. Under this identification the additive and multiplication structure of $\text{Vect}_{\mathbb{C}}(Y)$ and $\mathbb{N}$ are exactly the same.

We will first motivate the construction of $K(X)$ by first describing a procedure that turns $\mathbb{N}$ into a ring, namely $\mathbb{Z}$.

3.1 The group completion of $\mathbb{N}$

The canonical way of turning $\mathbb{N}$ into a group is to introduce the negative integers. This can be done by introducing an equivalence relation on the set $\mathbb{N} \times \mathbb{N}$ by

$$(m, n) \sim (m', n') \iff m + n' = m' + n.$$

The idea is to introduce each integer as a difference between two natural numbers, since $m + n' = m' + n \implies m - n = m' - n'$. Under the set $\mathbb{N} \times \mathbb{N}/\sim$ elements can be represented by $[m - n]$, which intuitively is $\mathbb{Z}$. Given two elements $[m - n]$, $[m', n'] \in \mathbb{N} \times \mathbb{N}/\sim$, we define addition by $[m - n] + [m' - n'] = [m + m' - (n + n')]$ and multiplication by $[m - n] \times [m' - n'] = [mm' + nn' - (mn' + m'n)]$. Then as a ring $\mathbb{N} \times \mathbb{N}/\sim$ is isomorphic to $\mathbb{Z}$ under the isomorphism

$$\begin{aligned} [n - 0] &\mapsto n \\ [0 - n] &\mapsto -n. \end{aligned}$$

The idea of turning $\text{Vect}(X)$ into a ring is similar to the previous construction. There is however more details involved.

First we write $p : \epsilon^n \to X$ to be the trivial bundle of rank n. Given two vector bundles E, F over X, we say E and F are **stably isomorphic**, written as $E \approx_s F$ if $E \oplus \epsilon^n = F \oplus \epsilon^n$ for some n. Similarly we define $E \sim F$ if $E \oplus \epsilon^n = F \oplus \epsilon^m$ for some m and n. Checking $\approx_s$ is an equivalence classes is easy. To see that the direct sum is well-defined, if $E' \in [E]$ and $F' \in [F]$ then suppose that $E \oplus \epsilon^m = E' \oplus \epsilon^m$, $F \oplus \epsilon^n = F' \oplus \epsilon^n$. We see that

$$E \oplus F \oplus \epsilon^{m+n} = E \oplus \epsilon^m \oplus F \oplus \epsilon^n = E' \oplus \epsilon^m \oplus F' \oplus \epsilon^n = \oplus E' \oplus F' \oplus \epsilon^{m+n}$$

since the direct sum is commutative. It also easy to see that direct sum is associative. These equivalnce relations defines us the K rings.

3.2 The ring $K(X)$

For $\approx_s$, only the trivial bundle has an inverse since if $[E \oplus F] = [\epsilon^0]$ then $E \oplus F \oplus \epsilon^n = \epsilon^n$ for some n which can happen only when $E = F = \epsilon^n$. However we have a cancellation property that $E \oplus E' \approx_s F \oplus E'$ implies $E \approx_s F$ since we can add a bundle E'' such that $E' \oplus E''$ is trivial.

Now, similar to the group completion construction of $\mathbb{N}$, we consider the set $\mathrm{Vect}_{\mathbb{C}}(X) \times \mathrm{Vect}_{\mathbb{C}}(X)$ modulo an equivalence relation $\approx$ defined as $(E, F) \approx (E', F')$ iff $E \oplus F' \approx_s E' \oplus F$. Checking reflecxive and symmetric is easy. TO see that $\approx$ is transitive, suppose that $(E, F) \approx (E', F')$ and $(E', F') \approx (E'', F'')$. Then by definition $E \oplus F' \approx_s E' \oplus F$ and $E' \oplus F'' \approx_s E'' \oplus F'$. Then we see that $E \oplus F' \approx_s E'' \oplus F'$. By adding both sides a bundle F_0 such that $F' \oplus F_0 = T^n$, we see that $E \oplus T^n \approx_s E''T^n$ which implies that $E \approx_s E''$ and similarly $F \approx_s F''$. Thus we see that $E \oplus F'' \approx_s E'' \oplus F$ which implies that $(E, F) \approx_0 (E'', F'')$.

We now define addition by $(E - F) + (E' - F') = (E \oplus E') - (F \oplus F')$. Then the set $\mathrm{Vect}_{\mathbb{C}}(X) \times \mathrm{Vect}_{\mathbb{C}}(X)/\approx$ is then a group with respect to the addition defined above. We denote this group by $K(X)$. The zero element is the equivalence class of $(\epsilon^0 - \epsilon^0)$ (or in general, $(E - E)$ for any vector bundle) and the inverse of $(E - F)$ is $(F - E)$. Notice that for any element $(E - F) \in K(X)$, let F' be the bundle such that $F \oplus F' = \epsilon^n$. Then since $(F' - F') = 0$, we see that $(E - F) = (E \oplus F' - F \oplus F') = (E \oplus F' - \epsilon^n)$. Hence any element in $K(X)$ can be representated as $(E - \epsilon^n)$ for some vector bundle E and some trivial bundle ϵ^n.

3.3 The ring $\widetilde{K}(X)$

Proposition 3.3.1. *If X is compact Hausdorff, then the set $\mathrm{Vect}_{\mathbb{C}}(X)$ modulo $\sim$ is an abelien group with respect to $\oplus$. We note this group by $\widetilde{K}(X)$.*

Proof. The identity is the trivial vector bundle ϵ^0. Clearly taking direct sum of any vector bundle is still a vector bundle so it remains to check that every element has an inverse. If all the fibers of X have the same dimension, then prop1.4(hatcher). Otherwise let $X_i = \{x \in X : \dim \pi^{-1}x = i\}$. To see that X_i is open, every $x \in X_i$ has a neighbourhood U_x such that $\pi^{-1}(U_x) = U_x \times \mathbb{C}^i$. It is obvious that $U_x \in X$. Then $X_i = \bigcup_{x \in X} U_x$ which is open. We see that $\{X_i\}_{i \in \mathbb{Z}}$ is an open cover for X which by the compactness of X must be finite. Hence over each X_i we can produce a suitable vector bundle which $E \oplus E'$ is trivial, and that all the fibers have the same dimension. $\square$

We can actually look at $\widetilde{K}$ as a subgroup of $K(X)$, to see this observe that there is a natural map

$$\begin{aligned} \phi : K(X) &\to \widetilde{K}(X) \\ (E - \epsilon^n) &\mapsto [E] \end{aligned}$$

Here $[E]$ denotes the $\sim$-class of E as an element in $\widetilde{K}(X)$. To see that this map is well-defined, suppose that $(E - \epsilon^n) = (F - \epsilon^m)$. Then by definition we see that $E \oplus \epsilon^m \approx_s F \oplus \epsilon^n \implies E \oplus \epsilon^m \oplus \epsilon^k = F \oplus \epsilon^n \oplus \epsilon^k \implies E \sim F \implies [E] = [F]$. Clearly ϕ is surjective, and if $(E - \epsilon^n) \in \ker \phi$ then $[E] = [\epsilon^0]$ which implies that $E \oplus \epsilon^m = \epsilon^0 \oplus \epsilon^n$. Now, the rank of E is non-negative, this implies $m \leq n$. Write $m = n + n'$ and we see that $E \oplus \epsilon^m \approx \epsilon^n \oplus \epsilon^{n'}$ which implies that $E \approx_s \epsilon^{n'}$. Hence the kernel of ϕ consists of the subset $\{\epsilon^m - \epsilon^n\} \subseteq K(X)$.

3.4 Ring structure on $K(X)$ and $\widetilde{K}(X)$

As promised, the groups $K(X)$ and $\widetilde{K}(X)$ has a well-definedd ring structure with respect to the tensor product. For two elements $(E - F), (E' - F') \in K(X)$, we define

$$(E - F)(E' - F') = E \otimes E' - E \otimes F' - E' \otimes F + F \otimes F'$$

The rings $K(X)$, $\widetilde{K}(X)$ can also be thought of as functors from the category of compact Hausdorff spaces to the category of rings. Suppose that we have a continuous map $f : X \to Y$. Then this induces a ring homomorphism

$$\begin{aligned} f^* : K(Y) &\to K(X) \\ E - F &\mapsto f^*(E) - f^*(F) \end{aligned}$$

3.5 Functoriality of $K(X)$ and $\widetilde{K}(X)$

Given a continuous map $f : X \to Y$, we can define an induced map

$$\begin{aligned} f^* : K(Y) &\longrightarrow K(X) \\ [E - F] &\mapsto [f^*(E) - f^*(F)]. \end{aligned}$$

Since $f^*(E \oplus F) \approx f^*(E) \oplus f^*(F)$. Given two elements $[E - F]$, $[E' - F']$, we see that

$$\begin{aligned} f^*([E - F] + [E' - F']) &= f^*([E \oplus E'] - [F \oplus F']) \\ &= [f^*(E \oplus E')] - [f^*(F \oplus F')] \\ &= [f^*(E) \oplus f^*(E')] - [f^*(F) \oplus f^*(F')] \\ &= [f^*(E) - f^*(F)] + [f^*(E') - f^*(F')] \\ &= f^*([E - F]) + f^*([E' - F']) \end{aligned}$$

The fact that f^* is multiplicative, $(fg)^* = g^* f^*$ and $1^* = 1$ easily follows from the other properties of pullback.

Now choose a base point $x_0 \in X$ and consider the projection map $X \to \{x_0\}$. This induces a ring homomorphism $i^* : K(x_0) \to K(X)$ where the image is the subgroup $\{\epsilon^m - \epsilon^n\}$, since the pullback of vector bundles over a point is trivial. This map is clearly injective and so we have a short exact sequence

$$0 \longrightarrow K(x_0) \xrightarrow{p^*} K(X) \xrightarrow{\phi} \widetilde{K}(X) \longrightarrow 0.$$

Now consider the map induced by $i : \{x_0\} \to X$

$$\begin{aligned} i^* : K(X) &\longrightarrow K(x_0) \\ (E - F) &\longrightarrow (E|_x - F|_x) \end{aligned}$$

This is clearly a ring homomorphism and moreover $\psi i^* = \text{id}$ and so the short exact sequence splits. In other words, $K(X) \cong \widetilde{K}(X) \oplus K(x_0) = \widetilde{K}(X) \oplus \mathbb{Z}$ depending on x_0. In this way, $\widetilde{K}(X)$ can be thought of as the kernel of the map $i^* : K(X) \to K(x_0)$.

We can now define a map called the **external product**

$$\begin{aligned} \mu : K(X) \otimes K(Y) &\to K(X \times Y) \\ \alpha \otimes \beta &\mapsto p_X^*(\alpha) \otimes p_Y^*(\beta) \end{aligned}$$

where $p_X : X \times Y \to X$, $p_Y : X \times Y \to Y$ are the projections. μ is a ring homomorphism since $\mu(a \otimes b)(c \otimes d) = \mu(ac \otimes bd) = p_X^*(ac)p_Y^*(bd) = p_X^*(a)p_X^*(c)p_Y^*(b)p_Y^*(d) = p_X^*(a)p_Y^*(b)p_X^*(c)p_Y^*(d) = \mu(a \otimes b)\mu(c \otimes d)$ by properties of pullback. If we take $Y = S^2$ we have the following

$$\mu : K(X) \otimes K(S^2) \to K(X \times S^2).$$

The product theorem, which we will state without proof, asserts that the above μ is an isomorphism.

Theorem 3.5.1. *The homomorphism $\mu : K(X) \otimes \frac{\mathbb{Z}[H]}{(H-1)^2} \to K(X \times S^2)$ is an isomorphism of rings for all compact Hausdorff spaces X.*

If we take X to be a point we get

Corollary 3.5.2. *The map $\frac{\mathbb{Z}[H]}{(H-1)^2} \to K(S^2)$ is an isomorphism of rings.*

The above theorem is non-trivial, and without surprise this theorem has many important applications. In particular, it allows us to deduce that the reduced K-theory of spheres are two periodic.

Theorem 3.5.3. *The homomorphism*

$$\begin{aligned} \beta : \widetilde{K}(X) &\longrightarrow \widetilde{K}(S^2 X) \\ a &\mapsto (H - 1) * a \end{aligned}$$

is an isomorpism for all compact Hausdorff spaces X. We now have Now, recall that $\widetilde{K}(S^1) = 0$ and $\widetilde{K}(S^2) = \mathbb{Z}$. Since $S^m S^n = S^{m+1}$, We have the following corollary.

Corollary 3.5.4. *$\widetilde{K}(S^{2n+1}) = 0$ and $\widetilde{K}(S^{2n}) = \mathbb{Z}$, generated by the n-fold reduced external product $(H - 1) * \cdots * (H - 1)$.*

In general, $K(X)$ and $\widetilde{K}(X)$ are very hard to compute, since the isomorphism classes of vector bundles on an arbitrary compact Hausdorff space is not very well understood. The power of the Bott periodicity theorem is that to understand the reduced K-theory of a sphere it is enough to compute the reduced K-theory of the circle and sphere, since the reduced K-theory of spheres are two periodic.

Chapter 4

Adam's operations

In this chapter we will primarily deal with a class of operations on $K(X)$ called the Adams operations. These operations turns out to be ring homomorphisms satisfying additional properties. We know that given any compact Hausdorff space, $K(X)$ is a ring. One natural question is that whether $K(X)$ can be any ring. Using the existence of these Adams operation we can show that there is no compact Hausdorff space X such that $K(X) = \mathbb{Z}[i]$. Hence the existence of these operations in some sense shows that the ring $K(X)$ has more 'structure'. The most important application of the Adam's operations is on the Hopf invariant problem, which is our main use of these ring homomorphisms.

Adam's operation were first introduced by Frank Adams to prove the Hopf invariant one problem. Adam's defined this class of ring homomorphism through the use of Newton polynomials. In 1960 Atiyah defined the Adams operations in terms of the symmetric group. We will follows Atiyah's definition and proceed to show the properties of these ring homomorphism.

One point we want to emphasize for the upcoming proofs involving isomorphism of vector bundles is that all the maps that are to be considered will be continuous, a fact which we will omit in the proofs. Hence by Lemma 2.1.10 we will only show that each fibers are isomorphic.

4.1 G-Bundles

We will now proceed by defining the notion of a finite group G acting on a topological space Y.

Definition 4.1.1. Let Y be a topological space and G be a finite group. We call Y a **G-space** if there is a continuous map

$$\begin{aligned} f : G \times Y &\longrightarrow Y \\ (g, y) &\mapsto g \cdot y. \end{aligned}$$

We will now define the notion of a finite group acting on a vector bundle over X.

Definition 4.1.2. Let $p : E \to X$ be a vector bundle over a compact Hausdorff space X and let G be a finite group. Suppose that E, X are G-spaces. We say E is a G-vector bundle if

(i) $g \cdot p(v) = p(g \cdot v)$ for all $g \in G$ and $v \in E$, that is the action of G commutes with the projection map.

(ii) For each $g \in G$ the map $E_x \to E_{g \cdot x}$ is linear.

In the remaining chapter we will only look at G-vector bundles with a trivial action on the base space X. That is $g \cdot x = x$ for all $x \in X$.

A G-vector bundle is therefore a vector bundle where each fiber is a representation space of G. We will now define a homomorphism between two G-bundles. Recall that a homomorphism between two vector bundles is a continuous map that is also a homomorphism when restricted to each fiber. For G-bundles, every fiber is a representation space of G, so we want the homomorphism to preserve the action of G.

Definition 4.1.3. Given two G-bundles E, F, a homomorphism $\phi : E \to F$ is G-linear if ϕ is a vector bundle homomorphism such that for all $v \in E$, $\phi(g \cdot v) = g \cdot \phi(v)$.

We can now make precise when two G-bundles are isomorphic.

Definition 4.1.4. Two G-bundles E, F are **isomorphic** as G-bundles if there exists a G-linear isomorphsim $\phi : E \to F$.

We can now consider the set of all isomorphisms classes of G-bundles which forms an abelian monoid. Similar to the contruction of $K(X)$ from $\mathrm{Vec}_{\mathbb{C}}(X)$ we can construct the Grothendieck group of the set of all isomorphisms classes of G-bundles, in which we will call $K_G(X)$.

In module theory we know that any representation of G over $\mathbb{C}$ can be decomposed into a direct sum of irreducible representations of G. Now a G-bundle E is a vector bundle where each fiber is a representation space of G. A natural question to ask is whether we can write E as a direct sum of subbundles, where the fibers of the sub-bundles are irreducible representation space of G. If a vector bundle is trivial, then we can indeed do this. In general this is more complicated.

We now let $\{[W_\pi]\}$ be the set of isomorphism classes of irreducible representations of G. We define $R(G) = \sum_\pi \mathbb{Z}[W_\pi]$, the free abelian group over $\{[W_\pi]\}$. This set $R(G)$ is a ring under the tensor product. To see this, the tensor prodcut of any two irreducible representations $V_\pi \otimes V_\rho$ is also a representation of G, and so $V_\pi \otimes V_\rho$ can be decomposed into a direct sum of irreducible representations.

Now suppose that V, E are G-bundles. We define $\mathrm{Hom}_G(V, E)$ to be the vector bundle where each fiber consists of $\mathrm{Hom}_G(V_x, E_x)$, the set of G-linear vector space homomorphisms from V_x to E_x. Then we have

Lemma 4.1.5. *The map*

$$\begin{aligned} \phi : V \otimes Hom_G(V, E) &\to E \\ v \otimes f &\mapsto f(v) \end{aligned}$$

is a G-linear homomorphism.

Proof. First we want to show that the map $\psi : V \times \mathrm{Hom}_G(V, E) \to E$ sending $(v, f) \mapsto f(v)$ is bilinear. We have

$$\psi(\lambda v + v', f) = f(\lambda(v + v')) = \lambda f(v) + f(v') = \lambda\psi(v, f) + \psi(v', f)$$

and similarly $\psi(v, \lambda f + f') = \lambda\psi(v, f) + \psi(v, f')$.

Since ψ is bilinear, the universal property of tensor product implies that there is a unique homomorphism sending $v \otimes f$ to $f(v)$. We see that the map is precisely ϕ. It remains to check that ϕ is G-linear. Notice that

$$\phi(g \cdot v \otimes f) = \phi(gv \otimes f) = f(gv) = g \cdot f(v) = g \cdot \phi(v \otimes f)$$

and this completes the proof. □

We will need an easy lemma which we will state without proof

Lemma 4.1.6. *Suppose A, B, C are representation space of G, and $\phi : B \to C$ is a G-linear isomorphism. Then the map*

$$\begin{aligned} \Phi : Hom_G(A, B) &\to Hom_G(A, C) \\ f &\mapsto \phi \circ f \end{aligned}$$

is a vector space isomorphism.

With the above lemma we can now establish

Lemma 4.1.7. *Suppose that E, F, H are G-vector bundles, and $\phi : F \to H$ is a G-linear isomorphism. Then the map*

$$\begin{aligned} \Phi : Hom_G(E, F) &\to Hom_G(E, H) \\ f &\mapsto \phi \circ f \end{aligned}$$

is a vector bundle isomorphism.

Proof. By Lemma 4.1.6, for each $x \in X$ the fibers $\mathrm{Hom}_G(E_x, F_x)$, $\mathrm{Hom}_G(E_x, H_x)$ are isomorphic. We assume Φ is continuous so by Lemma 2.1.10 we see that Φ is an vector bundle isomorphism. □

Lemma 4.1.8. *Suppose that $\{W_\pi\}$ is a set of irreducible representations of G. Let $V_\pi = X \times W_\pi$ be the trivial G-vector bundle. Then the following G-bundles are isomorphic:*

$$\sum_\pi V_\pi \otimes Hom_G(V_\pi, E) \cong E$$

Proof. We first want to show the fibers are isomorphic. Since E_x is a representation of G, we can write E_x as a direct sum of irreducible representations of G, that is $E_x \cong \sum_\pi W_\pi^{n_\pi}$. By Lemma 4.1.6 we can write

$$\begin{aligned} \sum_\pi W_\pi \otimes \mathrm{Hom}_G(W_\pi, E_x) &\cong \sum_\pi W_\pi \otimes \mathrm{Hom}_G(W_\pi, \sum_\rho W_\rho^{n_\rho}) \\ &\cong \sum_\pi W_\pi \otimes \sum_\rho \mathrm{Hom}_G(W_\pi, W_\rho)^{n_\rho} \quad \text{universal property of Hom,} \\ &\cong \sum_\pi W_\pi \otimes \mathbb{C}^{n_\pi} \quad \text{Schur's lemma,} \\ &\cong \sum_\pi W_\pi^{n_\pi} \\ &\cong E_x \end{aligned}$$

□

Recall that $R(G)$ is the free abelian group generated by the irreducible representations W_π of G. By letting $V_\pi = X \times W_\pi$ we see that the free abelian group generated by the isomorphism classes $\{[V_\pi]\}$ has the same ring structure as $R(G)$. Hence we can look at elements in $R(G)$ as trivial G-vector bundles $V_\pi = X \times W_\pi$.

We can now establish

Proposition 4.1.9. *The following rings are isomorphic*

$$\phi : R(G) \otimes_{\mathbb{Z}} K(X) \to K_G(X)$$

Proof. Let $V = X \times W$ where W is a representation of G. Consider the two maps

$$\begin{aligned} \Phi : R(G) \otimes_{\mathbb{Z}} K(X) &\to K_G(X) \qquad & \Psi : K_G(X) &\to R(G) \otimes_{\mathbb{Z}} K(X) \\ [V] \otimes [E] &\mapsto [V \otimes E] & [E] &\mapsto \sum_\pi [V_\pi] \otimes [\mathrm{Hom}_G(V_\pi, E)] \end{aligned}$$

We first need to define a G-action on $[V \otimes E]$. The obvious action is that over each fiber, we define $g \cdot (v \otimes e) = g(v) \otimes e \in V_x \otimes E_x$. With this set up we see that Φ, Ψ are inverses of each other.

□

With all of these setup, we can proceed to define Adam's operations. For any vector bundle E, we will first define a natural action of S_k on $E^{\otimes k}$. Given $\sigma \in S_k$, for any element $v \otimes \cdots \otimes v \in E^{\otimes k}$ we define

$$\sigma \cdot v_1 \otimes \cdots \otimes v_k = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}$$

Hence by eariler proposition we can look at $E^{\otimes k}$ as an S_k bundle and we get the following S_k-bundle isomorphism

$$E^{\otimes k} \simeq \sum_\pi V_\pi \otimes \mathrm{Hom}_{S_k}(V_\pi, E^{\otimes k})$$

Lemma 4.1.10. *Let $\{V_\pi\}$ be the isomorphism classes of the set of irreducible representations of G and let $\chi_\pi : G \to \mathbb{C}$ be the character of V_π. Then for any $g \in G$ the map*

$$\begin{aligned} tr(g) : R(G) &\to \mathbb{Z} \\ V_\pi &\mapsto \chi_\pi(g) \end{aligned}$$

is a ring homomorphism.

Proof. This follows from the fact that

$$\chi_{V_\pi \oplus V_\rho}(g) = \chi_{V_\pi}(g) + \chi_{V_\rho}(g)$$

and

$$\chi_{V_\pi \otimes V_\rho}(g) = \chi_{V_\pi}(g)\chi_{V_\rho}(g).$$

□

For any k, let $\sigma \in S_k$ be a k-cycle. Given a representation V of S_k with character χ, we define $c_k = \chi(\sigma)$. Notice that all the k-cycles in S_k are conjugate to each other and thus gives the same trace so this map is well-defined.

Another lemma which we will need but we won't prove is the following

Lemma 4.1.11. *Suppose that $\phi : A \to A'$ and $\psi : B \to B'$ are ring homomorphisms. Then the map*

$$\begin{aligned} \phi \otimes \psi : A \otimes B &\to A' \otimes B' \\ a \otimes b &\mapsto \phi(a) \otimes \psi(b) \end{aligned}$$

is well-defined and is a ring homomophism.

Now suppose that E is an S_k bundle and $\sigma \in S_k$ is a k-cycle. We know that $E \cong \sum_\pi V_\pi \otimes Hom_{S_k}(V_\pi, E)$. Now let $\iota : K(X) \to K(X)$ be the identity ring homomorphism. We now define the map

$$\begin{aligned} c_k \otimes \iota : R(S_k) \otimes K(X) &\to \mathbb{Z} \otimes K(X) \cong K(X) \\ \sum_\pi [V_\pi] \otimes [\mathrm{Hom}_{S_k}(V_\pi, E)] &\mapsto \sum_\pi c_k(V_\pi) \cdot [\mathrm{Hom}_{S_k}(V_\pi, E)] \end{aligned}$$

For this map to be well-defined, we need $c_k(V_\pi) \in \mathbb{Z}$ since $K(X)$ is an abelian group. Indeed, we have

Lemma 4.1.12. *All characters of S_k are integer-valued.*

To prove this lemma we need to introduce some Galois theory.

Lemma 4.1.13. *Suppose that ζ is an n-root of unity. Then the set of ring automorphisms from $\mathbb{Q}[\zeta] \to \mathbb{Q}$ that fixes $\mathbb{Q}$ is of the form*

$$\begin{aligned} \zeta^m : \mathbb{Q}[\zeta] &\longrightarrow \mathbb{Q} \\ \zeta &\mapsto \zeta^m \end{aligned}$$

for some $1 \leq m < n$ where m, n are coprime.

We can now go back and prove Lemma 4.1.12

Proof. [Lemma 4.1.12]. Suppose that $g \in S_k$, $|g| = n$. Let ρ be a representation of S_k with character χ. Since $\rho(\sigma)$ is diagonalisable, we can write $\rho(\sigma) = QDQ^{-1}$ for some diagonal matrix D, where the eigenvalues are roots of unity satisfying $\zeta^n = 1$. Now let $m < n$ be such that m is coprime to the cycles lengths of σ, so that g and g^m are conjugate in S_k. We then have $\rho(g^m) = \rho(g)^m = QD^mQ^{-1}$. Hence $\chi(\sigma^m)$ is the image of the element χ_σ under the ring automorphism $\zeta^m : \mathbb{Q}[\zeta] \to \mathbb{Q}[\zeta]$, $\zeta \mapsto \zeta^m$. That is $\zeta^m(\chi(\sigma)) = \chi(\sigma^m)$. But g and g^m are conjugate in S_k so $\chi(\sigma) = \chi(\sigma^m)$. This implies that $\chi(\sigma)$ is fixed by σ^m which implies that $\chi(\sigma) \in \mathbb{Q}$. Since $\chi(\sigma)$ is an algebraic integer, we conlude that $\chi(\sigma) \in \mathbb{Z}$. □

Remark 4.1.14. We see that $c_k = tr(\sigma) \otimes \iota$ so by lemma 2, c_k is a ring homomorphism. □

4.2 Adam's operations

Before defining the Adam's operations we need the following

Lemma 4.2.1. *The power map*

$$\begin{gathered}\otimes k : K(X) \longrightarrow K_{S_k}(X)\\ [E] \mapsto [E^{\otimes k}]\end{gathered}$$

is well-defined.

Definition 4.2.2. (Adam's operations) For each integer k and any $[E] \in K(X)$, define

$$\begin{gathered}\psi^k : K(X) \to R(S_k) \otimes K(X) \to K(X)\\ [E] \to [E^{\otimes k}] \to c_k \otimes \iota(E^{\otimes k})\end{gathered}$$

For a general element $[E - F]$, we take its tensor power $[(E - F)^{\otimes k}]$, where we get an integer combination of $E^{\otimes i} \otimes F^{\otimes k-i}$, decompose each of these vector bundles as an S_k vector bundle and apply $c_k \otimes \iota$ to each.

The Adam's operations have a number of properties. We will now proceed to prove the following proposition.

Proposition 4.2.3. *For each positive integer k, the map*

$$\psi^k : K(X) \to K(X)$$

satsifies the following properties:

(i) $\psi^k : K(X) \to K(X)$ *is a ring homomorphism.*
(ii) ψ^k *is natural, that is for any continuous map* $f : X \to Y$, $\psi^k f^* = f^* \psi^k$
(iii) $\psi^k([L]) = [L]^k$ *for any line bundle* $[L]$.
(iv) $\psi^k \circ \psi^l = \psi^l \circ \psi^k$ *for any positive integers* k, l.
(v) $\psi^p(\alpha) \equiv \alpha^p$ *for any prime* p.

Foe the last property, it means $\psi^p(\alpha) = \alpha^p + p\beta$ for some $\beta \in K(X)$. Atiyah proved that ψ^k is a ring homoorphism, that is $\psi^k()$ so we will omit his proof here. We will prove the remaining properties of the adams operations.

4.3 ψ^k is natural

Proof. [Proposition 4.2.3(ii)]

Let $\pi(E) = \mathrm{Hom}_{S_k}(V_\pi, E^{\otimes k})$ and let $c_k : R(G) \longrightarrow \mathbb{Z}$ be the trace of a k-cycle. We know that $E^{\otimes k} = \sum_\pi V_\pi \otimes \pi(E)$. Now, since $f^*(E \oplus F) = f^*(E) \oplus f^*(F)$, $f^*(E \otimes F) = f^*(E) \otimes f^*(F)$ and that the pull back of a trivial bundle is trivial, we see that

$$f^*(E^{\otimes k}) = f^*\left(\sum_\pi V_\pi \otimes \pi(E)\right) = \sum_\pi f^*(V_\pi) \otimes f^*(\pi(E)) = \sum_\pi V_\pi \otimes f^*(\pi(E))$$

and hence in $K(X)$ we have

$$\psi^k(f^*[E]) = \psi^k([f^*(E)]) = c_k([f^*(E)^{\otimes k}]) = c_k([f^*(E^{\otimes k})])$$
$$= c_k\left(\sum_\pi V_\pi \otimes f^*([\pi(E)])\right) = \sum_\pi \mathrm{tr}(V_\pi)[f^*(\pi(E)].$$

On the other hand,

$$f^*\psi^k([E]) = f^*c_k([E^{\otimes k}]) = f^*(\sum_\pi \mathrm{tr}(V_\pi)[\pi(E)]) = \sum_\pi \mathrm{tr}(V_\pi)[f^*(\pi(E))]$$

So we see that $\psi^k f^*([E]) = f^*\psi^k([E])$ for the element $[E] \in K(X)$. In general any element in $K(X)$ is of the form $[E]-[F]$ and since ψ^k, f^* are ring homomorphisms we see that $\psi^k f^*([E]-[F]) = f^*\psi^k([E]-[F])$ for any element $[E]-[F] \in K(X)$. □

4.4 $\psi^k(L) = L^k$ for any line bundle L

Proof. (Proposition 1(ii)) Since L is one-dimensional, $L^{\otimes k}$ is one-dimensional also. We see that S_k acts trivially on every fiber of $L^{\otimes k}$. Let V_1 be the trivial representation of S_k, and let $V = X \times V_1$ be the one-dimensional trivial bundle. We see that $L^{\otimes k} = V_1 \otimes \mathrm{Hom}_{S_k}(V, L^{\otimes k})$. But the trivial one dimensional vector bundle is the identity with respect to the tensor product in $\mathrm{Vect}_{\mathbb{C}}(X)$, so we see that $L^{\otimes k} = V_1 \otimes \mathrm{Hom}_{S_k}(V, L^{\otimes k}) = \mathrm{Hom}_{S_k}(V, L^{\otimes k})$. Thus $\psi(L) = [\mathrm{Hom}_{S_k}(V, L^{\otimes k})] = [L^{\otimes k}]$ □

4.5 $\psi^k(E) \cong E^k \mod p$

For any prime p, denote σ to be a p-cycle in S_p. We want to show that $p|\chi_V(1) - \chi_V(\sigma)$ for any irreducible representation of S_p.

Firstly we have

Lemma 4.5.1. *Suppose that $E = \sum_\pi V_\pi \otimes Hom_{S_k}(V_\pi, E^{\otimes k})$. Then*

$$[E^{\otimes}] = \sum_\pi \dim(V_\pi)[Hom_{S_k}(V_\pi, E^{\otimes k}] \in K(X)$$

Proof. We have

$$\begin{aligned}[E^{\otimes k}] &= \sum_\pi V_\pi \otimes \mathrm{Hom}_{S_k}(V_\pi, E^{\otimes k}) \\ &= \sum_\pi [\mathrm{Hom}_{S_k}(V_\pi, E^{\otimes k})]^{\dim(V_\pi)} \\ &= \sum_\pi \dim(V_\pi)[\mathrm{Hom}_{S_k}(V_\pi, E^{\otimes k})].\end{aligned}$$

□

Hence the element $[E^{\otimes k}]$ is the same if we took the trace of the identity element of V_π. To show that $\psi^k(E) \equiv E^{\otimes k} \mod p$ it suffices to show that $p|\chi_\pi(1) - \chi_\pi(\sigma)$, where χ_π is the character of V_π.

Lemma 4.5.2. *The map $\sigma^k : \mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta]$ sending $\zeta \mapsto \zeta^k$ is a ring automorphism for any $k \in \{1, ..., p-1\}$*

Proof. Since $(k,p) = 1$, σ^k is bijective. $\sigma^k(\zeta^m\zeta^n) = \zeta^{k(m+n)} = \zeta^{km}\zeta^{kn} = \sigma^k(\zeta^m)\sigma^k(\zeta^n)$. □

Lemma 4.5.3. *Let ζ be a p-th root of unity . Then $p = \prod_{k=1}^{p-1}(1-\zeta^k)$.*

Proof. By factorising the expression $x^p - 1$ we have $\prod_{k=1}^{p-1}(x-\zeta^k) = \sum_{k=0}^{p-1} x^k$. Substitute $x = 1$ and the result follows. □

We now come to the lemma

Lemma 4.5.4. *Suppose that $\rho : S_k \to GL(V)$ is an irreducible representation of S_k with character χ. Then $p|\chi_V(1) - \chi_V(\sigma)$.*

Proof. Suppose $\dim(V) = \chi_V(1) = d$. Let $\zeta \neq 1$ be a p-th root of unity. We know that the eigenvalues of $\rho(\sigma)$ are of the form ζ^k, $k \in \{1, .., p-1\}$. So we can write $\chi_V(1) - \chi_V(\sigma) = \sum_{k_d=1}^{d} 1 - \zeta^{k_d}$. But $1 - \zeta | 1 - \zeta^k$ for any k, so we can write $\chi_V(1) - \chi_V(\sigma) = (1-\zeta)\nu$ for some $\nu \in \mathbb{Z}[\zeta]$. Since the characters of S_n takes integer values, we can write $(1-\zeta)\nu = \chi_V(1) - \chi_V(\sigma) = n$ for some $n \in \mathbb{Z}$.

Now, apply the ring automorphism $\sigma^k : \mathbb{Z}[\zeta] \to \mathbb{Z}[\zeta]$ to the equation $(1-\zeta)\nu = n$ for each $k = \{1, ..., p-1\}$. We see that $\sigma^k((1-\zeta)\nu) = (1-\zeta^k)\sigma^k(\nu) = n$ for each k. Multiply all these together we get $\prod_{k=1}^{p-1}(1-\zeta^k)\sigma^k(\nu) = n^{p-1}$ which implies that $p\prod_{k=1}^{p-1}\sigma^k(\nu) = n^{p-1}$. Since p is prime, we conclude that $p|n = \chi_V(1) - \chi_V(\sigma)$. □

So now we have shown that $\psi^p([E]) \equiv [E^{\otimes k}] \mod p$ for $[E] \in K(X)$. We will now proceed to show

Proposition 4.5.5. *For a general element $[E] - [F] \in K(X)$, we have*

$$\psi^p([E] - [F]) \equiv ([E] - [F])^p \mod p$$

Proof. We first consider the case when $p = 2$. We then have

$$\begin{aligned}
\psi^2([E-F]) &= \psi^2([E]) - \psi^2([F]) \\
&\equiv [E^{\otimes 2}] - [F^{\otimes 2}] \\
&\equiv [E^{\otimes 2}] - 2[E \otimes F] - [F^{\otimes 2}] + 2[F^{\otimes 2}] \\
&= [(E-F)^{\otimes 2}] = [E-F]^2 \mod 2.
\end{aligned}$$

Now, for $p \geq 3$, we have

$$\begin{aligned}
\psi^p([E-F]) &= \psi^p([E]) - \psi^p([F]) \mod p \\
&\equiv [E^{\otimes p}] - [F^{\otimes p}] \mod p \\
&\equiv [E^{\otimes p}] + \sum_{i=1}^{p-1}(-1)^i \begin{pmatrix} p \\ i \end{pmatrix} [E^{\otimes i} \otimes F^{\otimes p-i}] - [F^{\otimes p}] \mod p \\
&\qquad \text{since } p \Big| \begin{pmatrix} p \\ i \end{pmatrix}, \\
&\equiv [(E-F)^{\otimes p}] = [E-F]^p \mod p
\end{aligned}$$

and this completes the proof.

□

4.6 $\psi^k \psi^l = \psi^l \psi^k$

Lemma 4.6.1. *For any finite groups G, H, we have*

$$R(G \times H) \cong R(G) \otimes_{\mathbb{Z}} R(H)$$

Proof. Firstly notice that for any $g, g' \in G$ and $h, h' \in H$, we have

$$(g', h')(g, h)((g')^{-1}, (h')^{-1}) = (g'g(g')^{-1}, h'h(h')^{-1})$$

so we see that the conjugacy class of (g, h), denoted by $C_{(g,h)}$ is precisely $C_g \times C_h$. Suppose that there are m, n number of conjugacy classes of G, H respectively. Then we see that the number of conjugacy classes of $G \times H$ is mn.

Suppose that $\rho : G \to GL(V), \pi : H \to GL(V)$ are irreducible representations of G, H respectively. Then we can define a representation space, $\rho \times \pi$, for $G \times H$ by

$$\begin{aligned} \rho \times \pi : G \times H &\to GL(V \otimes W) \\ (g, h) &\mapsto \rho(g) \otimes \pi(h) \end{aligned}$$

(check $\rho(g) \otimes \pi(h)$ is indeed an automorphism of $V \otimes W$)

Now, let χ be the character of $\rho \times \pi$. Then by definition we have $\chi(g, h) = \chi_\rho(g)\chi_\pi(h)$. Recall that $\langle \chi, \chi \rangle$ is the orthogonality relation. We then have

$$\begin{aligned} \langle \chi, \chi \rangle &= \frac{1}{|G \times H|} \sum_{(g,h) \in G \times H} \chi(g^{-1}, h^{-1})\chi(g, h) \\ &= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \chi_\rho(g^{-1})\chi_\rho(g)\chi_\pi(h^{-1})\chi_\pi(h) \\ &= \frac{1}{|G|} \left(\sum_{g \in G} \chi_\rho(g^{-1})\chi_\rho(g) \right) \frac{1}{|H|} \left(\sum_{h \in H} \chi_\pi(h^{-1})\chi_\pi(h) \right) \\ &= \langle \chi_\rho, \chi_\rho \rangle \langle \chi_\pi, \chi_\pi \rangle. \end{aligned}$$

Since ρ, π are irreducibles, we see that $\langle \chi, \chi \rangle = 1 \implies \rho \times \pi$ is an irreducible representation of $G \times H$.

Now, let $\{\rho_i\}, \{\pi_j\}$ be the set of irreducible representations of G, H respectively. From above we see that the set $\{\rho_i \times \pi_j\}$ is a set of irreducible representations of $G \times H$. Similar to the above formula $\langle \chi, \chi \rangle = \langle \chi_\rho, \chi_\rho \rangle \langle \chi_\pi, \chi_\pi \rangle$, if ρ_1, ρ_2 and π_1, π_2 are irreducible representations of G and H, then $\langle \chi_{\rho_1 \times \pi_1}, \chi_{\rho_2 \times \pi_2} \rangle = \langle \chi_{\rho_1}, \chi_{\rho_2} \rangle \langle \chi_{\pi_1}, \chi_{\pi_2} \rangle = 1$ iff $\rho_1 = \rho_2$ and $\pi_1 = \pi_2$. We also know that there are mn conjugacy classes of $G \times H$ so $\{\rho_i \times \pi_j\}$ is precisely the set of irreducible representations of $G \times H$. Hence we see that the map $\rho_i \otimes \pi_j \to \rho_i \times \pi_j$ is a bijection between the generators of $R(G) \otimes R(H)$ and $R(G \times H)$.

Now consider the map

$$\Phi : R(G) \times R(H) \to R(G \times H)$$
$$\rho_i \times \pi_j \to \rho_i \otimes \pi_j$$

Here we will look at $R(G), R(H), R(G \times H)$ as a subring of the ring of all complex class functions on $G, H, G \times H$ respectively. Thus, a basis of $R(G)$ is $\{\chi_{\rho_i}\}$ and similarly for $R(H)$. Now, by definition as above we see that a basis for $R(G \times H)$ is $\{\chi_{\rho_i}\chi_{\pi_j}\}$. It is now easy to check that Φ is a bilinear homomorphism so by the universal property of tensor product there is a ring isomorphism $\tilde{\Phi} : R(G) \otimes R(H) \to R(G \times H)$. □

We will now show that $\psi^k \circ \psi^l = \psi^l \circ \psi^k$ for all $k, l \in \mathbb{Z}$.

Proof. Recall that given an element $[E] \in K(X)$, $\psi^k(E)$ is obtained by

$$\psi^k : K(X) \to R(S_k) \otimes K(X) \to \mathbb{Z} \otimes K(X) \cong K(X)$$
$$E \mapsto E^{\otimes k} \mapsto c_k(E^{\otimes k})$$

Now consider the element $\psi^k \circ \psi^l(E)$. This element is obtained through

$$\psi^k \circ \psi^l : K(X) \to R(S_l) \otimes K(X) \to K(X) \to R(S_k) \otimes K(X) \to K(X)$$
$$E \mapsto E^{\otimes l} \mapsto c_l(E^{\otimes l}) \mapsto (c_l(E^{\otimes l}))^{\otimes k} \mapsto c_k((c_l(E^{\otimes l}))^{\otimes k})$$

Now since c_l is a ring homomorphism, we have $c_k((c_l(E^{\otimes l}))^{\otimes k}) = c_k \circ c_l((E^{\otimes l})^{\otimes k})$ and so we can redefine $\psi^k \circ \psi^l$ as

$$\psi^k \circ \psi^l : K(X) \to R(S_l) \otimes K(X) \to R(S_k) \otimes R(S_l) \otimes K(X) \to R(S_k) \otimes K(X) \to K(X)$$
$$E \mapsto E^{\otimes l} \mapsto (E^{\otimes l})^{\otimes k} \mapsto c_l((E^{\otimes l})^{\otimes k}) \mapsto c_k \circ c_l((E^{\otimes l})^{\otimes k})$$

Now, since $R(S_k) \otimes R(S_l) \cong R(S_k \times S_l)$, we can look at $(E^{\otimes l})^{\otimes k}$ as an $S_k \times S_l$-bundle. Suppose that $\{V_i\}, \{W_j\}$ are the set of irreducible representations of S_k, S_l respectively. We know that the set $V_i \times W_j$ are precisely the set of irreducible representations of $S_k \times S_l$ so we can decompose $(E^{\otimes l})^{\otimes k}$ as

$$E^{\otimes kl} \cong \sum_{i,j} V_i \times W_j \otimes Hom_{S_k \times S_l}(V_i \otimes W_j, (E^{\otimes l})^{\otimes k})$$

Pick the isomorphism $\phi : R(S_k \times S_l) \to R(S_k) \otimes R(S_l)$, $\rho_i \times \pi_j \to \rho_i \otimes \pi_j$. This induces an isomorphism

$$\tilde{\phi} : R(S_k \times S_l) \otimes K(X) \to R(S_k) \otimes R(S_l) \otimes K(X)$$

where

$$\sum_{i,j} V_i \times W_j \otimes Hom_{S_k \times S_l}(V_i \otimes W_j, (E^{\otimes l})^{\otimes k}) \mapsto \sum_{i,j} V_i \otimes W_j \otimes Hom_{S_k \times S_l}(V_i \otimes W_j, (E^{\otimes l})^{\otimes k})$$

Here we assume V_i is the representation space of ρ_i and W_i be the representation space of π_j. If we now take the trace of W_j and V_i we get $\psi^k \circ \psi^l(E)$. The same is true for $\psi^l \circ \psi^k$ however there is a difference. To obtain $\psi^k \circ \psi^l(E)$ we first take $E^{\otimes l}$ and then $(E^{\otimes l})^{\otimes k}$. On the other hand if we were to compute $\psi^l \circ \psi^k$ we first take $E^{\otimes k}$ then $(E^{\otimes k})^{\otimes l}$. As vector bundles both are isomorphic to $E^{\otimes kl}$. However the action of $S_k \times S_l$ on these vector bundles are different and hence $(E^{\otimes l})^{\otimes k}$ and $(E^{\otimes k})^{\otimes l}$ are not necessarily isomorphic as $S_k \times S_k$-bundles.

We will now proceed to show $(E^{\otimes l})^{\otimes k}$ and $(E^{\otimes k})^{\otimes l}$ are indeed isomorphic as $S_k \times S_l$-bundles and thus by Lemma 4.1.7

$$\sum_{i,j} V_i \otimes W_j \otimes Hom_{S_k \times S_l}(V_i \otimes W_j, (E^{\otimes l})^{\otimes k}) \cong \sum_{i,j} V_i \otimes W_j \otimes Hom_{S_k \times S_l}(V_i \otimes W_j, (E^{\otimes k})^{\otimes l})$$

To do this it suffices to show that every fiber of $(E^{\otimes l})^{\otimes k}$ and $(E^{\otimes k})^{\otimes l}$ are isomorphic as $S_k \times S_l$-modules and that the isomorphism is continuous. Let V be the fiber of E, so that $V^{\otimes kl}$ is the fiber of $E^{\otimes kl}$. For convenience we write $(V^{\otimes l})^{\otimes k} = \bigotimes_{i=1,j=1}^{l,k} V_{ij}$ where each $V_ij = V$ and similarly $(V^{\otimes k})^{\otimes l} = \bigotimes_{i=1,j=1}^{l,k} V_{ji}$. The action of $S_k \times S_l$ on $(V^{\otimes l})^{\otimes k}$ and $(V^{\otimes k})^{\otimes l}$ are illustrated below.

$$\begin{pmatrix} V_{11} & \cdots & V_{1k} \\ \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots \\ V_{l1} & \cdots & V_{lk} \end{pmatrix} \qquad \begin{pmatrix} V_{11} & \cdots & \cdots & V_{1l} \\ \vdots & \ddots & \cdots & \vdots \\ V_{k1} & \cdots & \cdots & V_{kl} \end{pmatrix}$$

For the left one $(V^{\otimes l})^{\otimes k}$, S_l acts on the columns while S_k acts on the rows. On the right side we have $(V^{\otimes k})^{\otimes l}$, where S_l acts on the rows and S_k acts on the columns. Now let v_{ij} be vectors in V_{ij}. We let

$$\Phi : \bigotimes_{i=1,j=1}^{l,k} V_{ij} \to \bigotimes_{i=1,j=1}^{l,k} V_{ji}$$
$$v_{ij} \mapsto v_{ji}$$

Clearly this is a vector space homomophism, so it remains to show that Φ is $S_k \times S_l$-linear. So suppose that $\sigma \in S_k, \tau \in S_l$. Then we have

$$\Phi((\sigma, \tau) \cdot V_{ij}) = \Phi(V_{\sigma(i)\tau(j)}) = V_{\tau(j)\sigma(i)} = (\sigma, \tau)V_{ji} = (\sigma, \tau)\Phi(V_{ij}).$$

□

In some sense, $\psi^k \circ \psi^l$ and $\psi^l \circ \psi^k$ differs only in the order of taking traces. We will now conclude this chapter by showing the following remark.

Proposition 4.6.2. *There is no compact Hausdorff space X such that $K(X) = \mathbb{Z}[i]$.*

Proof. Suppose for a contradiction that X is compact Hausdorff and $K(X) = \mathbb{Z}[i]$. We want to apply the ring homomorphism ψ^2 to elements of $\mathbb{Z}[i]$ and get a contradiction using the extra properties of ψ^2. So suppose that $\psi^2(i) = a + bi$ for some $a, b \in \mathbb{Z}$. Since $\psi^2(i) \equiv i^2 = -1 \mod 2$, we see that a is odd and b is even. Now

$$0 = \psi^2(0) = \psi^2(1 + i^2) = \psi^2(1) + (\psi^2(i))^2 = 1 + (a + bi)^2 = 1 + a^2 + 2abi - b^2$$

which implies that $2ab = 0 \implies a = 0$ or $b = 0$. But a is odd, so we must have $b = 0$. This implies that $1 + a^2 = 0$ which is a contradiction. $\square$

Chapter 5

Division algebras

Some examples of division algebras are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$,x the Quaternions $\mathbb{H}$ and the Octonions $\mathbb{O}$, corresponding to dimensions $1, 2, 4, 8$.

In fact we have

Theorem 5.0.3. *$\mathbb{R}^n$ is a division algebra only when $n = 1, 2, 4, 8$*

5.1 Divison algebra

An **Division algebra** D over $\mathbb{R}$ is the vector space $\mathbb{R}^n$ with a continuous multiplication map $\phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ such that if we fix an $a \in \mathbb{R}^n$ then the maps $x \mapsto \phi(a, x)$ and $x \mapsto \phi(x, a)$ are linear in x and invertible if $a \neq 0$. Since the maps are linear, invertibility implies that the multiplication map has no zero divisors. We can assume that the multiplcation map has a two-sided identity element as follows. Choose a unit vector $e \in \mathbb{R}^n$. Suppose that $M \in GL_n(\mathbb{R})$ takes e^2 to e. Then letting $\phi = M \circ \phi$ we can assume $\phi(e, e) = e$. Now, let $p, q : \mathbb{R}^n \to \mathbb{R}^n$ be the maps $x \mapsto \phi(x, e)$, $x \mapsto \phi(e, x)$ respectively. We see that $p(e) = q(e) = \phi(e^2) = e$. Then the map

$$\begin{aligned} \psi : \mathbb{R}^n \times \mathbb{R}^n &\longrightarrow \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \\ (x, y) &\mapsto (p^{-1}(x), q^{-1}(y)) \mapsto \phi(p^{-1}(x), q^{-1}(y)) \end{aligned}$$

maps (x, e) to $(p^{-1}(x), e)$ to x and similarly it sends (e, y) to y. For any a in the divisions algebra, the maps $(a, x) \mapsto e$ and $(x, a) \mapsto e$ are bijective, hence every element a has multiplicative inverse on both left and right.

The strategy here is that we can consider the sphere $S^{n-1} \subset \mathbb{R}^n$. By rescaling the multiplication map from $\mathbb{R}$ into the sphere we get a continuous multiplication map on the sphere, defined by

$$\begin{aligned} \mu : S^{n-1} &\times S^{n-1} \\ (x, y) &\mapsto \frac{xy}{|xy|} \end{aligned}$$

The bulk of this chapter is to show that such a map does not exists on the sphere if $n \neq 1, 2, 4, 8$. The strategy is that instead of showing directly that such a map does not exist, we pass it to an an induced homomorphism on the K rings and show that such a homomorphism does not exist. This is similar to showing that there is no deformation retract from the disk onto the circle.

We will first deduce some consequences of the Bott periodicity theorem

(i) We know that $\widetilde{K}(S^{2n-1}) = 0$ for n odd and S^2n. This follows from that fact that $\widetilde{K}(S^1) = 0$, $\widetilde{K}(S^2) = \mathbb{Z}$ and the isomorphism $\widetilde{K}(X) = \widetilde{K}(S^2X)$. The isomorphism is the map $a \mapsto a * (H-1)$. Hence we see that a generator of $\widetilde{K}(S^{2n})$ is the k-fold external product $(H-1) * \cdots * (H-1)$.

(ii) $\widetilde{K}(S^{2k}) \otimes \widetilde{K}(X) = \widetilde{K}(S^{2k}X) = \widetilde{K}(S^{2k} \wedge X)$ by iterating the bott periodicity theorem

(iii) Recall that

$$K(S^{2k}) \otimes K(X) = (\widetilde{K}(S^{2k}) \otimes \widetilde{K}(X)) \oplus \widetilde{K}(S^{2k}) \oplus \widetilde{K}(X) \oplus \mathbb{Z}$$
$$K(S^{2k} \times X) = \widetilde{K}(S^{2k} \wedge X) \oplus \widetilde{K}(S^{2k}) \oplus \widetilde{K}(X) \oplus \mathbb{Z}.$$

From (ii) we see that $K(S^{2k} \times X) = K(S^{2k})$, and the isomorphism is given by the external product.

Proposition 5.1.1. *$\mathbb{R}^n$ is not a division algebra for any odd $n > 1$.*

We will outline the proof here. To show this proposition it suffices to show that $\mu : S^{n-1} \times S^{n-1} \to S^{n-1}$ for any odd $n > 1$. In this case $n-1$ is even so we can replace S^{n-1} with S^{2n}. So suppose that $\mu : S^{2n} \times S^{2n} \to S^{2n}$ is a continuous map with a two sided identity. We then get a induced homomorphism $\mu^* : K(S^{2n}) \to K(S^{2n} \times S^{2n})$ which by eariler proposition, a homomorphism between the following rings

$$\mu^* : \frac{\mathbb{Z}[\gamma]}{\langle \gamma^2 \rangle} \longrightarrow \frac{\mathbb{Z}[\alpha, \beta]}{\langle \alpha^2, \beta^2 \rangle}.$$

It turns out that $\mu^*(\gamma) = \alpha + \beta + k\alpha\beta$ for some $k \in \mathbb{Z}$, which is a contradiction, as $\mu^*(\gamma^2) = (\alpha + \beta + k\alpha\beta)^2 = 2\alpha\beta$ since $\alpha^2 = \beta^2 = 0$. But this is a contradiction as $\mu^*(\gamma^2) = 0 \neq 2\alpha\beta$.

This is a good example of turning a topological problem into an algebraic problem. Often it is easier to show the non-existence of homomorphisms since they must preserve structure, which is easier to show the non-existence of continuous maps.

The previous theorem takes care of $\mathbb{R}^n$ for odd n. We cannot use the above idea because the crux of the proof above was the isomorphism

$$K(S^{2n}) \otimes K(S^{2n}) \cong K(S^{2n} \times S^{2n})$$

which allowed us to compute the ring $K(S^{2n} \times S^{2n})$. We do not have a similar isomorphism theorem for $K(S^{2n-1} \times S^{2n-1})$ and K groups in general are very hard to compute. What we do now is to associate any continuous map $g : S^{n-1} \times S^{n-1}$ to a map $\hat{g} : S^{2n-1} \to S^n$. In the end we will construct a map between spheres of even dimensions and get a contradiction.

We first regard S^{2n-1} as $\partial(D^n \times D^n)$. To see this, notice that D^n is homeomorphic to I^n, where I is the closed unit interval. Then $D^n \times D^n$ is homeomorphic to I^{2n} which is D^{2n} and so S^{2n-1} is precisely the boundary of D^{2n}. SO now we write $S^{2n-1} = \partial(D^n \times D^n) = \partial D^n \times D^n \cup D^n \times \partial D^n$ and $S^n = D^n_+ \cup D^n_-$. We then define

$$\hat{g}_+ : \partial D^n \times D^n \to D^n_+ \qquad\qquad \hat{g}_- : \partial D^n \times D^n \to D^n_-$$
$$(x,y) \mapsto |y| g\left(x, \frac{y}{|y|}\right) \qquad\qquad (x,y) \mapsto |x| g\left(\frac{x}{|x|}, y\right)$$

this gives us a map $\hat{g} : S^{2n-1} \to S^n$.

We now define a Hopf invariant associated to $\hat{g}$. We now replace n with $2n$ since we are interested in spheres of odd dimensions. Given a map $f : S^{4n-1} \to S^{2n}$, let C_f be S^{2n} with a cell e^{4n} attached by f. The quotient C_f/S^{2n} is then S^{4n}, and since $\widetilde{K}(S^{4n}) = \widetilde{K}(S^{2n}) = 0$, the exact sequence becomes a short exact sequence

$$0 \to \widetilde{K}(S^{4n}) \xrightarrow{q^*} \widetilde{K}(C_f) \xrightarrow{i^*} \widetilde{K}(S^{2n}) \to 0$$

Suppose that $q^*((H-1)^{2n}) = \alpha \in \widetilde{K}(C_f)$ and $i^*(\beta) = (H-1)^n$. Then $i^*(\beta^2) = i^*(\beta)^2 = ((H-1)^n)^2 = 0 \implies \beta^2 \in \ker i^* = \operatorname{Im} q^*$ by exactness. Thus we see that $\beta^2 = h\alpha$ for some integer h. To see that h is well defined, we need to show that for any other β' such that $i^*(\beta') = (H-1)^n$. Then since $i^*(\beta - \beta') = 0 \implies \beta - \beta' \in \ker i^* = \operatorname{Im} q^* \implies \beta' = \beta + m\alpha$. It follows that $(\beta + m\alpha)^2 = \beta^2 + 2m\alpha\beta$ since $\alpha^2 = 0$. Since $\alpha \in \operatorname{Im} q^* = \ker i^*$, we see that $i^*(\alpha) = 0$ and so does $\alpha\beta$. Hence $\alpha\beta = k\alpha$ for some k. Then $k\alpha\beta = \alpha\beta^2 = h\alpha^2 = 0$. Thus $k\alpha\beta = 0$ which implies $\alpha\beta = 0$ since $\alpha\beta$ belongs to the cyclic subgroup of $\widetilde{K}(C_f)$ generated by the image of α.

A key theorem that solves the problem is

Lemma 5.1.2. *[4] If $g : S^{2n-1} \times S^{2n-1} \to S^{2n-1}$ is a H-space multiplication, then the associated map $\hat{g} : S^{4n-1} \to S^{2n}$ has Hopf invariant ± 1*

Theorem 5.1.3. *There exists a map $f : S^{4n-1} \to S^{2n}$ of Hopf invariant ± 1 only when $n = 1, 2$ or 4.*

To prove this theorem we will need certain homomorphisms on $K(X)$. The ones we need are precisely the Adam's operations defined in Chapter 3.

We will now show that ψ^k restricts to a ring homomophism on the subring $\widetilde{K}(X)$. To see that recall that $\widetilde{K}(X)$ is the kernel of the homomorphism $i^* : K(X) \to K(x_0)$ induced by the inclusion map $\{x_0\} \hookrightarrow X$. By the naturality of ψ^k,

$$\alpha \in \widetilde{K}(X) \implies i^*(\alpha) = 0 \implies \psi^k i^*(\alpha) = 0 \implies i^*\psi^k(\alpha) = 0 \implies \psi^k(\alpha) \in \widetilde{K}(X)$$

Also, for the external product, we have

$$\begin{aligned}\psi^k(\alpha * \beta) &= \psi^k(p_1^*(\alpha)p_2^*(\beta)) = \psi^k(p_1^*(\alpha))\psi^k(p_2^*(\beta)) \\ &= p_1^*\psi^k(\alpha)p_2^*\psi^k(\beta) = \psi^k(\alpha) * \psi^k(\beta).\end{aligned}$$

We can now determine the map $\psi^k : \widetilde{K}(X) \to \widetilde{K}(X)$ explicitly.

Proposition 5.1.4. *$\psi^k : \widetilde{K}(S^{2n}) \to \widetilde{K}(S^{2n})$ is multiplication by k^n.*

Proof. We first consider the case when $n = 1$. Since ψ^k is additive and $\widetilde{K}(S^2)$ is generated by $\alpha = H - 1$, it suffices to show $\psi^k(\alpha) = k^n(\alpha)$. Now we have

$$\begin{aligned} \psi^k(\alpha) &= \psi^k(H) - 1 = H^k - 1 = (1+\alpha)^k - 1 \\ &= 1 + \alpha - 1 && \text{since } a^i = 0 \text{ for all } i > 1 \\ &= k\alpha \end{aligned}$$

For $n > 1$ we use the external product $\widetilde{K}(S^2) \otimes \widetilde{K}(S^{2n-2}) \to \widetilde{K}(S^2 S^{2n-1}) = \widetilde{K}(S^{2n})$ and prove by induction. Suppose that $\psi^k : \widetilde{K}(S^{2n-2}) \to \widetilde{K}(S^{2n-2})$ is multiplication by k^{n-1}. Then $\phi^k(\alpha * \beta) = \psi^k(\alpha) * \psi^k(\beta) = k\alpha * k^{n-1} = k^n(\alpha * \beta)$ and this completes the proof. □

Proof. (Adams theorem) Recall that we have the exact sequence

$$0 \longrightarrow \widetilde{K}(S^{4n}) \xrightarrow{q^*} \widetilde{K}(C_f) \xrightarrow{i^*} \widetilde{K}(S^{2n}) \longrightarrow 0.$$

Suppose again that $q^*(H-1)^{2n} = \alpha$ and $i^*(\beta) = (H-1)^n$. By naturality we have

$$\psi^k(\alpha) = \psi^k(q^*(H-1)^{2n}) = q^*(\psi^k((H-1)^{2n})) = q^*(k^{2n}(H-1)^{2n}) = k^{2n}\alpha.$$

Also,

$$i^*\psi^k(\beta) = \psi^k(i^*(\beta)) = k^n(H-1)^n = i^*(k^n\beta)$$

so we see that $\psi^k(\beta) - k^n\beta \in \ker i^* = \operatorname{Im} q^*$. Hence $\psi^k(\beta) = k^n\beta + \mu_k\alpha$ for some $\mu_k \in \mathbb{Z}$ since $\operatorname{Im} q^*$ is generated by α. Hence

$$\psi^k\psi^l(\beta) = \psi^k(l^n\beta + \mu_l\alpha) = l^n(k^n\beta + \mu_k\alpha) + k^{2n}\mu_l\alpha = l^nk^n\beta + (k^{2n}\mu_l + l^n\mu_k)\alpha$$

But $\psi^k\psi^l(\beta) = \psi^l\psi^k(\beta)$, so we see that

$$l^nk^n\beta + (k^{2n}\mu_l + l^n\mu_k)\alpha = l^nk^n\beta + (l^{2n}\mu_k + k^n\mu_l)\alpha$$

which implies that $k^n(k^n - 1)\mu_l = l^n(l^n - 1)\mu_k$. Now, since $\psi^2(\beta) \equiv \beta^2 = h\alpha \mod 2$, from $\psi^2(\beta) = 2^n\beta + \mu_2\alpha$ we see that $\mu_2 \equiv h \mod 2$ so μ_2 is odd as $h = \pm 1$. From above we see that $2^n(2^n - 1)\mu_3 = 3^n(3^n - 1)\mu_2$ and since 3^n and μ_2 are odd, we see that $2^n | 3^n - 1$. The proof will be completed by the following lemma. □

Lemma 5.1.5. *[4] If 2^n divides $3^n - 1$, then $n = 1, 2$, or 4.*

In some sense the modulo property of the Adams operations turns these topological questions into number theory problems, which is often easier.

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