GEOMETRIC INVARIANT THEORY

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October 2012

Submitted in partial fulfillment of the requirements of the degree of
Bachelor of Science with Honours

Introduction

Geometric Invariant Theory is the study of quotients in the context of algebraic geometry. Many objects we would wish to take a quotient of have some sort of geometric structure and Geometric Invariant Theory (GIT) allows us to construct quotients that preserve geometric structure. Quotients are naturally arising objects in mathematics. Given an object with an equivalence relation on the elements, the quotient gives a simpler object which retains information about the original object whilst removing unnecessary data, by considering equivalent elements as the same.

When we study an object, we may have an equivalence of elements where we are not concerned with the distinction between equivalent elements. By way of example, an analogous situation areises when studying matrices - we often want to study the similarity classes of matrices and not distinguish similar matrices.

We let a group G act on a geometric object X. The action of G gives a partition of X in to G-orbits, which defines an equivalence relation on X. It is not always the case that the set of G-orbits has a geometric structure. The Geometric Invariant Theory quotient is a construction that partitions G-orbits to some extent, while preserving some desirable geometric properties and structure.

For affine sets, the construction of the GIT quotient is well understood and is determined uniquely. In the projective case, the natural way to construct a quotient is to glue together quotients of affine sets. This can not always be done simply, or in a unique way. The G-action must be extended to cover affine subsets correctly, which introduces a certain degree of choice in the construction. An obvious question to ask is: How is the quotient constructed dependent on this choice? And further: Are there any relations that exist between differently constructed quotients? I will set up the construction of the GIT quotient and explore the relations existing between quotients constructed in different ways.

In the first chapter, I will give some background on algebraic geometry. I will introduce the concept of a variety, the objects of study, and the topology used on varieties. I will introduce the relation between algebra and geometry in this setting, and how a variety can be identified with its coordinate ring, in both the affine and projective case.

The second chapter introduces the morphisms in the category of varieties. In algebraic geometry we are largely concerned with the set of functions that act on a variety. I will define these functions and use these to define morphisms of varieties. There is a strong duality between the categories of affine varieties and finitely generated algebras. I will set up this correspondence and prove some results that are necessary for studying morphisms of varieties.

Thirdly, I will describe the construction of the GIT quotient and show the difficulties that arise in the projective setting. Only certain groups and group actions are allowable in the construction of the GIT quotient. I will define what properties are required of a group and in particular show that the multiplicative group $\mathbb{C}^{\times}$ and any finite group satisfy the required conditions. The construction of the quotient in the projective setting is subtle. I will describe explicitly how a quotient satisfying desirable geometric properties can be constructed and give some examples of how the choice in the construction can give non-isomorphic quotients.

In the fourth chapter I will introduce rational maps. These are another type of weaker maps between varieties. When there are invertible rational maps between varieties, we say that the varieties are birationally equivalent. This is an equivalence of varieties weaker than that of isomorphisms, although it still shows a very strong similarity of varieties.

Lastly, I will explore how differently constructed quotients of affine varieties are related. This chapter is largely based on a paper by Thaddeus [13] and I will look at different constructions of quotients of affine varieties acted on by the multiplicative group $\mathbb{C}^{\times}$. There are strong relations existing between the different quotients in this case and I will look at a key examples, demonstrating the maps that relate the different quotients.

Contents

Chapter 1

Algebraic geometry

To study quotients in algebraic geometry we first need to set up exactly what we are studying. We need to define the objects we wish to construct quotients of and what sort of structure we have on these objects. We work in the setting of either affine or projective space and the objects we are interested in are *varieties*, which are sets of points described as the set of solutions to polynomial equations. In this sense we get a relation between varieties and the polynomials that vanish on them.

In this chapter we will define formally what a variety is in both the affine and projective setting. We will introduce the topology that we will use on these varieties, defined in terms of polynomial equations, called the *Zariski* topology. There is also an important correspondence between varieties and sets of polynomials that we will establish. This correspondence will allow us to study varieties in terms of the polynomials which vanish on them.

Note that we work over algebraically closed fields of characteristic zero. A field k will be assumed to have these properties.

Many of the definitions in this chapter concerning varieties, zero sets and the Zariski topology are based on the online article by Arapura [7].

1.1 Affine varieties

Affine varieties are subsets of affine space that satisfy certain properties. They are the zero set of some family of polynomials and *irreducible* in some sense, which we will define later.

We define *affine n-space* over a field k to be the set of tuples

$$\mathbb{A}_k^n = \{(x_1, ..., x_n) : x_i \in k\}.$$

If the field is clear from the context, we simply write $\mathbb{A}^n$.

Of particular interest in algebraic geometry is the *coordinate ring* of $\mathbb{A}_k^n$, which is the polynomial ring on n variables, $k[x_1, ..., x_n]$. Note that elements of $k[x_1, ..., x_n]$ define mappings from $\mathbb{A}^n$ to k by evaluation. This gives us the notion of *algebraic* *sets*, which are the zero sets of polynomials. These will be used to define a topology on $\mathbb{A}^n$. First we introduce notation for taking zero sets.

Definition 1.1. Let $S \subseteq k[x_1, ..., x_n]$ be a set of polynomials. The *zero set* of S, denoted $V(S)$ is the subset of $\mathbb{A}_k^n$ defined as

$$V(S) = \{x \in \mathbb{A}_k^n : f(x) = 0, \text{ for all } f \in S\}$$

A subset X of $\mathbb{A}_k^n$ is defined to be *algebraic* if it is the zero set of some family of polynomials in the coordinate ring of $\mathbb{A}_k^n$. That is, $X \subseteq \mathbb{A}_k^n$ is algebraic if there exists a subset $S \subseteq k[x_1, ..., x_n]$ such that $X = V(S)$.

The following lemma shows that we can take S to be an ideal in this definition.

Lemma 1.2. *A set $X \subseteq \mathbb{A}_k^n$ is algebraic if and only if $X = V(I)$ for some ideal $I \trianglelefteq k[x_1, ..., x_n]$.*

Proof.

By definition, if $X = V(I)$ for some ideal I, then X is algebraic.

If X is algebraic, then $X = V(S)$ for some $S \subseteq k[x_1, ..., x_n]$. Consider the ideal $\langle S \rangle$ generated by elements of S. We will show that $V(S) = V(\langle S \rangle)$. Let $f \in \langle S \rangle$, then f is of the form $\sum g_i h_i$ for some $g_i \in S$, $h_i \in k[x_1, ..., x_n]$. For all $x \in X$ we have $f(x) = \sum g_i(x) h_i(x) = 0$, as $g_i \in S$ for each g_i. Hence $X \subseteq V(\langle S \rangle)$. Conversely, suppose $x \in V(\langle S \rangle)$. Then for all $f \in \langle S \rangle$ we have $f(x) = 0$. In particular, we have $f(x) = 0$ for any $f \in S$, as $S \subseteq \langle S \rangle$, hence $V(\langle S \rangle) \subseteq V(S) = X$ and so $X = V(\langle S \rangle)$, and thus is the zero set of some ideal in $k[x_1, ..., x_n]$.

□

The *Zariski topology*, which we will be using in this setting of algebraic geometry, is defined in terms of algebraic sets. It is the topology where the closed sets are precisely the algebraic ones. To show this defines a topology we first need some basic properties of algebraic sets and ideals.

Lemma 1.3. *Let I_1, I_2 be ideals in $k[x_1, ..., x_n]$. Then*

1. *If $I_1 \subseteq I_2$ then $V(I_1) \supseteq V(I_2)$*
2. *$V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$*
3. *For any set $\{I_\alpha\}$ of ideals, $\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right)$*

Proof.

1. Suppose $I_1 \subseteq I_2$ and let $x \in V(I_2)$. Then $f(x) = 0$ for all $f \in I_1$, as $I_1 \subseteq I_2$ and so $x \in V(I_1)$, thus $V(I_2) \supseteq V(I_1)$.

2. Using part 1. and as $I_1, I_2 \supseteq I_1 \cap I_2$ we have $V(I_1 \cap I_2) \supseteq V(I_1)$ and $V(I_1 \cap I_2) \supseteq V(I_2)$. Thus we have $V(I_1 \cap I_2) \supseteq V(I_1) \cup V(I_2)$.
   Suppose $x \in V(I_1 \cap I_2)$, we aim to show $x \in V(I_1) \cup V(I_2)$. If $x \in V(I_1)$, then we are done. Otherwise, we assume $x \notin V(I_1)$ and so there exists $f_0 \in I_1$ with $f_0(x) \neq 0$. For all $g \in I_2$, we have $f_0 g \in I_1 \cap I_2$ and so $f_0 g(x) = 0$. But $f_0(x) \neq 0$, so we must have $g(x) = 0$ for all $g \in I_2$. Thus $x \in V(I_2)$. Hence we have $V(I_1 \cap I_2) \subseteq V(I_1) \cup V(I_2)$ and so $V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$
3. Suppose $x \in \bigcap_\alpha V(I_\alpha)$. Then $f(x) = 0$ for all $f \in I_\alpha$ for all I_α, hence $x \in V\left(\sum_\alpha I_\alpha\right)$ and so $\bigcap_\alpha V(I_\alpha) \subseteq V\left(\sum_\alpha I_\alpha\right)$
   Applying part 1. gives $V\left(\sum_\alpha I_\alpha\right) \subseteq V(I_\beta)$ for any $I_\beta \in \{I_\alpha\}$, as $I_\beta \subseteq \sum_\alpha I_\alpha$. Hence $V\left(\sum_\alpha I_\alpha\right) \subseteq \bigcap_\alpha V(I_\alpha)$ and so $\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right)$.

□

Lemma 1.4.

There is a topology on $\mathbb{A}^n$, defined by declaring the closed sets to be precisely the algebraic sets.

Proof.

We need to show that the union of any two algebraic sets is algebraic, that the arbitrary intersection of any family of algebraic sets is algebraic and that the empty set and $\mathbb{A}^n$ are algebraic. We have that the union of any two algebraic sets is algebraic, as well as that any arbitrary intersection is algebraic from parts 2 and 3 of the previous lemma. To complete the proof, we see that $V(1) = \emptyset$ and $V(0) = \mathbb{A}^n$ and so the empty set and $\mathbb{A}^n$ are both algebraic.

□

This topology is called the *Zariski toplogy*, and is the topology used in the study of algebraic geometry. Henceforth, this is the topology that will be meant when referring to topological properties such as open and closed.

To define a variety, one more property is needed. A non-empty subset Y of a topolological space X is defined to be *irreducible* if it is not the union of two proper closed subsets of Y.

Definition 1.5. An *affine variety* is an irreducible closed subset of affine space $\mathbb{A}^n$.

That is, a variety is an algebraic subset of $\mathbb{A}^n$ that is not the union of two smaller algebraic subsets.

These are the objects of interest in algebraic geometry. Some examples of affine varieties include curves in $\mathbb{A}^2$ such as the parabola defined by $y - ax^2 - bx - c = 0$, or the elliptic curve defined by $y^2 - x^3 - x - 1 = 0$. The sphere in $\mathbb{A}^3$ defined by $x^2 + y^2 + z^2 - 1 = 0$ is also an affine variety.

We can also have a geometric structure on matrices by embedding $n \times n$ matrices in $\mathbb{A}^{n \times n}$. We then have examples of varieties such as $SL(n, k)$, the space of $n \times n$ matrices satisfying $det(A) - 1 = 0$.

We study varieties by looking at the polynomials that vanish on them, so that we can reduce geometric problems into questions of algebra. We now introduce the ideal associated with a subset of $\mathbb{A}^n$.

Definition 1.6.

Let X be a subset of $\mathbb{A}^n_k$. The *ideal* associated with X is defined as

$$\mathcal{I}(X) = \{f \in k[x_1, ..., x_n] : f(x) = 0 \text{ for all } x \in X\}.$$

Note that this is indeed an ideal. Let $f, g \in \mathcal{I}(X)$, $h \in k[x_1, ..., x_n]$. For $x \in X$ we have $(f + g)(x) = f(x) + g(x) = 0$ as well as $fh(x) = f(x)h(x) = 0$

We now aim to establish a correspondence between varieties and prime ideals, using the maps $\mathcal{I}$ and V. These maps in fact give a bijection between varieties in $\mathbb{A}^n_k$ and prime ideals in $k[x_1, ..., x_n]$. We will prove this bijection by showing that the maps $\mathcal{I}$ and V are inverses of each other on these domains.

Lemma 1.7.

$V(\mathcal{I}(X)) = \overline{X}$, the closure of X

Proof.

As $\overline{X}$ is closed, $\overline{X} = V(I)$ for some ideal I. As $f(x) = 0$ for all $f \in I$, $x \in X$, we have $I \subseteq \mathcal{I}(X)$. Thus $V(\mathcal{I}(X)) \subseteq V(I)$ by Lemma 1.3. Also we have $X \subseteq V(\mathcal{I}(X))$, as X is clearly contained in the zero set of $\mathcal{I}(X) = \{f : f(x) = 0 \text{ for all } x \in X\}$. As $V(\mathcal{I}(X)$ is closed, by minimality of the closure we must have $\overline{X} \subseteq V(\mathcal{I}(X))$. Hence $V(\mathcal{I}(X)) = \overline{X}$.

$\square$

For the other direction we consider the *radical* of an ideal.

Definition 1.8. The *radical* of an ideal $I \trianglelefteq R$ is defined as

$$\sqrt{I} := \{f \in R : f^n \in I, \text{ for some } n \in \mathbb{N}\}$$

A *radical ideal* is an ideal I such that $\sqrt{I} = I$.

Note that we have $I \subseteq \sqrt{I}$ for any ideal I.

Theorem 1.9. *(Hilbert's Nullstellensatz)*

Let k be algebraically closed and $I \trianglelefteq k[x_1, ..., x_n]$.
Then $\mathcal{I}(V(I)) = \sqrt{I}$

We omit the proof of this well-known theorem, see [9] for a proof.

Note that if X is a closed subset of $\mathbb{A}^n$ then $\mathcal{I}(X)$ is radical, as $\sqrt{\mathcal{I}(X)} = \mathcal{I}(V(\mathcal{I}(X))) = \mathcal{I}(X)$ using Lemmas 1.7 and 1.9. Also $V(I)$ is closed for any ideal I by definition. Thus there is a bijection between closed subsets of $\mathbb{A}_k^n$ and radical ideals of $k[x_1, ..., x_n]$ by the maps

$$X \mapsto \mathcal{I}(X)$$
$$I \mapsto V(I)$$

because on these domains $\mathcal{I}$ and V are inverses of each other.

This bijection induces a bijective correspondence between varieties and prime ideals. We obtain this from the following lemma.

Lemma 1.10. *Let X be a closed subset of $\mathbb{A}^n$. Then X is a variety if and only if $\mathcal{I}(X)$ is prime.*

Proof.

Suppose X is not a variety. As X is closed but not irreducible we have that X is the union of two proper closed sets. By the correspondence between closed sets and radical ideals we thus have $X = V(\mathcal{I}(X)) = V(J_1) \cup V(J_2) = V(J_1 \cap J_2)$ for some radical ideals J_1, J_2 such that $V(J_i) \subset X$. Using Lemma 1.9 we have $\sqrt{J_1 \cap J_2} = \mathcal{I}(V(J_1 \cap J_2)) = \mathcal{I}(X)$, so $J_1 \cap J_2 \subseteq \mathcal{I}(X)$. Consider an element $x \in X - V(J_1)$. There exists $f_1 \in J_1$ such that $f_1(x) \neq 0$, so $f_1 \notin \mathcal{I}(X)$. Similarly, there exists $f_2 \in V(J_2)$ such that $f_2 \notin \mathcal{I}(X)$. However $f_1 f_2 \in J_1 \cap J_2 \subseteq \mathcal{I}(X)$, thus $\mathcal{I}(X)$ is not prime.

For the other direction, suppose that $\mathcal{I}(X) \trianglelefteq k[x_1, ..., x_n]$ is not prime. Then there exists $f_1, f_2 \in k[x_1, ..., x_n] \setminus \mathcal{I}(X)$ such that $f_1 f_2 \in \mathcal{I}(X)$. We have $X = V(\mathcal{I}(X)) \subseteq V(f_1 f_2) = V(f_1) \cup V(f_2)$, and so $X = X \cap V(f_1 f_2) = (X \cap V(f_1)) \cup (X \cap V(f_2))$. But the intersection of two closed sets is closed, so this decomposition of X is a union of closed sets. Also, $X \cap V(f_i)$ is a proper subset of X, as $f_i \notin \mathcal{I}(X)$ implies $f_i(x) \neq 0$ for some $x \in X$, so $x \notin V(f_i)$ for such an x. Thus we have X is the union of two closed proper subsets and so is not irreducible, hence X is not a variety.

□

So for a variety X, the corresponding ideal $\mathcal{I}(X)$ is prime. Similarly, as any prime ideal is radical, we have $V(I)$ is a variety for any prime ideal I. Thus the bijection between radical ideals and closed sets induces a bijection between varieties and prime ideals.

One last correspondence we wish to show is that between points in $\mathbb{A}^n_k$ and maximal ideals of $k[x_1, ...x_{,n}]$. First we show the following lemma, which is a direct consequence of Hilbert's Nullstellensatz.

Lemma 1.11.

Let I be an ideal of $k[x_1, ..., x_n]$. If $I \neq k[x_1, ..., x_n]$, then $V(I) \neq \emptyset$.

Proof.

Suppose $V(I) = \emptyset$. Then $\sqrt{I} = \mathcal{I}(V(I)) = \mathcal{I}(\emptyset) = k[x_1, ..., x_n]$. So $1 \in \sqrt{I}$ and thus $1^n \in I$ for some $n \in \mathbb{N}$, that is $1 \in I$ and therefore $I = k[x_1, ..., x_n]$.

$\square$

To obtain a bijection between points in $\mathbb{A}^n_k$ and maximal ideals in $k[x_1, ..., x_n]$, we demonstrate a bijection between sets consisting of a single point and the maximal ideals. This bijection will come from the previous bijections, as sets consisting of a single point are closed.

Lemma 1.12.

Let X be a closed subset of $\mathbb{A}^n_k$. Then X consists of exactly one point if and only if $\mathcal{I}(X)$ is maximal ideal.

Proof.

Suppose X consists of exactly one point, that is, $X = \{a\}$ for some $a \in \mathbb{A}^n$. For a contradiction, suppose that $\mathcal{I}(X)$ is not maximal. As $1 \neq 0$ at a we have $\mathcal{I}(X) \neq k[x_1, ..., x_n]$, so there exists a maximal ideal $J \supset \mathcal{I}(X)$. As J is maximal, it is therefore prime and radical and so by the previous correspondence, we have $V(J) \neq V(\mathcal{I}(X))$, as $J \neq \mathcal{I}(X)$. Thus, using Lemma 1.3, we have $V(J) \subset V(\mathcal{I}(X)) = \{a\}$. But then we must have $V(J) = \emptyset$ and so, by the previous lemma, $J = k[x_1, ..., x_n]$, which contradicts J being maximal. Thus $\mathcal{I}(X)$ must be maximal.

For the other direction, suppose that $\mathcal{I}(X)$ is maximal, then $X = V(\mathcal{I}(X)) \neq \emptyset$ as $\mathcal{I}(X) \neq k[x_1, ..., x_n]$. So there exists a point $a = (a_1, ..., a_n) \in X$. Note that $\{a\}$ is closed, as it is the zero set of $\langle x_1 - a_1, ..., x_n - a_n \rangle$. As $a \in X$, we have $\mathcal{I}(\{a\}) \supseteq \mathcal{I}(X)$ and $\mathcal{I}(\{a\}) \neq k[x_1, ..., x_n]$ as $1 \neq 0$ at a. By maximality of $\mathcal{I}(X)$ we must have $\mathcal{I}(\{a\}) = \mathcal{I}(X)$ and hence $\{a\} = V(\mathcal{I}(\{a\})) = V(\mathcal{I}(X)) = X$.

$\square$

Any set $\{a\} \subseteq \mathbb{A}^n$ consisting of a single point $a = (a_1, ..., a_n) \in \mathbb{A}^n$ is closed, as $\{a\} = V(\langle x_1 - a_1, ..., x_n - a_n \rangle)$, so the above lemma induces a bijective correspondence between sets in $\mathbb{A}^n_k$ consisting of a single point and maximal ideals in

$k[x_1, ..., x_n]$. Therefore we have a bijective correspondence between points in $\mathbb{A}_k^n$ and maximal ideals in $k[x_1, ..., x_n]$.

1.2 The Zariski topology

Let $X \subseteq \mathbb{A}^n$ be an affine variety. We have a topology on X induced by the Zariski topology on $\mathbb{A}^n$, whereby a set $S \subseteq X$ is closed precisely when S is closed in $\mathbb{A}^n$. That is, a set S is closed if it is the zero set of some family of polynomials which is also zero on X. Note that with this definition, X is closed in $\mathbb{A}^n$ and hence is closed in X, so $\emptyset$ is open. The other axioms of being a topology follow immediately from the topology on $\mathbb{A}^n$.

We wish to have a basis for the open sets of a variety X. We define the basic open sets to be

$$X_f = \{x \in X : f(x) \neq 0\}$$

Note that X_f is indeed an open set, as we have $V(f) \cap X = V(f) \cap V(\mathcal{I}(X)) = V(\langle f \rangle \cup \mathcal{I}(X))$, hence $\{x \in \mathbb{A}^n : f(x) = 0\} \cap X$ is closed in X. The complement of this set in X is X_f, so X_f is open in X. Moreover, we can show these sets form a basis for the topology on X.

To prove this, we need to know that any ideal in $k[x_1, ..., x_n]$ is finitely generated. This is a result of Hilbert's basis theorem, which states that if all ideals of a ring R are finitely generated, then all ideals of $R[x]$ are also finitely generated.

Theorem 1.13. *(Hilbert's Basis Theorem)*

Let R be a commutative ring such that all ideals $I \trianglelefteq R$ are finitely generated. Then all ideals of $R[x]$ are finitely generated.

Proof.

This proof is based on one available online [4]

Suppose for a contradiction that there exists an ideal $I \trianglelefteq R[x]$ which is not finitely generated. Then we can construct a sequence of finitely generated ideals contained within I as follows.

Choose $f_1 \in I$ of minimal degree and let $J_1 = \langle f_1 \rangle$. We have $J_1 \subset I$, as I is not finitely generated. Inductively for all $n \in \mathbb{N}$, we can find $f_{n+1} \in I \setminus J_n$ of minimal degree, as $J_n \subset I$. Then we let $J_{n+1} = \langle f_1, ..., f_{n+1} \rangle$, which is finitely generated with $J_{n+1} \subset I$.

Consider the leading co-efficients a_i of the f_i in the above construction of the J_i. Let $H \trianglelefteq R$ be the ideal generated by all of the a_i as i goes to infinity, that is, $H = \langle a_1, a_2, ... \rangle$. As H is finitely generated there exists some N such that $H = \langle a_1, ..., a_N \rangle$. As $a_{N+1} \in H$ we thus have $a_{N+1} = b_1 a_1 + ... b_N a_N$ for some $b_i \in R$.

Now consider the polynomial $g = b_1 f_1 x^{d_1} + ... + b_N f_N x^{d_N}$, where $d_i = deg(f_{N+1}) - deg(f_i)$. We have $deg(g) = deg(f_{N+1})$ and the co-efficient of the leading term of g is $b_1 a_1 + ... + b_N a_N = a_{N+1}$. Thus we have $deg(f_{N+1} - g) < deg(f_{N+1})$. As $g \in J_N$ and $f_{N+1} \in I - J_N$ we must also have $f_{N+1} - g \in I - J_N$. But this contradicts the construction of the J_N, as f_{N+1} is not of minimal degree in $I - J_N$. Thus I must be finitely generated.

$\square$

With this theorem we now know that any ideal of $k[x_1, ..., x_n]$ must be finitely generated. As k is a field, the only ideals are 0 and k, which are both finitely generated. Then Hilbert's basis theorem tells us that all ideals of $k[x_1]$ must be finitely generated. Continuing to apply Hilbert's basis theorem on the n variables we must have that all ideals of $k[x_1, ..., x_n]$ are finitely generated.

Lemma 1.14.

Let $U \subseteq X$ be open. Then there exist finitely many basic open sets $X_{f_1}, ..., X_{f_r}$ such that $U = \bigcup_{i=0}^{r} X_{f_i}$

Proof.

If $U \subseteq X$ is open then $U^c = X \setminus U$, the complement of U in X, is closed in X and hence in $\mathbb{A}^n$. Then we have $U^c = V(J)$ for some ideal $J \trianglelefteq k[x_1, ..., x_n]$. As J is finitely generated we can write $J = \langle f_1, ..., f_r \rangle$ for some polynomials $f_i \in k[x_1, ..., x_r]$, and it follows that $U^c = V(\langle f_1, ..., f_r \rangle) = V(f_1) \cap ... \cap V(f_r)$.

By De Morgans law, we can take the complement inside X to get $U = V(f_1)^c \cup ... \cup V(f_r)^c$. For each i, we have $V(f_i)^c = \{x \in X : f_i(x) \neq 0\} = X_{f_i}$, thus we have a decomposition of U into basic open sets as $U = X_{f_1} \cup ... \cup X_{f_r}$.

$\square$

1.3 Projective varieties

It is natural to wish to also work in the projective setting, as projective space is complete in the sense that it contains all of its limit points. Projective space is also locally affine and can be viewed as the gluing together of affine spaces. Often when studying projective space we will view it as the gluing together of affine varieties.

The n-dimensional projective space over k, written $\mathbb{P}^n_k$, is defined as the set of one-dimensional subspaces of $\mathbb{A}^{n+1}_k$.

An equivalent definition of $\mathbb{P}^n_k$ is the set of equivalence classes of points $\{(x_0, ..., x_n) : x_i \in k, \text{ not all zero}\} \subset \mathbb{A}^{n+1}$ under the equivalence relation given by $(x_0, ..., x_n) \sim (\lambda x_0, ..., \lambda x_n)$. We denote points of $\mathbb{P}^n_k$ as $(x_0 : ... : x_n)$, $x_i \in k$, not all zero, where we have equality of points $(x_0 : ... : x_n) = (y_0 : ... : y_n)$ if and only if $(y_0, ..., y_n) = (\lambda x_0, ..., \lambda x_n)$ for some $\lambda \in k^\times$.

Note that the subset $\mathbb{P}^n_j$ defined as the subset of $\mathbb{P}^n_k$ where the j'th coordinate is non-zero can be written as

$$\mathbb{P}^n_j = \{(\frac{x_0}{x_j} : ... : \frac{x_{j-1}}{x_j} : 1 : \frac{x_{j+1}}{x_j} : ... \frac{x_n}{x_j}) : x_i \in k\}$$

This looks like $\mathbb{A}^n$, because $x_j \neq 0$ implies that $\frac{x_i}{x_j}$ ranges over all of the base field k for $i \neq j$. We then have $\mathbb{P}^n$ is the union of these subsets looking like affine space.

We aim to define what a variety is in the projective setting, as well as the Zariski topology we use on projective space. As in the affine setting, we study projective space by looking at how polynomials act on the space. To define which sets are algebraic, we need to consider the zero sets of polynomials. However this is more subtle in projective space, as we can not as simply evaluate polynomials at the coordinates of a point, due to there being multiple coordinate representations of the same point in projective space.

For example, consider $\mathbb{P}^1_{\mathbb{C}} = \{(x : y) : x, y \in \mathbb{C}, \text{ not both } 0\}$. The natural set of polynomials to study on this set would be $\mathbb{C}[x, y]$ and we would like to be able to evaluate a polynomial $f \in \mathbb{C}[x, y]$ at a point in $\mathbb{P}^1$ by substituting in the coordinates. However, such evaluation is undefined. For example, let $f = x + 1$, then the evaluation of f at $(1 : 0)$ would be $f(1, 0) = 2$. For any $a \in \mathbb{C}^\times$, we have $(a : 0) = (1 : 0)$ as representations of the same point in $\mathbb{P}^1$. But for $a \neq 1$, $f(a, 0)$ gives a different evaluation of the same point. However, we can still make some sense of how $\mathbb{C}[x, y]$ acts on $\mathbb{P}^1$ by looking at *homogeneous* polynomials. To study homogeneous polynomials we define a graded ring.

Definition 1.15.

An $\mathbb{N}$-graded ring is a ring R with decomposition $R = \bigoplus_{i=0}^{\infty} R_i$, such that for any elements $f_i \in R_i$, $f_j \in R_j$, we have $f_i f_j \in R_{i+j}$. The elements of R_d are said to be homogeneous of degree d.

A $\mathbb{Z}$-graded ring is a ring with the same type of decomposition, but where negative degrees are also allowed.

We see that the polynomial ring $k[x_0, ..., x_n]$ is graded by the degree of the terms. We set each of the x_i to be homogeneous of degree one and we see that for any $f \in k[x_0, ..., x_n]$ we have a unique decomposition of f in to its homogeneous parts. That is, we can write f uniquely as $f = f_1 + ... + f_r$ with each f_i homogeneous of distinct degree.

We define the coordinate ring of $\mathbb{P}^n_k$ to be $k[x_0, ..., x_n]$, the graded polynomial ring on $n+1$ variables. For $f \in k[x_0, ..., x_n]$, we cannot define a function $\mathbb{P}^n_k \to k$ by evaluating f at the coordinates of $x \in \mathbb{P}^n_k$, as we do not have $f(x_0, ..., x_n) = f(\lambda x_0, ..., \lambda x_n)$ in general.

However, for $x = (x_0 : ... : x_n) \in \mathbb{P}^k_n$ and f homogeneous in $k[x_0, ..., x_n]$, we can define a function $\mathbb{P}^n_k \to \{0, 1\}$, by $f(x) = 0$ if $f(x_0, ..., x_n) = 0$ and $f(x) = 1$ otherwise. This is well-defined, as if f is homogenous of degree d, then for any $\lambda \in k^\times$, we have $f(\lambda x_0, ... \lambda x_n) = \lambda^d f(x_0, ..., x_n) = 0$ if and only if $f(x_0, ..., x_n) = 0$.

We can extend this notion of a homogeneous polynomial being zero at a point to any polynomials. As any polynomial $f \in k[x_0, ..., x_n]$ can be written as the sum of its homogeneous parts $f = f_1 + ... + f_r$, we can define when any polynomial is 0 at a point $x \in \mathbb{P}^n$, by declaring that $f(x) = 0$ precisely when $f_i(x) = 0$ for all homogeneous parts f_i of f.

This leads us to the definition of algebraic sets in the projective setting.

Definition 1.16.

Let $S \subseteq k[x_0, ..., x_n]$ be a set of polynomials. We define the zero set of S in projective space to be

$$V(S) := \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in S\}$$

We define a subset of projective space to be *algebraic* if it the zero set of some family of polynomials.

Note that we also have an affine definition of V that takes a subset of $k[x_1, ..., x_n]$ and gives a subset of $\mathbb{A}^n$. Where it is not clear from the context, we will use the notation $V_{\mathbb{A}}$, $V_{\mathbb{P}}$ to denote whether the zero set is taken in affine or projective space.

We now consider ideals generated by homogeneous elements. These will be the ones used in the projective setting, as they correspond with subsets of projective space.

Definition 1.17.

A ideal I of a ring R is said to be *homogeneous* if it is generated by homogeneous elements.

As in the affine setting, we wish to have some sort of correspondence between algebraic sets and ideals. We can simplify the definition of an algebraic set to the following, in terms of ideals.

Lemma 1.18.

A set $Y \subseteq \mathbb{P}^n$ is algebraic if and only if it is the zero set of some homogeneous ideal $I \trianglelefteq R$.

Proof.

By definition we have that if Y is the zero set of some homogeneous ideal then it is algebraic.

Suppose Y is algebraic, then $Y = V(S)$ for some $S \subseteq k[x_0, ..., x_n]$. Let $S' \subseteq k[x_0, ..., x_n]$ be the set consisting of all homogeneous parts of elements of S. That is, if $f \in S$ with decomposition into homogeneous parts $f_1 + ... + f_r$ then we say each homogeneous f_i is in S'. Then by our definition of where a polynomial is zero, we have $V(S) = V(S')$. As in the proof of Lemma 1.2 for the affine case, we can see that $V(S') = V(\langle S' \rangle)$. Thus we have $Y = V(S) = V(\langle S' \rangle)$ and $\langle S' \rangle$ is generated by homogeneous polynomials, completing the proof.

$\square$

Similarly as in the affine case, we define the Zariski topology on $\mathbb{P}^n$ by declaring the closed sets to be precisely the algebraic sets. We need the following results for this to define a topology.

Lemma 1.19. *Let I_1, I_2 be homogeneous ideals in $k[x_0, ..., x_n]$. Then*

1. *If $I_1 \subseteq I_2$ then $V(I_1) \supseteq V(I_2)$*
2. *$V(I_1) \cup V(I_2) = V(I_1 \cap I_2)$*
3. *For any set $\{I_\alpha\}$ of ideals, $\bigcap_\alpha V(I_\alpha) = V\left(\sum_\alpha I_\alpha\right)$*

These results apply by the same arguments as in the proof of Lemma 1.3, the affine analogue of this lemma. As we also have $V(1) = \emptyset$ and $V(0) = \mathbb{P}^n$, the Zariski topology is indeed a topology. We can now define a variety in the projective setting.

Definition 1.20.

A projective variety is an irreducible algebraic set $X \subseteq \mathbb{P}^n$.

As in the affine case, we have the Zariski topology on a projective variety $X \subseteq \mathbb{P}^n$ induced by the Zariski topology on $\mathbb{P}^n$. We also have a basis for this topology with sets of the form

$$X_f := \{x \in X : f(x) \neq 0\}$$

for homogeneous $f \in k[x_0, ..., x_n]$. The proof that this forms a basis follows the same argument as for the affine case.

We wish to set up a correspondence between varieties and ideals in the projective setting. We will do this by looking at the correspondence there exists between sets in $\mathbb{P}^n$ and sets of one dimensional subspaces in $\mathbb{A}^{n+1}$.

As in the affine case, we can also take the *ideal* of a projective set

$$\mathcal{I}(X) := \{f \in k[x_0, ..., x_n] : f(x) = 0 \text{ for all } x \in X\}$$

We will denote by $\mathcal{I}_{\mathbb{A}}$ and $\mathcal{I}_{\mathbb{P}}$ the ideals of affine and projective sets respectively when the context is not clear.

So that we can use results about affine space in the projective setting, we will set up a correspondence between projective sets and certain affine sets.

Let $X \subseteq \mathbb{P}^n$ be non-empty. We define the *cone* over X, a subset of $\mathbb{A}^{n+1}$, to be

$$cone(X) := \{(x_0, ..., x_n) : (x_0 : ... : x_n) \in X\} \cup \{(0, ..., 0)\}$$

We also define $cone(\emptyset)$ to be the empty set in $\mathbb{A}^{n+1}$.

Considering projective space $\mathbb{P}^n$ as the set of lines through the origin in $\mathbb{A}^{n+1}$, the cone of a set $X \subseteq \mathbb{P}^n$ is simply the image of those lines in $\mathbb{A}^{n+1}$. If a set $Y \subseteq \mathbb{A}^{n+1}$ is the union of lines through the origin, then we call Y a *cone*. We see then that Y is a cone in $\mathbb{A}^{n+1}$ if and only if $Y = cone(X)$ for some $X \subseteq \mathbb{P}^n$. As taking the cone of a subset of $\mathbb{P}^n$ is an injective mapping into the set of cones in $\mathbb{A}^{n+1}$, we see that there is a bijective correspondence between subsets of $\mathbb{P}^n$ and cones in $\mathbb{A}^{n+1}$.

What's more, we can see that algebraic sets in $\mathbb{P}^n$ correspond to algebraic cones in $\mathbb{A}^{n+1}$, as for non-empty $X \subseteq \mathbb{P}^n$ and a homogeneous ideal $I \trianglelefteq k[x_0, ..., x_n]$, we have

$$X = V_{\mathbb{P}}(I) = \{x \in \mathbb{P}^n : f(x) = 0 \text{ for all } f \in I\}$$

if and only if

$$cone(X) = V_{\mathbb{A}}(I) = \{x \in \mathbb{A}^{n+1} : f(x) = 0 \text{ for all } f \in I\}.$$

We say here for X to be non-empty so as to avoid complications with the ideal $\langle x_0, ..., x_n \rangle$, which has empty zero set in $\mathbb{P}^n$, but contains the origin if the zero set is taken in affine $\mathbb{A}^{n+1}$.

It is also clear that we have $\mathcal{I}_{\mathbb{P}}(X) = \mathcal{I}_{\mathbb{A}}(cone(X))$, where $\mathcal{I}_A$ is the ideal of an affine set. By this correspondence with the zero sets and ideals in projective and affine setting, we see that some of the results about affine space carry over to analogous results about projective space.

Lemma 1.21.

Let $X \subseteq \mathbb{P}^n$. Then $V_{\mathbb{P}}(\mathcal{I}_{\mathbb{P}}(X)) = \overline{X}$

Proof.

As $\overline{X}$ is closed, we have $X = V(I)$ for some homogeneous ideal $I \trianglelefteq k[x_0, ..., x_n]$. If $f \in I$, then f is zero on $\overline{X}$, so is zero on X and hence $I \subseteq \mathcal{I}(X)$. Thus we have $V(\mathcal{I}(X)) \subseteq \overline{X}$. Suppose $x \in X$, then $f(x) = 0$ for all $f \in \mathcal{I}(X)$ and so $x \in V(\mathcal{I}(X)$. Thus $X \subseteq V(\mathcal{I}(X)$. As $V(\mathcal{I}(X)$ is closed, by minimality of the closure we must have $\overline{X} \subseteq V(\mathcal{I}(X))$ and so $V(\mathcal{I}(X))) = \overline{X}$.

□

Lemma 1.22.

Let $I \trianglelefteq k[x_0, ..., x_n]$ be a homogeneous ideal such that $V_{\mathbb{P}}(I) \neq \emptyset$.
Then $\mathcal{I}_{\mathbb{P}}(V_{\mathbb{P}}(I)) = \sqrt{I}$

Proof.

This follows from the correspondence of projective sets with affine cones. We have $\mathcal{I}_{\mathbb{P}}(V_{\mathbb{P}}(I)) = \mathcal{I}_{\mathbb{A}}(cone(V_{\mathbb{P}}(I))$. Also, as $V_{\mathbb{P}}(I)$ is non-empty we have $cone(V_{\mathbb{P}}(I)) = V_{\mathbb{A}}(I)$, and so $I_{\mathbb{P}}(V_{\mathbb{P}}(I)) = \mathcal{I}_{\mathbb{A}}(V_{\mathbb{A}}(I)) = \sqrt{I}$.

□

We now show the correspondence of projective varieties with prime homogeneous ideals.

Lemma 1.23.

Let X be a closed set in $\mathbb{P}^n$. Then $\mathcal{I}(X)$ is homogeneous and X is a variety if and only if $\mathcal{I}(X)$ is prime.

Proof.

To show that $\mathcal{I}(X)$ is homogeneous, we see that for any projective set X we have $\mathcal{I}(X) = \{f \in k[x_0, ..., x_n] : f(x) = 0\}$ consists of all its homogeneous parts. That is, given a decomposition of some $f \in \mathcal{I}(X)$ as $f = f_1 + ... + f_r$ with the f_i homogeneous, then all of its homogeneous parts are also in $\mathcal{I}(X)$. This follows from the definition of when a polynomial is zero on a projective set. As any $f \in \mathcal{I}(X)$ is the sum of its homogeneous parts we thus have $\mathcal{I}(X)$ is generated by homogeneous elements, so is a homogeneous ideal.

Suppose that X is not a variety. As X is closed but not irreducible, we have $X = Y \cup Z$ for some closed sets Y and Z such that $Y \subset X$ and $Z \subset X$. Then we have $cone(X) = cone(Y) \cup cone(Z)$ with $cone(Y)$ and $cone(Z)$ both proper closed subsets of $cone(X)$. But then we have $cone(X)$ is not irreducible, so $\mathcal{I}_{\mathbb{A}}(cone(X)) = \mathcal{I}_{\mathbb{P}}(X)$ is not prime.

For the other direction, suppose that $\mathcal{I}(X) \trianglelefteq k[x_0, ..., x_n]$ is not prime. Then there exists $f_1, f_2 \in k[x_0, ..., x_n] \setminus \mathcal{I}(X)$ such that $f_1 f_2 \in \mathcal{I}(X)$. As $X = V(\mathcal{I}(X)) \subseteq V(f_1 f_2) = V(f_1) \cup V(f_2)$, then we have $X = X \cap V(f_1 f_2) = (X \cap V(f_1)) \cup (X \cap V(f_2))$ which is a decomposition of X as a union of closed sets. Also, $X \cap V(f_i)$ is a proper subset of X, as $f_i \notin \mathcal{I}(X)$ implies $f_i(x) \neq 0$ for some $x \in X$, so $x \notin V(f_i)$ for such an x. Thus we have X is the union of two closed proper subsets and so is not irreducible, hence X is not a variety.

□

We thus have a bijective correspondence between prime homogeneous ideals and non-empty varieties, as we have shown that $V_{\mathbb{P}}$ and $\mathcal{I}_{\mathbb{P}}$ are inverse of each other on these domains. We now move on to studying morphisms between varieties.

Chapter 2

Morphisms of varieties

To study varieties we need to define and study the morphisms between them. In this chapter we will define morphisms of varieties and look at some of their important properties. We will give equivalent definitions of morphisms when mapping to affine or projective varieties, and these equivalent definitions are simpler and more natural in their construction. There is a correspondence between affine varieties over k and k-algebras, as well as a correspondence between morphisms of affine varieties and homomorphisms of k-algebras. We will set up these correspondences which can simplify studying morphisms of varieties considerably.

In algebraic geometry we are interested in the polynomials associated with varieties, so it makes sense to define morphisms in terms of polynomials. For this we will look at their coordinate rings. Firstly we will define what it means for a function $X \to k$ to be *regular*, in both the affine and projective case. These definitions come from Hartshorne's book on algebraic geometry [8].

Definition 2.1.

Let $X \subseteq \mathbb{A}^n$ be an affine variety. A partial map $f : X \dashrightarrow k$ is said to be *regular* at a point $P \in X$ if there exists an open neighbourhood $U \subseteq X$ of P and polynomials $g, h \in k[x_1, ..., x_n]$ such that $f = g/h$ on U and h is nowhere 0 on U.

Note that if $h(P) \neq 0$, then $X_h = \{x \in X : h(x) \neq 0\}$ defines an open neighbourhood of P in X on which h is never 0.

We also note for the projective case that if $g, h \in k[x_0, ..., x_n]$ are homogeneous of the same degree d, then $g/h : \mathbb{P}^n \dashrightarrow k$ gives a well-defined function whenever h is non-zero, as for any representation $(\lambda x_0 : ... : \lambda x_n)$ of the point $(x_0 : ... : x_n)$ we have

$$\frac{g}{h}(\lambda x_0, ..., \lambda x_n) = \frac{\lambda^d g(x_0, ..., x_n)}{\lambda^d h(x_0, ..., x_n)} = \frac{g}{h}(x_0, ..., x_n).$$

In the definition of regular functions on a projective variety we mention for a function to be regular on a *quasi-projective* varieties. A quasi-projective variety is any open subset of a projective variety. Note that affine varieties, as well as any open subsets of an affine variety are also quasi-projective, as any affine variety is an open subset of some projective variety.

Definition 2.2.

Let $X \subseteq \mathbb{P}^n$ be a quasi-projective variety. A partial map $f : X \dashrightarrow k$ is said to be *regular* at a point $P \in X$ if there exists an open neighbourhood $U \subseteq X$ of P and homogeneous polynomials $g, h \in k[x_0, ..., x_n]$ of the same degree, such that $f = g/h$ on U and h is nowhere 0 on U.

For any variety X, a partial map $f : X \to k$ is said to be regular on a subset $S \subseteq X$ if it is regular at all points of S. Here we say $f : S \to k$ is regular.

We write $\mathcal{O}(S)$ for the set of regular functions $S \to k$ of a subset S of a variety. This is a ring under polynomial multiplication and addition.

For example, the set of regular functions on $\mathbb{A}^n$ is just the coordinate ring $k[x_1, ..., x_n]$, as any non-constant polynomial is zero at some $x \in \mathbb{A}^n$, as k is assumed to be algebraically closed.

With this, we can now define a morphism between two varieties.

Definition 2.3.

A map $\phi : X \to Y$ is a morphism of varieties, if both

i. ϕ is continuous, that is, for every open subset $U \subseteq Y$, we have that $\phi^{-1}(U)$ is open in X, and
ii. For every open subset $U \subseteq Y$ and for every regular function $f : U \to k$, we have $f \circ \phi : \phi^{-1}(U) \to k$ is regular.

Note that with this definition, the composition of two morphisms is clearly a morphism.

This definition of a morphism is equivalent in some sense to a map described by polynomials. The reason we do not simply consider regular functions on the entire variety is because this is too restrictive in the projective case. Note that a function $g/h : \mathbb{P}^n \to k$ is regular only if h is nowhere 0 on $\mathbb{P}^n$, which implies that h must be constant, and hence g/h must be constant. The above definition lets us construct morphisms between projective varieties by considering how they act on affine subvarieties.

We now aim to give an equivalent definition for morphisms of affine varieties in a more natural way.

Let X be an affine variety over k. That is, $X = \{x \in \mathbb{A}^n_k : f(x) = 0 \text{ for all } f \in \mathcal{I}(X)\}$ for some n and for some prime ideal $\mathcal{I}(X) \trianglelefteq k[x_1, ..., x_n]$. Denote by $k[X]$ the coordinate ring

$$k[X] := k[x_1, ..., x_n]/\mathcal{I}(X).$$

This is the set of polynomials on n variables, with polynomials equal on X identified together. Similarly for a projective variety $Y \subseteq \mathbb{P}_k^n$ we define the coordinate ring $k[Y]$ to be

$$k[Y] := k[x_0, ..., x_n]/\mathcal{I}(Y).$$

We will show that for an affine variety X, this coordinate ring $k[X]$ gives us all the regular functions on X. First we need the following lemma, which is useful when comparing regular functions.

Lemma 2.4.

Let U be a non-empty open subset of a variety X. if two polynomials $g, h \in k[X]$ agree on all of U then $g = h$.

Proof.

As U is non-empty and open we have a cover of U by non-empty basic open sets $U = X_{f_1} \cup ... \cup X_{f_n}$. Note that we have $X = (V(f_1) \cap X) \cup \overline{X_{f_1}}$ as a union of closed sets. As X is irreducible, we can not have both $V(f_1)$ and $\overline{X_{f_1}}$ being proper subsets of X. As $X_{f_1} \neq \emptyset$, we have $(V(f_1) \cap X) \neq X$, thus we must have $\overline{X_{f_1}} = X$. But $U \supseteq X_{f_1}$, hence we must have $\overline{U} \supseteq \overline{X_{f_1}} = X$ and so $\overline{U} = X$.

Suppose that $g(x) = h(x)$ for all $x \in U$. We suppose now that X is affine with $X \subseteq \mathbb{A}^n$. Considering g and h as polynomials in $k[x_1, ..., x_n]$ we have $(g-h)(x) = 0$ for all $x \in U$ and so $g - h \in \mathcal{I}(U)$. From Lemma 1.7 we have $V(\mathcal{I}(U)) = \overline{U} = X$. Thus for all $x \in X$ we have $(g-h)(x) = 0$ and so $g = h$ as elements of $k[X]$.

If X is projective, the same argument applies with Lemma 1.21.

□

We can now show that the coordinate ring of an affine variety X precisely defines the regular functions on X.

Lemma 2.5.

Let $X \subseteq \mathbb{A}^n$ be an affine variety. Then $\mathcal{O}(X) = k[X]$.

Proof.

This proof is based on an equivalent proof in [12].

First note that if X is empty we have $\mathcal{O}(X) = k[X] = \{0\}$, as all functions are equal on the empty set. Suppose now that X is non-empty.

It is clear we have the inclusion $k[X] \subseteq \mathcal{O}(X)$ as $f = f/1$ is regular on X. Consider a regular function $F \in \mathcal{O}(X)$. Let $X = X_{f_1} \cup ... \cup X_{f_r}$ be a cover of X by non-empty basic open sets. For each i, F restricted to X_{f_i} is regular on that subset, so we have $F|_{X_{f_i}} = g_i/h_i$ for some $g_i, h_i \in k[x_1, ..., x_n]$ with $h_i \neq 0$ on X_{f_i}.

As every $x \in X$ is in some X_{f_i}, we cannot have all the h_i simultaneously 0. Thus $V(\langle h_1, ..., h_r\rangle) \cap X = \emptyset$. Using Lemma 1.3 and Hilbert's Nullstellensatz, we can rewrite this as

$$\begin{aligned}
\emptyset &= V(\langle h_1, ..., h_r\rangle) \cap V(\mathcal{I}(X)) \\
&= V(\langle h_1, ..., h_r\rangle + \mathcal{I}(X)), \qquad \text{and so} \\
\mathcal{I}(\emptyset) = k[x_1, ..., x_n] &= \sqrt{\langle h_1, ..., h_r\rangle + \mathcal{I}(X)}.
\end{aligned}$$

So there exists an element of the form $(\sum_{i=1}^{r} l_i h_i) + s \in k[x_1, ..., x_n]$ for some $l_i \in k[x_1, ..., x_n]$, $s \in \mathcal{I}(X)$, such that $((\sum_{i=1}^{r} l_i h_i) + s)^m = 1$ for some m. That is, there exist $l_i \in k[x_1, ..., x_n]$ such that $\sum_{i=1}^{r} l_i h_i = 1$ as a member of $k[X]$.

To complete the proof we will show that $F = \sum_{i=1} l_i g_i$ as functions on X which will imply that $F \in k[X]$. For each i, j, we have X_{f_i} and X_{f_j} are non-empty and hence their complements $V(f_i)$ and $V(f_j)$ are proper subsets of X. As X is irreducible, $V(f_i) \cup V(f_j) \neq X$. Noting that $V(f_i) \cup V(f_j) = V(f_i f_j)$ we then have the complement $X_{f_i f_j}$ is non-empty. On this subset $X_{f_i f_j} = X_{f_i} \cap X_{f_j}$ the regular functions $F|_{X_{f_i}}$ and $F|_{X_{f_j}}$ must agree, that is, $g_i/h_i = g_j/h_j$ on $X_{f_i f_j}$. From Lemma 2.4 we have $g_i h_j = h_i g_j$ as elements of $k[X]$.

Now, on any X_{f_i} we have

$$\begin{aligned}
F|_{X_{f_i}} &= \frac{g_i}{h_i} \\
&= (\sum_{j=1}^{r} l_j h_j) \frac{g_i}{h_i} \\
&= \sum_{j=1}^{r} l_j h_j g_i \frac{1}{h_i} \\
&= \sum_{j=1}^{r} l_j h_i g_j \frac{1}{h_i} \\
&= \sum_{j=1}^{r} l_j g_j.
\end{aligned}$$

Thus on all of X we have $F = \sum_{i=1} l_i g_i \in k[X]$, and so we have $\mathcal{O}(X) \subseteq k[X]$ and hence $\mathcal{O}(X) = k[X]$.

$\square$

The following lemma gives an equivalent definition of a morphism to an affine variety. This is somewhat of a more natural definition in the affine setting, as it is described explicitly by polynomials.

Lemma 2.6. *Let X be any variety over k and $Y \subseteq \mathbb{A}_k^m$ be an affine variety over k. A map $\phi : X \to Y$ is a morphism of varieties if and only if it can be written as*

$$\phi(x_1, ..., x_n) = (y_1, ..., y_m),$$

with $y_i = \phi_i(x_1, ..., x_n)$ for some regular function $\phi_i \in \mathcal{O}(X)$ for each i.

Proof.

Suppose ϕ is a morphism and let $\phi(x_1, ..., x_n) = (y_1, ..., y_m)$ with $y_i = \phi_i(x_1, ..., x_n)$ for some maps $\phi_i : X \to k$. For each i, the coordinate map $T_i : (y_1, .., y_m) \mapsto y_i$ is regular on Y, and hence the composition $T_i \circ \phi = \phi_i$ must be regular on $\phi^{-1}(Y) = X$.

Conversely, let $\phi(x_1, ..., x_n) = (y_1, ..., y_m)$ with $y_i = \phi_i(x_1, ..., x_n)$ for some regular functions $\phi_i : X \to k$. Let U be an open subset of Y and let f be a regular function on U. That is, $f = g/h$ for some $g, h \in k[y_1, ..., y_m]$ with $h(y) \neq 0$ for all $y \in U$. As the coordinate maps on ϕ are given by polynomials on X, for any $g \in k[y_1, ..., y_m]$ we have $g \circ \phi$ is a polynomial on the coordinates of X. Thus we have $(g/h) \circ \phi = (g \circ \phi)/(h \circ \phi)$ is a quotient of polynomials. Moreover, for any $x \in \phi^{-1}(U)$ we have $h \circ \phi(x) \neq 0$, as $\phi(x) \in U$, and so $(g/h) \circ \phi$ is regular on $\phi^{-1}(U)$.

Consider a decomposition of U in to basic open sets $U = Y_{f_1} \cup ... \cup Y_{f_r}$. Then we have

$$\begin{aligned}
\phi^{-1}(U) &= \phi^{-1}(Y_{f_1} \cup ... \cup Y_{f_r}) \\
&= \phi^{-1}(Y_{f_1}) \cup ... \cup \phi^{-1}(Y_{f_r}) \\
&= X_{f_1 \circ \phi} \cup ... \cup X_{f_r \circ \phi}
\end{aligned}$$

which is open, as each $f_i \circ \phi$ is regular on X, thus is an element of $k[X]$. Thus ϕ is a morphism of varieties.

□

In particular, combining this with the previous lemma gives us that the coordinates of a morphism of affine varieties are given by polynomials in the coordinate ring $k[X]$ of X.

Before describing morphisms between projective varieties more explicitly, we will look more closely at morphisms of affine varieties. For affine varieties, we are interested in their coordinate rings and will aim to study relations between varieties by relations between their coordinate rings.

The coordinate ring $k[X]$ of an affine variety is a k-algebra. Recall that a k-algebra is a ring which also has the structure of a vector space over k. A homomorphism of k-algebras is a map $\phi : R \to S$ such that ϕ is a homomorphism of rings and also a morphism of vector spaces in the sense that it preserves scalar multiplication. That is, a map $\phi : R \to S$ is a morphism of k-algebras if, for all $f, g \in R$, $\lambda \in k$, we have

$$\begin{aligned}
\phi(fg) &= \phi(f)\phi(g) \\
\phi(f+g) &= \phi(f) + \phi(g) \\
\phi(1_R) &= 1_S \\
\phi(\lambda f) &= \lambda\phi(f).
\end{aligned}$$

Recall that a category is some class of objects, along with a class of composable maps between them, called morphisms. The morphisms we have described give us the categories we are interested in: varieties (any affine or projective variety), affine varieties and finitely generated integral algebras.

In any category, an isomorphism is a morphism $\varphi : X \to Y$ such that there exists an inverse morphism $\varphi^{-1} : Y \to X$ such that $\varphi^{-1}\varphi = id_X$ and $\varphi\varphi^{-1} = id_Y$. We say two objects X and Y are *isomorphic*, or $X \cong Y$, if there exists an isomorphism between them.

We can study morphisms of affine varieties by looking at the morphisms of their coordinate rings, but we first need to prove a duality that allows us to do so.

2.1 Duality of affine varieties

There is an antiequivalence between the categories of finitely generated integral k-algebras and affine varieties over k. By this, we mean that there is a bijective correspondence between objects in each of the categories, as well as a bijective correspondence between morphisms of the objects. The "anti" in antiequivalence says that the correspondence between morphisms is in the reverse direction of the corresponding objects. That is, if X and Y are objects in one category corresponding to objects $F(X)$ and $F(Y)$ in the other category, then there is a bijection between morphisms $\phi : X \to Y$ and morphisms $\phi^* : F(Y) \to F(X)$.

$$F(X) \xleftarrow{\phi^*} F(Y)$$
$$X \xrightarrow{\phi} Y$$

Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine varieties over k. Then X corresponds to the k-algebra $\mathcal{O}(X)$, and a morphism $\phi : X \to Y$ corresponds to its *pull-back*, $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$, defined by

$$\phi^* f(x) = f \circ \phi(x).$$

Lemma 2.7.

Let $\phi : X \to Y$ be a morphism of varieties. Then the pull-back ϕ^* maps $\mathcal{O}(Y)$ to $\mathcal{O}(X)$ and is a morphism of k-algebras

Proof.

As ϕ is a morphism, we have $\phi(x) = (\phi_1(x), ..., \phi_m(x))$ for some $\phi_i(x) \in \mathcal{O}(X)$, and then for $f \in \mathcal{O}(Y)$ we have $\phi^* f = f \circ \phi$. From the definition of a morphism we have $f \circ \phi$ is a regular function on X, that is, we have $\phi^* f \in \mathcal{O}(X)$. Also ϕ^* is a morphism of k-algebras as can be seen by checking the axioms pointwise. For all $x \in X$, $f, g \in \mathcal{O}(Y)$, $\lambda \in k$, we have

$$\begin{aligned}
(f+g)(\phi(x)) &= f(\phi(x)) + g(\phi(x)) \\
(fg)(\phi(x)) &= (f(\phi(x)))(g(\phi(x))) \\
(\lambda f)(\phi(x)) &= \lambda(f(\phi(x))) \\
\phi^*(1)(x) &= 1 \circ (\phi(x)) = 1, \text{ so } \phi^*(1) = 1.
\end{aligned}$$

□

This correspondence with the pull-back is one to one. We prove this in the following lemma.

Lemma 2.8.

Let $X \subseteq \mathbb{A}^n, Y \subseteq \mathbb{A}^m$ be affine varieties over k and $\varphi : \mathcal{O}(Y) \to \mathcal{O}(X)$ be a morphism of k-algebras. Then there exists a unique morphism $\phi : X \to Y$ satisfying $\phi^* = \varphi$.

This unique $\phi : X \to Y$ satisfying $\phi^* = \varphi$ is called the *push-forward* of φ.

Proof.

Let $\mathcal{I}(Y) = J$, so that $\mathcal{O}(Y) = k[Y] = k[y_1, ..., y_m]/J$. For $x = (a_1, ..., a_n) \in X$ we define the morphism $\phi : X \to \mathbb{A}^m$ by

$$\phi(x) = (\varphi y_1(x), ..., \varphi y_m(x)),$$

where the y_i are the coordinate functions in $k[y_1, ..., y_m]/J$. We will show that ϕ satisfies the desired property, that $\phi^* f = f \circ \varphi$ for all $f \in \mathcal{O}(Y)$.

For a polynomial $f \in \mathcal{O}(Y) = k[x_1, ..., x_m]/J$, consider the decomposition of f into its terms $f = \sum_{\alpha \in B} f_\alpha$, where B is the set of terms f_α of f, that are of the form $f_\alpha := \lambda_\alpha \prod_{j=1}^m y_j^{d_{\alpha_j}}$. From this decomposition, and the fact that φ is a k-algebra homomorphism, we can see that for any $x \in X$ and any term f_α of f, we have

$$\begin{aligned}
\varphi f_\alpha(x) &= \varphi(\lambda_\alpha \prod_{j=1}^m y_j^{d_{\alpha_j}})(x) \\
&= \lambda_\alpha \prod_{j=1}^m (\varphi y_j(x))^{d_{\alpha_j}} \\
&= f_\alpha(\phi(x))
\end{aligned}$$

and hence $\varphi f(x) = \sum_{\alpha \in B} f_\alpha(\phi(x)) = f(\phi(x))$. That is, we have $\varphi f = f \circ \phi$ for all $f \in \mathcal{O}(Y)$.

It is left to prove is that this is the only such morphism. Let $\phi_1 : X \to Y$, $\phi_2 : X \to Y$ be two morphisms with pull-back $\phi_1^* = \phi_2^* = \varphi$. Then we have $f(\phi_1(x)) = f(\phi_2(x))$ for all $x \in X$, $f \in \mathcal{O}(Y)$. Fix $x \in X$ and let $\mathfrak{m}$ and $\mathfrak{n}$ be the maximum ideals $\mathcal{I}(\{\phi_1(x)\})$ and $\mathcal{I}(\{\phi_2(x)\})$ respectively. Suppose $f \in \mathfrak{m}$ and regard f as an element of $\mathcal{O}(X)$. Then we have $f(\phi_1(x)) = 0 = f(\phi_2(x))$ and so $f \in \mathfrak{n}$. Thus we have $\mathfrak{m} \subseteq \mathfrak{n}$. Similarly we have $\mathfrak{n} \subseteq \mathfrak{m}$ and so $\mathfrak{m} = \mathfrak{n}$. In the first chapter we showed there is a bijective correspondence between points in affine space and the maximal ideals associated with them. As we have $\mathfrak{m} = \mathfrak{n}$ we must thus have $\phi_1(x) = \phi_2(x)$. Hence, as x was chosen arbitrarily we have $\phi_1 = \phi_2$ and so the push-forward is unique, completing the proof. $\square$

We have now shown a bijective correspondence between morphims of affine varieties $\phi : X \to Y$ and morphisms of the associated k-algebras $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$. We will prove two important properties of this correspondence.

Lemma 2.9. *Let $\phi : X \to Y$ and $\varphi : Y \to Z$ be morphisms of affine varieties over k. Then:*

1. *The pull-back of $\varphi \circ \phi : X \to Z$ is $\phi^* \circ \varphi^* : \mathcal{O}(Z) \to \mathcal{O}(X)$.*
2. *The morphism $\phi : X \to Y$ is an isomorphism if and only if $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an isomorphism.*

Proof.

1. For any $f \in \mathcal{O}(Z)$, $x \in X$, we have $(\phi^* \circ \varphi^* f)(x) = \varphi^* f(\phi(x)) = f(\varphi \circ \phi(x)) = \left((\varphi \circ \phi)^* f\right)(x)$. This shows that $(\varphi \circ \phi)^* = \phi^* \circ \varphi^*$.
2. Suppose ϕ is an isomorphism, that is, there exists an inverse $\phi^{-1} : Y \to X$ such that $\phi^{-1} \circ \phi = id_X$ and $\phi \circ \phi^{-1} = id_Y$. For any $y \in Y$, $f \in \mathcal{O}(Y)$, we have $(\phi^{-1^*} \circ \phi^*) f(y) = (\phi \circ \phi^{-1})^* f(y) = (id_Y)^* f(y) = f(y)$, using part 1. That is we have $\phi^{-1^*} \circ \phi^* = id_{\mathcal{O}(Y)}$. Similarly we get $\phi^* \circ \phi^{-1^*} = id_{\mathcal{O}(X)}$ and so we have $\phi^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is an isomorphism of k-algebras with inverse $\phi^{*-1} = \phi^{-1^*}$.

   Conversely, suppose ϕ^* is an isomorphism with inverse $\phi^{*-1} : \mathcal{O}(X) \to \mathcal{O}(Y)$. Let $\rho : Y \to X$ be the push-forward of ϕ^{*-1}. We have $(\phi \circ \rho)^* = \rho^* \circ \phi^* = \phi^{*-1} \circ \phi^* = id_{\mathcal{O}(Y)}$. We have just seen that $id_Y{}^* = id_{\mathcal{O}(Y)}$, and from the bijective correspondence of the pull-back, we must have $\phi \circ \rho = id_Y$. Similarly we get $\rho \circ \phi = id_X$ and thus $\phi : X \to Y$ is an isomorphism of varieties with inverse ρ.

$\square$

The second part of this above lemma immediately shows the following important result.

Lemma 2.10.

Let X and Y be affine varieties. Then $X \cong Y$ if and only if $\mathcal{O}(X) \cong \mathcal{O}(Y)$

We have seen that an affine variety $X \subseteq \mathbb{A}^n$ corresponds to the k-algebra $\mathcal{O}(X)$. We have $\mathcal{O}(X) = k[X] = k[x_1, ..., x_n]/\mathcal{I}(X)$ and $\mathcal{I}(X)$ is prime. This is a finitely generated k-algebra. Also, as $\mathcal{I}(X)$ is prime, for $f, g \in k[x_1, ..., x_n]$ we have $fg \in \mathcal{I}(X)$ if and only if $f \in \mathcal{I}(X)$ or $g \in \mathcal{I}(X)$. Thus for f, g as elements of $k[X]$, $fg = 0$ if and only if $f = 0$ or $g = 0$. That is, $k[X]$ has no zero-divisors and so $k[X] = \mathcal{O}(X)$ is a finitely generated integral k-algebra.

For any finitely generated integral algebra we can also find a corresponding affine variety. Let R be a finitely generated k-algebra. Then we have $R = \langle s_1, ..., s_n \rangle$ for some elements s_i. Consider the surjective homomorphism of k-algebras

$$\varphi : k[T_1, ..., T_n] \to R, \ T_i \mapsto s_i.$$

By the first isomorphism we have $R \cong k[T_1, ..., T_n]/I$, where I is the ideal $ker\ \varphi \trianglelefteq k[T_1, ..., T_n]$. If R is also an integral domain, that is, R has no zero-divisors, then we must have $I = ker\ \varphi$ is prime. This algebra $k[T_1, ..., T_n]/I$ then corresponds to the variety $X = \{x \in \mathbb{A}_k^n : f(x) = 0 \text{ for all } f \in I\}$.

Denote by Spec R the affine variety corresponding to the k-algebra R. If we have R of the form $k[x_1, ..., x_n]/I$ for some prime ideal I then we mean by Spec R the particular variety Spec $R = V(I) \subseteq \mathbb{A}^n$. If we do not have a particular specification of R with choice of generating elements, but R is a finitely generated integral algebra, then we say that Spec R is some affine variety with coordinate ring $\mathcal{O}(\text{Spec } R) = R$. Note that by Lemma 2.10 we have this operation is well-defined up to isomorphism. Also, we have $\mathcal{O}(\text{Spec } R) \cong R$ and $\text{Spec}\,(\mathcal{O}(X)) \cong X$. Thus we have a bijective correspondence of objects, up to isomorphism, in the categories of finitely generated integral k-algebras and affine varieties, as well as a bijective correspondence of morphisms between them. This shows the antiequivalence of these categories.

$$\mathcal{O}(X) \xleftarrow{\phi^*} \mathcal{O}(Y) \qquad\qquad R \xleftarrow{\phi^*} S$$

$$X \xrightarrow{\phi} Y \qquad\qquad \text{Spec } R \xrightarrow{\phi} \text{Spec } S$$

Note that the exact definition of "Spec R", the *spectrum* of a ring R, is different to the one we have given. It is formally the set of all prime ideals of the ring R, which can be given a topology and class of regular functions and considered as a variety isomorphic to the affine variety associated with the ring R. For our purposes, just considering Spec R to be an affine variety in $\mathbb{A}^n$ with $\mathcal{O}(\text{Spec } R) = R$ is enough.

2.2 Morphisms of projective varieties

Recall that a projective variety is a subset $X \subseteq \mathbb{P}^n$ such that X is the zero set of some set of some prime homogeneous ideal $I \trianglelefteq k[x_0, ..., x_n]$.

A projective variety X can be obtained by gluing together open affine varieties. By this we mean there exist open subsets $X_1, ..., X_n$ such that the X_i are affine, in the sense that they are isomorphic to affine varieties. With these decompositions, we can show an equivalent definition of a morphism of projective varieties involving the gluing together of finitely many affine morphisms.

ThIs next lemma shows the affine decomposition. In a graded ring R we denote by R_0 the homogeneous part of R of degree zero.

Lemma 2.11.

Let $X \subseteq \mathbb{P}^n$ be a projective variety. Then there exists a decomposition of X as a union of open affine varieties of the form $X_{f_i} = \{x \in X : f(x) \neq 0\}$ for some homogeneous f_i. That is, we can write $X = X_{f_1} \cup ... \cup X_{f_r}$ for some open affine varieties X_{f_i}. Moreover, for any $f \in k[X]$ we have $\mathcal{O}(X_f) = (k[X][f^{-1}])_0$.

Proof.

Hartshorne gives a proof in [8] showing that there exists such a decomposition. We will show that the set of regular functions on X_f is $(k[X][f^{-1}])_0$. We have that a function in $\mathcal{O}(X_f)$ has the form $\frac{g}{h}$, where $g, h \in k[X]$ are homogeneous of the same degree, and h is never zero on X_f. As the base field is algebraically closed, the only elements of $k[X]$ which are never zero on $X_f = \{x \in X : f(x) \neq 0\}$ are generated by f. So powers of f are the only polynomials allowable on the denominator of an element of $\mathcal{O}(X_f)$ and so we have $\mathcal{O}(X_f) = (k[X][f^{-1}])_0$.

□

To give the equivalent definition of morphisms to projective varieties, we need the following lemma concerning intersections of open sets.

Lemma 2.12.

Let X be a variety and U, V be non-empty open subsets of X. Then $U \cap V \neq \emptyset$

Proof.

For a contradiction, suppose $U \cap V = \emptyset$. Then $U^c \neq X$ and $V^c \neq X$ are closed with $X = U^c \cup V^c$. But this contradicts X being irreducible. Hence we have $U \cap V \neq \emptyset$.

□

We now show a lemma which will help us construct morphisms to projective varieties. It shows that we can construct a morphism to a projective variety Y by gluing together morphisms to open affine subsets of Y.

Lemma 2.13.

Let $\phi : X \to Y$ be a map of varieties, with $Y \subseteq \mathbb{P}^n$ projective with some decomposition into open affine varieties $Y = \bigcup_{i=0}^{n} Y_i$. For each i, let $\phi_i : \phi^{-1}(Y_i) \to Y_i$ be the restriction of ϕ to $\phi^{-1}(Y_i)$. Then ϕ is a morphism if and only if each ϕ_i is a morphism of varieties.

Proof.

Suppose ϕ is a morphism and let $U \subseteq Y_i$ be open. If $f \in \mathcal{O}(U)$, then $f \circ \phi_i = f \circ \phi$, which is regular on $\phi^{-1}(U)$, by the definition of a morphism. Also $\phi_i^{-1}(U) = \phi^{-1}(U)$ which is open as ϕ is continuous. Hence each ϕ_i is a morphism.

Conversely, suppose each $\phi_i : \phi^{-1}(Y_i) \to Y_i$ defines a morphism. Let $U \subseteq Y$ be an open subset of Y and for each i let $U_i = U \cap Y_i$ so that $U = U_1 \cup ... \cup U_n$. Let $f \in \mathcal{O}(U)$, then f is regular on each of the U_i, that is $f \in \mathcal{O}(U_i)$ for each i.

If $U = \emptyset$, then any $f \circ \phi$ is regular on $\phi^{-1}(U) = \emptyset$. Otherwise, pick a non-empty U_i, then $f \circ \phi_i$ is regular on $\phi^{-1}(U_i)$, as $U_i \subseteq Y_i$ and ϕ_i is a morphism. Thus for some $g, h \in k[X]$ with $h \neq 0$ on $\phi^{-1}(U_i)$, we have $f \circ \phi_i(x) = \frac{g}{h}(x)$ for all $x \in \phi^{-1}(U_i)$. We will show that $\frac{g}{h}$ is regular on $\phi^{-1}(U_j)$ for each j and so is regular on $\phi^{-1}(U)$.

For any U_j, consider the subset $U_i \cap U_j$. As $Y_j \neq \emptyset$ and $U \neq \emptyset$, by Lemma 2.12 we have $U_j = Y_j \cap U \neq \emptyset$ and so $U_i \cap U_j$ is non-empty and open. Thus $\phi^{-1}(U_i \cap U_j) = \phi^{-1}(U_i) \cap \phi^{-1}(U_j)$ is non-empty. By definition of ϕ_j we have $\phi^{-1}(Y_j) = \phi_j^{-1}(Y_j)$ and so $\phi^{-1}(U_j) = \phi_i^{-1}(U_j)$, as $U_j \subseteq Y_j$. As ϕ_j is a morphism, we thus have $\phi^{-1}(U_j)$ is open in $\phi^{-1}(Y_j)$ and thus is open in X, as $\phi^{-1}(Y_j)$ is open in X. Similarly we see that $\phi^{-1}(U_i)$ is open and that ϕ is continuous.

Thus we have $\phi^{-1}(U_i \cap U_j) = \phi^{-1}(U_i) \cap \phi^{-1}(U_j)$ is non-empty and open in X. We have $f \circ \phi(x) = f \circ \phi_j(x) = f \circ \phi_i(x) = \frac{g}{h}(x)$ on this subset. As $f \circ \phi_j$ is regular on $\phi^{-1}(U_j)$, there exist $g', h' \in k[X]$ with $h' \neq 0$ on $\phi^{-1}(U_j)$ such that $f \circ \phi_j(x) = \frac{g'}{h'}(x)$ for all $x \in \phi^{-1}(U_j)$. But we have $\frac{g}{h}(x) = \frac{g'}{h'}(x)$ for all x in the non-empty open set $\phi^{-1}(U_i) \cap \phi^{-1}(U_j)$, so by Lemma 2.4, we have $gh' = g'h$, that is, $\frac{g}{h} = \frac{g'}{h'}$. Thus $\frac{g}{h}$ is regular on $\phi^{-1}(U_j)$. As U_j was chosen arbitrarily, we thus have $f \circ \phi = \frac{g}{h}$ and is regular on $\phi^{-1}(U)$. Thus ϕ defines a morphism.

$\square$

Chapter 3

The GIT quotient

Quotients arise naturally in mathematics and are of important interest, as they identify elements that are equivalent in some way. For example, consider the set M_n of $n \times n$ matrices over $\mathbb{C}$. We can give M_n a geometric structure by viewing M_n as an affine variety, by the natural embedding in $\mathbb{A}^{n\times n}$. When studying matrices we might like to identify similar matrices together. Let $GL(n, \mathbb{C})$, the group of invertible $n \times n$ matrices, act on M_n by $A.M := AMA^{-1}$. Then the orbits of this action are the equivalence classes of similar matrices. We would like to know if we can still have some sort of geometric structure on these equivalence classes of similar matrices. That is, can we view the set of orbits as an affine variety in some way? We will look at how to construct quotients of varieties.

In the category of sets, a quotient of a set S by an equivalence relation $\sim$ is a mapping from S to the set of equivalence classes $S/\sim$, where equivalent points are identified together. If $x \sim y$ in S, then x and y map to the same point in $S/\sim$. If we have a group G acting on a set S then there is an equivalence relation defined by the G-orbits. The equivalence class of a point $x \in S$ will be its G-orbit $\{y \in S : g.x = y \text{ for some } g \in G\}$. This indeed separates S into distinct G-oribts. The quotient S/G is defined as the set of G-orbits in S and we have the quotient mapping $S \to S/G$ with $x \in S$ mapped to its G-orbit in S/G.

Example 3.1.

This example was used in Newstead's article [11].

Let $\mathbb{C}^\times$ act on $\mathbb{C}^2$ by $\lambda.(x, y) = (\lambda x, \lambda y)$. The $\mathbb{C}^\times$-orbits are the then the punctured lines $\{(\lambda x, \lambda y) : \lambda \in \mathbb{C}^\times\}$ for $(x, y) \neq (0, 0)$ as well as the origin $\{(0, 0)\}$ The set theoretical quotient is simply the set of these orbits. However, this set of orbits does not have a structure of variety. We have to be more careful in constructing quotients in the category of varieties.

For the construction of a quotient of a variety X by a group G which acts on X, we desire the quotient to have a natural structure of variety. In the above example, the set of orbits does not. The origin lies in the closure of every orbit, so a morphism of varieties $\mathbb{C}^2 \to Y$ can not separate orbits. So in general we can not construct a quotient which separates orbits. We need to define a more general quotient.

The key property of a quotient of some object by a group action is that it identifies all elements in an orbit. To identify orbits of X with the same point in Z we must have a morphism $X \to Z$ which is constant on orbits. We call such a morphism G-invariant. This leads us to ask: which G-invariant morphism, if any, would be most appropriate for a quotient? We want the quotient to be as closest to X in structure as possible, while still identifying orbits. This introduces the *categorical quotient*, which is a natural definition for a quotient in any category.

Definition 3.2. In any category, we call a G-invariant morphism $\pi : X \to Y$ a *categorical quotient* of X by G, when for any G-invariant morphism $f : X \to Z$, we have that f factors uniquely through π. That is, there exists a unique $\overline{f}$ such that $\overline{f} \circ \pi = f$, for any G-invariant morphism f.

The categorical quotient is unique when it exists. However, for a quotient in the context of varieties, simply being a categorical quotient may not have good geometric properties. A *good categorical quotient* is a categorical quotient of varieties which preserves certain reasonable geometric properties.

If a group G acts on a variety X then we get an induced action on the regular functions on X. For $f \in \mathcal{O}(U)$, $U \subseteq X$, we define $g.f(x) = f(g^{-1}.x)$. For the types of group action we are interested in, which we will mention later, we indeed have $g.f \in \mathcal{O}(U)$ for $f \in \mathcal{O}(U)$. For a ring R on which G acts, denote by R^G the subring of G-invariant elements, that is $R^G := \{f \in R : g.f = f \text{ for all } g \in G\}$.

Definition 3.3. A surjective G-invariant map of varieties $p : X \to Y$ is called a *good categorical quotient* of X by G if the following three properties hold:

(i) For all open $U \subseteq Y$, $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))^G$ is an isomorphism.
(ii) If $W \subseteq X$ is closed and G-invariant, then $p(W) \subseteq Y$ is closed.
(iii) If $V_1, V_2 \subseteq X$ are closed, G-invariant, and $V_1 \cap V_2 = \emptyset$,
then $p(V_1) \cap p(V_2) = \emptyset$

Note that the first condition implies a good categorical quotient must be a categorical one. If $f : X \to Z$ is a G-invariant morphism, then $f^* : \mathcal{O}(Z) \to \mathcal{O}(X)$ must embed in $\mathcal{O}(X)^G$. If p is a good categorical quotient, then p^* is an isomorphism on to $\mathcal{O}(X)^G$. Hence any factoring $f^* = p^* \circ \overline{f}^*$ must have $\overline{f}^*$ send $h \in \mathcal{O}(Z)$ to the unique element of $\mathcal{O}(Y)$ that is mapped to $f^*(h)$. Thus this factoring is unique, and by the anti-equivalence of categories, the dual $f = \overline{f} \circ p$ is a unique factoring of f through p.

We denote by $X//G$ the good categorical quotient, or GIT quotient, of a variety X by a group G, when it exists. We will first construct this quotient for affine varieties. The quotient for a projective variety is then constructed by linearising the group action and gluing together GIT quotients on open affine subsets.

We will describe what sort of group and group action is acceptable. The group action should act by a morphism of varieties. Indeed we require that if f is a polynomial, then the function $g.f$ defined by $g.f(x) = f(g^{-1}.x)$ is also a polynomial. The affine quotient is $X//G = \text{Spec } \mathcal{O}(X)^G$, but for this we require that $\mathcal{O}(X)^G$ is finitely generated. This is provided by the group G being *reductive*.

3.1 Reductive groups

We now introduce the groups we are interested in and for which we can construct the geometric invariant theory quotient. We will define when a *linear algebraic group* is *reductive* and give the properties that these groups have which are needed.

A *linear algebraic group* is a subgroup of $GL(n,k)$ which is algebraic, in the sense that it is defined by the solution to some set of polynomial equations.

Example 3.4. The set of unitary matrices with determinant 1,

$$SO(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc - 1 = 0 \right\}$$

is an algebraic group.

Also $k^\times$ is an algebraic group, by the embedding $\lambda \mapsto \lambda I$, for $\lambda \in k^\times$. There are many finite algebraic groups of interest, such as $G = \{I, -I\} \cong \mathbb{Z}/2Z$. We will be most interested in finite groups as well as $\mathbb{C}^\times$ when working over $\mathbb{C}$, though most of the results we will show hold for more general groups that satisfy being *reductive*. Before defining reductive, we recall some definitions regarding representations.

A *representation* of a group G is an embedding $\rho : G \hookrightarrow GL(n,k)$. A representation of a group G then induces a linear action of G on the vector space $V = \mathbb{A}^n_k$. We say that a representation of G is *reducible* if there exists a G-invariant proper subspace of V, that is, a proper subspace $U \subset V$ such that $\rho(g)u \in U$ for all $u \in U$. A representation is called *irreducibile* otherwise.

Given an inner product on V and a subspace $U \subset V$, we can write V as the direct sum $V = U \oplus U^\perp$. Then any $v \in V$ can be represented uniquely as $v_1 + v_2$, with $v_1 \in U$, $v_2 \in U^\perp$. If these subspaces U and $U^\perp$ are G-invariant, then we can consider the action of G restricted to U and $U^\perp$. For $v = v_1 + v_2$ with $v_1 \in U$, $v_2 \in U^\perp$ we then get $\rho(g)v = \rho|_U(g)v_1 + \rho|_{U^\perp}(g)v_2$ and we can write the representation as a direct sum $\rho = \rho|_U \oplus \rho|_{U^\perp}$ acting on $U \oplus U^\perp$.

Definition 3.5.

A linear algebraic group G over k is *reductive* if every representation $\rho : G \to GL(n,k)$ has a decomposition as a direct sum of irreducible representations.

Many classical groups such as $GL(n,\mathbb{C})$, $SO(n,\mathbb{C})$ are reductive. We will prove that $\mathbb{C}^\times$ as well as all finite groups are reductive.

Lemma 3.6.

The multiplicative group $\mathbb{C}^\times$ is reductive.

Proof.

This proof uses the "Weyl unitary trick". Some of the ideas used in this proof come from Le Potier's book [10].

Let $V = \mathbb{C}^n$ and $S = \{z \in \mathbb{C}^\times : |z| = 1\}$. Let $\rho : \mathbb{C}^\times \to GL(n,\mathbb{C})$ be a representation of $\mathbb{C}^\times$. We will show that ρ has a decomposition as a direct sum of irreducible representations. If ρ is irreducible we are done, otherwise we show that we can write ρ as the direct sum of two representations of lower dimension.

Let $\langle\ ,\ \rangle$ denote the standard inner product on $V = \mathbb{C}^n$. Denote by $\langle\ ;\ \rangle$ the form defined by $\langle x; y\rangle := \int_0^{2\pi} \langle \rho(e^{i\theta})x, \rho(e^{i\theta})y\rangle d\theta$. We have the useful property of this form that $\langle \rho(g)x; \rho(g)y\rangle = \langle x; y\rangle$ for any $x, y \in V$, $e^{i\psi} \in S$. See that

$$\begin{aligned}
\langle \rho(e^{i\psi})x; \rho(e^{i\psi})y\rangle &= \int_0^{2\pi} \langle \rho(e^{i\theta}\rho(e^{i\psi}))x, \rho(e^{i\theta})\rho(e^{i\psi})y\rangle d\theta \\
&= \int_0^{2\pi} \langle \rho(e^{i(\theta+\psi)})x, \rho(e^{i(\theta+\psi)})y\rangle d\theta \\
&= \int_0^{2\pi} \langle \rho(e^{i\phi})x, \rho(e^{i\phi})y\rangle d\phi, \text{ where } \phi = \theta + \psi \\
&= \langle x; y\rangle
\end{aligned}$$

Also note that $\langle\ ;\ \rangle$ defines an inner product. We see $\langle x; x\rangle = \int_0^{2\pi} \langle \rho(e^{i\theta})x, \rho(e^{i\theta})x\rangle d\theta \geq 0$ with equality if and only if $\rho(e^{i\theta})x = 0$ for all $e^{i\theta}$ if and only if $x = 0$. Suppose that ρ is not irreducible, then there exists some $\mathbb{C}^\times$-invariant subspace U of V, that is, a subspace U such that $\rho(g)u \in U$ for all $g \in S$, $u \in U$. Let $W = U^\perp$ be the orthogonal complement of U with respect to this new inner product, so we have $V = U \oplus W$. Then for all $w \in W$, $u \in U$, $g \in S$, we have

$$\begin{aligned}
\langle u; \rho(g)w\rangle &= \langle \rho(g^{-1})u; \rho(g^{-1})\rho(g)w\rangle \\
&= \langle \rho(g^{-1})u; w\rangle \\
&= 0
\end{aligned}$$

This is zero because U is S-invariant, that is $\rho(g)u \perp w$ for all $u \in U$, $w \in W$, $g \in S$. Thus we have $\rho(g)w \perp u$ for all $g \in S$, $w \in W$, $u \in U$ and so $\rho(g)w \in W$, that is, W is S-invariant. This can be extended to show that W is $\mathbb{C}^\times$-invariant.

Let N be the subset of $\mathbb{C}^\times$ which leaves W invariant, which contains S. I will show that this set is closed in the Zariski topology.

Let $W = \text{span } \{e_1, ..., e_r\}$. This basis extends to some basis $\{e_1, ..., e_n\}$ of V. Then W is defined as the subset of vectors $v \in V$ where for each $j \in \{r+1, ..., n\}$ we have $\langle v, e_j \rangle = 0$, with respect to the usual inner product. These define polynomials in the coordinates of v, which we will call f_i so that we have W is the zero set of $\{f_{r+1}, ..., f_n\}$.

For each $i \in \{1, ..., r\}$, $j \in \{r+1, ..., n\}$, consider the set $\{T \in GL(V) : f_j(Te_i) = 0\}$. As f_j and e_i is fixed, this set is the zero set of a polynomial in the coordinates of T. That is, it is a closed subset of $GL(V)$. Then we have $\{T \in GL(V) : Te_i \in U\} = \bigcap_{j=r+1}^{n} \{T \in GL(V) : f_j(Te_i) = 0\}$ so is closed and thus

$$\{T \in GL(V) : Te_i \in U \text{ for each } e_i\} = \bigcap_{i=1}^{r} \{T \in GL(V) : Te_i \in U\}$$

is also closed. Thus we have

$$\begin{aligned}
&\{T \in GL(V) : Tu \in U \text{ for all } u \in U\} \\
&\qquad = \{T \in GL(V) : T(\lambda_1 e_1 + ... + \lambda_n e_n) \in U \text{ for all } \lambda_i \in \mathbb{C}\} \\
&\qquad = \{T \in GL(V) : \lambda_1(Te_1) + ... + \lambda_n(Te_n) \in U \text{ for all } \lambda_i \in \mathbb{C}\} \\
&\qquad = \{T \in GL(V) : Te_i \text{ for each } e_i\}
\end{aligned}$$

is closed in the Zariski topology. Thus we have $N = \rho^{-1}(\{T \in GL(V) : Tu \in U \text{ for all } u \in U\})$ must be closed. However, the only proper closed subsets of $\mathbb{C}^\times$ are finite, so as $N \supseteq S$ we thus must have $N = \mathbb{C}^\times$. That is, W is invariant under the action of $\mathbb{C}^\times$. Hence ρ has subrepresentations ρ_U, ρ_W acting on U and W with ρ acting on $V = U \oplus W$ by the direct sum $\rho = \rho_U \oplus \rho_W$.

So if ρ is not irreducible, then it has a decomposition into a direct sum of representations of lower dimension. We can repeatedly apply this argument to these subrepresentations, which will terminate with a decomposition of ρ into a sum of irreducible representations.

$\square$

We also are interested in the actions of finite groups on varieties, so we now show that finite groups are reductive, a result of Maschke's Theorem.

Lemma 3.7.

Let G be a finite group. Then G is reductive.

Proof.

Similarly as in the proof for $\mathbb{C}^\times$, we define an inner product which is invariant to the group action.

Let $V = \mathbb{C}^n$ and $\rho : G \to GL(n, \mathbb{C})$ be a representation of G.

Let $\langle\ ;\ \rangle$ be the form defined by $\langle x; y\rangle := \sum_{g\in G}\langle \rho(g)x, \rho(g)y\rangle$. We see that

$$\begin{aligned}\langle \rho(h)x; \rho(h)y\rangle &= \sum_{g\in G}\langle \rho(g)\rho(h)x, \rho(g)\rho(h)y\rangle \\ &= \sum_{g\in G}\langle \rho(gh)x, \rho(gh)y\rangle \\ &= \sum_{g'\in G}\langle \rho(g')x, \rho(g')y\rangle, \text{ where } g' = gh \\ &= \langle x; y\rangle\end{aligned}$$

Also we have $\langle\ ;\ \rangle$ defines an inner product. Suppose that ρ is not irreducible, then there is some G-invariant subspace U of V, that is a subspace U such that $\rho(g)u \in U$ for all $g \in G$, $u \in U$. Let $W = U^\perp$ be the orthogonal complement of U with respect to this new inner product, so we have $V = U \oplus W$. Then for all $w \in W$, $u \in U$, $g \in G$, we have

$$\begin{aligned}\langle u; \rho(g)w\rangle &= \langle \rho(g^{-1})u; w\rangle \\ &= 0, \quad \text{as } U \text{ is } G\text{-invariant}\end{aligned}$$

Thus we have W is also G-invariant. Hence ρ has subrepresentations ρ_U, ρ_W acting on U and W with ρ acting on $V = U \oplus W$ by the direct sum $\rho = \rho_U \oplus \rho_W$.

So if ρ is not irreducible, then it has a decomposition into a direct sum of representations of lower dimension. Thus ρ has a decomposition as a sum of irreducible representations and so any finite group G is reductive.

$\square$

With linear algebraic groups we have a natural sort of action on varieties. There is a natural embedding of $n \times n$ matrices in to $\mathbb{A}^{n\times n}$ by their coordinates. This gives an embedding of $GL(n, k)$ in to $\mathbb{A}^{(n\times n)^2}$ by $A \mapsto (A, A^{-1})$. The image of this embedding is a variety, defined by $\{(A, B) \in \mathbb{A}^{(n \times n)^2} : AB = I\}$. Thus $GL(n, k)$ is an affine variety.

Let G be a linear algebraic group, so G is a closed subset of $GL(n, k)$ defined by polynomials, thus is a closed affine set. If G is also irreducible, for example $\mathbb{C}^{\times}$, we say that G acts *rationally* on a variety X if it acts by a morphism of varieties $G \times X \to X$. If G is a finite group, then each connected component of G is a single element, which is irreducible. We say that a finite group G acts rationally on a variety X if each connected component acts by a morphism of varieties $\{g\} \times X \to X$.

An important operator that exists with reductive groups is the *Reynolds operator*. It projects on to the G-invariant part of a ring.

Definition 3.8.

Let G be a reductive group acting on a vector space V. A *Reynolds operator* is a linear map $\natural : V \to V^G$ such that

1. $v^{\natural} = f$, for all $v \in V^G$
2. $(g.v)^{\natural} = v^{\natural}$, for all $v \in V$.

It is a linear projection map on to the subspace of G-invariant elements.

Lemma 3.9.

Let G be a reductive group acting rationally on an affine variety X. Then there exists a unique Reynolds operator $\natural : k[X] \to k[X]^G$.

See [6] for a complete proof. If G is reductive, then we can decompose any finite dimensional subspace V of $k[X]$ as the direct sum $V^G \oplus (V^G)^{\perp}$, on which there is a Reynolds operator projecting along $(V^G)^{\perp}$. This idea can be extended using the fact that G acts rationally on X to show that there is a unique Reynolds operator on the infinite dimensional vector space $k[X]$.

Thus for any reductive group acting rationally on an affine variety X we can write $k[X]$ as $k[X]^G \oplus ker\ \natural$, where $\natural$ is the unique Reynolds operator on $k[X]$.

There are two key properties of reductive groups we are interested in.

Lemma 3.10.

Let G be a reductive group acting rationally on an affine variety X. Then $k[X]^G$ is finitely generated.

See [5] for a proof.

The following lemma is used in the construction of the GIT quotient. It allows us to find a G-invariant function which separates disjoint G-invariant sets.

Lemma 3.11.

Let G be a reductive group acting rationally on an affine variety $X \subseteq \mathbb{A}^n$. Let Z_1 and Z_2 be two closed G-invariant subsets of X with $Z_1 \cap Z_2 = \emptyset$. Then there exists a G-invariant function $F \in k[X]^G$ such that $F(Z_1) = 1$ and $F(Z_2) = 0$.

Proof.

As $Z_1 \cap Z_2 = \emptyset$, we have $\mathcal{I}(\emptyset) = \mathcal{I}(Z_1 \cap Z_2) = \mathcal{I}(Z_1) + \mathcal{I}(Z_2) = k[x_1, ..., x_n]$. So there exists $f_1 \in \mathcal{I}(Z_1)$, $f_2 \in \mathcal{I}(Z_2)$ such that $f_1 + f_2 = 1$.

As G is reductive there exists a Reynolds operator $\natural : k[x_1, ..., x_n] \to k[x_1, ..., x_n]^G$. Applying this Reynolds operator we get $f_1^\natural + f_2^\natural = 1$.

As Z_1 is G-invariant we have that $\mathcal{I}(Z_1)$ is a G-invariant subspace of $k[x_1, ..., x_n]$. Thus there exists a projection on to the G-invariant part of $\mathcal{I}(Z_1)$. By uniqueness of the Reynolds operator, we must have $\natural$ is this projection. In particular we have $f_1^\natural \in \mathcal{I}(Z_1)$. Thus we have $f_1^\natural(Z_1) = 0$ and so $f_2^\natural(Z_1) = 1$. Similarly as $f_2^\natural \in \mathcal{I}(Z_2)$ we have $f_2^\natural(Z_2) = 0$. As $f_2^\natural$ is also G-invariant we have $F = f_2^\natural + \mathcal{I}(X) \in k[X]$ satisifes the required properties, completing the proof.

$\square$

3.2 The affine quotient

We now have enough to construct the quotient of an affine variety by a reductive group. For an affine variety X, the quotient of X by a reductive group G is Spec $\mathcal{O}(X)^G$, the affine variety corresponding to the subring of G-invariant elements of $\mathcal{O}(X)$. We will prove that this construction satisfies the required conditions of being a good categorical quotient. The following proof builds on the proof in Newstead's atricle [11].

Theorem 3.12. *Let X be an affine variety and G be a reductive group acting rationally on X. Let $p^* : \mathcal{O}(X)^G \to \mathcal{O}(X)$ be defined by the inclusion $\mathcal{O}(X)^G \subseteq \mathcal{O}(X)$. Then the dual of this map, $p : X \to Y :=$ Spec $\mathcal{O}(X)^G$ is a good categorical quotient.*

Proof.

We will first show that Y is an affine variety and that $p : X \to Y$ is a G-invariant surjective map. Let $R = \mathcal{O}(X)$. As X is affine, R is a finitely generated integral algebra. By Lemma 3.10, R^G is also finitely generated. Also, R^G is an integral domain by being a subset of R, noting that f is a zero divisor of R^G if and only if f is a zero divisor of R. So $Y =$ Spec R^G is an affine variety.

For an element $g \in G$, it's action on X is a automorphism of affine varieties, and corresponds to a dual action on R, by $g^*f(x) = f(g.x)$. For all $f \in R^G$, we have $g^* \circ p^*(f) = p^*f$, as p^*f is G-invariant. This corresponds by duality to $p \circ g(x) = p(x)$, hence p is G-invariant.

To show p is surjective, let y be a point in Y and $f_1, ...f_r$ generate the maximal ideal $\mathfrak{n} \trianglelefteq R^G$ corresponding to y. Considering the f_i as elements of R, the ideal generated by these f_i in R is proper. See that if it were not proper, then we would have $1 = \sum_{i=0}^r a_i f_i$ for some $a_i \in R$. As G is reductive, there is a Reynolds operator on R and we get $1^\natural = (\sum_{i=0}^r a_i f_i)^\natural$, that is, $1 = \sum a_i^\natural f_i$, which contradicts $\mathfrak{n}$ being a proper ideal of R^G, as $a_i^\natural \in R^G$. Thus the ideal generated by the f_i in R is proper, so is contained within some maximal ideal $\mathfrak{m} \trianglelefteq R$. Let x be the point in X corresponding to the ideal $\mathfrak{m}$, then $f(x) = 0$ for all $f \in \mathfrak{m}$, and so $f \circ p(x) = p^*f(x) = 0$ for all f such that $p^*f \in \mathfrak{m}$. Thus we have $f(p(x)) = 0$ for all $f \in \mathfrak{n}$, and so $p(x) = y$

We show that for any open $U \subseteq Y$, $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))^G$ is an isomorphism. We prove this first for the basic open sets Y_f of Y, for any $f \in \mathcal{O}(Y) = R^G$. Here we have $p^{-1}(Y_f) = \{x : f(p(x)) \neq 0\} = \{x : p^*f(x) \neq 0\} = X_f$, so we need to prove

$\mathcal{O}(Y_f) = \mathcal{O}(X_f)^G$, that is, that $(R^G)_f = (R_f)^G$.

$$\begin{aligned} \frac{h}{f^n} \in (R^G)_f &\iff h \in R^G \\ &\iff \frac{h}{f^n} \text{ is G-invariant} \\ &\iff \frac{h}{f^n} \in (R_f)^G \end{aligned}$$

So we have $(R^G)_f = (R_f)^G$.

Any open set $U \subseteq Y$ has decomposition $U = Y_{f_1} \cup ... \cup Y_{f_r}$ for some $f_i \in R^G$, and then we have $p^{-1}(U) = p^{-1}(Y_{f_1} \cup ... \cup Y_{f_r}) = X_{f_1} \cup ... \cup X_{f_r}$. A function is regular on a union of sets if and only if it is regular on each of those sets, that is, for a family of sets U_i we have $\mathcal{O}(\bigcup U_i) = \bigcap \mathcal{O}(U_i)$. Then

$$\begin{aligned} \mathcal{O}(U) = \mathcal{O}(Y_{f_1} \cup ... \cup \mathcal{O}(Y_{f_r})) &= \mathcal{O}(Y_{f_1}) \cap ... \cap \mathcal{O}(Y_{f_r}) \\ &= (R^G)_{f_1} \cap ... \cap (R^G)_{f_r} \\ &= (R_{f_1})^G \cap ... \cap (R_{f_r})^G \\ &= \mathcal{O}(X_{f_1})^G \cap ... \cap \mathcal{O}(X_{f_r})^G \\ &= \mathcal{O}(X_{f_1} \cup ... \cup X_{f_r})^G \\ &= \mathcal{O}(p^{-1}(U))^G \end{aligned}$$

For closed G-invariant $W \subseteq X$, we need to that prove $p(W)$ is closed in Y. For a contradiction, assume $p(W)$ is not closed, so there exists some $y \in \overline{p(W)} \setminus p(W)$. The set $Z := p^{-1}(\{y\})$ is closed, as $\{y\}$ is closed, and G-invariant due to p being G-invariant. Also $Z \cap W = \emptyset$, as their images under p are disjoint. Lemma 3.11 implies there exists $F \in R^G$ such that $F(W) = 1$ and $F(Z) = 0$. As F is G-invariant we have $F = p^*(F')$ for some $F' \in \mathcal{O}(Y)$. This gives $F'(W) = 1$ and $F'(y) = 0$, which is a contradiction, as $y \in \overline{p(W)}$, hence $p(W)$ must be closed.

Let $V_1, V_2 \in X$ be closed and G-invariant, with $V_1 \cap V_2 = \emptyset$. We prove that $p(V_1) \cap p(V_2) = \emptyset$. Lemma 3.11 shows that there exists $F \in R^G$ such that $F(V_1) = 1$ and $F(V_2) = 0$. Thus there is a $F' \in \mathcal{O}(Y)$ such that $F = p^*F'$, so $F'(p(V_1)) = 1$ and $F'(p(V_2)) = 0$ and hence $p(V_1) \cap p(V_2) = \emptyset$.

□

We now give an example, showing how a quotient of the set of matrices by the group action of conjugation can be constructed.

Example 3.13.

Consider the set X of 2×2 matrices over $\mathbb{C}$, embedded in $\mathbb{A}^4$ by

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix} \mapsto (w, x, y, z).$$

Often we just want to study the similarity classes of matrices. So we will construct the quotient object that identifies similar objects. Let $G = GL(2, \mathbb{C})$ act on X by conjugation. That is, for $A \in G$, $M \in X$, define $A.M = AMA^{-1}$. Then we have $X//G = k[w, x, y, z]^G$.

We know for matrices that the determinant and trace are invariant under conjugation. These are the polynomials $det = wz - xy$ and $tr = w + z$, so we have $k[wz - xy, w + z] \subseteq k[w, x, y, z]^G$. We will show that we infact have equality.

Let $\lambda \in \mathbb{C}^\times$ be arbitrary and consider the matrix $A = \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix}$. For all matrices $M = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$ we have

$$\begin{aligned} A^{-1}MA &= \begin{pmatrix} 0 & -\frac{1}{\lambda} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \lambda & 0 \end{pmatrix} \\ &= \begin{pmatrix} z & \frac{y}{\lambda} \\ \lambda x & w \end{pmatrix} \end{aligned}$$

Let $f \in k[w, x, y, z]^G$. Then for all $\lambda \in \mathbb{C}^\times$ we require that $f(w, x, y, z) = A.f(M) = f(A.M) = f(A^{-1}MA) = f(z, \frac{y}{\lambda}, \lambda x, w)$. That is,

$$f(w, x, y, z) = f(z, \frac{y}{\lambda}, \lambda x, w).$$

We can see from this that we thus require $f \in k[xy, wz, w + z]$, as any x appearing in f must be of the form xy to cancel the λ appearing in $A.f$, and any w must appear with a z.

Consider the matrix $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then we have

$$\begin{aligned} B^{-1}MB &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} w - y & w + x - y - z \\ y & y + z \end{pmatrix} \end{aligned}$$

As we have $f(M) = f(B.M)$ we thus require $f(w, x, y, z) = f(w - x, w + x - y - z, y, y + z)$. As we have $f \in k[xy, wz, w + z]$ we can simplify this to requiring

$$f(xy, wz, w + z) = f(wy + xy - y^2 - z, wy + wz - y^2 - yz, w + z)$$

We see that this is satisfied only when $f \in k[wz - xy, w + z]$, as the extra terms in $B.f$ must cancel with each other. So we have $k[w, x, y, z]^G \subseteq k[wz - xy, w + z]$. These two polynomials define the determinant and the trace, which we know are invariant under conjugation, so we have $k[w, x, y, z]^G = k[wz - xy, w + z]$.

Applying Theorem 3.12, we can construct the quotient

$$\begin{aligned} X//G &= \text{Spec } k[w, x, y, z]^G \\ &= \text{Spec } k[wz - xy, w + z] \\ &= \text{Spec } k[u, v] \\ &= \mathbb{C}^2, \end{aligned}$$

Notice that this is not a separated quotient, as any matrix with the same determinant and trace are identified together. For example, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ maps to the same point in the quotient as the identity matrix does, even though they are not conjugate. However this quotient does preserve a sensible geometric structure on the matrices, paramaterized by the determinant and trace.

More generally in higher dimensions, if $GL(n, \mathbb{C})$ acts on $M(n, \mathbb{C})$ by conjugation we have

$$M(n, \mathbb{C})//GL(n, \mathbb{C}) = \mathbb{C}^n.$$

A matrices characteristic polynomial is given by the polynomial in t, $\chi(M) = det(M - tI)$, where I is the identity matrix. This is of the form $a_0 + a_1 t + ... + a^{n-1} t^{n-1} + t^n$ for some $a_i \in \mathbb{C}$. The quotient $M(n, \mathbb{C}) \to \mathbb{C}^n$ maps a matrix to the n co-efficients of its characteristic polynomial. For example, in the 2×2 matrices example above, we have $\chi(M) = t^2 + tr(M)t + det(M)t$. See [3] for more details on this example in higher dimensions.

3.3 The GIT quotient

We will now construct the projective quotient by gluing together affine quotients.

Let X be a projective variety. Then X is the union of some finite number of affine varieties X_{f_i}. To construct a good categorical quotient of X by G, we wish to cover the quotient with affine quotients of the form $X_{f_i}//G = \mathrm{Spec}(\mathcal{O}(X_{f_i})^G)$. To do this, we need an action of G on the coordinates of X.

Our approach is to embed X in $\mathbb{P}^m$ for some m such that the action of G can be extended to a linear action on $\mathbb{A}^{m+1}$. This is called a *linearisation* of the action of G.

Definition 3.14.

Let the group G act rationally on a projective variety X. Let $\varphi : X \hookrightarrow \mathbb{P}^m$ be an embedding of X that extends the group action, in the sense that we have a rational group action on $\mathbb{P}^m$ such that $\varphi(g.x) = g.\varphi(x)$. Let $\pi : \mathbb{A}^{m+1} \to \mathbb{P}^m$ be the natural projection. A *linearisation* of the action of G with respect to φ is a linear action of G on $\mathbb{A}^{m+1}$ that is compatible with the action of G on X in the sense that

(i) For all $y \in \mathbb{A}^{m+1}$, $g \in G$,

$$\pi(g.y) = g.(\pi(y)).$$

(ii) For all $g \in G$, the map

$$\mathbb{A}^{m+1} \to \mathbb{A}^{m+1} : y \mapsto g.y$$

is linear.

We write φ_G for a linearisation of the action of G with respect to φ.

Note that this induces an action of G on the coordinate ring of X. We have X is isomorphic to the image $\varphi(X) \subseteq \mathbb{P}^m$ and so $\mathcal{O}(X) \cong k[x_0, ..., x_m]/I$ for some homogeneous ideal I. Then G acts on $k[x_0, ..., x_m]$ by $g.f(x_0, ..., x_m) := f(g^{-1}.(x_0, ..., x_m))$, and so acts on $\mathcal{O}(X)$, noting that $g.f' \in I$ for any $f' \in I$.

Example 3.15.

This example was used in Birkner's article [2].

Let $\mathbb{C}^\times$ act on $\mathbb{P}^1_{\mathbb{C}}$ by $\lambda.(x_0 : x_1) = (x_0 : \lambda x_1)$. A linearisation can be given by the obvious action on $\mathbb{A}^2$ with $\lambda.(x_0, x_1) = (x_0, \lambda x_1)$

This example illustrates an issue in the construction of the quotient. In the projective setting, good categorical quotients need not exist. In this example, there are three orbits: $\{(1 : t), (1 : 0), (0 : 1)\}$. The only possible G-invariant morphism sends all orbits to a point, as $(1 : 0)$ and $(0 : 1)$ are both in the closure of $(1 : t)$. But this fails to separate closed orbits, so is not a good categorical quotient.

The solution to this problem is to take an open G-invariant subset which has a good categorical quotient. We desire this subset to be covered by G-invariant open affine subsets so that we can cover the quotient by gluing together affine quotients. This leads us to the notion of semistablility.

Definition 3.16.

Let G be a reductive group acting rationally on a projective variety X which has an embedding $\varphi : X \hookrightarrow \mathbb{P}^m$. A point $x \in X$ is called *semistable* (with respect to the linearisation φ_G) if there exists some G-invariant homogeneous polynomial f of degree greater than 0 in the coordinate ring of X, such that $f(x) \neq 0$ and X_f is affine.

We write $X^{ss}(\varphi_G)$ for the set of semistable points of X with respect to φ_G, or just X^{ss} when the linearisation is clear from the context.

In Example 2.15 above, the set of semistable points with respect to the linearisation given is $X^{ss} = X_{x_0} = \mathbb{P}^1 \setminus \{(0:1)\}$. On this subset, the map to a point, $p : X^{ss} \to \mathbb{P}^0$ is indeed a good categorical quotient.

Lemma 3.17.

Let $X \subseteq \mathbb{P}^n$ be a projective variety with coordinate ring R. Then for all homogenous G-invariant $f \in R$ of positive degree we have $(R[f^{-1}]_0)^G = R^G[f^{-1}]_0$

Proof.

An element of $\frac{h}{f^d} \in (R[f^{-1}]_0)$ is G-invariant if and only if h is G-invariant, if and only if $\frac{h}{f^d} \in (R^G[f^{-1}]_0)$.

□

To construct the projective quotient, we need to have a projective analogue of the Spec operation so that we can regain a projective variety from its coordinate ring. If $R \cong k[x_0, ..., x_n]/I$ for some prime homogeneous ideal I, we define Proj R to be the projective variety corresponding to R, that is, Proj R is the projective variety $V(I) \subseteq \mathbb{P}^n$.

We are now ready to construct the quotient of a projective variety. Note that the construction also works for an affine variety by embedding $X \hookrightarrow \mathbb{P}^m$ for some $\mathbb{P}^m$ as an open affine subvariety.

Theorem 3.18. *Let G be a reductive group acting rationally on a projective variety X embedded in $\mathbb{P}^m$ with a linearisation φ_G. Let R be the coordinate ring of X. Then there is a good categorical quotient $p : X^{ss}(\varphi_G) \to X^{ss}(\varphi_G)//G \cong$ Proj R^G.*

Proof.

We construct the quotient object by gluing together affine subsets of the image, as in Birkner's proof [2].

From Lemma 3.10 we see that R^G is finitely generated. Take a set of generators $f_0, ..., f_n$. Then the X_{f_i} are an affine cover of $X^{ss}(\varphi_G)$, by the definition of semistable points. By Theorem 2.12, for each X_{f_i} we can take the affine quotient

$$\begin{aligned} p_i : X_{f_i} \to X_{f_i}//G \\ &= \text{Spec } (\mathcal{O}(X_{f_i})^G) \\ &= \text{Spec } (R[f_i^{-1}]_0)^G \\ &= \text{Spec } R^G[f_i^{-1}]_0 \end{aligned}$$

Consider the surjective map

$$k[T_0, ..., T_n] \to R^G, y_i \mapsto f_i$$

This has kernel $I \trianglelefteq k[T_0, ..., T_n]$ so $k[T_0, ..., T_n]/I \cong R^G$.

We then have Proj $R = V(I) \subseteq \mathbb{P}^n$ is a projective variety with $k[Y] = R^G$. Let $Y =$ Proj R. Any point $y \in Y$ has a non-zero coordinate, by definition of projective space, that is, $T_i(y) \neq 0$ for some T_i. But we have $T_i(y) = f_i(y)$ and so y must be in some Y_{f_i}. Thus we can write Y as the union of affine subvarieties

$$Y = \bigcup_{i=0}^{n} Y_{f_i}$$

and for each of the Y_{f_i} we have $Y_{f_i} \cong \text{Spec } \mathcal{O}(Y_{f_i}) = \text{Spec } R^G[f_i^{-1}]_0$. Thus we can glue the quotient morphisms p_i above to get the good categorical quotient

$$p : X \to Y = \bigcup_{i=0}^{n} \text{Spec } R^G[f_i^{-1}]_0$$

We will show this gluing of maps is well-defined and gives a good categorical quotient.

To show the gluing of these maps to well defined, we need to consider $x \in X_{f_i} \cap X_{f_j}$ and show that p_i and p_j send x to the same point. We note that $X_{f_i} \cap X_{f_j} = X_{f_i f_j}$

and $Y_{f_i} \cap Y_{f_j} = Y_{f_i f_j}$. Consider the quotient map

$$p_{ij} : X_{f_i f_j} \to Y_{f_i f_j} = \text{Spec } R^G[f_i^{-1} f_j^{-1}]_0$$

in the following diagram, with ι_i, ι_j and π_i, π_j being the inclusion maps from the intersections as shown.

$$\begin{tikzcd}
X_{f_i} \arrow[r, "p_i"] & Y_{f_i} = \text{Spec } R^G[f_i^{-1}]_0 \\
X_{f_i f_j} \arrow[u, "\pi_i"] \arrow[d, "\pi_j"'] \arrow[r, "p_{ij}"] & Y_{f_i f_j} = \text{Spec } R^G[f_i^{-1} f_j^{-1}]_0 \arrow[u, "\iota_i"'] \arrow[d, "\iota_j"] \\
X_{f_j} \arrow[r, "p_j"] & Y_{f_i} = \text{Spec } R^G[f_j^{-1}]_0
\end{tikzcd}$$

Showing that this diagram commutes would imply that p_{ij} sends any $x \in X_{f_i f_j}$ to the same point in $Y_{f_i f_j}$ as p_i and p_j, thus showing that the gluing of maps is well defined. So we need to show that $\iota_i \circ p_{ij} = p_i \circ \pi_i$ and $\iota_j \circ p_{ij} = p_j \circ \pi_j$.

As ι_i is an inclusion map, its pull-back satisfies $\iota_i^* f(y) = f(\iota_i(y)) = f(y)$ for $y \in Y_{f_i f_j}$. Similarly we have $\pi_i^* f(x) = f(x)$ for $x \in X_{f_i f_j}$ and the pull-backs of the affine quotient maps are inclusions by construction. Thus we have for all $x \in X_{f_i f_j}$, for all $f \in R^G[f_i^{-1}]_0$,

$$(p_{ij}^* \circ \iota_i^*) f(x) = \iota_i^* f(x) = f(x)$$

So by the correspondence between morphism of affine varieties and their coordinate rings we have $\iota_i \circ p_{ij} = p_i \circ \pi_i$. Similarly we have $\iota_j \circ p_{ij} = p_i \circ \pi_j$ and so the diagram commutes. Thus the map $p : X \to \text{Proj } R^G$ is a well-defined mapping of projective varieties. It is a morphism of varieties as it is the gluing together of morphisms to affine varieties, by Lemma 3.13.

The quotient morphism p is surjective as it is surjective on to each Y_{f_i}. Let $x \in X_{f_i}$, $g \in G$. f_i is G-invariant, so X_{f_i} is G-invariant and thus $g.x \in X_{f_i}$. This implies $p(g.x) = p_i(g.x) = p_i(x) = p(x)$, as p_i is p restricted to X_{f_i} which is a good affine quotient and so is G-invariant.

Any open subset $U \subseteq Y$ is the union of open subsets from each Y_i. Take an open subset $U = U_1 \cup ... \cup U_n$, where $U_i \subseteq Y_{f_i}$. We have $\mathcal{O}(U) = \mathcal{O}(U_1 \cup ... \cup U_n) = \bigcap_{i=0}^n \mathcal{O}(U_i)$, noting that a function is regular on a union of sets if and only if it is regular on each of the sets. We wish to show $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))^G$ is an isomorphism.

First note that $p^{-1}(Y_{f_i}) = p_i^{-1}(Y_{f_i})$. To see this we need to check that if $p(x) \in Y_{f_i}$ then we must have $x \in X_{f_i}$, as then we would have $p(x) = p_i(x)$. If $x \notin X_{f_i}$, then $f_i(x) = 0$ and for some j we have $f_i(p_j(x)) = p_j^* f(x) = 0$, as p_j^* is defined by the inclusion map. Thus $p(x) \notin Y_{f_i}$. Hence if $p(x) \in Y_{f_i}$, then $x \in X_{f_i}$ and so $p^{-1}(Y_{f_i}) \in X_{f_i}$ and so $p^{-1}(Y_{f_i}) = p_i^{-1}(Y_{f_i})$.

For each U_i we have $p^* : \mathcal{O}(U_i) \to \mathcal{O}(p_i^{-1}(U_i))^G$ is an isomorphism, as p_i is a good categorical quotient. As $U_i \subseteq Y_{f_i}$ we have $p^{-1}(U_i) = p_i^{-1}(U_i)$ and so $p^* : \mathcal{O}(U_i) \to \mathcal{O}(p^{-1}(U_i))^G$ is an isomorphism.

To show that the image of $\mathcal{O}(U)$ under p^* is $\mathcal{O}(p^{-1}(U))^G$ we need to use the fact that the image under p^* of an intersection of sets is equal to the intersection of the image of the sets. In particular we need that for any i, j we have

$$p^*(\mathcal{O}(U_i) \cap \mathcal{O}(U_j)) = p^*(\mathcal{O}(U_i)) \cap p^*(\mathcal{O}(U_j)).$$

This is satisfied if and only if $p^* : \mathcal{O}(Y) \to \mathcal{O}(X)$ is injective. This is true and results from the fact that p is surjective. Suppose we have $p^*h_1 = p^*h_2$ for some $h_1, h_2 \in \mathcal{O}(Y)$. Then we have $h_1(p(x)) = h_2(p(x))$ for all $x \in X$. But as p is surjective we thus have $h_1(y) = h_2(y)$ for all $y \in Y$, that is $h_1 = h_2$. So p^* is injective.

We then have the image of $\mathcal{O}(U)$ under p^* is

$$\begin{aligned}
p^*(\mathcal{O}(U)) &= p^*\left(\bigcap_{i=0}^{n} \mathcal{O}(U_i)\right) \\
&= \bigcap_{i=0}^{n} p^*(\mathcal{O}(U_i)) \\
&= \bigcap_{i=0}^{n} \mathcal{O}(p^{-1}(U_i))^G \\
&= \mathcal{O}(\bigcup_{i=0}^{n} p^{-1}(U_i))^G \\
&= \mathcal{O}(p^{-1}(U))^G
\end{aligned}$$

As we also have $p^* : \mathcal{O}(U_i) \to \mathcal{O}(p_i^{-1}(U))^G$ is an isomorphism on the larger domain $\mathcal{O}(U_i) \supseteq \mathcal{O}(U)$, it follows that $p^* : \mathcal{O}(U) \to \mathcal{O}(p^{-1}(U))^G$ is an isomorphism.

The other two properties of being a good categorical quotient are also satisfied, using the facts that any closed subset V of $X^{ss}(\varphi_G)$ is the union of closed subsets $V_i \subseteq X_{f_i}$ and that each $p_i : X_{f_i} \to Y_{f_i}$ is a good categorical quotient.

Let $V \subseteq X^{ss}(\varphi_G)$ be closed and G-invariant and write $V = V_1 \cup ... \cup V_n$ with $V_i = V \cap X_{f_n}$. Then $p(V) = p_i(V_i) \cup ... \cup p_n(V_n)$ which is closed, as each of the $p_i(V_i)$ is closed due to p_i being a good categorical quotient.

Let $V \subseteq X^{ss}(\varphi_G)$ and $W \subseteq X^{ss}(\varphi_G)$ with $V \cap W = \emptyset$. Write $V = V_1 \cup ... \cup V_n$ with $V_i = V \cap X_{f_n}$. Then for each V_i we have $p(V_i) \cap p(W) = p_i(V_i) \cap p(W) \subseteq Y_{f_i}$. We then have

$$\begin{aligned} p_i(V_i) \cap p(W) &= p_i(V_i) \cap (p(W) \cap Y_{f_i}) \\ &= p_i(V_i) \cap p_i(W) \end{aligned}$$

as $p^{-1}(Y_{f_i}) \subseteq X_{f_i}$ and so $p = p_i$ on $p^{-1}(p(W) \cap Y_{f_i})$. We must now have that $p(V_i) \cap p(W) = \emptyset$, as p_i is a good categorical quotient. As this holds for each V_i it follows that $p(V) \cap p(W) = \emptyset$. We thus have $p : X^{ss}(\varphi_G) \to \text{Proj } R^G$ is a good categorical quotient.

□

Example 3.19.

Let $G = C^{\times}$ act on $X = \mathbb{P}^2_{\mathbb{C}}$ by

$$g.(x : y : z) := (gx : gy : g^{-1}z)$$

for $g \in G$. Note this action is well defined, as $g.(\lambda x : \lambda y : \lambda z) = g.(x : y : z)$ for all $\lambda \in \mathbb{C}^{\times}$, by the linearity of the action.

The fixed point orbits of this action are those of the form $\{(t : u : 0)\}$ for fixed $t, u \in \mathbb{C}$ not both zero, as well as $\{(0 : 0 : 1)\}$. The other G-orbits are those of the form $\{(gt : gu : 1) : g \in G\}$ for fixed $t, u \in \mathbb{C}$ not both zero. We can see that no good categorical quotient exists on all of X, as $(0 : 0 : 1)$ and $(t : u : 0)$ lie in the closure of the orbit $\{(gt : gu : 1) : g \in G\}$ for any $t, u \in \mathbb{C}$ not both zero. Hence a categorical quotient can not separate closed orbits.

There is a linearisation of the G-action on $\mathbb{A}^3$ in the obvious way, by letting $g.(x, y, z) = (gx, gy, g^{-1}z)$. Let $R = \mathbb{C}[x, y, z]$, the coordinate ring of $\mathbb{A}^3$. Then with this linearisation, we have $R^G = \mathbb{C}[xz, yz]$. We have $X_{xz} \subset X_x$, which is affine, so X_{xz} is affine, and similarly X_{yz} is affine. The set of semistable points is then $X^{ss} = X_{xz} \cup X_{yz} = X \setminus (\{(t : u : 0) : t, u \in \mathbb{C}\} \cup \{(0 : 0 : 1)\})$, which removes all the fixed point orbits.

Using Theorem 2.18, we can construct the quotient

$$X^{ss}//G = \text{Proj } \mathbb{C}[xz, yz] = \text{Proj } \mathbb{C}[u, v] = \mathbb{P}^1_{\mathbb{C}},$$

the projective line, as would be expected from the picture.

Note that a different quotient can be constructed by a different linearisation. Consider linearising the G-action by letting $g.(x, y, z) = (g^2x, g^2y, z)$. Then $R^G = \mathbb{C}[z]$ and we have the semistable point $X^{ss} = X_z = X - \{(t : u : 0) : t, u \in \mathbb{C}\}$, and the quotient in this case is

$$X_z//G = \text{Proj } \mathbb{C}[z] = \mathbb{P}^0_{\mathbb{C}},$$

which is a point, as expected, as $(0 : 0 : 1)$ lies in the closure of every orbit.

We will now move on to studying how quotients constructed by different choices of linearisation are related. Before doing this, we will introduce *birational maps* which give an equivalence of varieties weaker than that of isomorphisms.

Chapter 4

Birational maps

Isomorphisms between varieties can be restrictive, so we'll introduce a weaker relation of varieties, by the use of *birational maps*. This will give the notion of when two varieties have non-empty open subsets which are isomorphic. This is a strong similarity of varieties, as any non-empty open subset of a variety X is dense in X. We use Hartshorne's definition of rational maps used in [8], as it clarifies that a rational map is a morphism on a non-empty open set.

Definition 4.1.

A *rational map* $\varphi : X \dashrightarrow Y$ of varieties is an equivalence class of pairs $\langle U, \varphi_U \rangle$, where U is a non-empty open subset of X, φ_U is a morphism from U to Y and where $\langle U, \varphi_U \rangle$ and $\langle V, \varphi_V \rangle$ are equivalent if φ_U and φ_V agree on $U \cap V$. This defines a partial map of varieties.

Note the equivalence mentioned in the definition is indeed an equivalence relation. To see this we need the following lemma.

Lemma 4.2.

Let $\varphi_1 : U \to Y$ and $\varphi_2 : U \to Y$ be morphisms of varieties. If φ_1 and φ_2 agree on a non-empty open subset V of U, then $\varphi_1 = \varphi_2$.

A proof for this can be seen in [8]. It extends the fact that two regular functions agreeing on a non-empty open set are equal.

In the equivalence relation in the above definition, we see that if $\langle U, \varphi_U \rangle \sim \langle V, \varphi_V \rangle$ and $\langle V, \varphi_V \rangle \sim \langle W, \varphi_W \rangle$ for non-empty open subsets U, V and W, then we have that φ_U agrees with φ_W on the non-empty open subset $U \cap V \cap W$ and hence by the previous lemma agrees on $U \cap W$.

The reason we call $\langle U, \varphi_U \rangle$ a rational map, is that if $\varphi = \langle U, \varphi_U \rangle : X \dashrightarrow Y$ is a morphism on an open affine subset of X, then its coordinates are given by rational functions on X. We do not require that φ be defined everywhere, so non-trivial denominators in the coordinate functions are allowed, they only need to be non-zero where φ is defined.

We wish to have a category of varieties with the morphisms being the rational maps. This requires the maps to be composable, which is not always the case with rational maps. We say a rational map $\varphi : X \dashrightarrow Y$ is *dominant* if the image of X under φ (where φ is defined) is dense in Y. We can compose dominant rational maps to get dominant rational maps, but first we need to show some necessary lemmas about the intersection of dense sets.

Lemma 4.3.

Let Z be a dense subset of X and U be a non-empty open subset of X. Then $U \cap Z \neq \emptyset$.

Proof.

For a contradiction suppose $U \cap Z = \emptyset$. Then we have $Z = X \cap Z = (U \cup U^c) \cap Z = U^c \cap Z$. That is, we have $Z \subseteq U^c$, and as U^c is closed we must hence have $\overline{Z} \subseteq U^c$. But U is non-empty, so $U^c \subset X$ and so $\overline{Z} \subset X$, contradicting Z being dense in X.

□

We can now show that this intersection is in fact dense.

Lemma 4.4.

Let Z be a dense subset of X and U be a non-empty open subset of X. Then $U \cap Z$ is dense in X.

Proof.

Suppose $U \cap Z$ is not dense in X. Then $(\overline{U \cap Z})^c$ is open and non-empty. We have $(\overline{U \cap Z})^c \cap U \cap Z \subseteq (\overline{U \cap Z})^c \cap (\overline{U \cap Z}) = \emptyset$. Thus $\big((\overline{U \cap Z})^c \cap U\big) \cap Z = \emptyset$, but this contradicts the previous lemma, as Z is dense in X and $(\overline{U \cap Z})^c \cap U$ is open. Hence $U \cap Z$ is dense in X.

□

We now show that dominant rational maps compose to give dominant rational maps.

Lemma 4.5.

Let $\phi : X \dashrightarrow Y$ and $\varphi : Y \dashrightarrow Z$ be dominant rational maps of varieties. Then the composition $\varphi \circ \phi : X \dashrightarrow Z$ is a dominant rational map.

Proof.

Consider representations $\langle U, \phi_U \rangle$ of ϕ and $\langle V, \varphi_V \rangle$ of φ. We have $\phi_U(U)$ is dense in Y, so by Lemma 4.4 we have $\phi_U \cap V$ is dense in Y and so the preimage $\phi_U - 1(\phi_U \cap V) = U \cap \phi_U^{-1}(V)$ is non-empty. We show that the composition $\langle U \cap \phi_U^{-1}(V), \varphi_V \circ \phi_U \rangle$ defines a dominant rational map. Let W denote the image of $U \cap \phi_U^{-1}$ by $\varphi_V \circ \phi_U$. Then $\varphi_V^{-1}(W) \supset \phi_U(U) \cap V$ and so $\varphi_V^{-1}(\overline{W})$ is closed in V and contains $\phi_U(U) \cap V$. But $\phi_U(U) \cap V$ is dense in Y and hence is dense in V. Thus we have $\varphi_V^{-1}(\overline{W}) = V$

and so $\overline{W} = Z$, as $\varphi_V(V)$ is dense in Z. Thus $\langle U \cap \phi_U^{-1}(V), \varphi_V \circ \phi_U \rangle$ defines a dominant rational map.

$\square$

For a simple example of rational maps that are not composable, consider the maps $f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ and $g : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ given by $f(x : y) := (0 : y)$ and $g(x : y) := (x : 0)$. These partial maps are morphisms on the open subsets $\{(x : y) \in \mathbb{P}^1 : y \neq 0\}$ and $\{(x : y) \in \mathbb{P}^1 : x \neq 0\}$ respectively, so define rational maps. We have g is undefined at the point $(0 : 1)$, but this is the only point in the image of f. Thus we have the composition $g \circ f$ is undefined everywhere, so cannot be a rational map.

Important to note in the composition of rational maps is that the domain of a composition of maps $g \circ f$ may be larger than the domain of f. A simple example of this is the composition of the inverse function on $\mathbb{A}^1$ with itself. The domain of $x \mapsto 1/x$ is $\mathbb{A}^1 - \{0\}$, however the composition of this map with itself $x \mapsto 1/x \mapsto 1/(1/x) = x$, is defined on all of $\mathbb{A}^1$. With composition of dominant rational maps, we are able to study varieties with dominant rational maps as a category..

We now define *birational maps*, which are the isomorphisms in this category of dominant rational maps.

Definition 4.6.

A *birational map* is a rational map $\phi : X \dashrightarrow Y$ for which there exists an inverse rational map $\varphi : Y \dashrightarrow X$ such that $\varphi \circ \phi = id_X$ and $\phi \circ \varphi = id_Y$.

If there exists a birational map between two varieties X and Y, we say that X and Y are *birationally equivalent*, or *birational*.

In morphisms of affine varieties we found that we could study the morphisms by looking at their coordinate rings. We can similarly study dominant rational maps by looking at the *field of rational functions* of varieties.

Definition 4.7.

The field of rational functions of an affine variety X over k is the field of fractions

$$k(X) := \left\{ \frac{f}{g} : f, g \in k[X], g \neq 0 \right\}.$$

The field of rational function of a projective variety $Y \subseteq \mathbb{P}_k^n$ is the field of fractions

$$k(Y) := \left\{ \frac{f}{g} : f, g \in k[Y], g \neq 0 \right\}$$

with the additional requirement that f, g be homogeneous of the same degree. Note these are indeed fields and have the usual fraction equality, that is $\frac{f}{g} = \frac{f'}{g'}$ if and only if $fg' = f'g$.

A rational function on a variety over k defines a partial function to k. From the definition of regular functions, we see that a rational function $f/g \in k(X)$ is regular precisely on the open subset X_g and so defines a map to k on this subset.

We saw that there exists a duality between morphisms of affine varieties and homomorphisms of their coordinate rings, by the pull-back. There is similarly a bijective relation between dominant rational maps $X \dashrightarrow Y$ and k-algebra homomorphisms $k(Y) \to k(X)$, by the pull-back.

Lemma 4.8.

If $\varphi : X \dashrightarrow Y$ is a dominant rational map, then the map $\varphi^* : k(Y) \to k(X)$ defined by $\varphi^* f(x) = f(\varphi(x))$ is a k-algebra morphism.

Proof.

We show that φ^* indeed maps $k(Y)$ to $k(X)$. Let $f/g \in k(Y)$, then f/g is regular on the non-empty open subset Y_g. Let $\langle U, \varphi_U \rangle$ be a representation of φ. As $\varphi_U(U)$ is dense in Y we have by Lemma 4.3 that $\varphi_U(U) \cap Y_g$ is non-empty and hence $\varphi^{-1}(Y_g) \cap U$ is open and non-empty. As φ_U is a morphism we have $\varphi_U^*(\frac{f}{g}) = \varphi^*(\frac{f}{g})$ is regular on $\varphi_U^{-1}(Y_g) \cap U$, which is non-empty and open. Thus $\varphi^*(\frac{f}{g})$ is a rational function in $k(X)$, so we have that φ^* maps $k(Y)$ to $k(X)$. It is simple to see that φ^* defines a k-algebra morphism by checking the axioms pointwise.

$\square$

We also need for this pull-back to be unique.

Lemma 4.9.

Let X and Y be varieties and $\psi : k(Y) \to k(X)$ be a homomorphism of k-algebras. Then there exists a unique rational map $\phi : X \to Y$ such that $\phi^* = \psi$. Also this ϕ is dominant.

The proof of this lemma follows a similar argument as for the argument used in proving the uniqueness of the push-forward for homomorphisms of the coordinate rings of affine varieties. See [8] for more details.

An important property of this correspondence is the pull-back of the composition.

Lemma 4.10.

Let $\phi : X \dashrightarrow Y$ and $\varphi : Y \dashrightarrow Z$ be dominant rational maps. Then $(\varphi \circ \phi)^* = \phi^* \circ \varphi^*$

Proof.

For any $f \in k(Z)$ we have $(\phi^* \circ \varphi^*)f(x) = \phi^*(\varphi^* f)(x) = \varphi^* f(\phi(x)) = f(\varphi \circ \phi(x) = (\varphi \circ \phi)^* f(x)$. This shows $(\varphi \circ \phi)^* = \phi^* \circ \varphi^*$. $\square$

With this we can now prove the following important equivalence.

Lemma 4.11.

The following statements are equivalent:

(i). X and Y are birational

(ii). $k(X) \cong k(Y)$

(iii). There exists non-empty open subsets $U \subseteq X$ and $V \subseteq Y$ such that $U \cong V$

Proof.

The first two statements being equivalent follow from the correspondence between rational maps $X \to Y$ and k-algebra morphism $k(Y) \to k(X)$.

(i) $\implies$ (ii)

If X and Y are birational, then there exist dominant rational maps $\phi : X \dashrightarrow Y$ and $\phi^{-1} : Y \dashrightarrow X$ such that $\phi^{-1} \circ \phi = id_X$ and $\phi \circ \phi^{-1} = id_Y$. We then have morphisms of k-algebras $\phi^* : k(Y) \to k(X)$ and $\phi^{-1*} : k(X) \to k(Y)$. Let $f \in k(Y)$ and $x \in Y$, then by the previous lemma,

$$(\phi^{-1*} \circ \phi^*)f(x) = (\phi \circ \phi^{-1})^* f(x) = f(id_Y(x)) = f(x)$$

and so $\phi^{-1*} \circ \phi^*$ is the identity on $k(Y)$. Similarly $\phi^* \circ \phi^{-1*}$ is the identity on $k(X)$, and thus $k(X) \cong k(Y)$.

(ii) $\implies$ (i)

If $k(X) \cong k(Y)$ then by the correspondence of k-algebra morphisms with rational maps, there exist dominant rational maps $\phi : X \dashrightarrow Y$ and $\varphi : Y \dashrightarrow X$ such that $\phi^* : k(Y) \to k(X)$ is an isomorphism with inverse $\varphi^* : k(X) \to k(Y)$. We have $id_{k(Y)} = \varphi^* \circ \phi^* = (\phi \circ \varphi)^*$. Note that for any $f \in k(Y)$, $x \in Y$, we have the trivial pull-back $f(id_Y(x)) = f(x)$, so $id_Y{}^* = id_{k(Y)}$. By the bijective correspondence we must then have $\phi \circ \varphi = id_Y$. Similarly we can show that $\varphi \circ \phi = id_X$ and hence X and Y are birational.

(i) $\implies$ (iii)

Let $\phi : X \dashrightarrow Y$ be a birational map, with inverse $\varphi : Y \dashrightarrow X$. Each of these maps is regular on some non-empty open subset. Let $U \subseteq X$ and $V \subseteq Y$ be the open subsets where ϕ and φ are defined and regular. As ϕ is dominant, $\phi(X) \cap V$ is non-empty, so $\phi^{-1}(V) \neq \emptyset$. As ϕ is continuous we have $\phi^{-1}(V)$ is open in U and hence in X(*top stuff*). Repeating this argument, the set $\varphi^{-1}(\phi^{-1}(V)) \subseteq Y$ is non-empty and open. Similarly $\phi^{-1}(\varphi^{-1}(U)) \subseteq X$ is non-empty and open. We take these sets as our isomorphic open sets.

These sets are isomorphic by the maps ϕ and φ. We have $\varphi \circ \phi$ and $\phi \circ \varphi$ are the identities on these sets, as they are the identities on X and Y. We also have ϕ is regular on the image of $\varphi^{-1}(\phi^{-1}(V))$ under φ, as well as φ being regular on the image of $\phi^{-1}(\varphi^{-1}(U))$ under ϕ. This satisifies the requirements for an isomorphism of varieties.

(iii) $\implies$ (i)

Let $U \subseteq X$, $V \subseteq Y$ be non-empty open isomorphic subvarieties. We can assume U and V are affine, as is they are not, they themselves must contain affine isomorphic subvarieties.

As $U \cong V$, there exists morphisms $\varphi : U \to V$ and $\varphi^{-1} : V \to U$ such that $\varphi^{-1} \circ \phi = id_U$ and $\varphi \circ \phi^{-1} = id_V$. Thus we have rational maps $\phi : X \dashrightarrow Y$ and $\phi^{-1} : Y \dashrightarrow X$ which extend these morphisms, given by $\langle U, \varphi \rangle$ and $\langle V, \varphi^{-1} \rangle$. These are dominant rational maps as they map to V and U, which are non-empty and open, hence dense, in Y and X. Composing these maps, we get $\phi^{-1} \circ \phi(x) = x$ for all $x \in U$. So $\phi^{-1} \circ \phi$ agrees with id_X on the non-empty open subset U, hence by Lemma something , $\phi^{-1} \circ \phi = id_X$. Similarly, we can show $\phi \circ \phi^{-1} = id_Y$, and so X and Y are birational.

□

Chapter 5

Linearisations of affine varieties

This chapter largely follows the first section of [13]. We will look at affine varieties over $\mathbb{C}$ embedded in projective space in a very degenerate way, and consider the different choices of linearisation. For affine $X = \text{Spec } R$, we will look at X as a projective variety, by $X = \mathbb{P}^0_R$.

With this notation, being projective over R, we have $\mathbb{P}^0_R = X \times \mathbb{P}^0$. For $X \subseteq \mathbb{A}^n_k$ this is projective by the obvious embedding in $\mathbb{P}^n_k$ by $(x_1, ..., x_n) \mapsto (x_1 : ... : x_n : z)$.

We have seen that $R = \mathbb{C}[x_1, ..., x_n]/\mathcal{I}(X)$ is an $\mathbb{N}$-graded ring, graded by the degree of the polynomials. We will now consider a $\mathbb{Z}$-grading of R, which will describe an action of $\mathbb{C}^\times$ on X. An action of $\mathbb{C}^\times$ on Spec R is equivalent to a $\mathbb{Z}$-grading of R, with $R_i = \{f \in R : \lambda.f = \lambda^i f \text{ for all } \lambda \in \mathbb{C}^\times\}$. Note here that we get the convenient notation that $R^G = R_0$

Lemma 5.1.

Let R be a finitely generated alegbra and $X = \text{Spec } R$ be an affine variety over $\mathbb{C}$. Then an action of $\mathbb{C}^\times$ on X is equivalent to a $\mathbb{Z}$-grading of R such that we can write $R = \bigoplus_{i=-\infty}^{\infty} R_i$ with $R_i = \{f \in R : \lambda.f = \lambda^i f \text{ for all } \lambda \in \mathbb{C}^\times\}$

Proof.

This proof is based on a proof in [3].

An action of $\mathbb{C}^\times$ on X is given by a morphism of varieties $\sigma : \mathbb{C}^\times \times X \to X$ such that for all $g, g' \in \mathbb{C}^\times$, $x \in X$, we have $\sigma(g, (\sigma(g', x)) = \sigma(gg', x)$ and $\sigma(1, x) = x$. We have $\mathbb{C}^\times$ as a variety by the isomorphism

$$\begin{aligned} &\mathbb{C}^\times \to \{(t, u) \in \mathbb{A}^2 : tu = 1\} \text{ , given by} \\ &\quad g \mapsto (g, g^{-1}), \end{aligned}$$

so an action of $\mathbb{C}^\times$ on X is of the form $\sigma(g, x) = (\sigma_1(g, x), ..., \sigma_n(g, x))$ where the σ_i are given by polynomials in g, g^{-1}, and the coordinates of X.

We then have the pull-back of this action $\sigma^* : \mathbb{C}[X] \to \mathbb{C}[\mathbb{C}^\times \times X]$ with $\sigma^* f(g, x) = f(\sigma(g, x))$ being a polynomial in t, t^{-1} and the coordinates of X, where $t \in \mathbb{C}[\mathbb{C}^\times \times X]$ is the degree one polynomial with $t(g, x) := g$. Thus we can group together the parts of $\phi^* f$ where t appears with degree $d_i \in \mathbb{Z}$ and write $\sigma^* f$ as $\sum_{i=0}^{n} t^{d_i} f_i$ for with the f_i determined by f. This can be $\mathbb{Z}$-graded by writing as

$$\sigma^* f = \sum_{n=-\infty}^{\infty} t^{-n} f_n$$

with the f_n determined uniquely by f, being the part of $\sigma^* f$ in which t appears with degree $-n$.

As $g.f(x) = f(\sigma(g^{-1}, x)) = \sigma^* f(g^{-1}, x)$ we can write $g.f$ as

$$g.f = \sum_{n=-\infty}^{\infty} g^n f_n$$

As $g_1 g_2.f = g_1.(g_2.f)$ for all $g_1, g_2 \in \mathbb{C}^\times$, we have

$$g_1 g_2.f = \sum_{n=-\infty}^{\infty} g_1^n g_2^n f_n = g_1.(g_2.f) = \sum_{n=-\infty}^{\infty} g_2^n (g_1.f_n),$$

and so we have $g.f_n = g^n f_n$ for all $g \in \mathbb{C}^n$. We thus have a decompositon $R = \bigoplus_{n=-\infty}^{\infty} R_n$ with $R_n = \{f \in R : g.f = g^n f \text{ for all } g \in k^\times\}$. We can further see that if $f_n \in R_n$, $f_m \in R_m$ we have $g.f_n f_m = \sigma^*(f_n f_m)(g^{-1}, x) = \sigma^* f_n(g^{-1}, x) \sigma^* f_m(g^{-1}, x) = (g.f_n)(g.f_m)$ and so $f_n f_m \in R_{mn}$. Thus we have a $\mathbb{Z}$-grading of R. □

Let R a finitely generated integral $\mathbb{C}$-algebra, so that $X = \text{Spec } R$ is an affine variety over $\mathbb{C}$. Let $G = \mathbb{C}^\times$, then an action of G on Spec R is equivalent to a $\mathbb{Z}$-grading of R, say $R = \bigoplus_{i \in \mathbb{Z}} R_i$.

Consider $X = \text{Spec } R = \mathbb{P}_R^0$. A linearisation of the G-action is an action on Spec $R \times \mathbb{A}^1$ preserving the action on Spec R. This is equivalent to a $\mathbb{Z}$-grading of $R[z]$ that preserves the given $\mathbb{Z}$-grading of R. That is, we write $R[z] = \bigoplus_{i \in \mathbb{Z}} R[z]_i$, with $R_i \subset R[z]_i$. We let $z \in R[z]_{-n}$ for some $n \in \mathbb{Z}$ to determine the linearisation.

For fixed $n \in Z$ determining the linearisation, the quotient is Proj $R[z]^G =$ Proj $R[z]_0 = \text{Proj} \bigoplus_{i \in \mathbb{N}} R_{ni} z^i$. For $n = 0$, this is $\text{Proj} \bigoplus_{i \in \mathbb{N}} R_0 z^i = \text{Proj } R_0[z] =$ Spec R_0, the regular affine quotient. For $n > 0$, the quotient is $\text{Proj} \bigoplus_{i \in \mathbb{N}} R_{ni} z^i$. In [8] we see at [II Ex. 5.13] that this is equal to $\text{Proj} \bigoplus_{i \in \mathbb{N}} R^i z^i$. Similarly, for $n < 0$, the quotient is $\text{Proj} \bigoplus_{i \in \mathbb{N}} R_{-i} z^i$.

We will denote these three quotients, when $n = 0$, $n > 0$, $n < 0$; by $X//0$, $X//+$ and $X//-$ respectively and study relations between them.

$$X//0 = \text{Proj } R_0[z] = \text{Spec } R_0$$
$$X//+ = \text{Proj} \bigoplus_{i\in\mathbb{N}} R_i z^i$$
$$X//- = \text{Proj} \bigoplus_{i\in\mathbb{N}} R_{-i} z^i$$

Indeed we find that these varieties are closely related. Note that $X//+$ is projective over $X//0 = \text{Spec } R_0$, as R_0 is the degree zero part of $\bigoplus_{i\in\mathbb{N}} R_i z^i$ and similarly $X//-$ is projective over $X//0$. In particular this gives the natural projection $X//+ \to X//0$ as the dual of the inclusion $R_0 \hookrightarrow \bigoplus_{i\in\mathbb{N}} R_i z^i$. This is taken as a birational map being the dual of a k-algebra morphism of function fields, and is indeed a morphism, as the inclusion always gives a regular function. Recall that a morphism $\varphi : X \to Y$ is a map such that φ^* maps functions regular on open sets $U \subseteq Y$ to regular functions on $\varphi^{-1}(U) \subseteq X$. There is simlarly a natural morphism $X//- \to X//0$.

Lemma 5.2.

If $X//- \neq \emptyset$ then the natural projection $X//+ \to X//0$ is birational.

Similarly, if $X//+ \neq \emptyset$ then the natural projection $X//- \to X//0$ is birational.

Proof.

As $X//- \neq \emptyset$, we have $\bigoplus_{i\in\mathbb{N}} R_{-i} z^i \neq R_0$ and so for some $d > 0$, R_{-d} contains a nonzero element t. The function field of $X//0$ is $\{r/s : r, s \in R_0\}$, while that of $X//+$ is $\{r/s : r, s \in R_{di}$ for some $i \geq 0\}$, noting that $\text{Proj} \bigoplus_{i\in\mathbb{N}} R_i z^i = \text{Proj} \bigoplus_{i\in\mathbb{N}} R_{ni} z^i$. But the map $\varphi : r/s \mapsto (rt^i)/(st^i)$ from $\mathbb{C}(X//+)$ to $\mathbb{C}(X//0)$ is an isomorphism, as we will show.

Note this is well-defined on rational functions, as for $r, s \in R_{di}$, $m \in R_{dj}$, we have $\varphi(\frac{rm}{sm}) = \frac{rmt^{i+j}}{smt^{i+j}} = \varphi(\frac{rt^i}{st^i})$. Also φ is a morphism of $\mathbb{C}$-algebras. For $r_1, s_1 \in R_{di}$, $r_2, s_2 \in R_{dj}$, we have $\varphi(\frac{r_1}{s_1} + \frac{r_2}{s_2}) = \varphi(\frac{r_1 s_2 + r_2 s_1}{s_1 s_2}) = \frac{r_1 s_2 t^{i+j} + r_2 s_1 t^{i+j}}{s_1 s_2 t^{i+j}} = \varphi(\frac{r_1}{s_1}) + \varphi(\frac{r_2}{s_2})$ and the other requirements of being a morphism of $\mathbb{C}$-algebras follow similarly.

We can take the inverse $\varphi^{-1} : \mathbb{C}(X//0) \to \mathbb{C}(X//+)$ as $\varphi^{-1}(\frac{m}{n}) = \frac{m}{n}$. We see that for $r, s \in R_{di}$ we have $\varphi^{-1} \circ \varphi(\frac{r}{s}) = \varphi^{-1}(\frac{rt^i}{st^i}) = \frac{rt^i}{st_i} = \frac{r}{s}$ and also for $r, s \in R_0$ we have $\varphi \circ \varphi^{-1}(\frac{r}{s}) = \varphi(\frac{r}{s}) = \frac{r}{s}$.

This inverse map is the dual of the natural projection $X//+ \to X//0$, so is birational. Similarly, if $X//+ \neq \emptyset$ then the natural projection $X//- \to X//0$ is birational. $\square$

Let X^+ and X^- denote the subvarieties of X corresponding to the ideals $\langle R_i : i > 0\rangle$ and $\langle R_i : i < 0\rangle$ respectively. That is, $X^+ = \{x \in X : f(x) = 0$, for all $f \in R_i, i > 0\}$, and similarly for X^-. We see the semistable points with respect to the different linearisations in the following lemma. Recall the semistable points is where we define the quotient.

Lemma 5.3. *The semistable points are:*

$$
\begin{aligned}
X^{ss}(0) &= X \\
X^{ss}(+) &= X - X^+ \\
X^{ss}(-) &= X - X^-
\end{aligned}
$$

Proof. $X^{ss}(0) = X$, as $1 \in R_0$, and 1 is never zero. For $X^{ss}(+)$, the G-invariant functions are $f \in R_i z^i$ for some $i > 0$, so it is clear by the definition of X^+ that $X^{ss}(+) = X - X^+$. Similarly we get $X^{ss}(-) = X - X^-$.

□

We have seen that a birational map $X \dashrightarrow Y$ is isomorphic on some open subsets $U \subseteq X$ and $V \subseteq Y$. So we have that these quotients constructed by different choices of linearisations must be isomorphic on some open subsets. The following is another result in [13], which I omit the proof of.

Lemma 5.4.

The morphism $X//+ \to X//0$ is isomorphic on the complements of the images of X^- under the quotient maps. Also, $X//- \to X//0$ is isomorphic on the complements of the images of X^+ under the quotient maps.

We will look at an interesting example where the birational maps between the different quotients are not trivial. We first mention two important types of birational transformations called the *blow-up* and the inverse transformation, called a *blow-down.* For a variety X, we can blow-up X at a point by replacing that point with a copy of all lines coming out of that point. This is like sticking a copy of projective space in at that point. A blow-up is indeed a birational transformation.

For example, the blow-up of $\mathbb{A}^2$ at the origin is the subset of $\mathbb{A}^2 \times \mathbb{P}^1$ where the part of $\mathbb{P}^1$ associated for a point $(x, y) \in \mathbb{A}^2$ is all lines going through (x, y). For $(x, y) \neq (0, 0)$ this is a one-to-one correspondence (x, y) with $(x, y; x : y)$. At the origin, we include all of $\mathbb{P}^1$ and have the points $(0, 0; a : b)$ for all a, b not both zero. Notice it is clear that this is a birational transformation as the two varieties are isomorphic away from the origin.

We now look at a non-trivial example looked at by Atiyah [1]. It demonstrates an interesting succession of birational transformation known as the Atiyah flop, which consists of a blow-down at a point followed by a blow-up at the same point in a different orientation.

Example 5.5.

Let $X = \mathbb{A}^4$ and let $G = \mathbb{C}^\times$ act on X by $\lambda.(v, w, x, y) = (\lambda v, \lambda w, \lambda^{-1}x, \lambda^{-1}y)$. We have $X//0 = \text{Spec } \mathbb{C}[vw, vy, wx, wy] = \text{Spec } \mathbb{C}[a, b, c, d]/ \langle ad - bc\rangle$. This is the affine cone over a smooth surface in $\mathbb{P}^3$.

We have $X//+ = \text{Proj } \mathbb{C}[vx, vy, wx, wy, zv, zw] = \text{Proj } \mathbb{C}[a, b, c, d, za, zc]/ \langle ad - bc\rangle$, which looks somewhat simililar to $X//0$. It has the affine part of $X//0$ by the natural projection $X//+ \to X//0$, but also includes a projective part at the origin. It has $\mathbb{P}^1$ lying over the origin, this is the blow-up of $X//0$ at the origin, oriented in some sense to the plane defined by $a = c = 0$. Also, $X//- = \text{Proj } \mathbb{C}[a, b, c, d, zb, zd]/ \langle ad - bc\rangle$, which is also blown up at the origin, but oriented with respect to a different plane, where $b = d = 0$.

We can see that the blow-down $X//+ \dashrightarrow X//0$ is an isomorphism on the open set not containing the origin, and similarly for $X//-$. Of interest is the birational map we have from $X//+$ to $X//-$ through $X//0$. Indeed we have an isomorphism between these varieties $X//+$ and $X//-$ directly, by simply flipping the coordinates a, c with b, d, but more interesting is the inequality of these varieties, and the birational map through $X//0$ consisting of a blow-down then a blow-up. Both of these two quotients consist of $\mathbb{P}^1$ stuck into $X//0$ at the origin, a blow-up, but it is with different orientations. This is birational map $X//+ \dashrightarrow X//-$ is an example of an Atiyah flop.

What I have presented in this chapter is the strong relations that exist between the different possible constructions of the GIT quotient of affine varieties. This is somewhat of a degenerate case, embedding an affine variety X in to the projective point above X, however it is still interesting seeing that different quotients can be constructed, and that they are all birational. It is much less of a simple case when considering the different possible linearisations on a projective variety and it would be interesting to see what sort of relations hold between the different possible quotients of more complicated projective varieties.

References

[1] M. F. Atiyah. On analytic surfaces with double points. *Proc. Roy. Soc. London. Ser. A*, 247:237–244, 1958.

[2] René Birkner. Introduction to GIT. Freie Universitaet Berlin, 2008.

[3] Michel Brion. Introduction to actions of algebraic groups. Institut Fourier, 2009.

[4] bwebste and Matt Piaktus. "proof of hilbert basis theorem" (version 3). PlanetMath.org.

[5] H. E. A. Eddy Campbell and David L. Wehlau, editors. *Invariant theory in all characteristics*, volume 35 of *CRM Proceedings & Lecture Notes*, Providence, RI, 2004. American Mathematical Society.

[6] Harm Derksen and Gregor Kemper. *Computational invariant theory*. Invariant Theory and Algebraic Transformation Groups, I. Springer-Verlag, Berlin, 2002. Encyclopaedia of Mathematical Sciences, 130.

[7] Arapura Donu. Notes on basic algebraic geometry. Department of Mathematics, Purdue University, 2008.

[8] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[9] Serge Lang. *Algebra*, volume 211 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, third edition, 2002.

[10] J. Le Potier. *Lectures on vector bundles*, volume 54 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1997. Translated by A. Maciocia.

[11] P. E. Newstead. Geometric invariant theory. CIMAT, Guanajuato, Mexico, 2006.

[12] Karen E. Smith, Lauri Kahanpää, Pekka Kekäläinen, and William Traves. *An invitation to algebraic geometry*. Universitext. Springer-Verlag, New York, 2000.

[13] Michael Thaddeus. Geometric invariant theory and flips. *J. Amer. Math. Soc.*, 9(3):691–723, 1996.