AN INTRODUCTION TO A_∞-ALGEBRAS

Nathan Menzies

Supervisor: Dr. Daniel Chan

School of Mathematics,

The University of New South Wales.

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Introduction

The notion of an A_∞-algebra was first introduced by J. D. Stasheff in 1963, in a pair of papers titled 'Homotopy Associativity of H-Spaces'. It took a number of years for A_∞-algebras to be investigated outside the area of homotopy theory, however, this structure has recently been found to have importance in areas such as algebra, geometry and mathematical physics [8]. According to both B. Keller [8] and Palmieri et al [5], an important factor in the development of the theory of A_∞-algebras was a talk [11] given by M. Kontsevich in 1994 which prompted investigation of these structures outside of the area of homotopy theory.

At the beginning of the year, I started with the goal simply to understand something of A_∞-algebras; as a generalisation of the more familiar concept of a differential graded algebra and as an algebra with a multiplication map that is not associative, A_∞-algebras have an intriguing structure as well as applications to other areas, notably in ring theory and homological algebra.

The aim of this thesis is to introduce the definition and some basic properties of A_∞-algebras, along with providing an intuitive understanding of the defining identities. The two main results included are a method of constructing A_∞-algebras and an alternative interpretation of the higher homotopies.

The first chapter provides some necessary background concepts and terminology from homological algebra. In chapter 2 we will build up to the definition of an A_∞-algebra including the Stasheff identities which are the defining feature of an A_∞-algebra. We will also discuss the topological motivation that was originally explored by Stasheff. In chapter 3 we will discuss and prove Merkulov's construction of an A_∞-algebra which is a particular case of an important property of A_∞-algebras. This section is linked with chapter 4 where we will discuss the different structures of the Ext-algebra. This will also provide us with one of the connections between A_∞-algebras and ring theory. Lastly, in Chapter 5 we will formulate an alternative way to think about the Stasheff identities.

Contents

Chapter 1

Basic Homological Algebra

The idea of a homology initially arose in the study of algebraic topology, though the concepts and methods devised in this context have long since been generalised and are now studied in the abstract algebra setting.

In this chapter, we will begin to discuss some of the basic ideas from homological algebra which will be helpful in attempting to understand A_∞-algebras and which will also be useful in constructing examples of A_∞-algebras. Firstly, we will recall the following definitions.

Definition 1.1. Let R be a ring. A **module over** R, or R**-module**, is an abelian group M, with a map $\mu : R \times M \to M$ sending $(r, m) \mapsto rm := \mu(r, m)$ (say that R acts on M) such that

1. $1m = m$ for all $m \in M$ where $1 \in R$ is the multiplicative identity.
2. $r(m_1 + m_2) = rm_1 + rm_2$ for all $r \in R$, $m_1, m_2 \in M$.
3. $(r_1 + r_2)m = r_1m + r_2m$ for all $r_1, r_2 \in R$, $m \in M$.
4. $(r_1r_2)(m) = r_1(r_2m)$ for all $r_1, r_2 \in R$,$m \in M$.

Example 1.2. A ring R is itself an R module, with the action given by ring multiplication.

Definition 1.3. Let R be a ring and let M,N be R-modules. An **R-module homomorphism** is a group homomorphism $\varphi : M \to N$ such that

$$\varphi(rm) = r\varphi(m) \quad \text{for all } r \in R, m \in M.$$

A module is essentially a generalisation of a vector space, in the sense that if R is a field then an R-module is a vector space and R-module homomorphisms are linear maps.

1.1 Chain Complexes

The basis of homological algebra lies in studying sequences of modules and module homomorphisms of a particular form. As previously mentioned, the motivation for this comes from topology, which will be discussed as our first example.

Definition 1.4. Let R be a ring and let $\{M_i\}_{i\in\mathbb{Z}}$ be a sequence of modules over R, $\{\partial_i\}_{i\in\mathbb{Z}}$ a sequence of module homomorphisms $\partial_i : M_i \to M_{i+1}$. Then this sequence

$$M_\bullet : \ldots \longrightarrow M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \longrightarrow \ldots \tag{1.1}$$

is a **chain complex** (or **complex**) if $\partial_{i-1} \circ \partial_i = 0$ for all i.

Remark 1.5. The sequence of morphisms $\{\partial_i\}_{i\in\mathbb{Z}}$ are often denoted simply by ∂, where it is understood that ∂ acts on the different modules in different ways. Correspondingly, the identity $\partial_{i-1} \circ \partial_i = 0$ is often written simply as $\partial^2 = 0$.

We will now consider the quintessential example of a complex, that is, a simplicial complex. We will take a bit of time to develop this example as the structure introduced will be of use in the motivating example of an A_∞-algebra. First, we will need some definitions. Let $N \in \mathbb{N}$ and fix the Euclidean space $\mathbb{R}^N$.

Definition 1.6. Consider the set of points $\{a_0, a_1, \ldots, a_n\} \subseteq \mathbb{R}^N$. Suppose that the vectors $\{a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0\}$ form a linearly independent set over $\mathbb{R}$, in which case we say that $a_0, a_1, \ldots, a_n$ are **geometrically independent**. We then define the n-**simplex** $a_0a_1 \ldots a_n$ as the convex hull of $\{a_0, a_1, \ldots, a_n\}$. Explicitly,

$$a_0a_1 \ldots a_n = \big\{\sum_{i=0}^{n} \lambda_i a_i | \lambda_i \geq 0, \lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\big\} \tag{1.2}$$

If the convex hull of the points $b_0, b_1, \ldots, b_m$ is equal to the simplex $a_0a_1 \ldots a_n$, then we say that $b_0, b_1, \ldots, b_m$ **span** $a_0a_1 \ldots a_n$.

Example 1.7. A 1-simplex is a line segment, a 2-simplex is a triangle (including the interior), a 3-simplex is a tetrahedron etc.

Example 1.8. Consider the set $S = \{, e_1, e_2, \ldots, e_{n+1}\} \in \mathbb{R}^{n+1}$, where e_i is the i-th standard basis vector for $\mathbb{R}^{n+1}$.

Then S is geometrically independent and $e_1e_2 \ldots e_{n+1}$ is an n-simplex in $\mathbb{R}^{n+1}$. We call it the **standard n-simplex** in $\mathbb{R}^{n+1}$ and denote it by Δ^n.

$\Delta^n = \{(\lambda_0, \lambda_1, \ldots, \lambda_n) \in \mathbb{R}^{n+1} | \lambda_i \geq 0, \lambda_0 + \lambda_1 + \cdots + \lambda_n = 1\}$.

Definition 1.9. A **face** of a simplex $a_0a_1 \ldots a_n$ is a simplex $a'_0a'_1 \ldots a'_n$, where $\{a'_0, a'_1, \ldots, a'_n\}$ is a subset of $\{a_0, a_1, \ldots, a_n\}$. If $\{a'_0, a'_1, \ldots, a'_n\}$ is a proper subset of $\{a_0, a_1, \ldots, a_n\}$ then $a'_0a'_1 \ldots a'_n$ is a **proper face**.

Example 1.10. The proper faces of a 2-simplex $a_0a_1a_2$ are the 1-simplices a_0a_1, a_0a_2, a_1a_2 and the 0-simplices a_0, a_1, a_2.

Definition 1.11. A **simplicial complex** K in $\mathbb{R}$ is a set of simplices such that

1. If σ is a simplex in K, then so are all of the faces of σ
2. If $\sigma, \tau \in K$ are simplices, then $\sigma \cap \tau$ is a face of both σ and τ.

If K is a finite set then we say that K is a **finite simplicial complex**.

For $p \in \mathbb{N}$, the p**-skeleton** of K, $K^{(p)}$ is the subcomplex of K consisting of the m-simplices of K where $m \leq p$. In particular, $K^{(0)}$ is the set of vertices of K.

Example 1.12. If we consider the surface of an octahedron as the union of 8 triangles (2-simplices) with its 12 edges (1-simplices) and 6 vertices (0-simplices) then this is a simplicial complex.

Example 1.13. The union of an n-simplex with all of its faces is a simplicial complex, as is the disjoint union of any 2 such complexes.

Definition 1.14. Let K be a finite simplicial complex in $\mathbb{R}^N$. The **polytope** $|K|$ of K is the union of the simplices of K with the topology induced from $\mathbb{R}^N$.

Lemma 1.15. *Suppose* $\{a_0, a_1, \ldots, a_n\}$ *is a geometrically independent set. Then any element of the simplex* $a_0a_1 \ldots a_n$ *can be written uniquely in the form* $\sum_{i=0}^{n} \lambda_i a_i$ *where* $\lambda_i \geq 0$ *for all* i *and* $\lambda_1 + \cdots + \lambda_n = 1$

Proof. Since $a_0a_1\dots a_n = \left\{\sum_{i=0}^{n}\lambda_i a_i | \lambda_i \geq 0, \lambda_1+\lambda_2+\cdots+\lambda_n = 1\right\}$, we only need to show uniqueness. As $\lambda_1+\lambda_2+\cdots+\lambda_n = 1$,

$$\begin{aligned}\sum_{i=0}^{n}\lambda_i a_i &= \left(1-\sum_{i=1}^{n}\lambda_i\right)a_0+\sum_{i=1}^{n}\lambda_i a_i\\ &= a_0+\sum_{i=1}^{n}\lambda_i\left(a_i-a_0\right).\end{aligned}$$

Then $\sum_{i=0}^{n}\lambda_i a_i = \sum_{i=0}^{n}\mu_i a_i$ implies that $\sum_{i=1}^{n}\left(\lambda_i-\mu_i\right)\left(a_i-a_0\right) = 0$. Then $\lambda_i = \mu_i$ as $a_0, a_1, \dots, a_n$ are geometrically independent. Hence, the expression is unique. □

Lemma 1.16. *Let K and L be simplicial complexes and let $f : K^{(0)} \to L^{(0)}$ be a map such that, if $a_0, a_1, \dots, a_n$ span a simplex in K, then $f(a_0), f(a_1), \dots, f(a_n)$ spans a simplex in L. Then f can be extended to a continuous function $\tilde{f} : |K| \to |L|$*

Proof. Essentially, we extend f linearly. For any set $\{a_0, a_1, \dots, a_n\} \subseteq \mathbb{R}^N$, fix $\Delta^n \in \mathbb{R}^{n+1}$ and consider the function

$$\beta_{a_0\dots a_n} : \Delta^n \to \mathbb{R}^N$$

$$(\lambda_0, \lambda_1, \dots, \lambda_n) \mapsto \lambda_0 a_0 + \cdots + \lambda_n a_n.$$

This is clearly a continuous map onto the convex hull of $a_0a_1\dots a_n$. If $a_0a_1\dots a_n$ is a simplex then, by Lemma 1.15, $\beta_{a_0\dots a_n}$ has an inverse which is also continuous. We then define $\tilde{f}$ on each simplex $a_0\dots a_n$, and hence on all of $|K|$ by

$$\tilde{f} : a_0a_1\dots a_n \xrightarrow{\beta^{-1}_{a_0\dots a_n}} \Delta^{n+1} \xrightarrow{\beta_{f(a_0)\dots f(a_n)}} f(a_0)\dots f(a_n)$$

which is continuous as it is the composition of continuous functions. $\square$

Corollary 1.17. *Let $|K|, |L|$ be simplicial complexes. Then, suppose there is a bijection $f : K^{(0)} \to L^{(0)}$ such that $a_0, a_1, \ldots, a_n$ span an m-simplex in K if and only if $f(a_0), f(a_1), \ldots, f(a_n)$ span an m-simplex in L. Then $|K|$ and $|L|$ are homeomorphic.*

Proof. In the construction of $\tilde{f}$ in the above lemma, $a_0a_1 \ldots a_n$ is an n-simplex, hence $f(a_0)f(a_1) \ldots f(a_n)$ is also an n-simplex. Therefore, $\beta_{f(a_0) \ldots f(a_n)}$ has a continuous inverse and hence so does $\tilde{f}$. $\square$

From the way in which we have defined simplices, an n-simplex $a_0a_1 \ldots a_n$ is independent of the order of the vertices. We will now want to place a group structure on the set of simplices, and the first part of this will be to define an orientation for a simplex.

Definition 1.18. Given a set of vertices $a_0, \ldots, a_n$ we define two orderings of $a_0, \ldots, a_n$ to be equivalent if they differ by an even permutation. An **orientation** is the equivalence class of an ordering. An **oriented n-simplex** is an n-simplex $a_0 \ldots a_n$ with an orientation of its vertices.

If the orientation is given by $a_0 < a_1 < \cdots < a_n$ then we denote the oriented n-simplex by $[a_0a_1 \ldots a_n]$. Also, as the equivalence relation implies that there are two different orientations for an n-simplex, we write $[a_1a_0a_2 \ldots a_n] = -[a_0a_1 \ldots a_n]$.

Note that the term oriented n-simplex refers to the simplex whose ordering is equivalent to the given orientation. That is, given the ordering $a_0 < a_1 < \cdots < a_n$, the simplex $[a_1 a_0 a_2 \ldots a_n]$ is not an oriented n-simplex.

An orientation can be thought of as giving a direction to the elements of the n-simplex.

Example 1.19. The oriented 1-simplex $[a_0 a_1]$ is the directed line segment from a_0 to a_1 while $[a_1 a_0] = -[a_0 a_1]$ is the directed line segment from a_1 to a_0. So $[a_1 a_0]$ is in some sense the inverse of $[a_0 a_1]$ as it contains the same points, but is traversed in the opposite direction.

Example 1.20. Similarly, let $a_0, a_1, a_2 \in \mathbb{R}^2$ be the points $a_0 = 0, a_1 = e_1, a_2 = e_2$ where $\{e_i\}_{i=1}^2$ are the standard basis vectors in $\mathbb{R}^2$. Then the oriented 2-simplex $[a_0 a_1 a_2]$ is the triangle (with interior) whose vertices are the origin, e_1 and e_2, and which is traversed in an anti-clockwise direction.

Given a simplicial complex $|K|$, if we define a total ordering on the vertex set $K^{(0)}$ by $<$, then this will induce an orientation on each of the simplices.

Definition 1.21. The **free abelian group** F with basis $\{s_i\}_{i \in I}$ is the direct sum

$$F = \bigoplus_{i \in I} \mathbb{Z} s_i \tag{1.3}$$

Example 1.22. The free abelian group F with basis $\{x, y\}$ is $\mathbb{Z}x \oplus \mathbb{Z}y \simeq \mathbb{Z} \oplus \mathbb{Z}$

Definition 1.23. Let $|K|$ be a simplicial complex with a total ordering of its vertices. The **group of oriented** p**-chains**, $C_p(K)$ is the free abelian group whose basis is the set of oriented p-simplices. We define $C_{-1}(K) = 0$.

Example 1.24. Given the orientation $a_0 < a_1 < \cdots < a_n$, the non-oriented n-simplex $[a_1 a_0 a_2 \ldots a_n] = -1[a_0 a_1 \ldots a_n]$ is an element of $C_n(K)$.

Proposition 1.25. *Define a map $\partial_p : C_p(K) \to C_{p-1}(K)$ on the basis of $C_p(K)$ by*

$$\partial_p([a_0 a_1 \ldots a_p]) = \sum_{i=0}^{p} (-1)^i [a_0 \ldots \hat{a}_i \ldots a_p] \tag{1.4}$$

where the notation $\hat{a}_i$ means that the vertex a_i is omitted. We then extend ∂_p linearly to all of $C_p(K)$.

Then ∂_p is a group homomorphism, which we call the ***boundary operator****.*

Proof. Since $C_p(K)$ is a free abelian group, we only need to check that ∂_p is the same on n-simplices with the same vertices and equivalent orderings, and that ∂_p changes sign on n-simplices with inequivalent orderings. That is,

1. $\partial_p([a_0 \ldots a_n]) = \partial_p([a'_0 \ldots a'_n])$ if $a'_0 \ldots a'_n$ is an even permutation of $a_0 \ldots a_n$.
2. $\partial_p([a_0 \ldots a_n]) = -\partial_p([a'_0 \ldots a'_n])$ if $a'_0 \ldots a'_n$ is an odd permutation of $a_0 \ldots a_n$.

Note that the set of all permutations of $a_0 \ldots a_n$ is generated by transpositions of the form $(j \quad j+1)$ so we only need to check that

$$\partial_p([a_0 \ldots a_j a_{j+1} \ldots a_n]) = -\partial_p([a_0 \ldots a_{j-1} a_{j+1} a_j a_{j+2} \ldots a_n]). \tag{1.5}$$

Now, consider the i-th term in the sum (1.4) for both sides of the above expression (1.5). Then for $i < j$ we get

$$RHS = -(-1)^i[a_0 \dots \hat{a}_i \dots a_{j-1}a_{j+1}a_ja_{j+2} \dots a_p]$$

$$= (-1)^i[a_0 \dots \hat{a}_i \dots a_p]$$

$$= LHS$$

and similarly for $i > j + 1$.

If we now take the sum of the j-th and $(j + 1)$-th terms we get

$$LHS = (-1)^j([a_0 \dots \hat{a}_j \dots a_p] - [a_0 \dots \hat{a}_{j+1} \dots a_p])$$

$$RHS = -(-1)^j([a_0 \dots a_{j-1}\hat{a}_{j+1}a_ja_{j+2} \dots a_p] - [a_0 \dots a_{j-1}a_{j+1}\hat{a}_ja_{j+2} \dots a_p])$$

$$= (-1)^j(-[a_0 \dots \hat{a}_{j+1} \dots a_p] + [a_0 \dots \hat{a}_j \dots a_p])$$

$$= LHS$$

So $\partial_p(K)$ is well-defined and is a group homomorphism. □

The boundary operator acting on an oriented n-simplex $[a_0a_1 \dots a_n]$ gives the faces of $(a_0a_1 \dots a_n)$ with orientation equivalent to the one on $[a_0a_1 \dots a_n]$.

Example 1.26. Let $[a_0, a_1, a_2]$ be an oriented 2-simplex. Then,

$$\partial[a_0a_1a_2] = [a_1a_2] - [a_0a_2] + [a_0a_1]$$

$$= [a_0a_1] + [a_1a_2] + [a_2, a_1].$$

Note that if $[a_0, a_1, a_2]$ has an anti-clockwise orientation, then the line segments $[a_0a_1], [a_1a_2]$ and $[a_2, a_1]$ are also traversed in an anti-clockwise direction.

Following the convention mentioned in Remark 1.5 we denote the sequence of morphisms $\{\partial_p\}_{p=0}^N$ by ∂. We now have the following lemma.

Lemma 1.27. *The boundary operator ∂ on a simplicial complex K satisfies $\partial^2 = 0$.*

Proof. Since ∂ is a group homomorphism on a free abelian group, we only need to check that $\partial^2 = 0$ on the basis of $C_p(K)$ for all p. That is, $\partial^2([a_0 \dots a_p]) = 0$ for all oriented simplices $[a_0 \dots a_p]$.

$$\begin{aligned}
\partial(\partial([a_0 \dots a_p])) &= \partial\left(\sum_{i=0}^{p}(-1)^i[a_0 \dots \hat{a}_i \dots a_p]\right) \\
&= \sum_{i=0}^{p}(-1)^i\left(\sum_{j\leq i}(-1)^j[a_0 \dots \hat{a}_j \dots \hat{a}_i \dots a_p]\right. \\
&\qquad\qquad\qquad \left. + \sum_{j\geq i}(-1)^{j-1}[a_0 \dots \hat{a}_i \dots \hat{a}_j \dots a_p]\right) \\
&= 0.
\end{aligned}$$

□

We have in fact now proven the following, thus completing our first example of a chain complex.

Corollary 1.28. *Let K be a simplicial complex in $\mathbb{R}^N$. The sequence of abelian groups $\{C_p(K)\}_{p=0}^N$ and group homomorphisms $\partial := \{\partial_p\}_{p=0}^N$ form a chain complex $C_\bullet = C_\bullet(K)$ over the ring $\mathbb{Z}$.*

$$C_\bullet(K) : \dots \longrightarrow C_{p+1}(K) \xrightarrow{\partial_{p+1}} C_p(K) \xrightarrow{\partial_p} C_{p-1}(K) \longrightarrow \dots \longrightarrow C_0(K) \longrightarrow 0$$

We would like to generalise the concept of a simplicial complex and the action of the boundary operator to a more general setting and we will now do this by considering maps from simplices to a topological space X. Note first, that by Corollary 1.17, we only need to consider maps from the standard n-simplex Δ^n.

Definition 1.29. A **singular n-simplex** is a continuous map $T : \Delta^n \to X$.

Remark 1.30. The concept of the orientation of a singular n-simplex is induced by the orientation on Δ^n.

Example 1.31. A singular 0-simplex is a map from the origin to a point in X. A singular 1-simplex will in general be a map from the interval $[0, 1]$ to a segment of some one-dimensional curve in X, though it could also be the constant map $[0, 1] \mapsto x_0$ for some point $x_0 \in X$.

Let $S_n(X)$ be the free abelian group generated by the set of all singular n-simplices $T : \Delta^n \to X$. Corollary 1.17 gives an isomorphism $\tilde{f}_i : \Delta^{n-1} \xrightarrow{\sim} (-1)^i[e_0 \dots \hat{e}_i \dots e_n]$ for each oriented $(n-1)$-simplex in $\partial\Delta^n$.

Proposition 1.32. *With the notation above, let d_i be the map given by the composition*

$$d_i : \Delta^{n-1} \xrightarrow{\tilde{f}_i} \partial\Delta^n \hookrightarrow \Delta^n \xrightarrow{T} X$$

and define the action of ∂ on the basis of $S_n(X)$ by

$$\partial T := \sum_{i=0}^{n} d_i. \tag{1.6}$$

Then $\partial : S_n(X) \to S_{n-1}(X)$ is a group homomorphism satisfying $\partial^2 = 0$.

Proof. The d_i are continuous so $\partial T \in S_{n-1}(X)$. That $\partial : S_n(X) \to S_{n-1}(X)$ is a group homomorphism follows immediately since $S_n(X)$ is a free abelian group and we have defined ∂ on its basis.

The proof that $\partial^2 = 0$ is similar to the proof of Proposition 1.25 and is left as an exercise. □

Remark 1.33. Since these two examples are the most typical examples of chain complexes, we will often use the term boundary operator to refer to the homomorphism ∂ in the case of a general chain complex.

Since the definition of a chain complex was given abstractly in terms of modules and module homomorphisms we are clearly interested in constructing chain complexes using modules arising more directly from ring theory.

Example 1.34. Let $\mathbb{K}$ be a field and $\mathbb{K}[x, y]$ be the ring of polynomials in x and y over $\mathbb{K}$. Let $M_2 = M_0 = \mathbb{K}[x, y]$ and $M_1 = \mathbb{K}[x, y] \oplus \mathbb{K}[x, y]$ and define $\{\partial_i\}_{i=1}^2$ by

$$\partial_2 : M_2 \to M_1 \qquad\qquad \partial_1 : M_1 \to M_0$$

$$p(x, y) \mapsto \begin{pmatrix} yp(x.y) \\ -xp(x, y) \end{pmatrix} \qquad\qquad \begin{pmatrix} p(x, y) \\ q(x, y) \end{pmatrix} \mapsto xp(x, y) + yq(x, y)$$

Then,

$$\begin{aligned} (\partial_1 \circ \partial_2)(p(x, y)) &= \partial_1 \begin{pmatrix} yp(x.y) \\ -xp(x, y) \end{pmatrix} \\ &= x(yp(x.y)) + y(-xp(x, y)) \\ &= 0. \end{aligned}$$

So

$$M_\bullet : 0 \to \mathbb{K}[x,y] \xrightarrow{\partial_2} \begin{matrix} \mathbb{K}[x,y] \\ \oplus \\ \mathbb{K}[x,y] \end{matrix} \xrightarrow{\partial_1} \mathbb{K}[x,y] \to 0 \tag{1.7}$$

is a chain complex.

1.2 Homology

The identity $\partial_{i-1} \circ \partial_i = 0$ leads to the result that, as submodules, $im(\partial_i) \subseteq ker(\partial_{i-1})$. We are therefore able to take quotients of these submodules to try to explore some of the properties of a complex.

Definition 1.35. Let $M_\bullet$ be a chain complex as in (1.1). The i^{th} **homology group**, denoted $H_i(M_\bullet)$, of $M_\bullet$ is defined to be the quotient group $ker(\partial_i)/im(\partial_{i+1})$. We say that $M_\bullet$ is **exact** at M_i if $H_i(M_\bullet) = 0$ or, equivalently, $im(\partial_{i+1}) = ker(\partial_i)$, and that $M_\bullet$ is **exact** if $H_i = 0$ for all i.

Remark 1.36. Motivated by the terminology of the simplicial complex, we will often refer to elements of the image of ∂ as **boundaries** and write $\text{im}(\partial_{i+1}) = B_i$. Elements of the kernel of ∂ are called **cycles** and we write $\ker(\partial_i) = Z_i$, so $H_i = Z_i/B_i$.

Example 1.37. Let $M_\bullet$ be a chain complex over R. The homology groups $H_i(M_\bullet)$ are in fact R-modules, and the sequence

$$H_\bullet(M_\bullet) : \cdots \to H_{i+1}(M_\bullet) \xrightarrow{\partial_{i+1}} H_i(M_\bullet) \xrightarrow{\partial_i} H_{i-1}(M_\bullet) \to \dots$$

is a chain complex. The complex of R-modules $H_\bullet(M_\bullet)$ is called the **homology** of $M_\bullet$. Note that $\text{im}(\partial_{i+1}) = 0_{H_i}$ and so ∂ is in fact the zero map on $H_\bullet(M_\bullet)$. This also implies that taking the homology a second time doesn't provide us with any new structure, that is, $H_\bullet\left(H_\bullet(M_\bullet)\right) \simeq H_\bullet(M_\bullet)$.

Remark 1.38. Although the definition of a chain complex was given in terms of sequences of modules with decreasing indices, it is clear that we could simply relabel the modules and obtain a sequence with increasing indices. A complex of the form

$$M^\bullet : \ldots \longrightarrow M^{i-1} \xrightarrow{\partial^{i-1}} M^i \xrightarrow{\partial^i} M^{i+1} \longrightarrow \ldots$$

is called a **cochain complex** and the group $H^i(M^\bullet) :=\ker(\partial^i)/\text{im}(\partial^{i-1})$ is called the i^{th} **cohomology group** of $M^\bullet$. A cochain complex has essentially all of the same properties as a chain complex. The differences between them are in the ways they are constructed; a cochain complex will often arise as a construction from a chain complex and will therefore usually have greater algebraic structure. There is usually be a more natural way of labelling the modules of a complex, for instance, the boundary operator makes intuitive sense as a morphism which decreases the indices in a simplicial complex. Most of the following propositions will be stated in terms of chain complexes, but can be immediately applied to cochain complexes. Note that in a cochain complex the indices are written as superscripts; we can identify M^i with M_{-i} to change between chain and cochain complexes.

Example 1.39. Let

$$M_\bullet : 0 \to \mathbb{K}[x,y] \xrightarrow{\partial_2} \begin{smallmatrix} \mathbb{K}[x,y] \\ \oplus \\ \mathbb{K}[x,y] \end{smallmatrix} \xrightarrow{\partial_1} \mathbb{K}[x,y] \to 0$$

be the complex (1.7) in example (1.34). The homology groups are $H_2(M_\bullet) = H_1(M_\bullet) = 0$ and $H_0(M_\bullet) = \mathbb{K}[x,y]/\langle xp(x,y) + yq(x,y)\rangle \simeq \mathbb{K}$. Define ∂_0 by

$$\begin{aligned} \partial_0 : \mathbb{K}[x,y] &\to \mathbb{K} \\ x &\mapsto 0 \\ y &\mapsto 0. \end{aligned}$$

Then the complex

$$\tilde{M}_\bullet : 0 \to \mathbb{K}[x,y] \xrightarrow{\partial_2} \begin{smallmatrix} \mathbb{K}[x,y] \\ \oplus \\ \mathbb{K}[x,y] \end{smallmatrix} \xrightarrow{\partial_1} \mathbb{K}[x,y] \xrightarrow{\partial_0} \mathbb{K} \to 0$$

is exact.

Definition 1.40. Let R be a ring and let A, B, C be R-modules. If the complex

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact, then we say this is a **short exact sequence**.

Short exact sequences form an important part of homological algebra. Many properties are stated in terms of short exact sequences which is usually a strong enough statement for them to hold for longer sequences. This is often done by splicing together short exact sequences, for example, see [4] Section 22. Note that with the notation as above, the exactness implies that f is injective, and that g is surjective.

The following lemma is a standard result in most treatments of homological algebra

Lemma 1.41. *The Short* 5 *Lemma Consider the following commutative diagram where the rows are short exact sequences.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \longrightarrow & 0. \end{array}$$

If α and γ are isomorphisms, then so is β.

Proof. Let $b \in \ker(\beta)$. Then $g'\beta(b) = 0$, and by commutativity of the diagram, $\gamma g(b) = 0$, so $g(b) = 0$ as γ is an isomorphism. By exactness at B, there is an a in A such that $b = f(a)$ and $f'\alpha(a) = \beta f(a) = \beta(b) = 0$. Now exactness at A' implies that f' is injective, so $f'\alpha(a) = 0$ implies that $\alpha(a) = 0$. Then $a = 0$ as α is an isomorphism. So $b = f(a) = 0$ so $\ker(\beta)$ is injective.

Now suppose that $b' \in B'$. Then $g'(b') \in C'$, so as γ is an isomorphism there is a $c \in C$ such that $g'(b') = \gamma(c)$. Exactness at C implies that g is surjective so there is a $b \in B$ such that $c = g(b)$ and then $g'(b') = \gamma g(b)$. Now, $g'(\beta(b) - b') = g'\beta(b) - g'(b') = \gamma g(b) - \gamma g(b) = 0$, so by exactness at B', there is an $a' \in A'$ such that $f'(a') = \beta(b) - b'$ and as α is an isomorphism, there is an $a \in A$ such that $a' = \alpha(a)$. By commutativity of the diagram, $\beta f(a) = f'\alpha(a) = f'(a') = \beta(b) - b'$. Then as β is a morphism, $\beta\left(b - f(a)\right) = b'$ so β is surjective. □

Whenever we define an algebraic object, one of the first things we are naturally interested in is morphisms between them. That is, maps between objects which in some way preserve their structure. The short 5 Lemma provides the motivation for the way in which we wish to preserve the structure of a complex.

Definition 1.42. Let $M_\bullet$,$M'_\bullet$ be chain complexes. A **chain map** $\psi : M_\bullet \to M'_\bullet$ is a sequence of maps $(\psi_i)_{i\in\mathbb{Z}}$ such that each $\psi_i : M_i \to M'_i$ is a module homomorphism and the following diagram commutes.

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M_{i+1} & \xrightarrow{\partial_{i+1}} & M_i & \xrightarrow{\partial_i} & M_{i-1} & \xrightarrow{\partial_{i-1}} & \cdots \\
 & & \downarrow{\scriptstyle \psi_{i+1}} & & \downarrow{\scriptstyle \psi_i} & & \downarrow{\scriptstyle \psi_{i-1}} & & \\
\cdots & \longrightarrow & M'_{i+1} & \xrightarrow{\partial'_{i+1}} & M'_i & \xrightarrow{\partial'_i} & M'_{i-1} & \xrightarrow{\partial'_{i-1}} & \cdots
\end{array}$$

That is, $\partial'_i \circ \psi_i = \psi_{i-1} \circ \partial_i$ for all i.

Proposition 1.43. *Let $M_\bullet$,$M'_\bullet$ be chain complexes. Then a chain map $\psi : M_\bullet \to M'_\bullet$ induces a map on homology, $\psi_* : H_i(M_\bullet) \to H_i(M'_\bullet)$ given by*

$$\psi_* = H_i(\psi) : H_i(M_\bullet) \to H_i(M'_\bullet)$$

$$a + im(\partial_{i+1}) \mapsto \psi_i(a) + im(\partial'_{i+1}).$$

Proof. For ψ_* to be well-defined we need to first show that if $a \in \ker(\partial_i)$ then $\psi_i(a) \in \ker(\partial'_i)$ so that $\psi_i(a) + \operatorname{im}(\partial'_{i+1}) \in H_i(M'_\bullet)$. Let $a \in \ker(\partial_i)$ so $\partial_i(a) = 0$. Then by the commutativity of the diagram, $\partial'_i\psi_i(a) = \psi_{i-1}\partial_i(a) = \psi_{i-1}(0) = 0$ since ψ_{i-1} is a module homomorphism, so $\psi_i(a) \in \ker(\partial'_i)$.

Secondly, we need to show that if $b \in \operatorname{im}(\partial_{i+1})$ then $\psi_i(b) \in \operatorname{im}(\partial'_{i+1})$ so that different coset representatives are sent to the same cosets in $H_i(M'_\bullet)$. Let $b \in \operatorname{im}(\partial_{i+1})$.

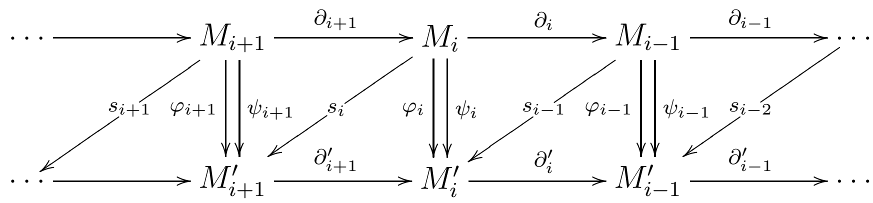
Then $b = \partial_{i+1}(c)$ for some $c \in M_{i+1}$, and by the commutativity of the diagram, $\psi_i(b) = \psi_i \partial_{i+1}(c) = \partial'_{i+1} \psi_{i+1}(c)$ so $\psi_i(b) \in \text{im}(\partial'_{i+1})$. $\square$

One of the strongest motivations for considering morphisms of any object is so that we can define isomorphisms between objects. However, when dealing with complexes and chain maps the emphasis is often on the homology of the complex, and so we would like a notion of isomorphism which takes into account the homology groups.

Definition 1.44. Let $M_\bullet, M'_\bullet$ be complexes and let $\psi : M_\bullet \to M'_\bullet$ be a chain map. ψ is a **quasi-isomorphism** if the induced map $H_i(\psi) : H_i(M_\bullet) \to H_i(M'_\bullet)$ is an isomorphism for all i.

In addition to using chain maps to define when we consider two complexes to be equivalent, we are also able to define an equivalence relation on the chain maps themselves.

Definition 1.45. Let $M_\bullet, M'_\bullet$ be two chain complexes and let $\psi, \varphi : M_\bullet \to M'_\bullet$ be chain maps between them. A **chain homotopy** between ψ and φ is a sequence of module homomorphisms $s_i : M_i \to M'_{i-1}$,



such that $\psi_i - \varphi_i = s_{i-1} \circ \partial_i + \partial'_{i+1} \circ s_i$. If such a sequence of homomorphisms exists, then we say that ψ and φ are **homotopy equivalent**.

The reason for defining an equivalence relation on chain maps in this way is due to the following result.

Proposition 1.46. *If two chain maps* $\psi, \varphi : M_\bullet \to M'_\bullet$ *are homotopy equivalent, then they induce the same map on homology.*

Proof. Let $a + \mathrm{im}(\partial_{i+1}) \in H_i(M_\bullet)$. Then,

$$\begin{aligned}\psi_*(a + \mathrm{im}(\partial_{i+1})) &= \psi(a) + \mathrm{im}(\partial'_{i+1}) \\ &= \varphi(a) + s_{i-1}\partial_i(a) + \partial'_{i+1}s_i(a) + \mathrm{im}(\partial'_{i+1}).\end{aligned}$$

Now, $a \in \ker(\partial_i)$ so $s_{i-1}\partial_i(a) = 0$ and $\partial'_{i+1}s_i(a) \in \mathrm{im}(\partial'_{i+1})$ and

$$\begin{aligned}\psi_*(a + \mathrm{im}(\partial_{i+1})) &= \varphi(a) + \mathrm{im}(\partial'_{i+1}) \\ &= \varphi_*(a + \mathrm{im}(\partial_{i+1})).\end{aligned}$$

□

Chapter 2

A_∞-Algebras

In this Chapter we will state the definition of an A_∞-algebra and of morphisms between them. However, as the definition of an A_∞-algebra is inherently complicated we will begin by looking at the more basic structures of which an A_∞-algebra is a generalisation.

Definition 2.1. An **algebra** A is a vector space over a field $\mathbb{F}$, together with a multiplication map

$$\mu : A \times A \to A$$
$$(x, y) \mapsto xy$$

such that μ is distributive and is linear in both variables. That is, for all $x, y, z \in A$ and $\lambda \in \mathbb{F}$,

1. $x(y + z) = xy + xz$
2. $(x + y)z = xz + yz$ and,
3. $\lambda(xy) = (\lambda x)y = x(\lambda y)$.

Example 2.2. The set of $n \times n$ matrices over the complex numbers, $M_n(\mathbb{C})$, is an algebra with matrix multiplication as the multiplication map. In this example, the multiplication is also associative, hence we say that $M_n(\mathbb{C})$ is an **associative algebra**.

Definition 2.3. Let I be some index set. An **I-graded vector space** is a vector space V which can be written in the form

$$V = \bigoplus_{i \in I} V_i$$

where V_i is a vector space for each i. We then refer to the elements $v \in V_i$ as the **homogeneous elements of degree** i and write deg $v = i$ or $|v| = i$.

Example 2.4. Let V be a vector space and define $V_i = V$ if $i = 0$ and $V_i = 0$ otherwise. Then $V = \bigoplus_{i \in \mathbb{N}} V_i = V_0$ is a graded vector space. We say that V is **concentrated in degree** 0. We will sometimes consider a field as a graded vector space, concentrated in degree 0.

Example 2.5. Let $\mathbb{K}$ be a field and let $\mathbb{K}[x, y]$ be the ring of polynomials in x and y over K. Let P_n =span($\{x^i y^{n-i} | n \in \mathbb{N}\}$) be the set of homogeneous polynomials of degree n. That is, elements of P_n are polynomials where the degree of each term is equal to n. Then $\mathbb{K}[x, y] = \bigoplus_{n=0}^{\infty} P_n$ is a graded vector space.

Remark 2.6. The act of placing a grading on a vector space does not intrinsically have any immediate significance. For example, let V be a finite dimensional vector space over a field $\mathbb{F}$, with basis $B = \{e_i\}_{i=1}^{n}$. Then we can write $V = \bigoplus_{v \in B} \mathbb{F}v$. A grading can then be placed on V by indexing the set B in a completely arbitrary way, without actually providing any additional information about the structure of V. The case which we are more interested in is where the graded vector space has some additional structure which 'respects' the grading. In order to better define what we mean by this we will make use of tensor products.

2.1 Tensor Products

Definition 2.7. Let V and W be vector spaces over a field $\mathbb{K}$ with bases $\{v_i\}_{i\in I}$ and $\{w_j\}_{j\in J}$ respectively. Define the **tensor product** $V \otimes W$ to be the vector space over $\mathbb{K}$ whose basis consists of the symbols $\{v_i \otimes w_j | i \in I, j \in J\}$. We also define the **tensor product** of two vectors $v = \sum_i a_i v_i$ and $w = \sum_j b_j w_j$ to be the bilinear map $V \times W \to V \otimes W$ given by

$$v \otimes w = \left(\sum_i a_i v_i \right) \otimes \left(\sum_j b_j w_j \right) = \sum_{i,j} a_i b_j (v_i \otimes w_j). \tag{2.1}$$

This gives an embedding of $V \times W$ in $V \otimes W$.

Remark 2.8. Note that taking the tensor product of basis elements gives $(v_i, w_j) \mapsto v_i \otimes w_j$ so these definitions are consistent. Also, all of the above sums are finite since elements of a vector space are finite linear combinations of the basis elements.

The vector space $V \otimes W$ with the bilinear map $(v, w) \mapsto v \otimes w$ satisfies the following universal property.

Proposition 2.9. *Let X be a vector space. Given any bilinear map $\beta : V \times W \to X$, there is a unique linear map $\beta' : V \otimes W \to X$ such that $\beta = \beta' \circ \varphi$ where φ is the natural embedding of $V \times W$ in $V \otimes W$.*

Also, any vector space and bilinear map with this universal property is unique up to a unique isomorphism.

Proof. Define β' on the basis of $V \otimes W$ by $\beta'(v_i \otimes w_j) = \beta(v_i, w_j)$. This is also a necessary condition for $\beta = \beta' \circ \varphi$ so this map is unique.

Now, suppose Y is a vector space with a bilinear map $\psi : V \times W \to Y$ such that given any bilinear map $\beta : V \times W \to X$ there is a unique linear map β' such that $\beta = \beta' \circ \psi$. Let $X = V \otimes W$ and let β be the map $(v, w) \mapsto v \otimes w$. Then by the universal property of Y, there is a unique map $\beta' : Y \to V \otimes W$ such that $\varphi = \beta' \circ \psi$. Similarly, by the universal property of $V \otimes W$ there is a unique map $\alpha' : V \otimes W \to Y$ such that $\psi = \alpha' \circ \varphi$ so $\varphi = (\beta' \circ \alpha') \circ \varphi$. Therefore, $\beta' \circ \alpha' = id$ and β' is the unique isomorphism $\beta' : Y \xrightarrow{\sim} V \otimes W$. □

Remark 2.10. The universal property of $V \otimes W$ also implies that while our definition was in terms of a basis, $V \otimes W$ is independent of the chosen bases.

We can also now consider the multiplication map $\mu : A \times A \to A$ as a linear map $m : A \otimes A \to A$. If A is a graded vector space then we define the degree of an element $v \otimes w$ to be the sum of the degrees of v and w in A, and for m to 'respect' the grading we mean that deg $m(v \otimes w)$ =deg $v \otimes w$.

Definition 2.11. Let $\mathbb{K}$ be a field. A **graded algebra** over $\mathbb{K}$ is an algebra A which is also a graded vector space over $\mathbb{K}$, $A = \bigoplus_{i \in I} A_i$ such that for all $x, y \in A$, the multiplication map satisfies

$$\deg xy = \deg x + \deg y. \tag{2.2}$$

Example 2.12. The graded vector space $\mathbb{K}[x, y] = \bigoplus_{n=0}^{\infty} P_n$ in example 2.5 is a graded algebra since $(x^i y^{n-i})(x^j y^{m-j}) = x^{i+j} y^{(n+m)-(i+j)} \in P_{n+m}$.

For vector spaces U, V, W, we define the tensor product $U \otimes V \otimes W$ inductively, noting that the tensor product (either of vector spaces or of vectors) is associative. In particular, we define $V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}$.

Proposition 2.13. *Let $\mathbb{K}$ be a field and let V be a vector space. Then the vector space*

$$T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \ldots$$

*is a graded algebra with the multiplication given by the tensor product. It is called the **tensor algebra** of V.*

Proof. Follows immediately from the definitions. The vector space $\mathbb{K} \otimes V$ is identified with the isomorphic space V, so multiplication by elements in $\mathbb{K}$ is just scalar multiplication. □

Example 2.14. Let V be an n-dimensional vector space with basis $\{e_1, e_2, \ldots, e_n\}$. Then the elements of $T(V)$ are sums of terms of the form $\lambda e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$, where $i_j \in \{1, 2, \ldots, n\}$. That is, elements of $T(V)$ are non-commuting polynomials in n variables.

We can also form two other graded algebras by taking quotient spaces. The first of these is the usual polynomial ring in n variables, which we get if we take $T(V) / \langle x \otimes y - y \otimes x \rangle$.

The second graded algebra which we denote by $\Lambda^*(V)$ is obtained by modding out by the anticommutative relations $\langle x \otimes y + y \otimes x \rangle$. We write an element of

$T(V)/\langle x \otimes y + y \otimes x \rangle$ as $x \wedge y$, and we call this the **wedge product** of x and y. Then $x \wedge y = -y \wedge x$ and $x \wedge x = 0$ for all $x, y \in \Lambda^*(V)$. The latter of these two identities also implies that $\Lambda^*(V) = \mathbb{K} \oplus V \oplus \Lambda^2 \oplus \cdots \oplus \Lambda^n V$, where $\Lambda^m V = \bigoplus_{i_l \leq i_{l+1}} \mathbb{K} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_m}$.

We could clearly have given the definition of a graded algebra with a multiplication which 'respects' the grading without any mention of tensor products. However, it is now a straightforward matter to generalise to other multilinear maps on a vector space.

Definition 2.15. Let $V = \otimes_{n \in \mathbb{Z}} V_n$ be a $\mathbb{Z}$-graded vector space. Let $f : V^{\otimes n} \to V$ be a homomorphism of vector spaces (that is, a linear map). We say that the **degree** of f, $|f|$, is i if deg $f(v) = (\deg v) + i$ for all $v \in V$. Explicitly,

$$|f(v_1 \otimes v_2 \otimes \cdots \otimes v_n)| = (|v_1| + |v_2| + \cdots + |v_n|) + i. \tag{2.3}$$

Example 2.16. Let A be a graded algebra. Then the degree of the multiplication map is 0.

Example 2.17. Let $M_\bullet : \ldots \longrightarrow M_{n+1} \xrightarrow{\partial_{n+1}} M_n \xrightarrow{\partial_n} M_{n-1} \longrightarrow \ldots$ be a complex over a field, so the modules M_n are actually vector spaces. Then $M = \bigoplus_n M_n$ is a graded vector space, and the degree of $\partial : M^{\otimes 1} \to M$ is -1.

Definition 2.18. Let V,W be graded vector spaces and let $\varphi : V \to V$ and $\psi : W \to W$ be linear maps. Then we define the **tensor product** of φ and ψ by

$$\begin{aligned} \varphi \otimes \psi : V \otimes W &\to V \otimes W \\ (v \otimes w) &\mapsto (-1)^{|\psi||v|}(\varphi(v) \otimes \psi(w)). \end{aligned} \tag{2.4}$$

Note that the degree of $(\varphi \otimes \psi)$ is $|\varphi| + |\psi|$.

Remark 2.19. The change of sign in the above expression is in fact a consequence of the Koszul sign rule which states that when the positions of two symbols a and b are swapped then the result is multiplied by $(-1)^{|a||b|}$. This is applied in the above expression as $(\varphi \otimes \psi)(v \otimes w) = (-1)^{|\psi||v|}(\varphi(v) \otimes \psi(w))$ where the symbols v and ψ are swapped in order to match the elements with the morphisms that are acting on them.

Given two complexes over a field $\mathbb{K}$ (so that the modules are vector spaces), $M_\bullet$ and $N_\bullet$, with boundary operators ∂_M and ∂_N respectively, as in (1.1) we can define a new complex $(M \otimes N)_\bullet$ as follows. We define $(M \otimes N)_n = \bigoplus_{i \in \mathbb{Z}} M_i \otimes N_{n-i}$, then the elements of $(M \otimes N)_n$ have degree n in agreement with our previous definition of the degree of $v \otimes w$. We define the boundary operator ∂ by

$$\partial = \partial_M \otimes 1 + 1 \otimes \partial_N. \tag{2.5}$$

Proposition 2.20. *The map ∂ defined on $(M \otimes N)_\bullet$ as above satisfies $\partial^2 = 0$ and is a vector space homomorphism, so $(M \otimes N)_\bullet$ does form a complex.*

Proof. By the bilinearity of the tensor product, ∂ is a sum of linear maps so is linear. Now,

$$
\begin{aligned}
\partial^2 &= \partial(\partial_M \otimes 1 + 1 \otimes \partial_N) \\
&= (\partial_M \otimes 1 + 1 \otimes \partial_N) \circ (\partial_M \otimes 1 + 1 \otimes \partial_N) \\
&= \partial_M^2 \otimes 1 + \partial_M \otimes \partial_N + (-1)^{|\partial_M||\partial_N|} \partial_M \otimes \partial_N + 1 \otimes \partial_N^2 \quad \text{by the Koszul sign rule,} \\
&= 0, \qquad \text{since } \partial_M \text{ and } \partial_N \text{ are boundary operators and } |\partial_M| = |\partial_N| = -1.
\end{aligned}
$$

□

Remark 2.21. The equation (2.5) can easily be applied to larger tensor products by induction. In particular, for a complex $M_\bullet$, the action of ∂ on $M_\bullet^{\otimes n}$ is given by

$$\sum_{i=0}^{n-1} 1^{\otimes i} \otimes \partial \otimes 1^{\otimes n-i-1}. \tag{2.6}$$

2.2 Differential Graded Algebras

Motivated by the form of equation (2.5), we now consider how a boundary operator should act when applied to a product of elements.

Definition 2.22. A **differential graded algebra** (or **DGA**) $A = (A, \mu, \partial)$, is a graded algebra (A, μ) over a field $\mathbb{K}$, with a homomorphism ∂ of degree -1 satisfying

1. $\partial^2 = 0$
2. $\partial \circ \mu = \mu \circ (\partial \otimes 1) + \mu \circ (1 \otimes \partial)$.

Remark 2.23. The first condition is simply that A is a chain complex and hence we can consider the homology of A, $H(A)$. The second condition is referred to as a graded Leibniz rule and a map satisfying this condition is called a **derivation**. Note that when it is applied to elements x and y it has the form $\partial(xy) = (\partial x)y + (-1)^{|x|}x(\partial y)$.

Example 2.24. Let $V_\bullet$ be a complex over a field $\mathbb{K}$ and define $A = \bigoplus_{n\in\mathbb{Z}} A_n$ where $A_n = \bigoplus_{i_1+\cdots+i_m=n} V_{i_1} \otimes V_{i_2} \otimes \cdots \otimes V_{i_m}$. Then $(A, \otimes, \partial)$ is a DGA with the differential given by equation (2.5).

Example 2.25. The tensor algebra $T(V)$ can be made into a DGA, with differential defined by

$$\partial(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = \sum_{i=1}^{n} (-1)^i (v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_n)$$

where $\hat{v}_i$ indicates that the term v_i is omitted, as in the definition of the boundary operator (1.4).

Example 2.26. With the wedge product, we can redefine the complex (1.7) to have the structure of a DGA by using the symbols dx, dy and $dx \wedge dy$ as basis elements for the modules. Then the complex becomes

$$M_\bullet : 0 \to \mathbb{K}[x,y]dx \wedge dy \xrightarrow{\partial_2} \begin{matrix} \mathbb{K}[x,y]dx \\ \oplus \\ \mathbb{K}[x,y]dy \end{matrix} \xrightarrow{\partial_1} \mathbb{K}[x,y] \to 0.$$

Then the boundary operator acts by sending $dx \wedge dy \mapsto ydx - xdy$, $dx \mapsto x$ and $dy \mapsto y$, and the multiplication is given by the wedge product.

2.3 A_∞-Algebras

Definition 2.27. Let $\mathbb{K}$ be a field. An A_∞**-algebra**, A, is a $\mathbb{Z}$-graded vector space

$$A = \bigoplus_{i \in \mathbb{Z}} A^i$$

together with a family of graded linear maps,

$$m_n : A^{\otimes n} \to A, \quad n \geq 1,$$

where the degree of m_n is $n-2$, such that the following identities (known as the Stasheff identities) hold for all $n \geq 1$.

$$\sum_{\substack{r+s+t=n \\ r,t\geq 0 \\ s \geq 1}} (-1)^{rs+t} m_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0. \qquad \text{SI(n)}$$

The Stasheff identities initially seem rather complicated to deal with computationally, and even harder to develop any intuition about the relations that a particular identity SI(n) is attempting to express. The first few identities, however, turn out to be properties that we are quite familiar with.

For $n = 1$, we must have $r = t = 0$ and $s = 1$, so SI(n) is

$$m_1 m_1 = 0.$$

The degree of m_1 is 1, so the identity SI(n) is saying that A is a (cochain) complex with boundary operator m_1. In particular, as in the case of a DGA, this means that we can define the homology of an A_∞-algebra A.

For $n = 2$, in the decomposition of n we can take $\{s = 2, r = t = 0\}$, $\{s = r = 1, t = 0\}$ or $\{s = 1, r = 0, t = 1\}$. Summing up the relevant terms gives us

$$m_1 m_2 - m_2(id \otimes m_1) - m_2(m_1 \otimes id) = 0$$
$$or \quad m_1 m_2 = m_2(id \otimes m_1) + m_2(m_1 \otimes id).$$

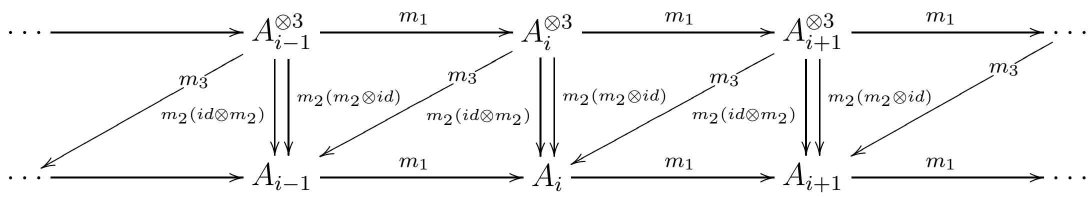
So m_1 is a graded derivation with respect to m_2. We can now see that m_2 appears to act like a multiplication map; it is a bilinear map and satisfies the graded Leibniz rule with m_1 as the differential. However, m_2 is not associative in general, as we shall see from the identity SI(3), which can be written as follows.

$$m_2(m_2 \otimes id) - m_2(id \otimes m_2) = m_1 m_3 + m_3(id^{\otimes 2} \otimes m_1 + id \otimes m_1 \otimes id + m_1 \otimes id^{\otimes 2}). \quad (2.7)$$

The left side of this equation is known as the **associator** for m_2. If we represent the 'multiplication' m_2 by juxtapositon and apply the associator to elements a, b, c, this becomes $(ab)c - a(bc)$ and so the condition for associativity is precisely that the right hand side of SI(3) is 0. Using the definition for the action (2.6) of a differential on $A^{\otimes 3}$, we can rewrite the identity SI(3) as

$$m_2(m_2 \otimes id) - m_2(id \otimes m_2) = m_1 m_3 + m_3 m_1.$$

If we now consider the two complexes $A^\bullet$ and $(A^{\otimes 3})^\bullet$, both $m_2(m_2 \otimes id)$ and $m_2(id \otimes m_2)$ are cochain maps from $(A^{\otimes 3})^\bullet$ to $A^\bullet$ and m_3, which is of degree -1, is a cochain homotopy between them.



So, while in general the multiplication m_2 is not directly associative, it is associative up to a homotopy m_3 which is itself part of the A_∞-algebra structure.

The Stasheff identities for $n \geq 4$ do not have an intuitive reinterpretation in the same way that SI(1),SI(2) and SI(3) do, though it is worth noting that they can always be rearranged so that the right hand side of SI(n) is

$$m_1 m_n + \sum_{r=0}^{n-1} (-1)^{n-1} m_n (id^{\otimes r} \otimes m_1 \otimes id^{\otimes n-1-r}) = m_1 m_n + (-1)^{n-1} m_n m_1.$$

The left hand side then consists of sums of all the compositions of maps m_u with m_s where $u, s \in \{2, 3, \ldots, n-2\}$ and $u + s = n + 1$. This can therefore be thought of as a generalisation of the homotopic associativity of m_2, where in this case, the homotopy is given by m_n if n is odd, and by

$$h_i : (A^{\otimes n})^i \to A^{i-1}$$
$$h_i = (-1)^i m_n$$

if n is even.

Remark 2.28. In the definition of the Stasheff identities, there are two different conventions for the signs. The convention used here is the same as in [7], which was chosen in order to simplify the calculations in Chapter 5. The difference between the conventions essentially just changes the direction of the sums.

Example 2.29. If $m_n = 0$ for $n \neq 2$, then A is just an associative algebra. If $m_n = 0$ for $n \geq 3$ then A is a differential graded algebra. Note that while this shows that a DGA is an A_∞-algebra in a rather trivial way, we will later look at a method for constructing an A_∞-algebra from a differential graded algebra such that the higher multiplications may not be trivial.

Remark 2.30. As in the case of rings, there are different conventions as to whether or not an A_∞-algebra must have a multiplicative identity; in the definition there is no necessity for a multiplicative identity or unit. Due to the greater amount of structure that an A_∞-algebra has, if an A_∞-algebra does have a unit, then we will require extra conditions on the way that the unit interacts with the higher multiplications.

Definition 2.31. Let $A = \bigoplus_{i \in \mathbb{Z}}$ be an A_∞-algebra. An element $1 \in A^0$ is called a **strict unit** if it satisfies the following conditions.

1. $m_2(1 \otimes a) = a = m_2(a \otimes 1)$ for all $a \in A$.
2. If $n \neq 2$, then $m_n(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = 0$ if $a_i = 1$ for some $i \in \{1, \ldots, n\}$.

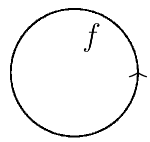
If such an element exists then we say that A is **strictly unital**.

2.4 Loop Spaces

Though we have given a definition of an A_∞-algebra and are able to see some motivation for the first few Stasheff identities, it is not immediately obvious why there might be a structure satisfying the general Stasheff identities in a non-trivial way. Before we continue to explore A_∞-algebras as an abstract algebraic structure, we will spend a short time looking at the motivating example arising from loop spaces, which is given in [8]. This example is expounded in detail in the context of homotopy theory by Stasheff in [9] and [10].

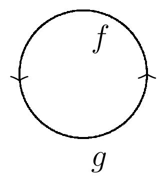
Example 2.32. Let X be a topological space and let x_0 be a fixed point in x_0. A **base fixed loop** is a continuous function on the circle, $f : S^1 \to X$ such that $f(1,0) = x_0$. We define the **loop space** ΩX to be the set of all base fixed loops on X. We will henceforth assume that all loops are base fixed and we represent the loop f with the following diagram.

f :



Because all loops in ΩX start and finish at the same point we can take the composition of two loops f and g to get another loop $f * g$. The image of $f * g$ is clearly just the union of the images of f and g, but because a loop is defined as a function, we also need to consider what part of the domain maps to which loop. We will define $f * g$ to be the loop that passes through the image of f on the first half of S^1 and through the image of g on the second half of S^1.

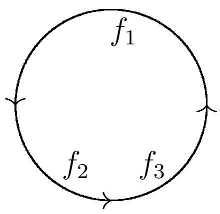
$f * g$:



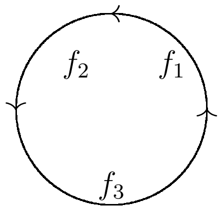
When we apply this 'multiplication' to three elements, we find that it is not associative. The two different loops obtained by multiplying the three loops f_1, f_2, f_3

together are depicted as follows.

$f_1 * (f_2 * f_3)$:

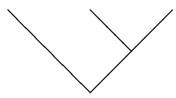


$(f_1 * f_2) * f_3$:

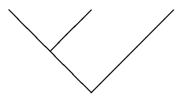


We will also represent the two different ways of multiplying three elements using tree diagrams.

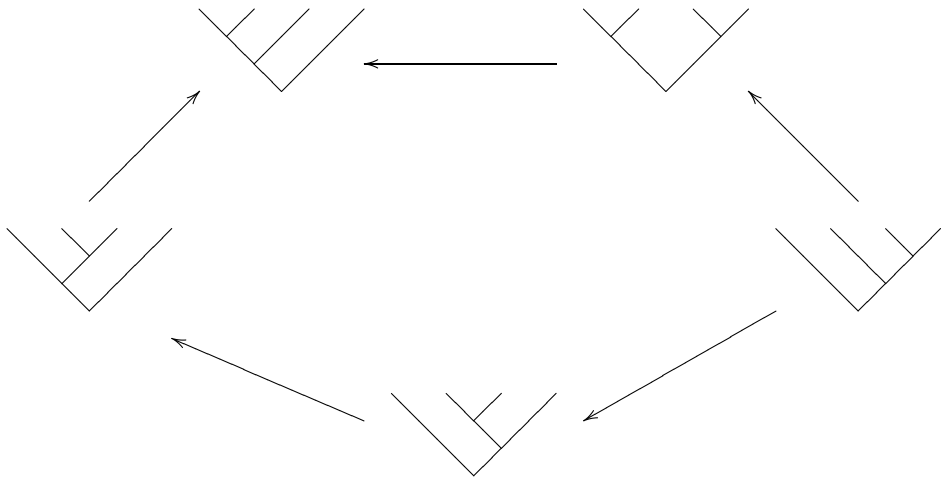
$f_1 * (f_2 * f_3)$:



$(f_1 * f_2) * f_3$:



While these two loops are not the same, they are homotopic in the topological sense. That is, there is a continuous map $M_3 : [0, 1] \times \Omega X \times \Omega X \times \Omega X \to \Omega X$ such that $M_3(0, f_1, f_2, f_3) = f_1 * (f_2 * f_3)$ and $M_3(1, f_1, f_2, f_3) = (f_1 * f_2) * f_3$. When we consider the composition of 4 loops, there are 5 different loops obtained, with the extreme cases being $((f_1 * f_2) * f_3) * f_4$ and $f_1 * (f_2 * (f_3 * f_4))$. Using M_3 repeatedly, we can show that these 5 loops are homotopic. Further, by using M_3 we get two different paths of homotopies between $((f_1 * f_2) * f_3) * f_4$ and $f_1 * (f_2 * (f_3 * f_4))$.



These two paths are themselves homotopic, with the homotopy given by a map $M_4 : K_4 \times (\Omega X)^4 \to \Omega X$, where K_4 is the pentagon whose boundary given by the maps M_3 in the above diagram.

We now have maps $M_2 = *$, M_3 and M_4 satisfying certain relations, with polytopes K_4, $K_3 = \Delta^1$ and $K_2 = x_0$. In [9], Stasheff continues this process to define polytopes K_n and maps $M_n : K_n \times (\Omega X)^n$ for $n \geq 2$. At this point however, there is no additive structure on the space. In order to obtain an A_∞-algebra structure from this, we take the singular homology of X, $S_\bullet(X)$. The maps $m_n : S_\bullet(X)^{\otimes n} \to S_\bullet(X)$ for $n \geq 2$ are induced by the maps M_n and m_1 is the boundary operator. To see how this corresponds to our definition of an A_∞-algebra we will consider SI(3) and SI(4). We will restrict to the case where we apply the maps m_n to elements from $S_0(X)$, that is maps of the form $g : \Delta^0 \to \Omega X$. In particular, the map m_2 will act on these elements of $S_\bullet(X)$ in essentially the same way that M_2 acts on the loops which they represent.

Write SI(3) as in equation (2.7). The left hand side of this is a sum of maps corresponding to

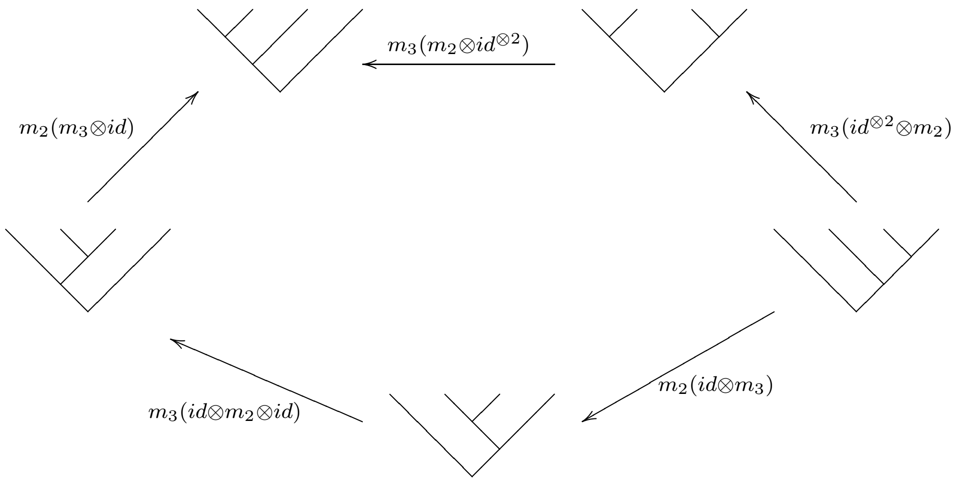
$$(f_1 * f_2) * f_3 - f_1 * (f_2 * f_3).$$

Now, $m_3(f_1, f_2, f_3)$ is the map $f : \Delta^1 \to \Omega X$ such that $f(0) = f_1 * (f_2 * f_3)$ and $f(1) = (f_1 * f_2) * f_3$. Then, from equation (1.6) in proposition 1.32

$$m_1 f = f(1) - f(0) = (f_1 * f_2) * f_3 - f_1 * (f_2 * f_3).$$

An element corresponding to a loop is a map $g : \Delta^0 \to S^1 \xrightarrow{f_1} X$, so $m_1(g) = 0$. Then the right hand side of SI(3) is $m_1 f$ and SI(3) is satisfied.

When we consider SI(4), note that M_4 is defined such that its boundary is given by the connecting homomorphisms (which are now elements of $S_1(X)$) in the following diagram.



When we take the boundary of this, note that K_4 is oriented in an anti-clockwise direction and so the signs are given by whether the homotopies are in the same or opposite orientation to K_4. We define m_4 to be induced by the homotopy M_4 such that m_3 is oriented clockwise so that the boundary of K_4 is

$$-m_1m_4 = m_3(id^{\otimes 2}\otimes m_2)+m_3(m_2\otimes id^{\otimes 2})-m_2(m_3\otimes id)-m_3(id\otimes m_2\otimes id)-m_2(id\otimes m_3)$$

and as $m_1(f_i) = 0$ for, this is the Stasheff identity SI(4).

2.5 A_∞-morphisms

We will now consider what we would mean by a structure preserving map on an A_∞-algebra. Since the definition of an A_∞-algebra involves a family of maps and identities, we would expect that the definition of an A_∞-morphism would reflect some of this structure.

Definition 2.33. An A_∞**-morphism**, $f : A \to B$, is a family of linear maps

$$f_n : A^{\otimes n} \to B \qquad n \geq 1$$

of degree $1 - n$ such that the following identities hold for each $n \geq 1$.

$$\sum_{\substack{r+s+t=1\\ r,t\geq 0\\ s\geq 1}} (-1)^{rs+t} f_{r+1+t}(id^{\otimes r}\otimes m_s\otimes id^{\otimes t}) = \sum_{j=1}^{n} \sum_{i_1+\cdots+i_j=n} (-1)^u m'_j(f_{i_1}\otimes f_{i_2}\otimes\cdots\otimes f_{i_j}), \tag{MI(n)}$$

where $u = (i_{j-1} - 1) + 2(i_{j-2} - 1) + \cdots + (j - 2)(i_2 - 1) + (j - 1)(i_1 - 1)$.

The first of these identities is that $f_1 m_1 = m'_1 f_1$. This says that f_1 is a morphism of complexes, that is, a (co)chain map. The morphism identities are essentially generalisations of this, where on the left hand side, the maps m_s act on $A^{\otimes n}$ in a way analogous to the action of a boundary operator on tensor products as in equation (2.6). On the right hand side, we are applying homomorphisms to tensor products, and this is consistent with equation (2.4).

If A and B are unital A_∞-algebras with strict units 1_A and 1_B respectively, then f needs to also satisfy the following unital morphism conditions.

1. $f_1(1_A) = 1_B$ such as we also require for ring morphisms.
2. $f_n(a_1 \otimes \cdots \otimes a_n) = 0$ for $n \geq 2$ if $a_i = 1_A$, for some $i \in \{1, \ldots, n\}$.

If $f_n = 0$ for all $n \geq 2$ then $f : A \to B$ is a **strict morphism** of A_∞-algebras. If $f : A \to B$ is a strict morphism, then the morphism identity MI(n) becomes

$$f_1 m_n = m_n(f_1 \otimes \cdots \otimes f_1).$$

So strict morphisms of A_∞-algebras are directly analogous to ring homomorphisms. If f is a strict morphism and f_1 is an isomorphism of vector spaces, then we say that f is a **strict isomorphism**. However, just as with chain complexes, it turns out to be more important to consider the homology of the A_∞-algebras.

Definition 2.34. Let A and B be A_∞-algebras and let $f : A \to B$ be an A_∞-morphism. Then we say f is a **quasi-isomorphism** if f_1 is a quasi-isomorphism of complexes. That is, the induced map $H(f_1) : H(A^\bullet) \to H(B^\bullet)$ is an isomorphism. In this case, we write $A \simeq B$.

If two A_∞-algebra are quasi-isomorphic, then we will consider them to be the same A_∞-algebra. That is, we will be considering quasi-isomorphism classes of A_∞-algebras. A representative of a class of A_∞-algebras quasi-isomorphic to A (usually a representative satisfying some nice property) is called a **model of** A. For example, a representative A_∞-algebra with a zero m_1 is called a **minimal model of** A. We can therefore study an A_∞-algebra A by considering various models of A.

Chapter 3

Examples and Merkulov's Construction

In most areas of algebra, one way to form a better intuitive understanding of a new definition is to look at some examples. There are various examples of A_∞-algebras that have been constructed (such as in [5], section 3), however, many of these examples essentially just 'look like' a ring, and the intuition gained in understanding the higher homotopies is that they are multilinear maps satisfying certain conditions. This is what we already knew from the abstract definition and so, to understand A_∞-algebras, the examples that we need to consider are examples of how to deal with the Stasheff identities which are the central feature of A_∞-algebras.

In this chapter we will see some particular cases where some simplifications can be made to the Stasheff identities, and we will also look at a method for constructing A_∞-algebras from existing structures, namely, differential graded algebras.

The first of the cases where the Stasheff identities are easier to deal with is where A has a limited number of components as a graded vector space.

Definition 3.1. Let $\mathbb{K}$ be a field and let A be an associative graded algebra over $\mathbb{K}$. If A is of the form $A = \mathbb{K} \oplus A^1 \oplus A^2$, then we say that A is **connected** and **cubic zero**.

Proposition 3.2. *Let (A, m_2) be a connected and cubic zero associative graded algebra. For $n \neq 2$, let $m_n : A^{\otimes n} \to A$ be linear maps of degree $n-2$ such that*

$$m_n(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = 0 \qquad \text{if } a_i \notin A^1 \text{ for some } i \in \{1, \ldots, n\} \tag{3.1}$$

Then $(A, m_1, m_2, m_3, \ldots)$ is an A_∞-algebra.

Proof. Since m_n is of degree $2-n$, the image of $m_n : (A^1)^{\otimes n} \to A$ is in A^2 for $n \neq 2$. This means that by (3.1), the composition of any maps m_n and m_k is 0 unless n or k is 2. That is, $m_n(id^{\otimes j} \otimes m_k \otimes id^{\otimes n-k-j}) = 0$ for all $j = 0, 1 \ldots n-k$, where $n, k \neq 2$.

When we consider the Stasheff identity SI(n),

$$\sum_{\substack{r+s+t=n \\ r,t \geq 0 \\ s \geq 1}} (-1)^{rs+t} m_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0$$

the only non-zero terms are where $s = 2$ or $r + t = 1$, then the left hand side becomes

$$\sum_{j=0}^{n-2} (-1)^j m_{n-1}(id^{\otimes n-j-2} \otimes m_2 \otimes id^{\otimes j}) - m_2(m_{n-1} \otimes id) + (-1)^{n-1} m_2(id \otimes m_{n-1}).$$

To get anything non-zero we need to apply m_{n-1} to elements of $(A^1)^{\otimes n-1}$ and the image of this will be in A^2. In the last two terms this means that the other term must be in A^0 as otherwise the product would be in $A^{\geq 3} = 0$, and by linearity we may assume this element is 1. Similarly, in the sum, we need the image of m_2 to be in A^1 and every other element must also be in A^1. So we only need to check that the identity is satisfied for elements where one entry is 1 and the rest are in A^1. Since 1 is the multiplicative identity with respect to m_2, it is easy to check that in each case this gives only two terms which cancel. □

The second case where the Stasheff identities are easy to check is where there is only one non-zero higher multiplication m_p for some $p \neq 2$. Note that in this case at least one of m_1 or m_3 is zero, so m_2 is associative.

Lemma 3.3. *Let A be a $\mathbb{Z}$-graded vector spaces with an associative multiplication map m_2 and a linear map $m_p : A^{\otimes p} \to A$. Then $\{m_2, m_p\}$ automatically satisfy all of the Stasheff identities except for SI(p) and SI$(2p-1)$.*

Proof. In the Stasheff identity $\sum(-1)^{rs+t} m_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t} = 0$, since there are only two non-zero multiplications, we must have $r+1+t$ and s each equal to either p or 2. From this, since $r+t = n-s$, we immediately get $n = 3, p+1, 2p-1$ and SI(3) is already satisfied as m_2 is associative and m_1 or m_3 is zero. □

3.1 Merkulov's Construction

We have already seen that a DG-algebra has the structure of an A_∞-algebra, if in a slightly trivial way. However, we are in fact able to use a DG-algebra to construct an A_∞-algebra with non-trivial higher multiplications. The method we will be considering here is Merkulov's Construction [2]. This construction is also discussed in [6]. Note that in Merkulov's paper the degree of an element is simply considered to be even or odd. That is, the algebra has a $\mathbb{Z}/2\mathbb{Z}$-grading. Also, the homotopies m_n in [2] are described explicitly in terms of their action on elements, and so there are additional signs introduced which are here implicit due to the Koszul sign rule.

Let $(A, \partial, \cdot)$ be a DG-algebra.

Definition 3.4. Let φ, ψ be two graded linear maps on A. The **supercommutator** of φ and ψ is given by

$$[\varphi, \psi] = \varphi\psi - (-1)^{|\varphi||\psi|}\psi\varphi.$$

We start with a subcomplex W of A satisfying the following condition.

Assumption 3.5. There exists a vector space homomorphism $Q : A \to A$ of degree -1 such that the image of $(1 - [\partial, Q])$ is in W where $[\cdot, \cdot]$ is the supercommutator.

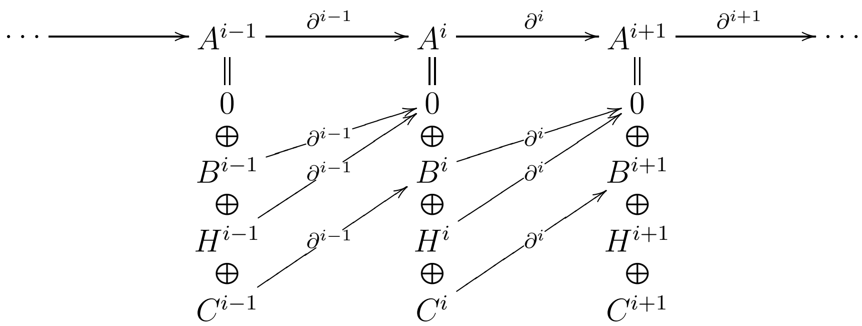
Example 3.6. Let $A = \bigoplus_{i \in \mathbb{Z}} A^i$ be a DG-algebra with differential ∂. Let Z^i be the cycles in A^i and B^i be the boundaries in A^i. Now, B^i is a subspace of Z^i so there exists a vector space complement to B^i in Z^i, call this H^i, so

$$Z^i = B^i \oplus H^i.$$

Similarly, there is a vector space C^i such that

$$A^i = Z^i \oplus C^i = B^i \oplus H^i \oplus C^i.$$

We can now view the complex (A, ∂) as follows.



Since $H^i \simeq Z^i/B^i$, we will identify the homology of A, $H^\bullet A$ with $\bigoplus_{i\in\mathbb{Z}} H^i$ and consider the subcomplex

$$H^\bullet : \ldots \longrightarrow H^{i-1} \xrightarrow{\partial^{i-1}} H^i \xrightarrow{\partial^i} H^{i-1} \longrightarrow \ldots$$

Now, $C^i \simeq A^i/Z^i = A^i/\ker(\partial^i)$ so by the first isomorphism theorem for vector spaces, $C^i \simeq \mathrm{im}(\partial^i) = B^{i+1}$. In particular, $\partial^i|_{C^i}$ has an inverse $(\partial^i|_{C^i})^{-1} : B^{i+1} \to C^i$. We now define the linear map Q by the composition

$$Q : A^{i+1} = B^{i+1} \oplus H^{i+1} \oplus C^{i+1} \xrightarrow{\pi} B^{i+1} \xrightarrow{(\partial^i|_{C^i})^{-1}} C^i \hookrightarrow B^i \oplus H^i \oplus C^i = A^i$$

where π is the projection map.

We now need to check that Q satisfies the condition that the image of $(1 - [\partial, Q])$ is in H. Note that $|\partial| = +1$ and $|Q| = -1$ so $[\partial, Q] = \partial Q + Q\partial$. Let $v^i \in A^i = B^i \oplus H^i \oplus C^i$, so we can write $v^i = (db^i, h^i, c^i)$. Then $\partial v^i = (\partial c^i, 0, 0)$ and $Qv^i = (0, 0, b^i)$. Now,

$$\begin{aligned}(1 - [\partial, Q])v^i &= (\partial b^i, h^i, c^i) - \partial(0, 0, b^i) - Q(\partial c^i, 0, 0)\\ &= (\partial b^i, h^i, c^i) - (\partial b^i, 0, 0) - (0, 0, c^i)\\ &= (0, h^i, 0) \in H^i.\end{aligned}$$

This holds for all elements of A by linearity, so $H^\bullet$ is a suitable subcomplex with Q an appropriate homomorphism of degree -1.

Given a subcomplex W and a linear map Q of degree -1 as above, we now build up the structure of an A_∞-algebra. First, we define a sequence of linear maps $\lambda_n : A^{\otimes n} \to A$ as follows. There is no λ_1, but we define $Q\lambda_1 = -\mathrm{id}$ and λ_2 is given by the multiplication from A. That is, $\lambda_2(v \otimes w) = v \cdot w$. The linear maps λ_n for $n \geq 3$ are defined recursively by

$$\lambda_n = \sum_{\substack{s+t=n \\ s,t\geq 1}} (-1)^{s+1} \lambda_2[Q\lambda_s \otimes Q\lambda_t]. \tag{3.2}$$

We will be using these maps to define an A_∞-algebra structure on the homology $H^\bullet A$, but in order to show that the resulting linear maps satisfy the Stasheff identities SI(n) we will first show that the linear maps λ_n satisfy the following equations.

Lemma 3.7. *Define the linear map $\varphi : A^{\otimes n} \to A$ by*

$$\varphi_n = \sum_{\substack{k+l=n+1 \\ k,l\geq 2}} \sum_{j=0}^{k-1} (-1)^{j(l-1)+(k-1)l} \lambda_k (id^{\otimes(n-j-l)} \otimes \lambda_l \otimes id^{\otimes j}). \tag{3.3}$$

Then, for the λ_n defined above, $\varphi_n = 0$ for all $n \geq 3$.

Proof. We will show that φ_n satisfies the recursive formula

$$\varphi_n = \sum_{\substack{k+l=n \\ k\geq 3 \\ l\geq 1}} (-1)^k \lambda_2(Q\varphi_k \otimes Q\lambda_l) - \sum_{\substack{k+l=n \\ k\geq 1 \\ l\geq 3}} \lambda_2(Q\lambda_k \otimes Q\varphi_l). \tag{3.4}$$

By substituting (3.3) for φ_k and φ_l this equation becomes

$$\sum_{\substack{k+l=n\\k\geq 3\\l\geq 1}}\sum_{\substack{s+t=k+1\\s,t\geq 2}}\sum_{j=0}^{s-1}(-1)^{k+j(t-1)+(s-1)t}\lambda_2\left(Q\lambda_s(id^{\otimes k-j-t}\otimes\lambda_t\otimes id^{\otimes j})\otimes Q\lambda_l\right)$$
$$-\sum_{\substack{k+l=n\\k\geq 1\\l\geq 3}}\sum_{\substack{s+t=l+1\\s,t\geq 2}}\sum_{j=0}^{s-1}(-1)^{j(t-1)+(s-1)t}\lambda_2\left(Q\lambda_k\otimes Q\lambda_s(id^{\otimes l-j-t}\otimes\lambda_t\otimes id^{\otimes j})\right)$$
$$=\sum_{\substack{s+t+l=n+1\\t,s\geq 2\\l\geq 1}}\sum_{j=0}^{s-1}(-1)^{(s+t-1)+j(t-1)+(s-1)t}\lambda_2\left(Q\lambda_s(id^{\otimes s-j-1}\otimes\lambda_t\otimes id^{\otimes j})\otimes Q\lambda_l\right)$$
$$+\sum_{\substack{k+s+t=n+1\\s,t\geq 2\\k\geq 1}}\sum_{j=0}^{s-1}(-1)^{j(t-1)+(s-1)t+1}\lambda_2\left(Q\lambda_k\otimes Q\lambda_s(id^{\otimes s-j-1}\otimes\lambda_t\otimes id^{\otimes j})\right)$$
$$=\sum_{\substack{s+t+l=n+1\\t,s\geq 2\\l\geq 1}}\sum_{j=0}^{s-1}(-1)^{(s+t-1)+j(t-1)+(s-1)t+(l-1)t}\lambda_2\left(Q\lambda_s\otimes Q\lambda_l\right)\circ\left(id^{\otimes n-t-j}\otimes\lambda_t\otimes id^{\otimes j}\right)$$
$$+\sum_{\substack{s+t+k=n+1\\t,s\geq 2\\k\geq 1}}\sum_{j=k}^{k+s-1}(-1)^{(j-k)(t-1)+(s-1)t+1}\lambda_2\left(Q\lambda_k\otimes Q\lambda_s\right)\circ\left(id^{\otimes n-t-j}\otimes\lambda_t\otimes id^{\otimes j}\right)$$
$$(3.5)$$

Note that the Koszul sign rule was used to expand the composition of functions.

We now use (3.2) to expand (3.3).

$$\varphi_n=\sum_{\substack{k+l=n+1\\k,l\geq 2}}\sum_{j=0}^{k-1}(-1)^{j(l-1)+(k-1)l}\lambda_k\circ\left(id^{\otimes n-j-l}\otimes\lambda_l\otimes id^{\otimes j}\right)$$
$$=\sum_{\substack{k+l=n+1\\k,l\geq 2}}\sum_{j=0}^{k-1}\sum_{\substack{s+t=k\\s,t\geq 1}}(-1)^{j(l-1)+(k-1)l+s+1}\lambda_2\left(Q\lambda_s\otimes Q\lambda_t\right)\circ\left(id^{\otimes n-j-l}\otimes\lambda_l\otimes id^{\otimes j}\right).$$
$$(3.6)$$

Every term in (3.5) and (3.6) is of the form $\lambda_2 (Q\lambda_a \otimes Q\lambda_b) \circ \left(id^{\otimes n-c-j} \otimes \lambda_c \otimes id^{\otimes j}\right)$. From these two expressions it is now straightforward to check that the coefficients of all such terms are the same in both equations, so φ_n does satisfy the recursive formula (3.5). Finally,

$$\begin{aligned}\varphi_3 &= \sum_{j=0}^{1}(-1)^j \lambda_2 \left(id^{\otimes 1-j} \otimes \lambda_2 \otimes id^{\otimes j}\right) \\ &= \lambda_2 (id \otimes \lambda_2) - \lambda_2 (\lambda_2 \otimes id) \\ &= 0 \qquad\qquad \text{since } \lambda_2 \text{ is associative.}\end{aligned}$$

So $\varphi_n = 0$ for all $n \geq 3$. □

Lemma 3.8. *Define the linear map $\theta : A^{\otimes n} \to A$ by*

$$\begin{aligned}\theta_n = \partial\lambda_n &+ \sum_{j=0}^{n-1}(-1)^{n-1}\lambda_n \left(id^{\otimes n-j-1} \otimes \partial \otimes id^{\otimes j}\right) \\ &- \sum_{\substack{k+l=n+1 \\ k,l\geq 2}} \sum_{j=0}^{k-1}(-1)^{j(l-1)+(k-1)l}\lambda_k \left(id^{\otimes n-j-l} \otimes [\partial, Q]\lambda_l \otimes id^{\otimes j}\right).\end{aligned}$$

Then, for the λ_n defined above, $\theta_n = 0$ for all $n \geq 2$.

Proof. The proof is similar to Lemma 3.7 after using, in the expansion of θ_n, the fact that since λ_2 is associative and ∂ is a differential, $\partial\lambda_2 = \lambda_2(\partial \otimes id) + \lambda_2(id \otimes \partial)$.

□

Theorem 3.9. *Let $(A, \partial, \cdot)$ be a DG-algebra and let W be a subcomplex satisfying 3.5, with Q the relevant linear map. Let $\{\lambda_n\}_{n=2}^{\infty}$ be the vector space homomorphisms given by (3.2). Define the linear maps $m_n : W^{\otimes n} \to W$ by*

$$m_1 := \partial$$
$$m_n := (1 - [\partial, Q])\lambda_n, \quad \textit{for } n \geq 2.$$

Then $\{m_n\}_{n=1}^{\infty}$ satisfy the Stasheff identities SI(n), and so form the structure of an A_∞-algebra on W.

Proof. The maps $\{m_n\}_{n=1}^{\infty}$ are well defined since W is a subcomplex and the image of $(1-[\partial, Q])$ is in W. Now, SI(1) is the condition that $\partial^2 = 0$ which holds as (W, ∂) is a complex. For SI(2) we need $m_1 m_2 = m_2(m_1 \otimes id + id \otimes m_1)$ which becomes

$$\partial(1 - [\partial, Q])\lambda_2 = (1 - [\partial, Q])\lambda_2(\partial \otimes id + id \otimes \partial).$$

Since λ_2 is associative and ∂ is a differential, the right hand side of this becomes $(1 - [\partial, Q])\partial\lambda_2$. Then,

$$\begin{aligned}
\partial(1 - [\partial, Q]) &= \partial - \partial^2 Q - \partial Q \partial, \\
&= \partial - \partial Q \partial - Q\partial^2, \qquad \text{as } \partial^2 = 0, \\
&= (1 - [\partial, Q])\partial. \qquad (3.7)
\end{aligned}$$

So SI(2) holds. For SI(n) where $n \geq 3$ we make use of the identities we have just proven. Write the left hand side of SI(n) as ψ_n. Then,

$$
\begin{aligned}
\psi_n &= \sum_{\substack{r+s+t=n \\ r,t\geq 0 \\ s\geq 1}} (-1)^{rs+t} m_{r+t+1} \left(id^{\otimes r} \otimes m_s \otimes id^{\otimes t} \right) \\
&= \sum_{\substack{k+l=n+1 \\ k,l\geq 1}} \sum_{j=0}^{k-1} (-1)^{j(l-1)+(k-1)l} m_k \left(id^{\otimes n-j-l} \otimes m_l \otimes id^{\otimes j} \right).
\end{aligned}
$$

Splitting off the terms where $k = 1$ and $k = n$ gives

$$
\begin{aligned}
&= m_1(m_n) + \sum_{j=0}^{n-1} (-1)^{n-1} m_n \left(id^{\otimes n-j-1} \otimes m_1 \otimes id^{\otimes j} \right) \\
&\quad + \sum_{\substack{k+l=n+1 \\ k,l\geq 2}} \sum_{j=0}^{k-1} (-1)^{j(l-1)+(k-1)l} m_k \left(id^{\otimes n-j-l} \otimes m_l \otimes id^{\otimes j} \right) \\
&= \partial(1-[\partial,Q])\lambda_n + (1-[\partial,Q]) \sum_{j=0}^{n-1} (-1)^{n-1} \lambda_n \left(id^{\otimes n-j-1} \otimes \partial \otimes id^{\otimes j} \right) \\
&\quad + (1-[\partial,Q]) \sum_{\substack{k+l=n+1 \\ k,l\geq 2}} \sum_{j=0}^{k-1} (-1)^{j(l-1)+(k-1)l} \lambda_k \left(id^{\otimes n-j-l} \otimes (1-[\partial,Q])\lambda_l \otimes id^{\otimes j} \right)
\end{aligned}
$$

Lastly, after using (3.7) to swap ∂ and $(1-[\partial,Q])$ in the first term we get

$$
\begin{aligned}
&= (1-[\partial,Q])\left(\partial\lambda_n + \sum_{j=0}^{n-1}(-1)^{n-1}\lambda_n\left(id^{\otimes n-j-1}\otimes\partial\otimes id^{\otimes j}\right)\right.\\
&\quad - \sum_{\substack{k+l=n+1\\ k,l\geq 2}}\sum_{j=0}^{k-1}(-1)^{j(l-1)+(k-1)l}\lambda_k\left(id^{\otimes n-j-l}\otimes[\partial,Q]\lambda_l\otimes id^{\otimes j}\right)\\
&\quad \left. + \sum_{\substack{k+l=n+1\\ k,l\geq 2}}\sum_{j=0}^{k-1}(-1)^{j(l-1)+(k-1)l}\lambda_k\left(id^{\otimes n-j-l}\otimes\lambda_l\otimes id^{\otimes j}\right)\right)\\
&= (1-[\partial,Q])(\theta_n+\varphi_n)\\
&= 0 \qquad \text{by Lemma 3.7 and 3.8.}
\end{aligned}
$$

□

While we now have a way to construct A_∞-algebras, in order to apply this construction we need to first construct a differential graded algebra. We might first think of applying the construction to a DGA such as the one given in example 2.26, however, this complex is exact everywhere except at $\mathbb{K}[x,y]$, and so the homology of this DGA will have only one component. Since Merkulov's construction, with the subcomplex and morphism as in example 3.6, forms the structure of an A_∞-algebra on the homology of the DGA, the resulting A_∞-algebra will just be an associative algebra. In the next chapter we will see one way to construct a DGA with potentially non-trivial homology from exact sequences.

Chapter 4

Resolutions and the Ext-Algebra

The Ext-algebra $\mathrm{Ext}^\bullet_R(M, M)$ for a ring R and an R-module M is an object which is of great interest, particularly to ring theorists. The proper theoretical development of this algebra is usually done in the context of category theory, and while multiple chapters on this subject would be of interest in and of themselves, the Ext-algebra will be used in this thesis merely as part of the construction of an A_∞-algebra and so the discussion on this topic will be kept relatively brief.

Thus, in this chapter, while we will be discussing the ways in which the various structures of the Ext-algebra can be described, we will not be proving the equivalences of the different constructions. The material in this chapter is mostly taken from [3], and we will be referring the reader to various chapters of this text for details.

Definition 4.1. Let R be a ring and A,B be R-modules. An ***n*-fold extension** of A by B is an exact sequence of R-modules of the form

$$0 \to A \to M_{n-1} \to M_n \to \cdots \to M_1 \to M_0 \to B \to 0. \tag{4.1}$$

An equivalence relation can be placed on the set of all n-fold extensions for each n. We will demonstrate this in the case where $n = 1$ and so an extension of A by B is a short exact sequence;

$$E: \quad 0 \to A \xrightarrow{f} M \xrightarrow{g} B \to 0. \tag{4.2}$$

Proposition 4.2. *Fix a ring R and R-modules A,B. Let E, E' be two extensions of A by B. We define an equivalence relation on the set of 1-fold extensions by saying that $E \equiv E'$ if there is a morphism φ such that the following diagram commutes.*

$$\begin{array}{llllllllll} E: & 0 & \longrightarrow & A & \xrightarrow{f} & M & \xrightarrow{g} & B & \longrightarrow & 0 \\ & & & \downarrow{\scriptstyle id_A} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle id_B} & & \\ E': & 0 & \longrightarrow & A & \xrightarrow{f'} & M' & \xrightarrow{g'} & B & \longrightarrow & 0 \end{array}$$

Proof. By the short 5 lemma, the map φ is an isomorphism, so this is indeed an equivalence relation. □

For $n > 1$, similar notions of equivalence can be defined for n-fold extensions of A by B over R and the set of equivalence classes is denoted by $\mathrm{Ext}^n_R(B, A)$. This set can be given the structure of a group with the operation known as the Baer Sum. Defining this sum and verifying that it makes $\mathrm{Ext}^n_R(B, A)$ into an abelion group is not immediately obvious. However, in the case of $\mathrm{Ext}^n_R(A, A)$, it is straightforward to introduce a multiplication by taking the composition of extensions as follows.

Given n-fold and m-fold extensions of A by A,

$$0 \to A \to M_{m-1} \to \cdots \to M_0 \xrightarrow{\varphi} A \to 0$$

$$\text{and} \quad 0 \to A \xrightarrow{\psi} N_{n-1} \to \cdots \to N_0 \to A \to 0$$

we can form an $(n+m)$-fold extension

$$0 \to A \to M_{m-1} \to \cdots \to M_0 \xrightarrow{\psi \circ \varphi} N_{n-1} \to \cdots \to N_0 \to A \to 0.$$

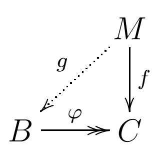
This sequence is exact since $\mathrm{im}(\psi \circ \varphi) \subseteq \mathrm{im}(\psi)$.

While the study of module extensions is one of the motivations for studying the group $\mathrm{Ext}^n_R(B, A)$, it is most commonly defined (equivalently) by the use of resolutions which we will consider below. See [3], Chapter III for the connections between these definitions.

We will first define some particular types of modules.

Definition 4.3. Let R be a ring and M an R-module. M is a **free module** over R if it is isomorphic to a direct sum of copies of R, considered as an R-module.

Definition 4.4. Let R be a ring and let M be an R-module. M is a **projective module** if, given two R-modules B and C and a surjective homomorphism $\varphi : B \to C$, there is a homomorphism $g : M \to B$ such that the following diagram commutes.



That is, $\varphi \circ g = f$.

Lemma 4.5. *Let R be a ring and let M be a free R-module. Then M is a projective module.*

The proof is straightforward and is left as an exercise.

Definition 4.6. Let R be a ring and M an R-module. A **projective resolution** of M is an exact complex of the form

$$\cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \xrightarrow{\varphi_0} M \to 0.$$

If the modules P_i are free modules then the resolution is a **free resolution** of M. We will often write a projective resolution as

$$P_\bullet : \quad \cdots \to P_2 \xrightarrow{\varphi_2} P_1 \xrightarrow{\varphi_1} P_0 \to 0 \tag{4.3}$$

where the complex is exact everywhere except at P_0.

Given R-modules M and N, recall that $\mathrm{Hom}_R(M, N)$ is the set of all module homomorphisms from M to N, which is an abelian group under addition of functions. Then, for a chain complex,

$$M_\bullet : \ldots \longrightarrow M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\partial_i} M_{i-1} \longrightarrow \ldots$$

consider the group $\mathrm{Hom}_R(M_\bullet, N) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_R(M_i, N)$.

We can define the action of ∂ on $\mathrm{Hom}_R(M_\bullet, N)$ as follows. Given a morphism

$\varphi \in \mathrm{Hom}_R(M_i, N)$, we define the morphism $\partial^i\varphi \in \mathrm{Hom}(M_{i+1}, N)$ by the composition

$$M_{i+1} \xrightarrow{\partial_{i+1}} M_i \xrightarrow{\varphi} N.$$

We then obtain the cochain complex

$$\mathrm{Hom}(M_\bullet, N): \quad \cdots \to \mathrm{Hom}(M_{i-1}, N) \xrightarrow{\partial^{i-1}} \mathrm{Hom}(M_i, N) \xrightarrow{\partial^i} \mathrm{Hom}(M_{i+1}, N) \to \dots$$

We can therefore think of $\mathrm{Hom}_R(-, N)$ acting on a chain complex to give a cochain complex. Since $\mathrm{Hom}_R(-, N)$ changes the direction of the morphisms in this way, we say that $\mathrm{Hom}_R(-, N)$ is **contravariant**. Also note that $\mathrm{Hom}_R(-, N)$ does not preserve exactness. So in particular, given a projective resolution $P_\bullet$ of M, $\mathrm{Hom}_R(P_\bullet, N)$ may have non-zero homology. In fact, the homology of $\mathrm{Hom}_R(P_\bullet, N)$ is the complex $\mathrm{Ext}^\bullet_R(M, N)$, that is, $\mathrm{Ext}^i_R(M, N) = H^i(\mathrm{Hom}(P_\bullet, N))$.

Example 4.7. Let $\mathbb{K}$ be a field and consider the ring of polynomials $\mathbb{K}[x]$. Then $\mathbb{K} \simeq \mathbb{K}[x]/\langle x \rangle$ so $\mathbb{K}$ is itself a $\mathbb{K}[x]$ module. We then have the following free resolution of $\mathbb{K}$.

$$0 \to \mathbb{K}[x] \xrightarrow{\partial_1} \mathbb{K}[x] \xrightarrow{\partial_0} \mathbb{K} \to 0$$

where ∂_1 acts by multiplication by x and ∂_0 maps $x \mapsto 0$. Applying $\mathrm{Hom}(-, \mathbb{K})$ to the free resolution gives

$$0 \leftarrow \mathrm{Hom}(\mathbb{K}[x], \mathbb{K}) \xleftarrow{\partial^1} \mathrm{Hom}(\mathbb{K}[x], \mathbb{K}).$$

Given an element $\varphi \in \mathrm{Hom}(\mathbb{K}[x], \mathbb{K})$, $\partial^1\varphi(p(x)) = \varphi(\partial_1(p(x))) = \varphi(xp(x)) = x\varphi(p(x)$ since φ is a module homomorphism. So ∂^1 also acts as multiplication by x. Also, the $\mathbb{K}[x]$ modules $\mathrm{Hom}(\mathbb{K}[x], \mathbb{K})$ are in fact isomorphic to $\mathbb{K}$, under the isomorphism

$$\mathrm{Hom}(\mathbb{K}[x], \mathbb{K}) \xrightarrow{\sim} \mathbb{K}$$
$$\varphi \mapsto \varphi(1).$$

The complex now becomes

$$0 \to \mathbb{K} \xrightarrow{x} \mathbb{K} \to 0.$$

Since we have identified $\mathbb{K}$ as the quotient module $\mathbb{K}[x]/\langle x \rangle$, the module homomorpihism x acts as the zero map on $\mathbb{K}$. We can now easily see that $\mathrm{Ext}^\bullet_{\mathbb{K}[x]}(\mathbb{K}, \mathbb{K})$ is the cochain complex

$$0 \to \mathbb{K} \to \mathbb{K} \to 0$$

where the map $\mathbb{K} \to \mathbb{K}$ is the zero map, which is the map induced by ∂^1.

When we consider $\mathrm{Ext}^n_R(M, N)$ in this way it is easy to see that it has an abelian group structure with the group operation given by addition of functions.

To form the ring structure of $\mathrm{Ext}^n_R(M, M)$ by using resolutions requires a bit more work. We will demonstrate how to construct this multiplicative structure; for some details on showing that this does again give the same complex as the previous definitions, see [3], Chapter VIII, section 4.

Let M be a module over the ring R, such that M has a finite projective resolution of the form

$$0 \to P_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \to M \to 0.$$

We form the complex

$$\dots \xrightarrow{\partial} \bigoplus_{i=0}^{n-1} \mathrm{Hom}(P_i, P_{i+1}) \xrightarrow{\partial} \bigoplus_{i=0}^{n} \mathrm{Hom}(P_i, P_i) \xrightarrow{\partial} \bigoplus_{i=0}^{n-1} \mathrm{Hom}(P_{i+1}, P_i) \xrightarrow{\partial} \dots \quad (4.4)$$

Where the boundary operator ∂ is defined on $\varphi \in \mathrm{Hom}(P_j, P_k)$ by

$$\partial\varphi = \varphi \circ \partial_{j+1} - \partial_k \circ \varphi \in \mathrm{Hom}(P_{j+1}, P_k) \oplus \mathrm{Hom}(P_j, P_{k-1}).$$

The multiplicative structure of $\mathrm{Ext}_R^n(M, M)$ is given by composition of functions with the composition defined to be 0 when the domain and range of the homomorphisms do not coincide.

We will make this more explicit in the case where M has the following projective resolution.

$$0 \to P_1 \xrightarrow{\partial} P_0 \to M \to 0.$$

When we consider homomorphisms from $P_\bullet$ to itself, the complex (4.4) becomes

$$0 \longrightarrow \mathrm{Hom}(P_0, P_1) \xrightarrow{\partial} \begin{matrix} \mathrm{Hom}(P_0, P_0) \\ \bigoplus \\ \mathrm{Hom}(P_1, P_1) \end{matrix} \xrightarrow{\partial} \mathrm{Hom}(P_1, P_0) \longrightarrow 0. \tag{4.5}$$

The action of ∂ on $\mathrm{Hom}(P_0, P_1)$ is given by $\varphi \mapsto \left(\begin{smallmatrix} \partial\varphi \\ -\varphi\partial \end{smallmatrix}\right)$ and on $\left(\begin{smallmatrix} \mathrm{Hom}(P_0,P_0) \\ \mathrm{Hom}(P_1,P_1) \end{smallmatrix}\right)$ by $\left(\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix}\right) \mapsto \varphi\partial + \partial\psi$.

Example 4.8. Consider the free resolution of $\mathbb{K}$ given in example 4.7

$$0 \to \mathbb{K}[x] \xrightarrow{\partial_1} \mathbb{K}[x] \xrightarrow{\partial_0} \mathbb{K} \to 0.$$

To keep track of which module $\mathbb{K}[x]$ we are talking about, we will denote the first module by P_1 and the second by P_0 as in the sections above.

Noting that the modules $\mathrm{Hom}(\mathbb{K}[x], \mathbb{K}[x])$ are isomorphic to $\mathbb{K}[x]$, with composition of homomorphisms now given by multiplication in the polynomial ring, we rewrite the complex (4.5) as

$$0 \longrightarrow \mathbb{K}[x]u \xrightarrow{\partial} \begin{matrix} \mathbb{K}[x] \\ \bigoplus \\ \mathbb{K}[x] \end{matrix} \xrightarrow{\partial} \mathbb{K}[x]v \longrightarrow 0. \tag{4.6}$$

where the symbols u and v are introduced as basis elements to make the notation easier.

The action of the boundary operator is

$$\partial : \mathbb{K}[x]u \to \begin{pmatrix} \mathbb{K}[x] \\ \\ \mathbb{K}[x] \end{pmatrix} \qquad\qquad \partial : \begin{pmatrix} \mathbb{K}[x] \\ \\ \mathbb{K}[x] \end{pmatrix} \to \mathbb{K}[x]v$$

$$p(x)u \mapsto \begin{pmatrix} xp(x) \\ \\ -xp(x) \end{pmatrix} \qquad\qquad \begin{pmatrix} p(x) \\ \\ q(x) \end{pmatrix} \to (xp(x) + xq(x))\, v.$$

In order to describe how the multiplication works, we will need to use both equation (4.5) and (4.6).

For $p(x)u \in \mathbb{K}[x]u$ and $\binom{q_1(x)}{q_2(x)} \in \binom{\mathbb{K}[x]}{\mathbb{K}[x]}$, the product is in $\mathbb{K}[x]u$, given by the composition

$$\begin{aligned} & p(x)q_1(x) : P_0 \xrightarrow{q_1(x)} P_0 \xrightarrow{p(x)} P_1 \\ \text{or} \quad & q_2(x)p(x) : P_0 \xrightarrow{p(x)} P_1 \xrightarrow{q_2(x)} P_1. \end{aligned}$$

Similarly, for $\binom{q_1(x)}{q_2(x)} \in \binom{\mathbb{K}[x]}{\mathbb{K}[x]}$ and $p(x)v \in \mathbb{K}[x]v$ we get

$$\begin{aligned} & p(x)q_2(x) : P_1 \xrightarrow{q_2(x)} P_1 \xrightarrow{p(x)} P_0 \\ \text{or} \quad & q_1(x)p(x) : P_1 \xrightarrow{p(x)} P_0 \xrightarrow{q_1(x)} P_0. \end{aligned}$$

Lastly, for $p(x)u \in \mathbb{K}[x]u$ and $q(x)v \in \mathbb{K}[x]v$ we are taking the compositions of maps from $P_0 \to P_1$ and $P_1 \to P_0$ so we will either get an element of $\mathrm{Hom}(P_0, P_0)$ or $\mathrm{Hom}(P_1, P_1)$ depending on the order we take the composition. So the products are

$$p(x)q(x) : P_1 \xrightarrow{q_2(x)} P_0 \xrightarrow{p(x)} P_1$$

$$\text{and} \quad q(x)p(x) : P_0 \xrightarrow{p(x)} P_1 \xrightarrow{q_1(x)} P_0.$$

$$p(x)u \cdot q(x)v = \left(\begin{smallmatrix} 0 \\ p(x)q(x) \end{smallmatrix}\right)$$

$$q(x)v \cdot p(x)u = \left(\begin{smallmatrix} q(x)p(x) \\ 0 \end{smallmatrix}\right).$$

We will now make sure that the homology of this complex which is $\mathrm{Ext}^{\bullet}_{\mathbb{K}[x]}(\mathbb{K},\mathbb{K})$ is the same as that which we calculated in example 4.7.

At $\mathbb{K}[x]v$, the kernel of ∂ is all of $\mathbb{K}[x]v$, while the image of ∂ in $\mathbb{K}[x]v$ is generated by elements of the form $x\,(p(x)+q(x))$ so $\mathrm{Ext}^0_{\mathbb{K}[x]}(\mathbb{K}[x],\mathbb{K}[x]) = \mathbb{K}[x]/x\mathbb{K}[x] \simeq \mathbb{K}$.

At $\left(\begin{smallmatrix} \mathbb{K}[x] \\ \mathbb{K}[x] \end{smallmatrix}\right)$ the kernel of ∂ is $\left\{\left(\begin{smallmatrix} q_1(x) \\ q_2(x) \end{smallmatrix}\right) \mid xq_1(x) + xq_2(x) = 0\right\} = \mathbb{K}[x]\left\{\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right\}$. The image of ∂ in $\left(\begin{smallmatrix} \mathbb{K}[x] \\ \mathbb{K}[x] \end{smallmatrix}\right)$ is $x\mathbb{K}[x]\left\{\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right\}$ so $\mathrm{Ext}^1_{\mathbb{K}[x]}(\mathbb{K},\mathbb{K}) = \mathbb{K}$. The complex is exact at $\mathbb{K}[x]u$ so $\mathrm{Ext}^{\bullet}_{\mathbb{K}[x]}(\mathbb{K},\mathbb{K})$ is the same complex that we calculated before, except that we can now also see the ring structure of $\mathrm{Ext}^{\bullet}_{\mathbb{K}[x]}(\mathbb{K},\mathbb{K})$, induced by the multiplication on the complex (4.6).

Remark 4.9. Note that $\mathrm{Ext}^{\bullet}_{R}(\mathbb{K},\mathbb{K})$ is also a $\mathbb{K}$-vector space, and therefore is an algebra. In fact, $\mathrm{Ext}^{\bullet}_{R}(\mathbb{K},\mathbb{K})$ is the homology of a differential graded algebra, so we can now use Merkulov's construction to derive the structure of an A_∞-algebra on $\mathrm{Ext}^{\bullet}_{R}(\mathbb{K},\mathbb{K})$.

While we are now able to construct A_∞-algebras on, $\mathrm{Ext}^{\bullet}_{\mathbb{K}[x]}(\mathbb{K},\mathbb{K})$ for example, even in this rather simple example, if we attempt to apply the construction naïvely, then the calculations to find any of the higher multiplications quickly become very involved, even just to find if the map m_3 is non-zero. Also, despite the computational difficulty, since $\text{Ext}^\bullet_{\mathbb{K}[x]}(\mathbb{K}, \mathbb{K}) = \mathbb{K} \oplus \mathbb{K}$, this is a connected, cubic zero A_∞-algebra, so if we are wanting to find specific examples of A_∞-algebras, then this construction does not appear to be particularly useful.

Merkulov's construction does however, have significant importance in studying A_∞-algebras. This construction is in fact a case of the following theorem of Kadeishvili [1].

Theorem 4.10. *Let A be an A_∞-algebra and let $H^\bullet(A)$ be the cohomology ring of A. Then there is an A_∞-algebra structure on $H^\bullet(A)$ with $m_1 = 0$ and m_2 induced by the multiplication on A, which is constructed from the A_∞-structure of A, such that there is a quasi-isomorphism of A_∞-algebras $H^\bullet(A) \to A$ lifting the identity of $H^\bullet(A)$.*

This A_∞-algebra structure on $H^\bullet(A)$ is unique up to quasi-isomorphism.

This theorem states that every A_∞-algebra has a minimal model (that is, a model with zero m_1).

One application of the particular construction due to Merkulov is not so much to study A_∞-algebras, but to study the Ext-algebra by using properties of A_∞-algebras. In particular, this is discussed in [6].

Chapter 5

Bar Construction

To understand A_∞-algebras, we need to really understand the structure that the Stasheff identities are trying to encompass. In this chapter, we will look at an alternative way of thinking about these identities through the process known as the bar construction.

The bar construction is a basic technique used in the study of differential graded algebras to move from a DGA to the dual notion of a differential coalgebra. We will be specifically discussing the bar construction for an A_∞-algebra.

Definition 5.1. Let $\mathbb{K}$ be a field. A **coalgebra** over $\mathbb{K}$, is a vector space C over K together with a linear map $\Delta : C \to C \otimes C$ called a **comultiplication**, such that

$$(id_C \otimes \Delta) \circ \Delta = (\Delta \otimes id_C) \circ \Delta. \tag{5.1}$$

Remark 5.2. A coalgebra is dual to an algebra in the sense that the comultiplication is the dual notion of a multiplication $\mu : A \otimes A \to A$ and the equation (5.1) is dual to the associative law $\mu(id_A \otimes \mu) = \mu(\mu \otimes id_A)$. Coalgebras are also usually defined with the dual notion to a multiplicative identity, called the **counit**, $\varepsilon : C \to \mathbb{K}$ which satisfies

$$(\varepsilon \otimes id) \circ \Delta = id_C = (id \otimes \varepsilon) \circ \Delta.$$

We will be making use of a coalgebra without a counit and so this will not be part of our definition.

Remark 5.3. Note also that there are no simplicies in this chapter; all symbols involving Δ refer to maps on a coalgebra.

We place a grading on a coalgebra in the same way that we did for an algebra, with the meaning of the comultiplication having degree 0 being that if $\Delta(c) = c_1 \otimes c_2$, then the degree of c is $|c| = |c_1| + |c_2|$.

Example 5.4. Let $\mathbb{K}$ be a field and V be a vector space over $\mathbb{K}$. Let $T(V) = \mathbb{K} \oplus V \oplus V^{\otimes 2} \oplus \dots$ be the tensor algebra as in proposition 2.13. We can also define a coalgebra structure on $T(V)$ with the comultiplication

$$\begin{aligned} \Delta : T(V) &\to T(V) \\ v_1 \otimes v_2 \otimes \cdots \otimes v_n &\to 1 \otimes (v_1 \otimes \cdots \otimes v_n) + \sum_{i=1}^{n-1} (v_1 \otimes \dots v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_n) \\ &\qquad + (v_1 \otimes \cdots \otimes v_n) \otimes 1. \end{aligned}$$

We call this the **tensor coalgebra on** V and it is in fact a coalgebra with counit $\varepsilon : C \to \mathbb{K}$ which is the identity on the subspace $\mathbb{K} \subseteq T(V)$ and is the zero map on $V^{\otimes n}$ for $n \geq 1$.

Remark 5.5. In order to make the notation clearer, we will write an element of $V^{\otimes n}$ in the tensor coalgebra as $(v_1|v_2|\dots|v_n)$ though we will still sometimes need to use the tensor notation, in particular to write $id^{\otimes n}$. It is from this notation that the term bar construction arises.

Example 5.6. Let V be a vector space over a field $\mathbb{K}$ and define

$$\bar{T}(V) = V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$$

with the comultiplication

$$\begin{aligned} \Delta : \bar{T}(V) &\longrightarrow \bar{T}(V) \otimes \bar{T}(V) \\ (v_1|v_2|\dots|v_n) &\longrightarrow \sum_{i=1}^{n-1} (v_1|\dots|v_i) \otimes (v_{i+1}|\dots|v_n) \end{aligned}$$

We call this the **reduced tensor coalgebra on** V

Definition 5.7. Let (C, Δ) be a coalgebra over a field $\mathbb{K}$. A **coderivation on** C is a map $b : C \to C$ such that

$$\Delta \circ b = (b \otimes id + id \otimes b) \circ \Delta.$$

A **differential on** C is a coderivation b of degree 1, such that $b^2 = 0$.

Lemma 5.8. *Let $\bar{T}(V)$ be the tensor coalgebra on the vector space V and let $b_s : \bar{T}(V) \to V$ be a graded map of degree 1. Then b_s lifts to a graded coderivation $b : \bar{T}(V) \to \bar{T}(V)$ of degree 1 and every coderivation b of degree 1 arises uniquely in this way.*

Proof. Let $b_s : \bar{T}(V) \to V$ be a graded map of degree 1, we define $b : \bar{T}(V) \to \bar{T}(V)$ by

$$b = \sum_{r,t \geq 0} id^{\otimes r} \otimes b_s \otimes id^{\otimes t}.$$

Then,

$$\begin{aligned}\Delta \circ b &= \sum_{r,t\geq 0} \left(\sum_{i=1}^{r} (id^{\otimes i}) \otimes (id^{\otimes r-i} | b_s | id^{\otimes t}) + \sum_{i=1}^{t} (id^{\otimes r} | b_s | id^{\otimes t-i}) \otimes (id^{\otimes i}) \right) \\ &= \sum_{\substack{r,t\geq 0 \\ i \geq 1}} (id^{\otimes i}) \otimes (id^{\otimes r} | b_s | id^{\otimes t}) + (id^{\otimes r} | b_s | id^{\otimes t}) \otimes (id^{\otimes i}).\end{aligned}$$

Now,

$$(b \otimes 1 + 1 \otimes b) = \sum_{r,t\geq 0} (id^{\otimes r} | b_s | id^{\otimes t}) \otimes 1 + 1 \otimes (id^{\otimes r} | b_s | id^{\otimes t}) \qquad \text{so,}$$

$$\begin{aligned}(b \otimes 1 + 1 \otimes b) \circ \Delta &= \sum_{r,t\geq 0} \Bigg((id^{\otimes r} | b_s | id^{\otimes t}) \otimes 1 \circ \sum_{i,j\geq 1} (id^{\otimes i} \otimes (id^{\otimes j})) \\ &\qquad + 1 \otimes (id^{\otimes r} | b_s | id^{\otimes t}) \circ \sum_{i,j \geq 1} (id^{\otimes i} \otimes (id^{\otimes j})) \Bigg) \\ &= \sum_{\substack{r,t\geq 0 \\ j \geq 1}} (id^{\otimes r} | b_s | id^{\otimes t}) \otimes (id^{\otimes j}) + (id^{\otimes j}) \otimes (id^{\otimes r} | b_s | id^{\otimes t}) \\ &= \Delta \circ b.\end{aligned}$$

So b is a coderivation. It is clearly of degree 1 and uniqueness is clear from considering the components of b.

Now suppose that $b : \bar{T}(V) \to \bar{T}(V)$ is a graded coderivation of degree 1. To show that b can be written in the form $b = \sum_{r,t\geq 0} id^{\otimes r} \otimes b_s \otimes id^{\otimes t}$ for a map b_s of degree 1, we will define a map $\Delta^n : \bar{T}(V) \to \bar{T}(V)^{\otimes n+1}$ recursively by $\Delta^1 = \Delta$, and for $n \geq 2$, $\Delta^n = \frac{1}{n}\left(\sum_{j=0}^{n-1} id^{\otimes j} \otimes \Delta \otimes id^{\otimes n-j-1}\right)\Delta^{n-1}$.

Then for $(v_1|v_2|\ldots|v_n) \in \bar{T}(V)$, $\Delta^n(v_1|v_2|\ldots|v_n) = v_1 \otimes v_2 \otimes v_n$. Since Δ is a coderivation $\Delta b = (b \otimes 1 + 1 \otimes bb)\Delta$ which leads to

$$\Delta^n b = \left(\sum_{j=0}^{n} id^{\otimes j} \otimes b \otimes id^{\otimes n-j}\right)\Delta^n.$$

Since Δ^n essentially breaks an element of $\bar{T}(V)$ down to a tensor product, this equation says that b acts on $V^{\otimes n}$ by $\sum_{j=0}^{n}(id^{\otimes j} \otimes b \otimes^{\otimes n-j})$ so every coderivation on $\bar{T}(V)$ can be lifted from a map $b : \bar{T}(V) \to V$. □

Now, given a $\mathbb{Z}$-graded vector space A, we define its **suspension** SA by $(SA)^i = A^{i+1}$ and the map $\sigma : A \to SA$ by $\sigma(a) = a \in SA$ so the degree of σ is -1. Given a map $m_n : A^{\otimes n} \to A$ of degree $2-n$, we lift m_n to a map $b_n : (SA)^{\otimes n} \to SA$ of degree 1 as in the following commutative diagram.

$$\begin{array}{ccc} (SA)^{\otimes n} & \xrightarrow{b_n} & SA \\ {\scriptstyle \sigma^{\otimes n}}\uparrow & & \uparrow{\scriptstyle \sigma} \\ A^{\otimes n} & \xrightarrow{m_n} & A \end{array}$$

So $b_n = \sigma \circ m_n \circ (\sigma^{-1})^{\otimes n}$. In particular, for $(A, m_1, m_2, \ldots)$ an A_∞-algebra, each homomorphism m_n can be lifted to a degree 1 map on the suspension of A, SA. These maps can then be lifted by lemma 5.8 to coderivations on the tensor coalgebra $\bar{T}(SA)$. A sum of coderivations is also a coderivation, so we finally have a bijection between families of maps $m_n : A^{\otimes n} \to A$ and coderivations $b : \bar{T}(SA) \to \bar{T}(SA)$ where the coderivation b is given by

$$b = \sum_{n=1}^{\infty} \sum_{\substack{r+s+t=n \\ r,t \geq 0 \\ s \geq 1}} (id^{\otimes r} \otimes b_s \otimes id^{\otimes t}). \tag{5.2}$$

where $b_s = \sigma \circ m_s \circ (\sigma^{-1})^{\otimes s}$.

Lemma 5.9. *With a family of maps $\{m_n\}_{n=1}^{\infty}$ on a $\mathbb{Z}$-graded vector space A, and a coderivation defined as above, the following conditions are equivalent.*

1. *The maps m_n form the structure of an A_∞-algebra on A.*
2. *The coderivation satisfies $b^2 = 0$, that is, b is a coalgebra differential.*
3. *For $n \geq 1$, the maps $\{b_n\}_{n=1}^{\infty}$ satisfy*

$$\sum_{\substack{r+s+t=n \\ r,t \geq 0 \\ s \geq 1}} b_{r+1+t}(id^{\otimes r} \otimes b_s \otimes id^{\otimes t}) = 0$$

Proof. First we will check that 2 is equivalent to 3.

$$\begin{aligned} b^2 &= \sum_{n=1}^{\infty} \sum_{\substack{r+s+t=n \\ s \geq 1 \\ r,t \geq 0}} (id^{\otimes r} \otimes b_s \otimes id^{\otimes t}) \circ \sum_{n=1}^{\infty} \sum_{\substack{j+k+l=n \\ k \geq 1 \\ j,l \geq 0}} (id^{\otimes j} \otimes b_k \otimes id^{\otimes l}) \\ &= \sum_{n=1}^{\infty} \Bigg(- \sum_{r \geq j} (id^{\otimes j} \otimes b_k \otimes id^{r-j-1} \otimes b_s \otimes id^{\otimes t}) \\ &\qquad + \sum_{r+s \geq j \geq r} (id^{\otimes r} \otimes b_s \left(id^{\otimes j-r} \otimes b_k \otimes id^{s-1-(j-r)}\right) \otimes id^{\otimes t} \\ &\qquad\qquad + \sum_{j \geq r+s} (id^{\otimes r} \otimes b_s \otimes id^{\otimes j-r-s} \otimes b_k \otimes id^{\otimes l}) \Bigg) \end{aligned}$$

Omitting the summation indices for clarity, this gives us

$$
\begin{aligned}
b^2 &= \sum (id^{\otimes r} \otimes b_s \left(id^{\otimes j-r} \otimes b_k \otimes id^{s-1-(j-r)}\right) \otimes id^{\otimes t}) \\
&= (id^{\otimes r} \otimes \sum \left(b_s \left(id^{\otimes j-r} \otimes b_k \otimes id^{s-1-(j-r)}\right)\right) \otimes id^{\otimes t})
\end{aligned}
$$

So 2 is equivalent to 3.

Finally,

$$
\begin{aligned}
b_{r+1+t}(id^{\otimes r} \otimes b_s \otimes id^{\otimes t}) &= \sigma m_{r+1+t}(\sigma^{-1})^{\otimes r+1+t}(id^{\otimes r} \otimes \sigma m_s(\sigma^{-1})^{\otimes s} \otimes id^{\otimes t}) \\
&= (-1)^t \sigma m_{r+1+t} \left((\sigma^{-1})^{\otimes r} \otimes m_s(\sigma^{-1})^{\otimes s} \otimes (\sigma^{-1})^{\otimes t}\right) \\
&= (-1)^{rs+t} \sigma m_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t})(\sigma^{-1})^{\otimes n}.
\end{aligned}
$$

Since σ and $\sigma^{\otimes n}$ are invertible, this implies that $\sum\limits_{\substack{r+s+t=n \\ r,t\geq 0 \\ s\geq 1}} b_{r+1+t}(id^{\otimes r} \otimes b_s \otimes id^{\otimes t}) = 0$ if and only if $\sum\limits_{\substack{r+s+t=n \\ r,t\geq 0 \\ s\geq 1}} (-1)^{rs+t} m_{r+1+t}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0$. So 1 and 3 are equivalent.

□

This gives a bijection between the structure of an A_∞-algebra on A, and a coalgebra differential on $\bar{T}(SA)$.

Remark 5.10. The bar construction provides us with an elegant way of thinking about the structure of an A_∞-algebra; the identity $b^2 = 0$ for a coalgebra differential is in some sense more natural. We are also able to lift A_∞-morphisms to a coalgebra homomorphism of degree 0 in a similar way, which provides additional motivation for the definition of an A_∞-algebra.

Just as a coalgebra is the dual notion to an associative algebra, and the bar construction maps an algebra to a coalgebra, there is a dual notion to the bar construction taking a coalgebra to an algebra. This leads to the following theorem, which is cited in [5] and is proven in [7] as lemma 1.3.6.6.

Theorem 5.11. *Every A_∞-algebra A is quasi-isomorphic to a free differential graded algebra.*

We refer to this DGA as a **DGA model** or an **anti-minimal model**.This theorem is in some sense the complement to Merkulov's construction of an A_∞-structure on a DGA. We can therefore study A_∞-algebras by considering their DGA-models and using the tools that we have from the more established area of differential graded homological algebra.

Chapter 6

Closing Remarks

The study of A_∞-algebras is both a relatively new area, and a subject which can be quite computationally difficult, or at least fickle in sorting out the details, and this can at times obscure the simplicity and elegance of some of the results which can be obtained. In this thesis I have attempted to provide an introduction to the notion of an A_∞-algebra which is hopefully both motivating and somewhat intuitive.

As this is an introduction however, there are many aspects of A_∞-algebras which I have not been able to discuss. Some of these are generalisations of common ring theoretic concepts such as a module, or viewing A_∞-algebras from the context of category theory. A more unusual topic is Massey products, which are discussed in [6] and provide another interpretation of the higher homotopies, in addition to the Stasheff identities and the bar construction which we discussed.

To conclude my thesis, I would like to mention a result which provides an answer to a natural question arising from homological algebra. As homological algebra seeks to study complexes by taking quotient modules to obtain the homology, the question arises as to whether any information is lost in this process. This is equivalent to asking whether it is possible to reconstruct the original complex (up to quasi-isomorphism) given its homology or whether some additional structure is needed in order to be able to do this. This is presented as one of the motivating problems and is proven in [8], and the answer, which should come as no surprise, is that the additional structure needed is an A_∞-structure constructed on the homology of the complex.

References

[1] T. Kadeishvili, *On the homology theory of fibre spaces*, Russ. Math. Surv. **35** (1980), no. 3, 231-238.

[2] S. A. Merkulov, *Strong homotopy algebras of a Kähler manifold*, Internat. Math. Res. Notices 1999, no. **3**, 153-164

[3] Maclane S., *Homology*, Springer-Verlag: Berlin, 1963.

[4] Chan, D., Homology and Homological Algebra, *Notes typeset by Kenneth Chan* (2005). `http://www.maths.unsw.edu.au/~danielch/homology05b/halec.pdf`

[5] D.-M. Lu, J.H. Palmieri, Q.-S. Wu, and J.J. Zhang, A_∞-algebras for ring theorists, Algebra Colloquium, **11** (2004), no. 1, 91-128.

[6] D.-M. Lu, J.H. Palmieri, Q.-S. Wu, and J.J. Zhang, A_∞-structure on Ext-algebras, available online as arXiv:math/0606144v1, 2006.

[7] K. Lefèvre-Hasegawa, Sur Les A_∞-Catégories, Université Paris 7, Thése de Doctorat, Année 2002.

[8] B. Keller, Introduction to A-infinity algebras and modules, *Homology Homotopy Appl.* **3** (1) (2001) 1-35 (electronic).

[9] J. D. Stasheff, *Homotopy associativity of H-spaces*, I, Trans. Amer. Math. Soc. **108** (1963), 275-292.

[10] J. D. Stasheff, *Homotopy associativity of H-spaces*, II, Trans. Amer. Math. Soc. **108** (1963), 293-312.

[11] M. Kontsevich, *Homological algebra of Mirror Symmetry*, Proceedings of the International Congress of Mathematicians, Zürich, Switzerland 1994, Vol. 1, Birkhäuser Basel, 1995, 120-139.