TWISTED RINGS OF DIFFERENTIAL OPERATORS ON THE PROJECTIVE LINE AND THE BEILINSON-BERNSTEIN THEOREM

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November 2007

Submitted in partial fulfillment of the requirements of the degree of
Bachelor of Science with Honours

Acknowledgements

One year ago, I had the option to finish my Bachelor's degree in Commerce and Science with majors in Actuarial Studies and Pure Mathematics. Long before then, I had decided to do Honours in Pure Mathematics. I had definite plans for Honours and tentative plans for a PhD for quite some time before my Honours year actually began. It would represent the first time in my life where I could spend all my time solely concentrating on mathematics.

The year began a little earlier than usual, with an AMSI course on Lie Algebras at Sydney University in January 2007. After suffering from a personal setback, I threw myself at this course and spent as much time as possible learning about Lie Algebras. I was extremely happy with my result in this course, and my goals for 2007 suddenly changed. I decided then and there, in February 2007, that my goal was to give this Honours year my absolute best effort. I decided to make any necessary sacrifice to achieve this goal, my Honours year was my highest priority.

My priorities for the year became, in their respective order, my Honours year, my health and fitness, learning more about my family, and spending time with my closest friends. As the year progressed, I realised that sacrificing things to maintain my priorities simplified my life and gave me a sense of clarity and strength, as I broke habits that I never believed I was able to break. None of this would have happened if it wasn't for the people that I spent my time with during this year, and that is what I'd like to acknowledge.

I would not have had the strength to push myself, without my family. To my sister, who is by far the most special person to me in my life. I am grateful for her undying loyalty, love and support, and the fact that no matter what, we can always make each other laugh. A special thanks goes to my dad, I feel that this year I have learnt more about him than I have previously in my life. I'd like to thank him for the faith and confidence that he has shown in me. I want to thank my mum, because she encapsulates my determination to succeed in every aspect of my life, I doubt that I will ever meet anyone more driven and focused than she is.

I would not have learnt anything without the generous help of the academic staff at UNSW. A special thanks must go to my supervisor, Daniel Chan. Words cannot express what he has given me. I will always admire Daniel, for his mathematical mind, but aside from that, for his kindness, generosity and his way of life. I've never learnt more about anything from anyone else, and I doubt I ever will. I'd like to also give a grateful thanks to Hendrik Grundling, Daniel Chan (again), Jie Du, Norman Wildberger, Ian Doust, Catherine Greenhill, Colin Sutherlan and Anthoney Henderson for their *style* of teaching. I will also never forget the teaching style of David Hunt, who is strict, but effective. I'd like to thank David for giving me my first taste of algebra.

Lastly, I'd like to thank my friends for giving me balance in my life and keeping me sane. This year has seen some friendships end, some begin and some slowly drift into indifference. To my best mates, Robbie and Stuart, I want to thank you guys for sticking around and going to the effort to keep in touch this year, and also for the great times. I owe a special thanks to all my buddies in the honours room, Boris, Mandy, Tim, Beren, Charles, Cuong and Michelle, thanks for giving me great reasons to procrastinate. I'd like to thank Kenneth and Hugo, for their help during the year. Lastly I'd also like to thank Elmar, Nitesh, Wim and the rest of my squash buddies, for giving me a weekly release and giving me a great start to the week with monday night squash.

I look back at this year, and now I see that my motivation to work hard and spend time learning about mathematics was my *interest* in mathematics and nothing else. I think that is the way it should be.

Introduction

Consider the following differential equation,

$$\frac{d^2 f}{dt^2} + at^2 \frac{df}{dt} + bf = 0 \tag{0.1}$$

where $a, b \in \mathbb{C}$. It is possible to think of this equation as $Pf = 0$ where,

$$P := \frac{d^2}{dt^2} + at^2 \frac{d}{dt} + bI \tag{0.2}$$

is a *differential operator* on some function $f \in \mathbb{C}[t]$. The algebraic properties of (0.2) will give us extra information about (0.1) and help us to further understand these differential operators. Another example of a differential operator on an affine space is that of a vector field acting on the space. In this thesis, we aim to describe how differential operators over projective spaces can be perceived. In particular, we will see how to classify rings of differential operators on the complex projective line $\mathbb{P}^1_{\mathbb{C}}$, and we will also attempt to understand the Beilinson-Bernstein Theorem.

In 1981, Beilison and Bernstein, produced a much celebrated article entitled *Localisation de* $\mathfrak{g}$*-modules*, [17]. In this article, they proved an equivalence of categories and also proved the Kazhdan-Lusztig multiplicity conjecture. This article provided great insight into the study of the geometry of flag varieties. In 1989, Masaki Kashiwara, produced work on the geometry of flag varieties in [12]. He used the results of Beilinson and Bernstein and used an approach that is similar to the approach taken in this thesis.

Following Kashiwara we proceed down a path inside Algebraic Geometry. Algebraic Geometry is a rich and diverse field in mathematics which can be approached many different ways. The most sensible approach for this thesis was that of Robin Hartshorne's text, Algebraic Geometry, [6]. We need concepts such as projective varieties, quasi-coherent sheaves and schemes, which allow us to fully analyse exactly how differential operators work on projective spaces. Since we can break $\mathbb{P}^1_{\mathbb{C}}$ into affine patches, we can study differential operators acting on each patch. Sheaves allow us to glue the information from each patch together so that we can obtain differential operators that act on the whole of $\mathbb{P}^1_{\mathbb{C}}$. Most of this theory will be covered in Chapter 2. It is important to mention that Algebraic Geometry, due to its depth, has urged us to seek alternative methods in this thesis. This is why we take an alternative (but well known) approach covered by Richard Swan in [9]. The concept of a sheaf is vital in this thesis.

We begin the thesis with a description of differential operators through algebraic eyes, as we construct various rings of differential operators. After covering the necessary topics of Algebraic Geometry mentioned above, we use Swan's approach, in the third chapter, to convert our problem of Algebraic Geometry into a problem involving categories and modules. This new problem is, evidently, much easier to solve and it allows us to classify the *sheaves of rings of differential operators on* $\mathbb{P}^1_{\mathbb{C}}$, which introduces the concept of *twisting*. With the knowledge of all the sheaves of twisted rings of differential operators, we will form a *quantised moment map* which forms a connection between differential operators that are defined on the whole of $\mathbb{P}^1_{\mathbb{C}}$ and a quotient of $U(\mathfrak{sl}_2)$, the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. This result is one of the most remarkable results of the thesis, as we see that these complicated sheaves of differential operators tell us something about $U(\mathfrak{sl}_2)$.

After this, we learn more about these *twisted sheaves of differential operators* by understanding the Beilinson-Bernstein Theorem. Initially, a primary objective for this thesis was to prove this theorem in its simplest case, but unfortunately, it didn't eventuate. Instead we provide the necessary theory to understand the theorem and demonstrate its uses as we prove the translation principle for $U(\mathfrak{sl}_2)$-modules. Finally we will see how the thesis really describes the work of Kashiwara in [12], for the simplest non-trivial case.

In this thesis, the only assumed knowledge is that of rings, modules and their basic properties, such as tensoring of modules (in particular). The thesis itself is "proof heavy" as I personally dislike skipping proofs, but in an effort to be concise, proofs that do not add any intuition to the concepts being described have been omitted.

Contents

Chapter 1

Rings of Differential Operators and Localisation

Differential operators are maps with two primary properties, namely they are $\mathbb{F}$-linear (where $\mathbb{F}$ is a field like $\mathbb{C}$ or $\mathbb{R}$), and they satisfy the Liebnitz rule. Studying their algebraic properties gives us extra insight in their nature, and allows us to formalise them through a different perspective. This chapter contains the elementary definitions and results related to the ring of differential operators. We will also describe the technique of localisation of a non-commutative ring. The definitions and results in the next two sections can be found in [1] and [2].

1.1 Rings of Differential Operators

The algebraic properties of differential operators provide us with valuable information and intuition of their nature. Before we define the ring of differential operators we begin by defining a K-algebra over a general ring, K.

Definition 1.1. *Suppose R is a ring (not necessarily commutative with unity) and K is a subring of R. Then R is a K-**algebra** if the centre of R contains K.*

Suppose R is a K-algebra, since elements of K commute with every element of R we can think of R as a K-module. Let $End_K(R)$ denote the set of all K-module homomorphisms from R to R. The following proposition shows us that R embeds itself neatly inside $End_K(R)$.

Proposition 1.2. *Let R be a K-algebra, then the map $\phi : R \hookrightarrow End_K(R)$ which takes $a \mapsto (\phi_a : x \mapsto ax)$ is an injective ring homomorphism.*

Proof Suppose $a, b, x \in R$ then $\phi_{a+b}(x) = (a+b)x = ax + bx = \phi_a(x) + \phi_b(x) = (\phi_a + \phi_b)(x)$ and $\phi_{ab}(x) = (ab)x = a(bx) = \phi_a(bx) = \phi_a\phi_b(x)$ so ϕ is a ring homomorphism. To prove that ϕ is injective, suppose $a \in ker(\phi)$ then ϕ_a is the zero map so $ax = 0$ for all $x \in R$ but $1 \in R$ so $a = a1 = 0$. $\square$

The following example of Proposition 1.2 is also a motivation for the definition of the ring of differential operators.

Example 1.3. We can think of the ring of complex polynomials, $\mathbb{C}[t]$, as a $\mathbb{C}$-algebra. Furthermore, $\phi(\mathbb{C}[t])$ is precisely the set of all maps $\phi_{a(t)} \in End_{\mathbb{C}}(\mathbb{C}[t])$ that send $f(t) \mapsto a(t)f(t)$, where $a(t) \in \mathbb{C}[t]$. This corresponds to how $\mathbb{C}[t]$ acts on itself via its ring multiplication operation.

Consider the following two maps in $End_{\mathbb{C}}(\mathbb{C}[t])$,

$$t : f(t) \mapsto tf(t)$$

$$\delta : f(t) \mapsto f'(t)$$

It is important to realise that we are not distinguishing between $t \in \mathbb{C}[t]$ and $\phi(t) \in End_{\mathbb{C}}(\mathbb{C}[t])$ as it does not create any confusion. These two maps are two more examples of differential operators. We now observe that these two operators do not commute. For any $f \in \mathbb{C}[t]$ we have,

$$\delta(tf) = (tf)' = f + tf' = f + t\delta(f)$$

hence,

$$(\delta t - t\delta - 1)f = 0$$

Interestingly enough, $\{t, \delta\}$ generates a ring called the ***first Weyl Algebra*** on $\mathbb{C}$,

$$A_1(\mathbb{C}) = \frac{\mathbb{C}\langle t, \delta \rangle}{(\delta t - t\delta - 1)}$$

The ring $A_1(\mathbb{C})$ is an example of a ring of differential operators. For a full treatment on the Weyl Algebras, see [1] (Chapters 1 and 2 are both devoted to the structure of the Weyl Algebras).

As a motivation for the definition of the ring of differential operators we make the following observations. We see that in $A_1(\mathbb{C})$,

$$\phi_{a(t)}\phi_{b(t)} - \phi_{b(t)}\phi_{a(t)} = 0 \text{ for all } \phi_{a(t)}, \phi_{b(t)} \in \phi(\mathbb{C}[t]) \subseteq A_1(\mathbb{C})$$

In other words the elements of $\phi(\mathbb{C}[t])$ commute. Furthermore,

$$\delta t - t\delta = 1 \in \phi(\mathbb{C}[t])$$

This suggests that measuring how close maps are to commuting allows us to distinguish between different substructures of $A_1(\mathbb{C})$.

For the remainder of this chapter let R be a commutative K-algebra where K is a non-trivial field.

Definition 1.4. *Define $D^0(R) := \{P \in End_K(R) \mid [\phi_a, P] = 0$ for all $a \in R\}$ where $[A, B] = AB - BA$ is called the **commutator bracket** of A and B. For all $n \geq 0$, let $D^{n+1}(R) := \{P \in End_K(R) \mid [\phi_a, P] \in D^n(R)$ for all $a \in R\}$. Also, define the **ring of differential operators** over R to be*

$$D(R) := \bigcup_{n \geq 0} D^n(R)$$

*and say that the elements of $D^n(R)$ are **differential operators** on R of degree $\leq n$.*

Continuing Example 1.3, $t \in D^0(\mathbb{C}[t])$ and it can be shown that $\delta \in D^1(\mathbb{C}[t])$. This result will follow from Proposition 1.6. Our major goal for this section is to prove that $D(R)$ is in fact a ring. In Section 1.2, we will prove that $A_1(\mathbb{C})$ is identical to $D(\mathbb{C}[t])$. This will be the first ring of differential operators we compute. Due to the inductive definition of $D(\mathbb{C}[t])$, this will take us a considerable amount of effort to prove. Before we begin proving this result, we will try to analyse $D^n(R)$ a little more closely.

Proposition 1.5. *For each $n \geq 0$, $D^n(R)$ is a vector space over K.*

Proof We first prove that $D^n(R)$ is closed under addition by induction on n. Due to the linearity of the commutator bracket, closure under addition is trivial for the case where $n = 0$. Suppose $P, Q \in D^n(R)$, then $[\phi_a, P + Q] = [\phi_a, P] + [\phi_a, Q]$, again by the linearity of the commutator bracket. We know that $[\phi_a, P]$, $[\phi_a, Q] \in D^{n-1}(R)$, thus via the inductive assumption, $[\phi_a, P] + [\phi_a, Q] \in D^{n-1}(R)$. Hence, $D^n(R)$ is closed under addition. Proving closure under scalar multiplication is similar to the previous argument since $[\phi_a, kP] = \phi_a kP - kP\phi_a = k(\phi_a P - P\phi_a) = k[\phi_a, P]$ for all $a \in R$, $P \in D^n(R)$ and $k \in K$. Thus via induction on n again, $D^n(R)$ is closed under scalar multiplication. Since the zero map is an element of $D^n(R)$ for all $n \geq 0$, $D^n(R)$ is a vector space over K for each $n \geq 0$. □

Call $P \in End_K(R)$ a ***derivation*** if $P(ab) = (Pa)b + a(Pb)$ for all $a, b \in R$. Denote the set of derivations by $Der_K(R)$. The map $\delta : f(t) \mapsto f'(t)$ from Example 1.3 is clearly a derivation as this property is just the product rule for differentiation. We can actually compute $D^n(R)$ for small n. We consider the case for $n = 0, 1$ here.

Proposition 1.6. *$D^0(R) = \phi(R)$ and, if R is a domain then $D^1(R) = \phi(R) \oplus Der_K(R)$.*

Proof We first prove $D^0(R) = \phi(R)$. Since R is a commutative ring, $[\phi_a, \phi_b] = 0$ for all $a, b \in R$, so $\phi(R) \subseteq D^0(R)$. Conversely, suppose $P \in D^0(R)$, then $P(a) = P(a1) = P(\phi_a(1)) = (P\phi_a)(1) = (\phi_a P)(1) = a(P(1)) = P(1)a$ for all $a \in R$. Hence $P = \phi_{P(1)} \in \phi(R)$.

For the second part, let us first show that the right hand side is indeed a direct sum. Suppose $P \in \phi(R) \cap Der_K(R)$, then $P = \phi_x$ for some $x \in R$. Hence, $xab = P(ab) = (Pa)b + a(Pb) = 2axb$ for all $a, b \in R$, as R is commutative. This tells us that $x = 0$ (as R is a domain), and hence $P = 0$. We already know $\phi(R) \subseteq D^1(R)$, so let us now show that $Der_K(R) \subseteq D^1(R)$. Let $P \in Der_K(R)$, then $P(ab) = (Pa)b + a(Pb)$. In other words $P(\phi_a(b)) = \phi_{Pa}(b) + \phi_a(Pb)$, since this holds for all $b \in R$, $(P \circ \phi_a - \phi_a \circ P - \phi_{Pa})b = 0$. This tells us that $[P, \phi_a] = \phi_{Pa} \in D^0(R)$ for all $a \in R$, so $P \in D^1(R)$ and hence $\phi(R) \oplus Der_K(R) \subseteq D^1(R)$ from Proposition 1.5. Suppose $P \in D^1(R)$, we already know that $\phi_{P(1)} \in \phi(R)$ so it suffices to prove that $Q := P - \phi_{P(1)} \in Der_K(R)$. We note that $Q(1) = P(1) - \phi_{P(1)}(1) = 0$ and that $Q \in D^1(R)$. Thus, $[Q, \phi_a] \in D^0(R)$ for all $a \in R$. Hence $[[Q, \phi_a], \phi_b] = 0$ for all $a, b \in R$. Explicitly,

$$Q\phi_a\phi_b - \phi_a Q\phi_b - \phi_b Q\phi_a + \phi_b\phi_a Q = 0$$

Applying this map to $1 \in R$ and using $Q(1) = 0$ we find that

$$Q(ab) - \phi_a(Qb) - \phi_b(Qa) = 0$$

So $Q \in Der_K(R)$ and the result follows. □

We now prove that $D(R)$ is a graded ring.

Proposition 1.7. *Let $P \in D^n(R)$ and $Q \in D^m(R)$ for $n, m \geq 0$. Then,*

1. $PQ \in D^{n+m}(R)$
2. $0 \subseteq D^0(R) \subseteq D^1(R) \subseteq D^2(R) \subseteq \cdots$

Proof The proof of (1) is by induction on $m+n$. If $m+n=0$ the result is clear from Proposition 1.6. Suppose $m+n=k$ and $a \in R$, then,

$$[\phi_a, PQ] = P[\phi_a, Q] + [\phi_a, P]Q$$

Since $P \in D^n(R)$ and $Q \in D^m(R)$, we have $[\phi_a, P] \in D^{n-1}(R)$ and $[\phi_a, Q] \in D^{m-1}(R)$. The inductive hypothesis tells us that both $P[\phi_a, Q]$ and $[\phi_a, P]Q$ are elements of $D^{n+m-1}(R)$. Thus $[\phi_a, PQ] \in D^{n+m-1}(R)$ as $D^{n+m-1}(R)$ is a vector space from Proposition 1.5. Thus $PQ \in D^{n+m}(R)$. For part (2), suppose $P \in D^n(R)$ and $a \in R$. Since $\phi_a \in \phi(R) = D^0(R)$ and $P \in D^n(R)$, part (1) implies that $P\phi_a, \phi_a P \in D^n(R)$ and hence $[\phi_a, P] \in D^n(R)$. Hence $P \in D^{n+1}(R)$. □

Corollary 1.8. *$D(R)$ is a ring.*

Proof Proposition 1.7 part (1) proves closure under multiplication. Part (2) tells us that if $P \in D^n(R)$ and $Q \in D^m(R)$ for $n, m \geq 0$ then $P + Q \in D^{max\{m,n\}}(R)$. Hence $D(R)$ is a subring of $End_K(R)$. □

1.2 Rings of Differential Operators Associated to Polynomial Rings

Now that we have shown that $D(R)$ is a ring we are ready to prove that $D(\mathbb{C}[t]) = A_1(\mathbb{C})$.

Proposition 1.9. *Let $P \in D(\mathbb{C}[t])$, if $[P, t] = 0$ then $P \in \mathbb{C}[t]$.*

Proof Suppose $P \in D(\mathbb{C}[t])$ and $[P, t] = 0$. Then we claim that $P \in D^0(\mathbb{C}[t])$. By the linearity of the commutator we only need to check that $[P, t^n] = 0$ for all $n \geq 1$ which is clear via induction on n since $[P, t^n] = t[P, t^{n-1}]$. □

Note that we are no longer distinguishing between $\phi(\mathbb{C}[t])$ and $\mathbb{C}[t]$. We need to define the subsets of $A_1(\mathbb{C})$ which will correspond to $D^n(\mathbb{C}[t])$ for each $n \geq 0$. Before we can do this though, we need to learn more about the structure of $A_1(\mathbb{C})$. In particular, seeing as we can view $D(\mathbb{C}[t])$ as a vector space over $\mathbb{C}$, we can try to do the same with $A_1(\mathbb{C})$. This raises questions about the possibility of a natural choice for a basis of $A_1(\mathbb{C})$, which leads us to the next result.

Proposition 1.10. *The set $B = \{t^a \delta^b \mid a, b \geq 0\}$ is a basis for $A_1(\mathbb{C})$.*[1]

Proving this result amounts to showing that this set spans $A_1(\mathbb{C})$ and is linearly independent. The only interesting technique used in the proof is the use of the

[1] There is in fact a more general result related to the basis of the *nth Weyl Algebra*, A_n in [1].

relation $\delta t - t\delta = 1$ to move all the t's to the left of the δ's in each monomial. We display this technique in the following example.

Example 1.11. Consider the element $t^2\delta^2 t\delta \in A_1(\mathbb{C})$. We can use the relation $\delta t - t\delta = 1$ to rewrite this element in the form $\sum_{j=0}^{3} f_j(t)\delta^j$ where $f_j(t) \in \mathbb{C}[t]$ for all $j = 0, 1, 2, 3$. Thus we obtain, $t^2\delta^2 t\delta = t^2\delta(1 + t\delta)\delta = t^2\delta^2 + t^2\delta t\delta^2 = t^2(1 + t\delta)\delta^2 + t^2\delta^2 = t^3\delta^3 + 2t^2\delta^2$.

Definition 1.12. *Define the subset $B_n \subseteq B$ as,*

$$B_n = \{t^a\delta^b \in B \mid a \geq 0,\ b \leq n\}$$

*and the subspace $C_n \subseteq A_1(\mathbb{C})$ as $C_n := span_{\mathbb{C}}(B_n)$. We say that elements of C_n are of **order** $\leq n$.*

It is clear that $A_1(\mathbb{C}) = \bigcup_{n \geq 0} C_n$.

Lemma 1.13. *For all $j \geq 1$, $[\delta^j, t] = j\delta^{j-1}$. Hence, if $P \in C_{r-1}$ then there exists a $Q \in C_r$ such that $P = [Q, t]$.*

Proof We first prove the identity $[\delta^j, t] = j\delta^{j-1}$ by induction on j. For the base case we observe that $[\delta, t] = 1$. Suppose $[\delta^{j-1}, t] = (j-1)\delta^{j-2}$, this means that $\delta^{j-1}t = (j-1)\delta^{j-2} + t\delta^{j-1}$. Then $[\delta^j, t] = \delta((j-1)\delta^{j-2} + t\delta^{j-1}) - t\delta^j = (j-1)\delta^{j-1} + (t\delta + 1)\delta^{j-1} - t\delta^j = j\delta^{j-1}$. Now we prove the remainder of the lemma. Suppose $P \in C_{r-1}$, then $P = \sum_{j=0}^{r-1} f_j(t)\delta^j$ for some $f_j(t) \in \mathbb{C}[t]$. Then $Q = \sum_{j=1}^{r} \frac{1}{j} f_{j-1}(t)\delta^j \in C_r$ is the required element using the identity $[\delta^j, t] = j\delta^{j-1}$ and the fact that $[Q, t] = \sum_{j=1}^{r} \frac{1}{j} f_{j-1}(t)[\delta^j, t]$. □

Theorem 1.14. *The ring of differential operators on $\mathbb{C}[t]$, $D(\mathbb{C}[t])$, is precisely $A_1(\mathbb{C})$, the first Weyl algebra. Furthermore, $D^n(\mathbb{C}[t]) = C_n$ for all $n \geq 0$.*

Proof We first prove $D^n(\mathbb{C}[t]) \subseteq C_n$ by induction on n. It is clear that $C_0 = D^0(\mathbb{C}[t])$. Suppose $Q \in D^n(\mathbb{C}[t])$ and that $D^{n-1}(\mathbb{C}[t]) \subseteq C_{n-1}$. By the definition of $D^n(\mathbb{C}[t])$, $[Q, t] \in D^{n-1}(\mathbb{C}[t]) \subseteq C_{n-1}$. Thus by Lemma 1.13 there exists an $S \in C_n$ such that $[S, t] = [Q, t]$. This means that $[S - Q, t] = 0$ so that by Proposition 1.9, $S - Q \in \mathbb{C}[t]$. Hence the orders of S and Q are the same so that $Q \in C_n$.

To prove the reverse inclusion we will first need to show $[\delta, t^i] \in C_0$ and $[\delta^j, t^i] \in C_{j-1}$ for all $i, j \geq 1$. We now prove $[\delta, t^i] \in C_0$ by induction on i. For the base case, $[\delta, t] = 1 \in C_0$. The induction step follows from the fact that $[\delta, t^i] = t[\delta, t^{i-1}] + t^{i-1}$ and by definition, $C_0 = \mathbb{C}[t]$. Next we show $[\delta^j, t^i] \in C_{j-1}$ by induction on j. The base case is the first result, that is $[\delta, t^i] \in C_0$. For the induction step we use the identity, $[\delta^j, t^i] = \delta[\delta^{j-1}, t^i] + [\delta, t^i]\delta^{j-1}$. By the inductive hypothesis, $[\delta^{j-1}, t^i] \in C_{j-2}$, so the order of $[\delta^{j-1}, t^i]$ is at most $j-2$. Thus $\delta[\delta^{j-1}, t^i]$ has order of at most $j-1$ because we use $\delta t = t\delta + 1$ to move all the t's to the left of the δ's we are only adding terms of lower order. In other words, $\delta[\delta^{j-1}, t^i] \in C_{r-1}$. Similarly $[\delta, t^i]\delta^{j-1} \in C_{r-1}$ and hence $[\delta^j, t^i] \in C_{j-1}$.

Lastly we prove $C_n \subseteq D^n(\mathbb{C}[t])$ by induction on n. The base case is clear as $C_0 \subseteq D^0(\mathbb{C}[t]) = \mathbb{C}[t]$. For the induction step, suppose $Q = \sum_{j=0}^{n} f_j(t)\delta^j \in C_n$ and $f(t) = \sum_{i=0}^{m} \alpha_i t^i \in \mathbb{C}[t]$. Then $[Q, f(t)] = \sum_{j=0}^{n}\sum_{i=0}^{m} f_j(t)\alpha_i[\delta^j, t^i]$ by the linearity of the commutator bracket. The latter of the two previous results implies that $[Q, f(t)] \in D^{n-1}(\mathbb{C}[t])$ for all $f(t) \in \mathbb{C}[t]$. Hence $D^n(\mathbb{C}[t]) = C_n$ for all $n \geq 0$. This tells us that $D(C[t])$ is precisely $A_1(\mathbb{C})$. □

This theorem allows us to compute another ring of differential operators quite cheaply. Namely, the ring $D(\mathbb{C}[t^{-1}])$.

Corollary 1.15. *The ring of differential operators of* $\mathbb{C}[t^{-1}]$ *is,*

$$D(\mathbb{C}[t^{-1}]) = \frac{\mathbb{C}\langle t^{-1}, -t^2\delta\rangle}{(t^{-1}\delta - \delta t^{-1} - t^{-2})}$$

Proof By Theorem 1.14, $D(\mathbb{C}[s]) = \frac{\mathbb{C}\langle s, \frac{d}{ds}\rangle}{(\frac{d}{ds}s - s\frac{d}{ds} - 1)}$. Now let $s = t^{-1}$, then $\frac{d}{ds} = \frac{dt}{ds}\frac{d}{dt} = -t^2\frac{d}{dt}$. Furthermore, we have the relation $\frac{d}{ds}s - s\frac{d}{ds} - 1 = 0$, this is equivalent to $-t^2\frac{d}{dt}t^{-1} + t^{-1}t^2\frac{d}{dt} - 1 = 0$. We multiply by t^{-2} on the left to obtain,

$$t^{-1}\delta - \delta t^{-1} - t^{-2} = 0$$

Thus,

$$D(\mathbb{C}[t^{-1}]) = \frac{\mathbb{C}\langle t^{-1}, -t^2\delta\rangle}{(t^{-1}\delta - \delta t^{-1} - t^{-2})}$$

which is precisely the result. □

In this section we have explicitly computed the ring of differential operators of both $C[t]$ and $C[t^{-1}]$. Without too much effort we can also construct $D(\mathbb{C}[t, t^{-1}])$, the ring of differential operators on $\mathbb{C}[t, t^{-1}]$. To do this we will need the technique of localisation of a non-commutative ring.

1.3 Localisation

Localisation is a tool which allows one to invert elements of a ring to create a new ring of 'fractions'. The simplest example of localisation is that of a domain, where we obtain its fraction field. In this section we discover how to localise both commutative and non-commutative rings and use this tool to compute yet another ring of differential operators, $D(\mathbb{C}[t, t^{-1}])$.

We begin by describing the set whose elements we will invert. We define a multiplicative set for a general ring R.

Definition 1.16. *Let R be a ring and $S \subseteq R$. Then S is a* ***multiplicative*** *set if*

1. *If $s, t \in S$, then $st \in S$*
2. *$1 \in S$ and $0 \notin S$*

The following results describe localisation of a commutative ring. Details for the localisation of a commutative ring can be found in [3].

Proposition 1.17. *Let R be a commutative ring, and let S be a multiplicative subset of R. Consider the following relation on elements of $R \times S$,*

$$(a, b) \sim (c, d) \Leftrightarrow \text{ there exists } s \in S \text{ such that } s(ad - bc) = 0$$

Then $\sim$ is an equivalence relation.

Proof Reflexivity and symmetry are already clear. For transitivity, suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then there exists an $s, t \in S$ such that $s(ad-bc) = 0$ and $t(cf - de) = 0$. Then $tfsad = tfsbc$ and $sbtcf = sbtde$. Combining these expressions and using commutativity in R we see that $std(af - be) = 0$ and since $s, t, d \in S$, $std \in S$. Hence $(a, b) \sim (e, f)$. □

Definition 1.18. *Let R be a commutative ring and S be a multiplicative subset. If $a \in R$ and $b \in S$ denote the equivalence class containing (a, b) by $\frac{a}{b}$. Denote the set of all equivalence classes by RS^{-1}. Furthermore, define the following for all elements of RS^{-1},*

1. *addition : $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$*
2. *multiplication : $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$*
3. *additive identity : $\frac{0}{1}$*
4. *additive inverse : $-(\frac{a}{b}) = \frac{-a}{b}$*
5. *multiplicative identity : $\frac{1}{1}$*

We need to make sure that these operations are well defined, and that the equivalence classes defined obey the axioms of a commutative ring. Fortunately, this is all just routine checking.

Theorem 1.19. *The operations in Definition 1.18 are well defined and RS^{-1} is a commutative ring with these operations.*

Example 1.20. We can localise $\mathbb{C}[t]$ with an appropriate multiplicative set to obtain $\mathbb{C}[t, t^{-1}]$. Let $R = \mathbb{C}[t]$, so R is a commutative ring. Let $S = \{1, t, t^2, t^3, \ldots\}$, so that $S \subseteq R$ is a multiplicative set. Then $RS^{-1} = \{\frac{a}{b} \mid a \in \mathbb{C}[t], b \in S\}$. Due to the addition law in $\mathbb{C}[t]$, we can split $a \in \mathbb{C}[t]$ into monomials so that $RS^{-1} = \{\frac{\alpha t^n}{t^j} \mid \alpha \in \mathbb{C}, j \geq 0\}$. If $n > j$, then $\frac{\alpha t^n}{t^j} = \frac{\alpha t^{n-j}}{1}$ and if $n < j$, then $\frac{\alpha t^n}{t^j} = \frac{\alpha}{t^{j-n}}$. Hence $RS^{-1} = \mathbb{C}[t, \frac{1}{t}] = \mathbb{C}[t, t^{-1}]$.

We are now ready to localise a non-commutative ring. Let R be a general ring and $S \subseteq R$ a multiplicative subset of R. Similar to the commutative case, we can try to define a ring RS^{-1} which inverts the elements of S. The elements of RS^{-1} would be equivalence classes of the form rs^{-1}. It is evident that for multiplication to be well defined we need to be able to rewrite each $s^{-1}r$ as $r's'^{-1}$ for some $r' \in R$ and $s' \in S$. This leads us to the Öre conditions.

Definition 1.21. *Suppose R is a ring and $S \subseteq R$ is a multiplicative subset. Then S is* ***right Öre*** *if,*

1. *For all $r \in R$, $s \in S$, there exists $r' \in R$, $s' \in S$ such that $rs' = sr'$*
2. *S has no zero divisors*

To localise a non-commutative ring R, we will need a different equivalence relation from the commutative case and a couple of results to help us in the construction of RS^{-1}. For details in the localisation of non-commutative rings, see [4] and [5][2].

Definition 1.22. *Let R be a ring, and let S be a multiplicative subset of R. Consider the following relation on elements of $R \times S$,*

$$(a, b) \sim (c, d) \Leftrightarrow \text{ there exists } s, t \in R \text{ such that } as = ct \text{ and } bs = dt \in S$$

To help us prove that $\sim$ is an equivalence relation and also proving the ring structure of our localisation we need the following lemma.

Lemma 1.23. *Suppose R is a ring and $S \subseteq R$ is multiplicative and right Öre.*

1. *If $s_1, s_2, \ldots s_n \in S$ then there exists $r_1, r_2, \ldots r_n \in R, s \in S$ such that $s_1 r_1 = s_2 r_2 = \ldots = s_n r_n = s$.*
2. *Suppose $(a, b) \sim (c, d)$ and $s', t' \in R$ such that $bs' = dt' \in S$. Then $as' = ct'$.*

Proof (1) We proceed with induction on n. The case for $n = 1$ follows by taking $r_1 = 1$. Suppose $s_1, s_2, \ldots s_n \in S$. The induction hypothesis implies that there exists $r'_1, r'_2, \ldots, r'_{n-1} \in R$ such that $s_1 r'_1 = s_2 r'_2 = \ldots = s_{n-1} r'_{n-1} = r \in S \subseteq R$. Thus as S is right Öre, $r \in R$, $s_n \in S$, there exists $r_n \in R$, $s' \in S$ such that $s_n r_n = rs'$. Also

[2]Sections 2.1.1-2.1.12 in particular are related to localisation of non-commutative rings

since both $r, s' \in S$ and S is multiplicative, $rs' \in S$. Thus by setting $r_j := r'_j s'$ for $j = 1, 2, \ldots, n-1$ and $s := rs' \in S$ we obtain $s_1 r_1 = s_2 r_2 = \ldots = s_n r_n = s$ as required.

(2) Suppose we have $s, t \in R$ such that $as = ct$ and $bs = dt \in S$. Then as $bs' \in R$, $bs \in S$ and S is Öre, there exists $x \in R$, $x' \in S$ such that $(bs')x' = (bs)x$. Thus, $b(s'x' - sx) = 0$ and since S has no zero divisors and $b \in S$, $s'x' = sx$. Similarly $t'x' = tx$. Thus $as'x' = asx = ctx = ct'x'$ since $as = ct$. Thus $(as' - ct')x' = 0$ so as $x' \in S$ and S has no zero divisors, $as' = ct'$. □

The first part of Lemma 1.23 says that for a finite set of fractions in RS^{-1} we can always find a common denominator. We will require both parts of Lemma 1.23 to prove the equivalence of $\sim$.

Proposition 1.24. *Suppose R is a ring and $S \subseteq R$ is multiplicative and right Öre set. Then $\sim$ from Definition 1.22 is an equivalence relation.*

Proof Reflexivity and symmetry are clear. For transitivity, suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Since $b, d, f \in S$, Lemma 1.22(1) says that there exists $r_1, r_2, r_3 \in R$ such that $br_1 = dr_2 = fr_3$. Furthermore, Lemma 1.22(2) tells us that $ar_1 = cr_2$ and $cr_2 = er_3$, hence $ar_1 = er_3$ so that $(a, b) \sim (e, f)$. Thus $\sim$ is an equivalence relation. □

Let R be a ring and $S \subseteq R$ be multiplicative and right Öre. If $a \in R$ and $b \in S$ denote the equivalence class containing (a, b) by ab^{-1}. Denote the set of all equivalence classes by RS^{-1}. Suppose $ab^{-1}, cd^{-1} \in RS^{-1}$. By Lemma 1.23(1), there exists $r_1, r_2 \in R$ such that $br_1 = dr_2 = s \in S$. Also, as S is right Öre, there exists $r' \in R$, $s' \in S$ such that $cs' = br'$. We will require these two facts to define addition and multiplication in RS^{-1}.

Definition 1.25. *Define the following for all elements of RS^{-1},*

1. *addition :* $ab^{-1} + cd^{-1} = (ar_1 + cr_2)s^{-1}$
2. *multiplication :* $ab^{-1} \cdot cd^{-1} = (ar')(ds')^{-1}$
3. *additive identity :* $(0)(1)^{-1}$
4. *additive inverse :* $-(ab^{-1}) = (-a)b^{-1}$
5. *multiplicative identity :* $(1)(1)^{-1}$

Theorem 1.26. *Suppose R is a ring and $S \subseteq R$ is a multiplicative and right Öre. Then the operations in Definition 1.25 are well defined and RS^{-1} is a ring.*

Proof We will only show that addition is well defined here. For a complete proof, see [4]. Suppose $a', c', r'_1, r'_2 \in R$ and $b', d', s' \in S$ such that

$$ab^{-1} = a'b'^{-1},\ cd^{-1} = c'd'^{-1} \text{ and } b'r_1' = d'r_2' = s'$$

We want to show that $(ar_1 + cr_2)s^{-1} = (a'r_1' + c'r_2')s'^{-1}$, or in other words that these belong to the same equivalence class. We know $s, s' \in S$, so by Lemma 1.23(1), there exists $u, u' \in R$ such that $su = s'u'$. Thus $br_1u = b'r_1'u'$ and $dr_2u = d'r_2'u'$. Since $ab^{-1} = a'b'^{-1}$, by Lemma 1.23(2), $ar_1u = a'r_1'u'$ and similarly, $cr_2u = c'r_2'u'$. Hence $(ar_1 + cr_2)u = (a'r_1' + c'r_2')u'$ and we have already seen that $su = s'u'$. This implies that $(ar_1 + cr_2)s^{-1} = (a'r_1' + c'r_2')s'^{-1}$. □

We end this chapter by computing $D(\mathbb{C}[t, t^{-1}])$ as an example of Theorem 1.26. This will be the last ring of differential operators that we will need to compute. Just as $\mathbb{C}[t]$ and $\mathbb{C}[t^{-1}]$ can be naturally embedded in $\mathbb{C}[t, t^{-1}]$, we will see that $D(\mathbb{C}[t])$ and $D(\mathbb{C}[t^{-1}])$ can be naturally embedded in $D(\mathbb{C}[t, t^{-1}])$. It is this connection that motivates the computation of $D(\mathbb{C}[t, t^{-1}])$.

Example 1.27. In Example 1.20 we computed $\mathbb{C}[t, t^{-1}]$ by localising $\mathbb{C}[t]$ at the multiplicative set $S = \{1, t, t^2, \ldots\}$. We proceed with the same plan, except with $D(\mathbb{C}[t])$. Thus, let $R = D(\mathbb{C}[t])$, and $S = \{1, t, t^2, \ldots\} \subseteq R$ since we know there is a copy of $\mathbb{C}[t]$ in $D(\mathbb{C}[t])$. To guarantee that RS^{-1} exists, all we need to show is that S is right Öre.

We first prove an identity on elements of $D(\mathbb{C}[t])$. We claim that for $0 < k \leq j$,

$$\delta^k t^j = \sum_{r=0}^{k} {}^kC_r\, {}^jP_r\, t^{j-r}\delta^{k-r}$$

where ${}^kC_r = \frac{k!}{r!(k-r)!}$ and ${}^jP_k = \frac{j!}{(j-k)!}$. We proceed via induction on k. For the base case, suppose $1 = k \leq j$, then we want to show that $\delta t^j = \sum_{r=0}^{1} {}^1C_r\, {}^jP_r\, t^{j-r}\delta^{1-r}$. This is equivalent to proving that $\delta t^j = t^j\delta + jt^{j-1}$ for $1 \leq j$ which we prove via induction on j. For $j = 1$ this is the relation which defines $D(\mathbb{C}[t])$. Next, suppose $\delta t^{j-1} = t^{j-1}\delta + (j-1)t^{j-2}$. Then $\delta t^j = t^{j-1}\delta t + (j-1)t^{j-1} = t^{j-1}(1 + t\delta) + (j-1)t^{j-1} = t^j\delta + jt^{j-1}$, so the base case with $1 = k \leq j$ is proved. Thus for the induction step of the original proof, suppose $\delta^{k-1}t^j = \sum_{r=0}^{k-1} {}^{k-1}C_r\, {}^jP_r\, t^{j-r}\delta^{k-1-r}$. Then, $\delta^k t^j = \sum_{r=0}^{k-1} {}^{k-1}C_r\, {}^jP_r\, \delta t^{j-r}\delta^{k-1-r}$. After a simple calculation and using the combinatorial identity, ${}^{k-1}C_{r-1} + {}^{k-1}C_r = {}^kC_r$, we obtain the desired result.

To show that S is right Öre, suppose $r \in R$ and $t^l \in S$ for some $l \geq 0$. We can assume that r is a monomial of the form $t^j\delta^k$ by Proposition 1.10. Then by setting $s' = t^{k+l} \in S$, we see that $rs' = t^j\delta^k t^{k+l} = t^j \sum_{r=0}^{k} {}^kC_r\, {}^{k+l}P_r\, t^{k+l-r}\delta^{k-r} = t^l \sum_{r=0}^{k} {}^kC_r\, {}^{k+l}P_r\, t^{j+k-r}\delta^{k-r} = sr'$ for $r' = \sum_{r=0}^{k} {}^kC_r\, {}^{k+l}P_r\, t^{j+k-r}\delta^{k-r} \in R$. Thus for a polynomial in R we set $s' = t^{k+l}$ where k is the greatest power of δ in the polynomial and $s = t^l$. Since S only contains powers of t, S has no zero divisors. Thus, S is a right Öre set of R and we can localise R to obtain the ring $RS^{-1} = D(\mathbb{C}[t])S^{-1}$.

Elements of $D(\mathbb{C}[t])S^{-1}$ are of the form $t^j\delta^k t^{-l}$ for $j, k, l \geq 0$. From Example 1.15 we can use the relation $t^{-1}\delta - \delta t^{-1} - t^{-2}$ to rewrite every element of $D(\mathbb{C}[t])S^{-1}$ as a sum of monomials of the form $t^j\delta^k$ for $j \in \mathbb{Z}$ and $k \geq 0$. Hence using Theorem 1.14,

$$D(\mathbb{C}[t])S^{-1} = \frac{\mathbb{C}\langle t, t^{-1}, \delta\rangle}{(\delta t - t\delta - 1, t^{-1}\delta - \delta t^{-1} - t^{-2})}$$

All that remains is to see why $D(\mathbb{C}[t])S^{-1} = D(\mathbb{C}[t, t^{-1}])$. In [2][3], it is shown that for any commutative K-algebra, R, with multiplicative set $S \subseteq R$ which is also a right Öre set of $D(R)$, we have $D(R)S^{-1} = D(RS^{-1})$. Hence $D(\mathbb{C}[t])S^{-1} = D(\mathbb{C}[t]S^{-1}) = D(\mathbb{C}[t, t^{-1}])$.

In this chapter we have introduced rings of differential operators, covered a small amount of theory of their structure, and seen three examples of them. Namely,

$$D(\mathbb{C}[t]) = \frac{\mathbb{C}\langle t, \delta\rangle}{(\delta t - t\delta - 1)}$$

$$D(\mathbb{C}[t^{-1}]) = \frac{\mathbb{C}\langle t^{-1}, -t^2\delta\rangle}{(t^{-1}\delta - \delta t^{-1} - t^{-2})}$$

and by employing localisation,

$$D(\mathbb{C}[t, t^{-1}]) = \frac{\mathbb{C}\langle t, t^{-1}, \delta\rangle}{(\delta t - t\delta - 1, t^{-1}\delta - \delta t^{-1} - t^{-2})}$$

These three rings will appear frequently throughout this thesis and much effort will be spent in understanding how we can connect these three rings together. To begin to understand these connections we must cover some basic facts about sheaves. Sheaves are a tool used most commonly in algebraic geometry. Sheaves allow us to understand local properties of a topological space in more detail as we shall see.

[3]This result is a small part of a larger result in Section 5

Chapter 2

Algebraic Sets and Quasi-Coherent Sheaves

Sheaves are used to keep track of the relationship between local and global algebraic data on a topological space. We will define the *Zariski* topology on $\mathbb{P}^1$, the projective line. We will see that algebraic sets will be the closed sets in this topology. After equipping a topology on this space, the remainder of the chapter is devoted to introducing sheaves and also to describe quasi-coherent sheaves on $\mathbb{P}^1$, which are the main objects of study in this thesis.

2.1 Algebraic Sets

Projective spaces have an advantage over affine spaces in the sense that they have an extra 'point' at infinity. In this sense, studying geometry in a projective space provides a more complete picture of what is really going on. For example, affinely different curves such as the parabola, ellipse and hyperbola turn out to be affine parts of the projective conic. In this section we describe a famous topology on projective spaces, the *Zariski* topology. The Zariski topology is defined by describing the closed sets, which will be algebraic sets. We begin by first defining projective spaces. For more details on algebraic sets in projective spaces, see [6]. For the remainder of this thesis, let K be an algebraically closed field.

Definition 2.1. *A* ***projective*** n***-space*** *over* K*, denoted by* $\mathbb{P}^n_K$ *is defined as,*

$$\mathbb{P}^n_K := \frac{K^{n+1} - \{0\}}{\sim}$$

Where $(a_0, a_1, \dots, a_n) \sim (\lambda a_0, \lambda a_1, \dots, \lambda a_n)$ *for all* $\lambda \in K$ *with* $\lambda \neq 0$*. An element of* $\mathbb{P}^n_K$ *is called a* ***point****. If* $P \in \mathbb{P}^n_K$ *is a point then any* $(n+1)$*-tuple in the equivalence class* P *is called the* ***set of homogenous coordinates*** *for* P*.*

An alternative perspective of $\mathbb{P}^n_K$ is to view it as the set of all lines through the origin in K^{n+1}. Functions from $\mathbb{P}^n_K \to K$ will give us information about the structure of $\mathbb{P}^n_K$. For the remainder of this section, let S be the commutative polynomial ring $K[x_0, x_1, \ldots, x_n]$. We want to use elements of S to define a function $\mathbb{P}^n_K \to K$. Due to the homogeneous cooordinates in $\mathbb{P}^n_K$ an arbitrary polynomial in S won't be well defined. Hence we define homogeneous polynomials in S.

Definition 2.2. *Suppose $f \in S$. Then f is a **homogeneous polynomial of degree** d, if for all $\lambda \in K$ and all $(a_0, a_1, \ldots, a_n) \in K^{n+1}$,*

$$f(\lambda a_0, \lambda a_1, \ldots, \lambda a_n) = \lambda^d f(a_0, a_1, \ldots, a_n)$$

A simple example of a homogeneous polynomial in S of degree 2 is $f = x_j^2$, for any $j = 0, 1, \ldots n$. If $f \in K[x_0, x_1, \ldots, x_n]$ is a homogeneous polynomial, we can see that f preserves roots in $\mathbb{P}^n$. Thus we can define a function $f : \mathbb{P}^n \to \{0, 1\}$. Suppose $(a_0, a_1, \ldots, a_n) \in K^{n+1}$ is a homogenous set of coordinates for a point $P \in \mathbb{P}^n$. Then,

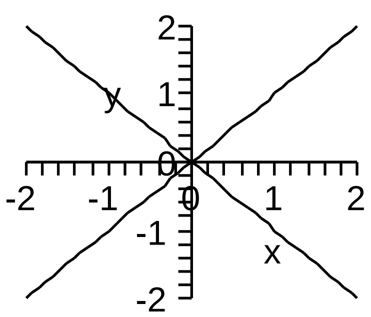
$$f(P) := \begin{cases} 0 & \text{if } f(a_0, a_1, \ldots, a_n) = 0; \\ 1 & \text{if } f(a_0, a_1, \ldots, a_n) \neq 0. \end{cases}$$

Thus, we can talk about the ***zeros*** of a homogeneous polynomial. Namely, they are the set $Z(f) = \{P \in \mathbb{P}^n \mid f(P) = 0\}$. This allows us to define the zero set of a *set* of homogeneous polynomials.

Definition 2.3. *Suppose $T \subseteq S$ is a set of homogeneous polynomials. Then the **zero set** of T is,*

$$Z(T) = \{P \in \mathbb{P}^n \mid f(P) = 0 \text{ for all } f \in T\}$$

Example 2.4. Consider the element $f(x, y) = y^2 - x^2 \in \mathbb{C}[x, y]$, then $f(x, y)$ is a homogenous element of degree 2. In this case $Z(f) = \{(1, -1), (1, 1)\}$ and the real part of the zero set can be thought of in the following diagram.



We are now ready to define algebraic sets in $\mathbb{P}^n$. These will be the closed sets in the Zariski topology of $\mathbb{P}^n$.

Definition 2.5. *Suppose $Y \subseteq \mathbb{P}^n_K$. Then Y is an* ***algebraic set*** *if there exists a set $T \subseteq S$ of homogeneous elements such that $Y = Z(T)$.*

Since we have claimed that algebraic sets will be the closed sets of a topology, we should prove that algebraic sets obey the axioms of closed sets in a topology.

Proposition 2.6. *The following holds in a projective n-space $\mathbb{P}^n$. The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and $\mathbb{P}^n$ are both algebraic sets.*

Proof Suppose $Y_1 = Z(T_1)$ and $Y_2 = Z(T_2)$ with $T_1, T_2 \in S$ homogeneous sets of polynomials. Then we claim that $Y_1 \cup Y_2 = Z(T_1T_2)$, where $T_1T_2 = \{f_1f_2 \mid f_1 \in T_1, f_2 \in T_2\}$, the set of all products of polynomials in T_1 and T_2. If $P \in Y_1 \cup Y_2$ then $P \in Y_1$ or $P \in Y_2$. Thus $f_1(P) = 0$ for all $f_1 \in T_1$ or $f_2(P) = 0$ for all $f_2 \in T_2$. Thus in any case, $f_1(P)f_2(P) = 0$ for all $f_1 \in T_1$, $f_2 \in T_2$. Conversely, if $P \in Z(T_1T_2)$ and $P \notin Y_1$ then there is an $f \in T_1$ such that $f(P) \neq 0$. But for any $g \in T_2$, $f(P)g(P) = 0$ which implies that $g(P) = 0$ for all $g \in T_2$. Hence $P \in Y_2$.

If $Y_\alpha = Z(T_\alpha)$ is a family of algebraic sets, then $\bigcap_\alpha Y_\alpha = Z(\bigcup_\alpha T_\alpha)$. Thus $\bigcap_\alpha Y_\alpha$ is an algebraic set. Also $\emptyset = Z(1)$ and $\mathbb{P}^n = Z(0)$. □

We have just successfully constructed the closed sets of a topology on $\mathbb{P}^n_K$. This topology is called the *Zariski* topology. For the rest of this thesis, when we refer to a topology on $\mathbb{P}^1_{\mathbb{C}}$ we will assume that it is the Zariski topology.

Definition 2.7. *We define the* ***Zariski*** *topology on $\mathbb{P}^n_K$ by taking the open sets to be the complements of the algebraic sets.*

It will be of great use to us in the preceding sections to distinguish between different algebraic sets. We want to be able to focus on the smallest or in some sense, simplest, algebraic sets. This leads us to the idea of irreducibility and allows us to define projective varieties.

Definition 2.8. *Suppose X is a topological space and Y is a nonempty subset of X. Then Y is* ***irreducible*** *if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y (i.e. in the induced topology on Y). The empty set is not considered to be irreducible.*

Definition 2.9. *A* ***projective variety*** *is an irreducible algebraic set in $\mathbb{P}^n$ with the Zariski topology.*

Example 2.10. Consider the projective 2-space $\mathbb{P}^2$. Let $f = y^2 - x^2$, $g = y - x$, and $h = y + x$. Then $Z(f)$ is not a projective variety, but both $Z(g)$ and $Z(h)$ are.

We have defined projective varieties, but we haven't discussed what maps are allowed between them. We will now define regular functions on a projective variety and this will give us a motivation for sheaves.

Definition 2.11. *Suppose $Y \subseteq \mathbb{P}^n_K$ is an open set. A function $f : Y \to K$ is* ***regular at a point*** *$P \in Y$ if there is an open neighbourhood U with $P \in U \subseteq Y$, and homogeneous polynomials $g, h \in S$, of the same degree, such that h is not zero on U and $f = g/h$ on U. We say that f is* **regular** *on Y if it is regular at each point $P \in Y$.*

It is important to note, that although $g, h \in S$ are not functions on $\mathbb{P}^n$, their quotient is a well-defined function whenever $h \neq 0$ as they are homogenous of the same degree. Furthermore, the set of all regular functions on some open set $Y \subseteq \mathbb{P}^n_K$ has a ring structure by defining addition and multiplication by,

$$g_1/h_1 + g_2/h_2 = (g_1h_2 + g_2h_1)/h_1h_2$$

$$(g_1/h_1) \cdot (g_2/h_2) = g_1g_2/h_1h_2$$

We motivate sheaves using an example of regular functions.

Example 2.12. If we want to study the global structure of $\mathbb{P}^1_{\mathbb{C}}$, we would need regular functions that are well defined on all of $\mathbb{P}^1_{\mathbb{C}}$. Thus we would need $g, h \in \mathbb{C}[x, y]$, homogeneous of the same degree, with h nonzero on all of $\mathbb{C}^2$. Louiville's theorem tells us that h must be a constant function, and hence so must g. Thus the only regular maps on all of $\mathbb{P}^1_{\mathbb{C}}$ are the constant functions. This doesn't give us enough data to study the structure of $\mathbb{P}^1_{\mathbb{C}}$. We will see that sheaves give us more information about the topological space, which makes them vastly more useful to study than regular functions.

2.2 Sheaves

A sheaf of rings associates an abelian group to each open set in a topological space, X. Thus by using sheaves we have a lot of data with which to work with. This makes sheaves extremely useful in the study of the structure of X. In this section, we define sheaves following [6] and also give some examples of them. For a category theoretical treatment of sheaves, see [7]. We begin by defining a presheaf of abelian groups.

Definition 2.13. *Let X be a topological space. A* ***presheaf*** *$\mathscr{F}$, of abelian groups on X consists of the following data.*

1. *For every open subset $U \subseteq X$, an abelian group $\mathscr{F}(U)$*

2. *For every inclusion $V \subseteq U$ of open subsets of X, a homomorphism of abelian groups $\rho_{UV} : \mathscr{F}(U) \to \mathscr{F}(V)$ subject to*
   (a) *$\mathscr{F}(\emptyset) = 0$, where $\emptyset$ denotes the empty set and 0 denotes trivial abelian group*
   (b) *ρ_{UU} is the identity map $\mathscr{F}(U) \to \mathscr{F}(U)$*
   (c) *If $W \subseteq V \subseteq U$ are three open subsets of X then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$. In other words the following diagram commutes,*

$$\mathscr{F}(U) \xrightarrow{\rho_{UV}} \mathscr{F}(V) \xrightarrow{\rho_{VW}} \mathscr{F}(W), \qquad \mathscr{F}(U) \xrightarrow{\rho_{UW}} \mathscr{F}(W)$$

We can define a presheaf of rings, simply by replacing 'abelian groups' in the definition with 'rings'. In this thesis when we refer to presheaves in general we will assume that they are presheaves of abelian groups. If $\mathscr{F}$ is a presheaf on a topological space X and $U \subseteq X$ is an open subset, then $\sigma \in \mathscr{F}(U)$ is a ***section*** of the presheaf $\mathscr{F}$ on U. If $U = X$ then $\sigma \in \mathscr{F}(X)$ is a ***global section***. Also, the maps ρ_{UV} for open subsets $V \subseteq U \subseteq X$ are called ***restriction maps***. Lastly, we sometimes write $\sigma|_V$ instead of $\rho_{UV}(\sigma)$ where $\sigma \in \mathscr{F}(U)$.

Roughly speaking, a sheaf is a presheaf whose sections are completely determined with local data. To be more precise we give the following definition.

Definition 2.14. *Suppose $\mathscr{F}$ is a presheaf on a topological space X. Then $\mathscr{F}$ is a **sheaf** if it satisfies the **sheaf condition** as follows.*
Suppose $U \subseteq X$ is an open subset and $\{V_i\}$ is an open covering of U. Furthermore, suppose we have sections $\sigma_i \in \mathscr{F}(V_i)$ for each i such that $\sigma_i|_{V_{ij}} = \sigma_j|_{V_{ij}}$ for all i, j[1]. Then there exists a unique section, $\sigma \in \mathscr{F}(U)$, such that $\sigma|_{V_i} = \sigma_i$ for each i.

Note that the sheaf condition requires that we use an open covering $\{V_i\}$ of U such that V_{ij} is nonempty for all i, j. A morphism of sheaves is also a morphism of presheaves, so we define a morphism of presheaves here.

Definition 2.15. *Suppose $\mathscr{F}, \mathscr{G}$ are two presheaves on the same topological space X. Then $\psi : \mathscr{F} \to \mathscr{G}$ is a **morphism of presheaves** if it is a morphism of abelian*

[1]Here we are using the notation $V_{ij} = V_i \cap V_j$

groups $\psi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ *for each open subset* $U \subseteq X$*. Furthermore, if* $V \subseteq U$ *are two open subsets of* X *then the following diagram commutes,*

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\psi(U)} & \mathscr{G}(U) \\ \downarrow{\scriptstyle \rho_{UV}} & & \downarrow{\scriptstyle \rho'_{UV}} \\ \mathscr{F}(V) & \xrightarrow{\psi(V)} & \mathscr{G}(V) \end{array}$$

In this diagram, ρ *and* ρ' *are the restriction maps in* $\mathscr{F}$ *and* $\mathscr{G}$ *respectively.*

We now give some example of sheaves of rings. We first describe the ***sheaf of regular functions*** on $\mathbb{P}^1_{\mathbb{C}}$. For the remainder of this section, let $X = \mathbb{P}^1_{\mathbb{C}}$.

Definition 2.16. *For each open subset* $U \subseteq X$*, let* $\mathcal{O}(U)$ *be the ring of regular functions from* U *to* $\mathbb{C}$*. Suppose* $V \subseteq U$ *are two open subsets of* X*. Define the restriction map (in the usual sense)* $\rho_{UV} : \mathcal{O}(U) \to \mathcal{O}(V)$ *which maps* $f \mapsto f|_V$ *for all regular functions* $f \in \mathscr{O}(U)$*.*

We need to check that ρ_{UV} is well defined. We check this and also show that $\mathscr{O}$ is a sheaf of rings on $\mathbb{P}^1_{\mathbb{C}}$. We call $\mathscr{O}$ the sheaf of regular functions.

Proposition 2.17. *Using the set up of Definition 2.16,* $\mathscr{O}$*, with restriction maps defined by* ρ_{UV}*, is a sheaf of rings on* $\mathbb{P}^1_{\mathbb{C}}$*.*

Proof Suppose $V \subseteq U \subseteq X$ are open subsets. We first show ρ_{UV} is well defined. Suppose f is a regular function on U, that is suppose $f \in \mathscr{O}(U)$. Then $f = g/h$ where g, h are homogenous polynomials of the same degree in $\mathbb{C}[x, y]$ and for each $P \in U$, there is an open neighbourhood, $N_P \subseteq U$, in which h is nonzero. Thus for each $P \in V$, $V \cap N_P$ is an open neighbourhood contained in V, in which h is nonzero. Hence f is regular on V as well, so we can restrict f to V, $f|_V$. Thus ρ_{UV} is well defined. It is also clear that $\rho_{UU}(f) = f|_U = f$ and that for all open $W \subseteq V \subseteq U$ and $f \in \mathscr{O}(U)$, $(f|_V)|_W = f|_W$. Thus $\mathscr{O}$ is a presheaf.

For the sheaf condition, suppose $U \subseteq X$ is an open subset and that $\{V_i\}$ is an open covering of U. Without any loss of generality, we can assume that each $V_i \subseteq U$. Suppose also that we have $f_i \in \mathscr{O}(V_i)$ such that $f_i|_{V_{ij}} = f_j|_{V_{ij}}$ for all i, j. Since f_i are quotients of homogeneous polynomials, if $f_i|_{V_{ij}} = f_j|_{V_{ij}}$ then $f_i = f_j$ whenever $V_{ij} \neq \emptyset$. Thus f_i are the same unique regular function for all i. Hence we can take $f = f_i$ for any i. Then $f|_{V_i} = f_i$ for all i. This shows that $\mathscr{O}$ satisfies the sheaf condition. □

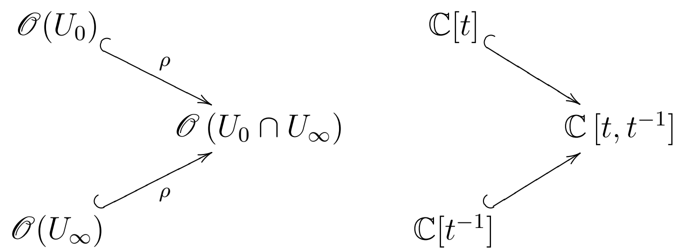
Let $U_0 = \mathbb{P}^1_{\mathbb{C}} - \{(1,0)\}$ and $U_\infty = \mathbb{P}^1_{\mathbb{C}} - \{(0,1)\}$. Interestingly, we can make a connection with Chapter 1, by identifying $\mathscr{O}(U_0)$ and $\mathscr{O}(U_\infty)$ with some polynomial rings that we have already seen.

Proposition 2.18. *Let $\mathscr{O}$ be the sheaf of regular functions on X. Then*

$$\mathscr{O}(U_0) \simeq \mathbb{C}[t] \text{ and } \mathscr{O}(U_\infty) \simeq \mathbb{C}[t^{-1}]$$

Proof Suppose $f(x,y) = g(x,y)/h(x,y) \in \mathscr{O}(U_0)$. Then $h(x,y)$ is nonzero for all $(x,y) \in \mathbb{C}^2$ with $y \neq 0$. Thus, if we fix a nonzero $y \in \mathbb{C}$, Liouville's Theorem implies that $h(x,y)$ is constant. Thus, $h(x,y)$ is a polynomial in y only. Since h is a homogeneous polynomial, $h(x,y) = ay^d$ for some $a \in \mathbb{C}$, $d \geq 0$. Thus, g must be a homogeneous polynomial of degree d. After splitting f into rational polynomials with monomial numerators and cancelling where possible, we see that f is generated over $\mathbb{C}$ by $\dfrac{x^n}{y^n}$ for $n \geq 0$. Thus $\mathscr{O}(U_0) = \mathbb{C}\left[\frac{x}{y}\right] \simeq \mathbb{C}[t]$ by setting $x = ty$. Similarly, $\mathscr{O}(U_\infty) = \mathbb{C}\left[\frac{y}{x}\right] \simeq \mathbb{C}[t^{-1}]$. □

Continuing this line of thought we see that similarly, $\mathscr{O}(U_0 \cap U_\infty) = \mathbb{C}\left[\frac{x}{y}, \frac{y}{x}\right] \simeq \mathbb{C}[t, t^{-1}]$. Furthermore, the restriction maps from $\mathscr{O}$ give us the natural embeddings of $\mathbb{C}[t]$ and $\mathbb{C}[t^{-1}]$ into $\mathbb{C}[t, t^{-1}]$.



Before we can describe the next example of a sheaf, we compute $D(\mathscr{O}(U))$ for each open subset $U \subseteq \mathbb{P}^1_{\mathbb{C}}$. These will be the sections for our next example of a sheaf, the sheaf of differential operators. This sheaf is the major object of study in this thesis. We describe constructions analogous to B_n and C_n in Definition 1.12 and use them to describe $D(\mathscr{O}(U))$.

Definition 2.19. *Suppose $U \subset \mathbb{P}^1_{\mathbb{C}}$ is a open subset. Then define,*

$$B^U_n = \{f(t)\frac{d^a}{dt^a} \mid f(t) \in \mathscr{O}(U),\ a \leq n\}$$

Also define $C^U_n := span_{\mathbb{C}}(B^U_n)$.

This definition is a generalisation of Definition 1.12. We can see that $B_n = B^{U_0}_n$ and that $C_n = C^{U_0}_n$. One important thing we must note in this definition is that we are expressing regular functions on U by functions of one variable. This was described in the above example, and is done so explicitly be letting $x = ty$. Take any open subset $U \subseteq \mathbb{P}^1_{\mathbb{C}}$ and $f = g/h \in \mathscr{O}(U)$. Then,

$$f(ty, y) = \frac{g(ty, y)}{h(ty, y)} = \frac{y^d g(t, 1)}{y^d h(t, 1)} = \frac{g(t, 1)}{h(t, 1)} = f(t, 1)$$

This shows us that every regular function defines a rational function in $\mathbb{C}(t)$. It is this rational function that is used in the definition of B_n^U. We now continue a path similar to the result in Theorem 1.14. After all, there is nothing special about U_0 that makes it different from other open subsets, except that computing $D(U_0)$ is simpler.

Theorem 2.20. *Suppose $U \subseteq X$ is open. Then,*

$$D^n(\mathscr{O}(U)) = C_n^U$$

Furthermore,

$$D(\mathscr{O}(U)) = \bigcup_{n \geq 0} C_n^U$$

We want to describe the sheaf, $\mathscr{D}$, of differential operators on $\mathbb{P}^1$. To do this, we need to understand a result related to the structure of the rings, $\mathscr{O}(U)$, for an open subset $U \subseteq X$. More specifically, if $V \subseteq U$ are open subsets in X, it follows that $U^c \subseteq V^c$, both of which are algebraic sets. Thus, we can find sets of homogeneous polynomials, $T_U \subset T_V$, such that $Z(T_U) = U^c$ and $Z(T_V) = V^c$. A primary subset of U is in some sense the simplest type of subset of U.

Definition 2.21. *Suppose $V \subseteq U \subseteq X$ are open subsets of X. Then V is a* ***primary subset*** *of U if V is a proper subset of U and there is a homogenous polynomial $h \in \mathbb{C}[x, y]$ such that $Z(T_V) = Z(T_U) \cup Z(h)$. In this case call h a* ***primary polynomial****.*

In the case where $V \subseteq U$, it is in fact simple to compute $\mathscr{O}(V)$ in terms of $\mathscr{O}(U)$. We see this computation in the following proposition. Again we will view sections of $\mathscr{O}(U)$ and $\mathscr{O}(V)$ as rational polynomials in $\mathbb{C}(t)$.

Proposition 2.22. *Suppose $V \subseteq U$ are open subsets of X such that V is a primary subset of U with primary polynomial h. In this case,*

$$\mathscr{O}(V) = \mathscr{O}(U)[f_0, f_1, \ldots, f_d]$$

where $f_i = t^i/h$ and d is the degree of h(as a polynomial in t).

Proof $\mathscr{O}(U)[f_0, f_1, \ldots, f_d]$ is contained in $\mathscr{O}(V)$ since sections in $\mathscr{O}(U)$ restrict to sections in $\mathscr{O}(V)$ and since h is nonzero on V. Conversely, suppose $f = g_1/g_2 \in$

$\mathscr{O}(V)$ such that $f \notin \mathscr{O}(U)$. Then there exists some $P \in U \setminus V$ such that $g_2(P) = 0$ (Here we are thinking of g_2 as a homogeneous polynomial in $\mathbb{C}[x, y]$). Since V is a primary subset of U, this implies that $h(P)$ is zero and hence $g_2 = h^n$ for some $n \geq 1$. Thus $f = g_1/h^n$, and hence by splitting f into rationals with monomials in the numerator we see that f is contained in $\mathbb{C}[f_0, f_1, \ldots, f_d]$ (Here we are thinking of f as a rational polynomial over $\mathbb{C}$). Hence $\mathscr{O}(V) = \mathscr{O}(U)[f_0, f_1, \ldots, f_d]$. □

If we have an open $U \subseteq \mathbb{P}^1_{\mathbb{C}}$, then we know we can always find a set of homogeneous polynomials, $T \subseteq \mathbb{C}[x, y]$, such that $Z(T) = U^c$. Evidently, we can replace T with the ideal, A, generated by all the elements of T as $Z(T) = Z(A)$. Since $\mathbb{C}[x, y]$ is a noetherian ring, it follows that A is finitely generated (More details on noetherian rings can be found in [5]). Hence it suffices to define the restriction maps for this sheaf just for $V \subseteq U$ with a primary subset V of U. To make sure this map is well defined we have the following proposition.

Proposition 2.23. *Suppose $V \subseteq U$ are open sets in $\mathbb{P}^1_{\mathbb{C}}$ and that V is a primary subset of U. Also, suppose that $Q \in D(\mathscr{O}(U))$. Then we can naturally extend Q, to an element of $D(\mathscr{O}(V))$.*

Proof Since V is a primary subset of U, we know from Proposition 2.22, $\mathscr{O}(V) = \mathscr{O}(U)[f_0, f_1, \ldots, f_d]$ where $f_i = t^i/h$ and d is the degree of h. We know that Q is a $\mathbb{C}$-linear map on $\mathscr{O}(U)$ and by Theorem 2.20, Q is a finite linear combination of terms of the form $f(t)\frac{d^a}{dt^a}$ where $f \in \mathscr{O}(U)$. Hence if we know how Q acts on f_i, we can extend it to $\mathscr{O}(V)$ via the product rule of differentiation. But Q acts naturally on each f_i via differentiation and multiplication of rational polynomials in $\mathbb{C}(t)$. It remains to check that the image of f_i, $Q(f_i)$, is a regular function on V. We show this by example, suppose $Q = \frac{d}{dt}$. Then,

$$Q(f_i) = \frac{d}{dt}\left(\frac{t^i}{h}\right) = \frac{hit^{i-1} - t^i h'}{h^2}$$

which is still regular on V as the denominator is h^2. Thus we can extend Q to $D(\mathscr{O}(V))$. □

We are now ready to define the sheaf of differential operators on $\mathbb{P}^1_{\mathbb{C}}$.

Definition 2.24. *Consider $X = \mathbb{P}^1_{\mathbb{C}}$. Let $\mathscr{O}$ be the sheaf of regular functions on X with restriction maps ρ_{UV} for open sets $V \subseteq U$, where V is a primary subset of U. Define $\mathscr{D}(U)$ to be the ring $D(\mathscr{O}(U))$. For open sets $V \subseteq U$, define the restriction map $\rho'_{UV} : \mathscr{D}(U) \to \mathscr{D}(V)$ which naturally extends elements of $\mathscr{D}(U)$ to elements of $\mathscr{D}(V)$ and allows the following diagram to commute.*

$$\begin{array}{ccc} \mathscr{D}(U) & \xrightarrow{\rho'_{UV}} & \mathscr{D}(V) \\ {\scriptstyle Q}\downarrow & & \downarrow{\scriptstyle \rho'_{UV}(Q)} \\ \mathscr{D}(U) & \xrightarrow{\rho'_{UV}} & \mathscr{D}(V) \end{array}$$

Proposition 2.25. *Using the set up in the previous definition, $\mathscr{D}$, together with restriction maps ρ'_{UV} for open sets $V \subseteq U$, with V a primary subset of U, is a sheaf on $\mathbb{P}^1_{\mathbb{C}}$. We name it the* ***sheaf of differential operators****.*

Now that we have defined the sheaf of differential operators on $\mathbb{P}^1_{\mathbb{C}}$, we can work on U_0 and U_∞ again. Just like in the previous example we have the natural embeddings of $D(\mathbb{C}[t])$ and $D(\mathbb{C}[t^{-1}])$ into $D(\mathbb{C}[t, t^{-1}])$.

$$\begin{array}{ccc} \mathscr{D}(U_0) & & \\ & \overset{\rho'}{\searrow} & \\ & & \mathscr{D}(U_0 \cap U_\infty) \\ & \underset{\rho'}{\nearrow} & \\ \mathscr{D}(U_\infty) & & \end{array} \qquad \begin{array}{ccc} D(\mathbb{C}[t]) & & \\ & \searrow & \\ & & D(\mathbb{C}[t, t^{-1}]) \\ & \nearrow & \\ D(\mathbb{C}[t^{-1}]) & & \end{array}$$

In this section we have defined sheaves, and constructed two examples of sheaves on $\mathbb{P}^1_{\mathbb{C}}$. Namely, the sheaf of regular functions and the sheaf of differential operators. These two sheaves provided us with sections that were isomorphic to rings that we had constructed in Chapter 1. In the next section, we delve into more algebraic geometry to see why $\mathscr{D}$ is in fact a quasi-coherent sheaf. This property of $\mathscr{D}$, allows us to study $\mathscr{D}$ in a different way, thanks to Richard Swan's article, [9], entitled 'Higher Algebraic K-theory'.

2.3 Quasi-Coherent Sheaves

In this section we will define the concept of quasi-coherence of a sheaf. We will see a speck of dust in the universe that is, algebraic geometry. The initial theory of algebraic geometry, that is, the language of schemes, is quite terse. In this sense, this section motivates the next chapter, which takes an alternative view on quasi-coherent sheaves, via modules, given in [9]. We will introduce $Spec\ A$, form a topology on it[2], and then define a sheaf on it, $\mathscr{O}_{Spec\ A}$, its structure sheaf. We will define the notion of a scheme, an $\mathscr{O}_X$-module for a scheme X, and finally a

[2]Interestingly, due to the nature of the closed sets, this topology is also called the Zariski topology

quasi-coherent sheaf. This section follows the derivation of a quasi-coherent sheaf from [6], but details can also be found in [8].

Let A be a commutative ring with an identity element 1. We will first construct the topological space *Spec* A, associated to A.

Definition 2.26. *Define Spec A to be the set of all prime ideals*[3] *of A. If $\mathfrak{a}$ is an ideal in A, define the subset $V(\mathfrak{a}) \subseteq$ Spec A as,*

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \textit{Spec } A \mid \mathfrak{a} \subseteq \mathfrak{p}\}$$

As in Proposition 2.6, we see that $V(\mathfrak{a})$ form the closed sets of a topology on *Spec* A.

Proposition 2.27. *Let $X =$ Spec A, then,*

1. *If $\mathfrak{a}$, $\mathfrak{b}$ are ideals of A, then $V(\mathfrak{a}) \cup V(\mathfrak{b}) = V(\mathfrak{a} \cap \mathfrak{b})$*
2. *If $\{\mathfrak{a}_i\}$ is any set of ideals of A, then $\bigcap_i V(\mathfrak{a}_i) = V\left(\sum_i \mathfrak{a}_i\right)$*

Hence $V(\mathfrak{a})$, for each ideal $\mathfrak{a}$ in A, form the closed sets of a topology on X, which we will call the ***Zariski*** *topology on X.*

In this proposition, by $\sum_i \mathfrak{a}_i$, we mean the smallest ideal of A containing all $\{\mathfrak{a}_i\}$. Note that $V(A) = \emptyset$ and that $V((0)) =$ *Spec* A. Thus we have another topological space, and it it slightly more abstract than what we saw on $\mathbb{P}^n_K$. As in Section 2.2 we continue along the same path to construct a sheaf of rings on *Spec* A. If $f \in A$, let $E(f) = X \setminus V((f))$. Note that $E(f)$ is an open set, and all the sets of this form, are a base for the Zariski topology on *Spec* A. Thus it suffices to define a sheaf of rings on *Spec* A just by defining the sections associated to the open sets in the base as opposed to arbitrary open sets. Before we can define the structure sheaf, we need to know the universal property of localisation.

Theorem 2.28. *Suppose A is a commutative ring and S is a multiplicative subset of A. Then there exists a ring denoted by AS^{-1} and a homomorphism $\phi : A \to AS^{-1}$ with the following properties,*

1. *$\phi(s)$ is invertible for all $s \in S$*
2. *If $\psi : A \to A'$ is a ring homomorphism such that $\psi(s)$ is invertible for all $s \in S$, then there exists a unique $\bar{\psi} : AS^{-1} \to A'$ such that the following diagram commutes,*

[3] Recall that a prime ideal I of A is a proper ideal of A with the property that if $ab \in I$ then $a \in I$ or $b \in I$

$$\begin{array}{ccc} A & \xrightarrow{\phi} & AS^{-1} \\ & \searrow^{\psi} & \vdots \; \bar{\psi} \\ & & A' \end{array}$$

Definition 2.29. *Let $X = Spec\ A$. We will define the* **structure sheaf***, denoted by $\mathscr{O}_X$, on X as follows. Suppose $f \in A$, then define,*

$$\mathscr{O}_X(E(f)) := A\left[\frac{1}{f}\right] := AS^{-1}$$

where $S = \{1, f, f^2, \ldots\}$. For the restriction map, suppose $f, g \in A$ such that $E(g) \subseteq E(f)$. We need a map, $r_{fg} : A\left[\frac{1}{f}\right] \to A\left[\frac{1}{g}\right]$. We define r_{fg} to be the unique map given from the universal property of localisation.

Since $E(g) \subseteq E(f)$, it follows (non-trivially) that f is a unit in $A\left[\frac{1}{g}\right]$. Thus from the universal property of localisation, we take $S = \{1, f, f^2, \ldots\}$, and $A' = A\left[\frac{1}{g}\right]$. Then we set $r_{fg} := \bar{\psi}$. With this data, $\mathscr{O}_X$ is in fact a sheaf, see [6] or [8] for a proof. We can now define the spectrum of A.

Definition 2.30. *The* **spectrum** *of A is the pair consisting of the topological space $Spec\ A$ together with its structure sheaf $\mathscr{O}_{Spec\ A}$.*

The spectrum of a ring is an integral part of the definition of a scheme. Other concepts we will need for the definition of a scheme is the notion of a ringed space, and morphisms of ringed spaces. In order to understand a morphism of a ringed space, we first need to define a direct image of a sheaf.

Definition 2.31. *Let $f : X \to Y$ be a continuous map of topological spaces. For any sheaf $\mathscr{F}$ on X, define the* **direct image** *sheaf $f_*\mathscr{F}$ on Y by $(f_*\mathscr{F})(V) = \mathscr{F}(f^{-1}(V))$ for any open set $V \subseteq Y$.*

The fact that the direct image $f_*\mathscr{F}$, of a sheaf $\mathscr{F}$, is in fact a sheaf amounts to simple checking. Keeping the definition of the direct image sheaf in mind, if we have topological spaces X, Y, equipped with the Zariski topology, we would like to obtain a continuous map $f : X \to Y$ so that we can obtain a new sheaf on Y given a sheaf on X.

Proposition 2.32. *Let $\phi : A \to A'$ be a homomorphism of rings. Let $X = Spec\ A'$, $Y = Spec\ A$, with structure sheaves $\mathscr{O}_X$, $\mathscr{O}_Y$ respectively. Let $f : X \to Y$, defined by $\mathfrak{p} \mapsto \phi^{-1}(\mathfrak{p})$. Then f is continuous with respect to the Zariski topology on X and Y.*

Proof Let $C \subset Y$ be a closed set. Then there exists $S \subseteq A$ such that $C = V(S) := \{\mathfrak{p} \in X \mid S \subseteq \mathfrak{p}\}$. Then $f^{-1}(C) = \{\mathfrak{p}' \in X \mid f(\mathfrak{p}') \in C\} = \{\mathfrak{p}' \in X \mid S \subseteq \phi^{-1}(\mathfrak{p}')\} = V(\phi(S))$. □

This proposition tells us that we can use ϕ to define a direct image sheaf. This is, in fact, exactly what we do when we define morphisms of ringed spaces.

Definition 2.33. *A **ringed space** is a pair $(X, \mathscr{O}_X)$, consisting of a topological space X and a sheaf of rings $\mathscr{O}_X$ on X. A **morphism** of ringed spaces from $(X, \mathscr{O}_X)$ to $(Y, \mathscr{O}_Y)$ is a pair $(f, f^{\#})$, consisting of a continuous map $f : X \to Y$ and a morphism $f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$ of sheaves of rings on Y. An **isomorphism** of ringed spaces is just a morphism with a two sided inverse.*

Before we can define a scheme, we need to define the stalk of a sheaf.

Definition 2.34. *Let $\mathscr{F}$ be a presheaf on a topological space X and let $x \in X$. Consider the set of all pair (s, U) where U is a neighbourhood of x and s is a section in $\mathscr{F}(U)$. We define a relation on this set as follows,*

$$(s, U) \sim (t, V) \Leftrightarrow \text{ there exists a neighbourhood } W \subset U \cap V \text{ of } x \text{ such that } s|_W = t|_W$$

*An equivalence class $[(s, U)]_\sim$ is called a **germ of a section** s at x. The **stalk** $\mathscr{F}_x$ of $\mathscr{F}$ at x is defined to be the set of all germs of sections of $\mathscr{F}$ at x.*

Definition 2.35. *A **scheme** is a ringed space $(X, \mathscr{O}_X)$ such that,*

1. *For each $x \in X$, the stalk $\mathscr{O}_{X,x}$ is a local ring*
2. *For all $x \in X$, there exists some open neighbourhood U of x such that $(U, \mathscr{O}_X|_U)$ is isomorphic to $(Spec\ A, \mathscr{O}_{Spec\ A})$ for some ring A that depends on x*

In this definition $\mathscr{O}_X|_U$ is the ***restriction*** of $\mathscr{O}_X$ to U. Also, a ***local ring*** is a ring with a unique maximal ideal.

Definition 2.36. *Let $(X, \mathscr{O}_X)$ be a ringed space. A **sheaf of $\mathscr{O}_X$-modules** (or simply an **$\mathscr{O}_X$-module**) is a sheaf $\mathscr{F}$ on X such that,*

1. *For any open $U \subseteq X$, $\mathscr{F}(U)$ is an $\mathscr{O}_X(U)$-module given by a map*

$$\mathscr{O}_X(U) \times \mathscr{F}(U) \xrightarrow{M_U} \mathscr{F}(U)$$

2. *For any open sets $V \subseteq U$ the following diagram commutes*

$$\begin{array}{ccc} \mathscr{O}_X(U) \times \mathscr{F}(U) & \xrightarrow{M_U} & \mathscr{F}(U) \\ {\scriptstyle r_{UV}^{\mathscr{O}_X} \times r_{UV}^{\mathscr{F}}} \downarrow & & \downarrow {\scriptstyle r_{UV}^{\mathscr{F}}} \\ \mathscr{O}_X(V) \times \mathscr{F}(V) & \xrightarrow{M_V} & \mathscr{F}(V) \end{array}$$

It is interesting to note that in this definition, as a sheaf, $\mathscr{F}$ is a sheaf of abelian groups on X. As before, when defining a sheaf on *Spec* A, it suffices to define the image of the sheaf only on open sets in the base of the Zariski topology, that is, sets of the form $E(f)$ for some $f \in A$.

Definition 2.37. *Let A be a ring and let M be an A-module. We define the **sheaf associated** to M on Spec A, denoted by $\tilde{M}$, as follows. For each open set in the base of the Zariski topology, $E(f)$ for some $f \in A$, define $\tilde{M}(E(f)) := \mathscr{O}_{Spec\ A}(E(f)) \otimes_A M = A\left[\frac{1}{f}\right] \otimes_A M$.*

Definition 2.38. *Let $(X, \mathscr{O}_X)$ be a scheme, and suppose $\mathscr{F}$ is an $\mathscr{O}_X$-module. Then $\mathscr{F}$ is **quasi-coherent** if X can be covered by affine open sets $U_i = Spec(A_i)$ such that for each i there is an A_i-module M_i satisfying $\mathscr{F}|_{U_i} \simeq \tilde{M}_i$.*

Theorem 2.39. *Let $X = \mathbb{P}^1_{\mathbb{C}}$, and $\mathscr{O}$ be the sheaf of regular functions on X. Then $(X, \mathscr{O})$ is a scheme. Furthermore, if we let $\mathscr{D}$ be the sheaf of differential operators on X, then $\mathscr{D}$ is an $\mathscr{O}$-module on X and $\mathscr{D}$ is a quasi-coherent sheaf.*

As we can see, even the initial definitions of algebraic geometry are quite complicated. Fortunately, there is an alternative to studying quasi-coherent sheaves on $\mathbb{P}^1_{\mathbb{C}}$. It involves modules and category theory.

Chapter 3

Twisted Rings of Differential Operators and Global Sections on $\mathbb{P}^1_{\mathbb{C}}$

Sheaves give us a connection between global data and local data, as a result, our major aim in this chapter is to compute the global sections of sheaves of differential operators. In the last chapter we learnt that $\mathscr{D}$, the sheaf of differential operators on $\mathbb{P}^1_{\mathbb{C}}$ is a quasi-coherent sheaf. We saw that indeed showing that $\mathscr{D}$ was quasi-coherent was a large task in itself. Hence we want an alternative method for working with $\mathscr{D}$, and in [9], Richard Swan describes how one can use modules to study $\mathscr{D}$ on $\mathbb{P}^1_{\mathbb{C}}$. We will apply Swan's approach to the sheaf of differential operators, and this will allow us to construct *sheaves of twisted rings of differential operators.* It may be of interest to note that Swan's technique is quite standard and is well known.

3.1 Modules - An Alternative

In this section we will be using tensoring of modules quite frequently. It arises from Definition 2.37, since the sections of a quasi-coherent sheaf are tensor products of modules. We will begin by defining what a category is, then we will define a category which will allow us to study the global sections of any quasi-coherent sheaf on $\mathbb{P}^1_K$.

Definition 3.1. *A* ***category*** *is a quintuple* $\mathscr{C} = (Ob(\mathscr{C}), Mor(\mathscr{C}), dom, cod, \circ)$ *where*

1. $Ob(\mathscr{C})$ *is a class whose members are called* $\mathscr{C}$***-objects***
2. $Mor(\mathscr{C})$ *is a class whose members are called* $\mathscr{C}$***-morphisms***
3. dom *and* cod *are functions from* $Mor(\mathscr{C})$ *to* $Ob(\mathscr{C})(dom(f)$ *is called the* ***domain*** *of* f *and* $cod(f)$ *is called the* ***codomain*** *of* f

4. $\circ$ *is a function from,*

$$D = \{(f,g) \mid f,g \in Mor(\mathscr{C}) \text{ and } dom(f) = cod(g)\}$$

into $Mor(\mathscr{C})$, called the ***composition law of*** $\mathscr{C}$ *($\circ(f,g)$ is usually written $f \circ g$ and is defined if and only if $(f,g) \in D$) such that the following conditions are satisfied,*

(a) ***Matching Condition:*** *If $f \circ g$ is defined then $dom(f \circ g) = dom(g)$ and $cod(f \circ g) = cod(f)$*

(b) ***Associativity Condition:*** *If $f \circ g$ and $h \circ f$ are defined then $h \circ (f \circ g) = (h \circ f) \circ g$*

(c) ***Identity Existence Condition:*** *For each $\mathscr{C}$-object A there exists a $\mathscr{C}$-morphism e such that $dom(e) = A = cod(e)$ and*

   i. *$f \circ e = f$ whenever $f \circ e$ is defined*

   ii. *$e \circ g = g$ whenever $e \circ g$ is defined*

(d) ***Smallness of Morphism Class Condition:*** *For any pair (A,B) of $\mathscr{C}$-objects, the class*

$$hom_{\mathscr{C}}(A,B) = \{f \mid f \in \mathscr{M},\ dom(f) = A,\ \text{and } cod(f) = B\}$$

is a set

Example 3.2. Some simple examples of categories include

1. R-**Mod**, the category of left R-modules and module homomorphisms for a ring R
2. **Rng**, the category of rings and ring homomorphisms
3. **Grp**, the category of groups and group homomorphisms

We define when a morphism in $Mor(\mathscr{C})$ is an isomorphism.

Definition 3.3. *A morphism $A \xrightarrow{f} B$ in a category $\mathscr{C}$ is said to be a* ***$\mathscr{C}$-section*** *if there exists some $\mathscr{C}$-morphism $B \xrightarrow{g} A$ such that $g \circ f = 1_A$. Similarly, $A \xrightarrow{f} B$ is said to be a* ***$\mathscr{C}$-retraction*** *if there exists some $\mathscr{C}$-morphism $B \xrightarrow{h} A$ such that $f \circ h = 1_B$. A $\mathscr{C}$-morphism $A \xrightarrow{f} B$ is said to be an* ***isomorphism in*** $\mathscr{C}$ *if it is a $\mathscr{C}$-section and a $\mathscr{C}$-retraction.*

Category theory is really an organisational language that allows us to generalise concepts, for example, kernels, cokernels, etc. We will follow Swan, and define a category which creates an analogue for the sections of the open sets we saw in Chapter 2, namely, U_0 and U_∞.

Definition 3.4. *If A is any ring (not necessarily commutative) let $\mathscr{M}od(\mathbb{P}^1_A)$ be the category whose objects are triples (M^+, M^-, θ), where M^+ is an $A[t]$-module, M^- is an $A[t^{-1}]$-module, and θ is an isomorphism, called the* ***gluing isomorphism****,*

$$\theta : A[t,t^{-1}] \otimes_{A[t]} M^+ \tilde{\to} A[t,t^{-1}] \otimes_{A[t^{-1}]} M^-$$

Here t is an indeterminate.

A morphism $(M^+, M^-, \theta) \to (N^+, N^-, \phi)$ is a pair (f^+, f^-) where $f^+ : M^+ \to N^+$ over $A[t]$, $f^- : M^- \to N^-$ over $A[t^{-1}]$, and the following diagram commutes.

$$\begin{array}{ccc} A[t,t^{-1}] \otimes_{A[t]} M^+ & \xrightarrow[\simeq]{\theta} & A[t,t^{-1}] \otimes_{A[t^{-1}]} M^- \\ \downarrow {\scriptstyle 1\otimes f^+} & & \downarrow {\scriptstyle 1\otimes f^-} \\ A[t,t^{-1}] \otimes_{A[t]} N^+ & \xrightarrow[\simeq]{\phi} & A[t,t^{-1}] \otimes_{A[t^{-1}]} N^- \end{array}$$

Swan then defines the Serre twist of a sheaf, which changes the gluing isomorphism θ. The Serre twist is an auto-equivalence of the category $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$, which is analogous to an automorphism of an algebraic structure.

Definition 3.5. *If $\mathscr{M} = (M^+, M^-, \theta)$ is an object in $\mathscr{M}od(\mathbb{P}^1_A)$, then define the* ***Serre twist*** *$\mathscr{M}(n)$ by*

$$\mathscr{M}(n) = (M^+, M^-, t^{-n}\theta)$$

Proposition 3.6. *Let $\mathscr{O} = (A[t], A[t^{-1}], 1)$ be an object of $\mathscr{M}od(\mathbb{P}^1_A)$. Suppose N is an A-module and define $\mathscr{O}(n) \otimes_A N = (A[t] \otimes_A N, A[t^{-1}] \otimes_A N, t^{-n})$. Then $\mathscr{O}(n) \otimes_A N \simeq (N[t], N[t^-1], t^{-n})$.*

Proof We give a morphism (f^+, f^-), with a two sided inverse in the category $\mathscr{M}od(\mathbb{P}^1_A)$ as follows. We want to define $(f^+, f^-) : (A[t] \otimes_A N, A[t^{-1}] \otimes_A N, t^{-n}) \to (N[t], N[t^-1], t^{-n})$. Thus, define $f^+ : A[t] \otimes_A N \to N[t]$ by $f^+((at^d) \otimes x) = (ax)t^d$ where $a \in A$, $x \in N$. We see that f^+ is well defined since $ax \in N$. Similarly, we define $f^- : A[t^{-1}] \otimes_A N \to N[t^{-1}]$ by $f^-(at^{-d} \otimes x) = (ax)t^{-d}$. The inverse morphism is given by $(g^+, g^-) : (N[t], N[t^-1], t^{-n}) \to (A[t] \otimes_A N, A[t^{-1}] \otimes_A N, t^{-n})$ defined by $g^+(xt^d) = t^d \otimes x$ and $g^-(xt^{-d}) = t^{-d} \otimes x$. Thus, since (g^+, g^-) is a two sided inverse for (f^+, f^-) we see that $\mathscr{O}(n) \otimes_A N \simeq (N[t], N[t^-1], t^{-n})$. □

We shall soon see that the following map will help us to compute the global sections of any sheaf on $\mathbb{P}^1_{\mathbb{C}}$.

Definition 3.7. *Suppose $\mathscr{M} = (M^+, M^-, \theta)$ is an object in $\mathscr{M}od(\mathbb{P}^1_A)$ and define the following sequence called the **Cech complex** (the cohomology of this complex is the Cech cohomology) of $\mathscr{M}$,*

$$0 \longrightarrow M^+ \oplus M^- \xrightarrow{\ d\ } A[t,t^{-1}] \otimes_{A[t^{-1}]} M^- \longrightarrow 0$$

*Where $d(x,y) = \theta(1 \otimes x) - (1 \otimes y)$ is called the **Cech map** (this is not standard notation).*

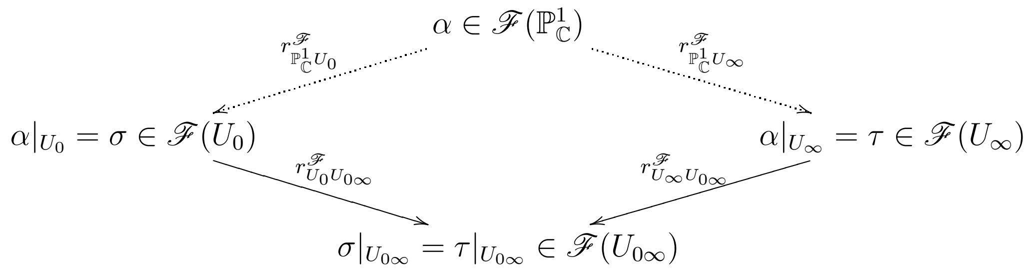
Our aim is to somehow use the Cech map to give us the global sections of sheaves on $\mathbb{P}^1_{\mathbb{C}}$. Before we can do this, we must apply Swan's technique to $\mathscr{D}$, the sheaf of differential operators on $\mathbb{P}^1_{\mathbb{C}}$.

3.2 Sheaves of Twisted Rings of Differential Operators

Our major aim in this section is to compute the objects of $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$ that correspond to sheaves of twisted rings of differential operators. We will apply Swan's technique, which uses an object in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$ to describe a quasi-coherent sheaf on $\mathbb{P}^1_{\mathbb{C}}$. Furthermore, of particular interest to us is the gluing isomorphism we use for each object, simply because a different gluing isomorphism describes a different quasi-coherent sheaf.

Let us recall that the primary reason for using sheaves is that the sheaf condition gives us a unique section that restricts to specific sections on smaller open sets. This means that we can use local data to understand something global. Our major goal for this chapter is to compute the ring of global sections for sheaves of twisted rings of differential operators. Suppose we have a sheaf, $\mathscr{F}$, of rings on $\mathbb{P}^1_{\mathbb{C}}$. Since we have our open covering, $\mathbb{P}^1_{\mathbb{C}} = U_0 \cup U_\infty$, if we can find sections $\sigma \in \mathscr{F}(U_0)$ and $\tau \in \mathscr{F}(U_\infty)$ that restrict to the same section in $\mathscr{F}(U_0 \cap U_\infty)$, the sheaf condition states that there exists a unique section $\alpha \in \mathscr{F}(\mathbb{P}^1_{\mathbb{C}})$ that restricts to σ and τ in $\mathscr{F}(U_0)$ and $\mathscr{F}(U_\infty)$ respectively.

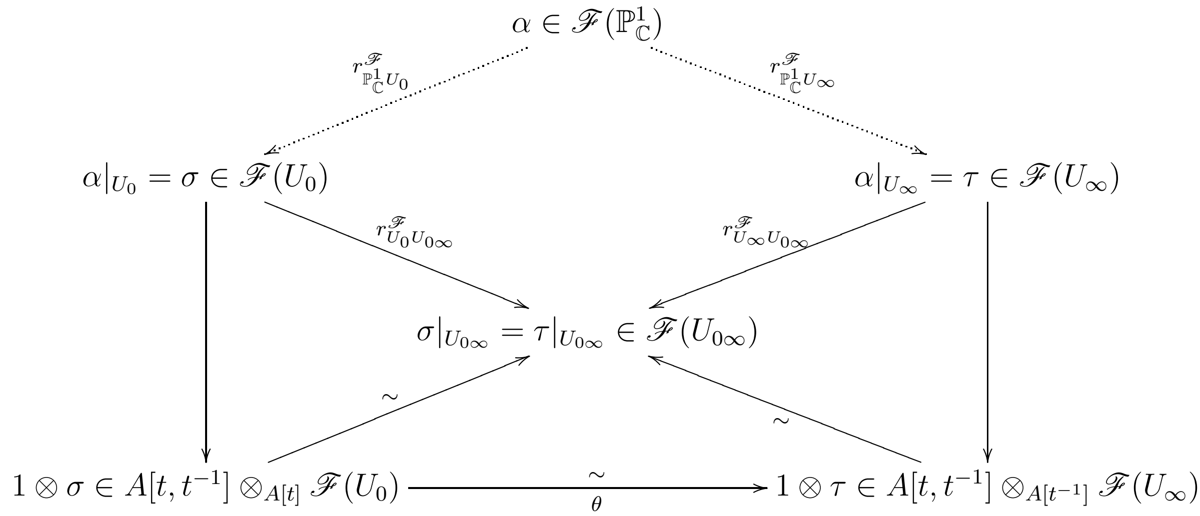
The sheaf condition can be seen more clearly in the following diagram,



We now discuss the relevance of Swan's alternative approach. In [9], Swan focuses on the sections $\mathscr{F}(U_0)$ and $\mathscr{F}(U_\infty)$. After writing $\mathbb{P}^1_{\mathbb{C}} = U_0 \cup U_\infty$ as a covering of affine open subsets (which is a condition for a quasi-coherent sheaf) and letting the object (M^+, M^-, θ) correspond to a quasi-coherent sheaf $\mathscr{F}$, Swan uses the following correspondence,

$$\mathscr{F}(U_0) = M^+$$
$$\mathscr{F}(U_\infty) = M^-$$
$$\mathscr{F}(U_0 \cap U_\infty) = A[t, t^{-1}] \otimes_{A[t]} M^+ \simeq A[t, t^{-1}] \otimes_{A[t^{-1}]} M^-$$

Thus, combining this information with the sheaf condition, we see that really, Swan is giving us isomorphic copies of $\mathscr{F}(U_{0\infty})$ to embed M^+ and M^- into. Thus, to include this information in our diagram, we obtain the following commutative diagram,



We now move our focus to two objects of the category $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$. Namely the object $(\mathbb{C}[t], \mathbb{C}[t^{-1}], 1)$ which corresponds to the sheaf of regular functions on $\mathbb{P}^1_{\mathbb{C}}$, and the object $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}], \theta)$ which corresponds to, $\mathscr{D}$, the sheaf of differential operators. In Chapter 1, we saw how $\mathbb{C}[t]$ embeds as a subring of $D(\mathbb{C}[t])$. Following this path, we'd like to say that $(\mathbb{C}[t], \mathbb{C}[t^{-1}], 1)$ is a 'subobject' of $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}], \theta)$. Hence we define what a subobject is and prove that this is the case.

Definition 3.8. *Suppose $\mathscr{C}$ is a category, $A, B \in Ob(\mathscr{C})$ and $f \in Hom_{\mathscr{C}}(A, B)$. We say that f is a* ***monomorphism in $\mathscr{C}$*** *(or a* ***$\mathscr{C}$-monomorphism****) if for all $h, k \in Mor(\mathscr{C})$ such that $f \circ h = f \circ k$, it follows that $h = k$. We say that (A, f) is a* ***subobject*** *of B if f is a monomorphism.*

Proposition 3.9. *Consider $(\mathbb{C}[t], \mathbb{C}[t^{-1}], 1)$, $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}], \theta)$, two objects in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$. Then $(\mathbb{C}[t], \mathbb{C}[t^{-1}], 1)$ is a subobject of $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}], \theta)$.*

Proof We already have ring homomorphisms, $f^+ : \mathbb{C}[t] \to D(\mathbb{C}[t])$ and $f^- : \mathbb{C}[t^{-1}] \to D(\mathbb{C}[t^{-1}])$. More precisely, these are ϕ, from Proposition 1.2, taking $R = \mathbb{C}[t]$ for f^+ and $R = \mathbb{C}[t^{-1}]$ for f^-. Furthermore, Proposition 1.2 says that both f^+ and f^- are injective ring homomorphisms. Hence suppose we have two morphisms $(h^+, h^-), (k^+, k^-) \in Mor(\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}}))$, such that $(f^+, f^-) \circ (h^+, h^-) = (f^+, f^-) \circ (k^+, k^-)$. Then we have $f^+ \circ h^+ = f^+ \circ k^+$ and $f^- \circ h^- = f^- \circ k^-$. Since both f^+ and f^- are injective, it follows that $h^+ = k^+$ and $h^- = k^-$. Hence (f^+, f^-) is a $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$-monomorphism. This shows that $(\mathbb{C}[t], \mathbb{C}[t^{-1}], 1)$ is a subobject of $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}], \theta)$. □

We are left with no choices when gluing $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} \mathbb{C}[t]$ to $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} \mathbb{C}[t^{-1}]$ since the gluing ismorphism of the object $(\mathbb{C}[t], \mathbb{C}[t^{-1}], 1)$ is the identity map. We need to analyse the isomorphism θ a little more deeply, because as we shall see, there are many choices for how we want to glue $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t])$ to $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}])$. Before we do this we must first show that the two isomorphic copies of $D(\mathbb{C}[t, t^{-1}])$ given by Swan are in fact isomorphic copies of $D(\mathbb{C}[t, t^{-1}])$.

Proposition 3.10. *Consider the two $\mathbb{C}[t, t^{-1}]$-modules, $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t])$ and $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}])$. Then,*

$$\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t]) \simeq D(\mathbb{C}[t, t^{-1}])$$
$$\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}]) \simeq D(\mathbb{C}[t, t^{-1}])$$

are isomorphic as $\mathbb{C}[t, t^{-1}]$ modules.

Proof Consider the $\mathbb{C}[t, t^{-1}]$-linear map $\phi_t : \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t]) \to D(\mathbb{C}[t, t^{-1}])$ defined by $\phi_t(f(t) \otimes g(t, \delta)) = f(t)g(t, \delta)$. It is clear that ϕ_t is a surjective $\mathbb{C}[t, t^{-1}]$-module homomorphism, hence we show that ϕ_t is injective. Suppose $f(t)g(t, \delta) = 0$. Then, either $f(t) = 0$ or $g(t, \delta) = 0$. In either case, $f(t) \otimes g(t, \delta) = 0$. Hence, $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t]) \simeq D(\mathbb{C}[t, t^{-1}])$ is an isomorphism of $\mathbb{C}[t, t^{-1}]$-modules. For the second isomorphism, we use $\phi_{t^{-1}} : \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t^{-1}]) \to D(\mathbb{C}[t, t^{-1}])$ defined by $\phi_{t^{-1}}(f(t) \otimes g(t,\delta)) = f(t)g(t,\delta)$. Similarly, $\phi_{t^{-1}}$ is an isomorphism of $\mathbb{C}[t,t^{-1}]$ modules. $\square$

We aim to show that for each $\lambda \in \mathbb{C}$ there is an isomorphism of $\mathbb{C}[t,t^{-1}]$-modules, $\theta_\lambda : \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t]) \tilde{\rightarrow} \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}])$. Thus we need to learn more properties of the gluing ismorphism, θ.

Proposition 3.11. *Consider $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta)$, an object of $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$. Then θ induces a $\mathbb{C}[t,t^{-1}]$-module automorphism, ψ, on $D(\mathbb{C}[t,t^{-1}])$.*

Proof From the previous proposition we see that we have just created the following commutative diagram.

$$\begin{array}{ccc} \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t]) & \xrightarrow{\ \theta\ } & \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}]) \\ \phi_t \downarrow & & \downarrow \phi_{t^{-1}} \\ D(\mathbb{C}[t,t^{-1}]) & \xrightarrow{\ \psi\ } & D(\mathbb{C}[t,t^{-1}]) \end{array}$$

Hence, ψ is an isomorphism as the other three maps are isomorphisms. $\square$

Proposition 3.12. *Consider $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta)$, an object of $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$. Then ψ, the $\mathbb{C}[t,t^{-1}]$-module automorphism induced by θ, is a ring isomorphism on $D(\mathbb{C}[t,t^{-1}])$ and maps*

$$t \mapsto t$$
$$\delta \mapsto \alpha(t,t^{-1})\delta + \beta(t,t^{-1})$$

for some polynomials $\alpha, \beta \in \mathbb{C}[t,t^{-1}]$.

Proof We know from Proposition 3.11 that ψ is a $\mathbb{C}[t,t^{-1}]$-module automorphism. Thus it is linear over $\mathbb{C}[t,t^{-1}]$. Hence we can think of ψ as a ring isomorphism that maps $t \mapsto t$ and $t^{-1} \mapsto t^{-1}$. Since ψ is a ring homomorphism, $\psi(\delta) = f(t,t^{-1},\delta)$, that is some polynomial in t, t^{-1}, δ. Using $\delta t - t\delta = 1$, we can move all the δ's to the right in each term, so we can think of $f(t,t^{-1},\delta)$ as a polynomial in δ with coefficients from $\mathbb{C}[t,t^{-1}]$. Let $deg(f)$ be the degree of $f(t,t^{-1},\delta)$ as a polynomial in δ. Suppose, $n := deg(f) > 1$. Since ψ is surjective, there exists a $g \in D(\mathbb{C}[t,t^{-1}])$ such that $\psi(g) = \delta$. Since ψ fixes $\mathbb{C}[t,t^{-1}]$, g must contain a term with δ in it, thus $deg(g) \geq 1$. Suppose $g_d(t,t^{-1})\delta^d$ for some $d \geq 1$, is the term with highest degree in g. Then $\psi(g_d(t,t^{-1})\delta^d) = g_d(t,t^{-1})f(t,t^{-1},\delta)^d$, which is of degree dn. As this is the only way to obtain a term of degree dn, $deg(\psi(g)) = dn > 1$, which contradicts $\psi(g) = \delta$. Hence, the surjectivity of ψ implies that $f(t,t^{-1},\delta) = \alpha(t,t^{-1})\delta + \beta(t,t^{-1})$ for $\alpha, \beta \in \mathbb{C}[t,t^{-1}]$. $\square$

Our aim is to find all the ways of gluing $\mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t])$ to $\mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}])$. This will give us all the objects of the form $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta)$ up to isomorphism. Thus we further analyse ψ, the $\mathbb{C}[t,t^{-1}]$-module automorphism induced by θ and show that $\alpha(t,t^{-1}) = 1$.

Proposition 3.13. *Using the set up of Proposition 3.12, $\alpha(t,t^{-1}) = 1$. That is, ψ maps $\delta \mapsto \delta + \beta(t,t^{-1})$.*

Proof We know that $\psi(t) = t$ and $\psi(\delta) = f = \alpha\delta + \beta$. Thus, $1 = \psi(1) = \psi(\delta t - t\delta) = ft - tf = \alpha\delta t - t\alpha\delta = \alpha(\delta t - t\delta) = \alpha$. This shows that $\alpha = 1$. □

We now explore what $\beta(t,t^{-1})$ could be. Suppose $\beta(t,t^{-1}) = \sum_{j=-M}^{N} b_j t^j$ with $b_j \in \mathbb{C}$ for all $j = -M, \dots, N$ and $b_{-M} \neq 0, b_N \neq 0$. We currently have the following commutative diagram for our gluing isomorphism θ,

$$\begin{array}{ccc} \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t]) & \xrightarrow{\theta} & \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}]) \\ \downarrow{\scriptstyle \phi_t} & & \downarrow{\scriptstyle \phi_{t^{-1}}} \\ D(\mathbb{C}[t,t^{-1}]) & \xrightarrow{\psi} & D(\mathbb{C}[t,t^{-1}]) \end{array}$$

We can see that θ maps $1 \otimes t \mapsto 1 \otimes t$ and $1 \otimes \delta \mapsto 1 \otimes \delta + \beta(t,t^{-1}) \otimes 1$. We now construct an isomorphism of objects in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$.

Theorem 3.14. *Suppose $\lambda \in \mathbb{C}$ and let $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta_\lambda)$ be an object in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$, with gluing isomorphism θ_λ mapping*

$$1 \otimes t \mapsto 1 \otimes t$$
$$1 \otimes \delta \mapsto 1 \otimes (\delta + \lambda t^{-1})$$

Then $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta) \tilde{\to} (D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta_\lambda)$ are isomorphic as objects of $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$ with $\lambda = b_{-1}$, the coefficient of t^{-1} in $\beta(t,t^{-1})$.

Proof Firstly, $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta_\lambda)$ is an object in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$ since θ_λ has inverse $\theta_{-\lambda}$. Consider the following morphism in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$,

$$(f^+, f^-) : (D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta) \to (D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta_\lambda)$$

defined by the $\mathbb{C}[t]$-module homomorphism, $f^+ : D(\mathbb{C}[t]) \to D(\mathbb{C}[t])$ where,

$$f^+(t) = t$$
$$f^+(\delta) = \delta + \textstyle\sum_{j=0}^{N} b_j t^j$$

and the $\mathbb{C}[t^{-1}]$-module homomorphism, $f^- : D(\mathbb{C}[t^{-1}]) \to D(\mathbb{C}[t^{-1}])$ where,

$$f^-(t) = t$$
$$f^-(\delta) = \delta - \textstyle\sum_{j=-M}^{-2} b_j t^j$$

It is vital to note at this stage the reason why the sum in $f^-(\delta)$ only goes up to $j = -2$. It is really a consequence of $t^2\delta$ being an element of $D(\mathbb{C}[t^{-1}])$. We see that $f^-(t^2\delta) = f^-(t^2)f^-(\delta) = t^2 f^-(\delta) \in D(\mathbb{C}[t^{-1}])$. This means that the highest power of t that can appear in $f^-(\delta)$ is t^{-2}. We can see that (f^+, f^-) is a $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$-section and a $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$-retraction, and is hence an isomorphism in $Mor(\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}}))$, with inverse (left and right),

$$(g^+, g^-) : (D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta_\lambda) \to (D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta)$$

defined by the $\mathbb{C}[t]$-module homomorphism, $g^+ : D(\mathbb{C}[t]) \to D(\mathbb{C}[t])$ where,

$$\begin{gathered} g^+(t) = t \\ g^+(\delta) = \delta - \textstyle\sum_{j=0}^{N} b_j t^j \end{gathered}$$

and the $\mathbb{C}[t^{-1}]$-module homomorphism, $g^- : D(\mathbb{C}[t^{-1}]) \to D(\mathbb{C}[t^{-1}])$ where,

$$\begin{gathered} g^-(t) = t \\ g^-(\delta) = \delta + \textstyle\sum_{j=-M}^{-2} b_j t^j \end{gathered}$$

Hence as objects in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$, we have an isomorphism,

$$(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta) \tilde{\to} (D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta_\lambda)$$

of objects in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$.

We now show that $\lambda = b_{-1}$. Our argument gives us a commutative diagram which allows us to compute $\theta_\lambda(1 \otimes t)$ and also $\theta_\lambda(1 \otimes \delta)$,

$$\begin{array}{ccc} \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t]) & \xrightarrow[\sim]{\theta} & \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}]) \\ {\scriptstyle 1\otimes g^+}\uparrow \downarrow{\scriptstyle 1\otimes f^+} & & {\scriptstyle 1\otimes g^-}\uparrow \downarrow{\scriptstyle 1\otimes f^-} \\ \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t]) & \xrightarrow[\sim]{\theta_\lambda} & \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}]) \end{array}$$

This means that,

$$\begin{aligned} \theta_\lambda(1 \otimes t) &= (1 \otimes f^- \circ \theta \circ 1 \otimes g^+)(1 \otimes t) \\ &= (1 \otimes f^- \circ \theta)(1 \otimes t) \\ &= (1 \otimes f^-)(1 \otimes t) \\ &= 1 \otimes t \end{aligned}$$

since all the maps send $t \mapsto t$. Also for $1 \otimes \delta$,

$$\begin{aligned}
\theta_\lambda(1 \otimes \delta) &= (1 \otimes f^- \circ \theta \circ 1 \otimes g^+)(1 \otimes \delta) \\
&= (1 \otimes f^- \circ \theta)(1 \otimes \delta - 1 \otimes \sum_{j=0}^{N} b_j t^j) \\
&= (1 \otimes f^-)(1 \otimes \delta + \sum_{j=-M}^{N} b_j t^j \otimes 1 - \sum_{j=0}^{N} b_j t^j \otimes 1) \\
&= (1 \otimes f^-)(1 \otimes \delta + \sum_{j=-M}^{-1} b_j t^j \otimes 1) \\
&= 1 \otimes \delta - \sum_{j=-M}^{-2} b_j t^j \otimes 1 + \sum_{j=-M}^{-1} b_j t^j \otimes 1 \\
&= 1 \otimes \delta + b_{-1} t^{-1} \otimes 1 \\
&= 1 \otimes (\delta + b_{-1} t^{-1})
\end{aligned}$$

So we can see that $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta) \simeq (D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta_\lambda)$ as objects in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$, with $\lambda = b_{-1}$. □

This theorem tells us that if $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta)$ is an object of $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$, then there exists a $\lambda \in \mathbb{C}$ such that $(D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta) \simeq (D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta_\lambda)$ and the gluing isomorphism, θ_λ, maps,

$$\begin{aligned}
1 \otimes t &\mapsto 1 \otimes t \\
1 \otimes \delta &\mapsto 1 \otimes (\delta + \lambda t^{-1})
\end{aligned}$$

Furthermore, θ_λ induces a ring isomorphism, ψ_λ, on $D(\mathbb{C}[t, t^{-1}])$ which maps,

$$\begin{aligned}
t &\mapsto t \\
\delta &\mapsto \delta + \lambda t^{-1}
\end{aligned}$$

Thus for each $\lambda \in \mathbb{C}$ we obtain a different gluing isomorphism. Hence we obtain distinct sheaves of rings of differential operators for each $\lambda \in \mathbb{C}$. These sheaves are the sheaves of twisted rings of differential operators.

Definition 3.15. *Suppose $\lambda \in \mathbb{C}$ and that $\mathscr{D}$ is a quasi-coherent sheaf of differential operators on $\mathbb{P}^1_{\mathbb{C}}$. Then $\mathscr{D}$ has a* ***twist*** *λ if*

$$(\mathscr{D}(U_0), \mathscr{D}(U_\infty), \theta) \simeq (D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta_\lambda)$$

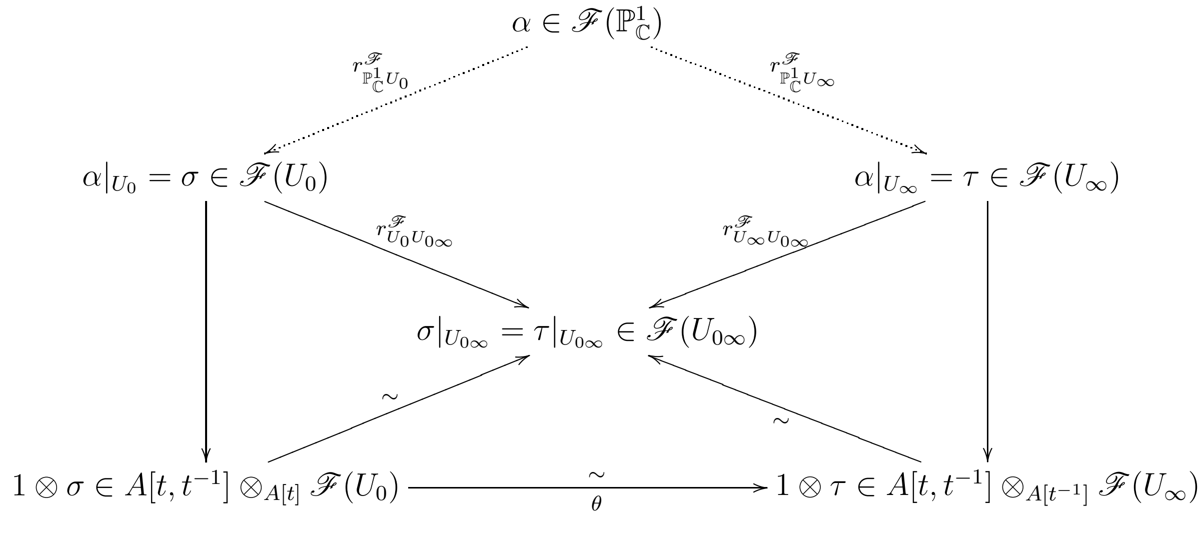
as objects in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$. Call $\mathscr{D}$ a ***sheaf of twisted rings of differential operators*** *and denote $\mathscr{D}$ by $\mathscr{D}^\lambda$. Furthermore if $\lambda = 0$, say that $\mathscr{D}^0$ is* ***untwisted****.*

In this section we discovered that there are in fact many sheaves of differential operators on $\mathbb{P}^1_{\mathbb{C}}$, all of which, up to isomorphism (of sheaves) have a twist, λ, for some $\lambda \in \mathbb{C}$. In the next section we will continue with our analysis on these sheaves, by computing generators for their global sections.

3.3 Generating Global Sections on $\mathbb{P}^1_{\mathbb{C}}$

Recall that our major goal for this chapter is to compute the global sections for all the sheaves of twisted rings of differential operators, $\mathscr{D}^\lambda$. The machinery of sheaves allows us to compute the global sections using local sections. In what follows, we first show that the global sections for $\mathscr{D}^\lambda$ form a ring, and then we find a generating set for them. All the results of this section can be found in [12], but the proofs are much more sophisticated than the ones we use. Swan, in [9], suggests that the Cech map d from Definition 3.7, is helpful in the search for global sections. The second commutative diagram in Section 3.2 motivates why the Cech map d is of any relevance to us at all. For clarity we repeat the diagram and Swan's correspondence here,

$$\mathscr{F}(U_0) = M^+$$
$$\mathscr{F}(U_\infty) = M^-$$
$$\mathscr{F}(U_0 \cap U_\infty) = A[t,t^{-1}] \otimes_{A[t]} M^+ \simeq A[t,t^{-1}] \otimes_{A[t^{-1}]} M^-$$



Proposition 3.16. *Suppose* $(\sigma, \tau) \in M^+ \oplus M^-$, *then* $(\sigma, \tau) \in ker\ d$ *if and only if* $\sigma|_{U_{0\infty}} = \tau|_{U_{0\infty}}$.

Proof Suppose $(\sigma, \tau) \in ker\ d$. Then we know that $\theta(1 \otimes \sigma) = 1 \otimes \tau$ both of which are elements of $A[t, t^{-1}] \otimes_{A[t^{-1}]} \mathscr{F}(U_\infty)$. Hence as $A[t, t^{-1}] \otimes_{A[t^{-1}]} \mathscr{F}(U_\infty) \simeq \mathscr{F}(U_{0\infty})$, the image of $\theta(1 \otimes \sigma)$ and $1 \otimes \tau$ are equal in $\mathscr{F}(U_{0\infty})$. Hence, by the commutativity of the above diagram, $\sigma|_{U_{0\infty}} = \tau|_{U_{0\infty}}$. Conversely, the commutativity of the diagram implies that if $\sigma|_{U_{0\infty}} = \tau|_{U_{0\infty}}$ then $(\sigma, \tau) \in ker\ d$. □

Excitingly, due to this proposition, we now have a method of finding global sections. Finding global sections of $\mathbb{P}^1_{\mathbb{C}}$ amounts to understanding $ker\ d$ completely, since, using Proposition 3.16, the sheaf condition will give us a global section whenever we can find $\sigma \in M^+$, $\tau \in M^-$ with $(\sigma, \tau) \in ker\ d$. We now show that $ker\ d$ is a ring.

Proposition 3.17. *Suppose* $\mathscr{F}$ *is a quasi-coherent sheaf of rings on* $\mathbb{P}^1_{\mathbb{C}}$ *such that* $(\mathscr{F}(U_0), \mathscr{F}(U_\infty), \theta)$ *is an object in* $\mathscr{M}od(\mathbb{P}^1_A)$. *Suppose* d *is defined as in Definition 3.7. Then* $ker\ d$ *has a ring structure with addition defined by* $(\sigma_1, \tau_1) + (\sigma_2, \tau_2) := (\sigma_1 + \sigma_2, \tau_1 + \tau_2)$ *and multiplication defined by* $(\sigma_1, \tau_1).(\sigma_2, \tau_2) := (\sigma_1.\sigma_2, \tau_1.\tau_2)$.

Proof Suppose (σ_1, τ_1), $(\sigma_2, \tau_2) \in ker\ d$. Then $\theta(1 \otimes (\sigma_1 + \sigma_2)) = \theta(1 \otimes \sigma_1 + 1 \otimes \sigma_2) = 1 \otimes \tau_1 + 1 \otimes \tau_2 = 1 \otimes (\tau_1 + \tau_2)$. Thus $(\sigma_1 + \sigma_2, \tau_1 + \tau_2) \in ker\ d$. For closure under multiplication, since $U_0 \cap U_\infty \subseteq U_0$ and $U_0 \cap U_\infty \subseteq U_\infty$, we have restriction maps, which are also ring homomorphisms, $r^{\mathscr{F}}_{U_0 U_{0\infty}}$ and $r^{\mathscr{F}}_{U_\infty U_{0\infty}}$. Thus, since (σ_1, τ_1), $(\sigma_2, \tau_2) \in ker\ d$, by Proposition 3.16, we see that $r^{\mathscr{F}}_{U_0 U_{0\infty}}(\sigma_i) = r^{\mathscr{F}}_{U_\infty U_{0\infty}}(\tau_i)$ for $i = 1, 2$. This means that $r^{\mathscr{F}}_{U_0 U_{0\infty}}(\sigma_1 \sigma_2) = r^{\mathscr{F}}_{U_0 U_{0\infty}}(\sigma_1) r^{\mathscr{F}}_{U_0 U_{0\infty}}(\sigma_2) = r^{\mathscr{F}}_{U_\infty U_{0\infty}}(\tau_1) r^{\mathscr{F}}_{U_\infty U_{0\infty}}(\tau_2) = r^{\mathscr{F}}_{U_\infty U_{0\infty}}(\tau_1 \tau_2)$. Thus using Proposition 3.16 again, $(\sigma_1 \sigma_2, \tau_1 \tau_2) \in ker\ d$. With additive identity $(0, 0)$ and multiplicative identity $(1, 1)$, which exists because $\mathscr{F}(U_0)$, $\mathscr{F}(U_\infty)$ are both rings, we see that $ker\ d$ has a ring structure. □

We consider the Cech map for sheaves of twisted rings of differential operators, and use this map to compute the global sections of the sheaf. Now that we know that $ker\ d$ is a ring whenever d is the Cech map for a quasi-coherent sheaf of rings, we can find a set of elements that generate $ker\ d$. To find a suitable generating set, we need to analyse the structure of $ker\ d$ more.

For the remainder of this thesis let d_λ be the Cech map of $\mathscr{D}^\lambda$, so that we have the Cech complex of $\mathscr{D}^\lambda$,

$$0 \longrightarrow D(\mathbb{C}[t]) \oplus D(\mathbb{C}[t^{-1}]) \xrightarrow{d_\lambda} \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}]) \longrightarrow 0$$

with $d_\lambda(x, y) = \theta_\lambda(1 \otimes x) - 1 \otimes y$. Denote the global sections of $\mathscr{D}^\lambda$ by U_λ so that $\mathscr{D}^\lambda(\mathbb{P}^1_{\mathbb{C}}) = U_\lambda = ker\ d_\lambda$.

Corollary 3.18. *We can write $U_\lambda = R^\lambda_t \oplus R^\lambda_{t^{-1}}$ for subrings R^λ_t of $D(\mathbb{C}[t])$, $R^\lambda_{t^{-1}}$ of $D(\mathbb{C}[t^{-1}])$.*

Proof Since U_λ is a ring by Proposition 3.17, and the operations on U_λ are defined co-ordinate wise, the result follows. □

The generating set will be easier to describe if we can generate the ring R^λ_t and map the generators into $R^\lambda_{t^{-1}}$, not surprisingly this map is an isomorphism.

Proposition 3.19. *The gluing isomorphism, θ_λ, induces a ring isomorphism*

$$\tilde{\theta}_\lambda : R^\lambda_t \tilde{\rightarrow} R^\lambda_{t^{-1}}$$

Proof Firstly, we can embed R^λ_t into $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t])$ and $R^\lambda_{t^{-1}}$ into $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t^{-1}])$. Secondly, we note that since $U_\lambda = R^\lambda_t \oplus R^\lambda_{t^{-1}}$, $\theta_\lambda(1 \otimes R^\lambda_t) = 1 \otimes R^\lambda_{t^{-1}}$. Lastly, we note that the embedding of $R^\lambda_{t^{-1}}$ into $\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t])$ surjects onto the image of $1 \otimes R^\lambda_t$ under θ_λ. Hence we can define a ring isomorphism $\tilde{\theta}_\lambda : R^\lambda_t \tilde{\rightarrow} R^\lambda_{t^{-1}}$ which makes the following diagram commute,

$$\begin{array}{ccc} R^\lambda_t & \xrightarrow{\tilde{\theta}_\lambda} & R^\lambda_{t^{-1}} \\ \big\downarrow & & \big\downarrow \\ \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t]) & \xrightarrow{\theta_\lambda} & \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t^{-1}]) \end{array}$$

□

Hence it suffices to find a generating set just for R^λ_t. Before we can find this set, we should obtain a clear understanding of the nature of the elements of R^λ_t. Suppose $f(t, \delta) \in R^\lambda_t$, then $f(t, \delta) \in D(\mathbb{C}[t])$, so by Theorem 1.14, $f(t, \delta) \in A_1(\mathbb{C})$, the first Weyl algebra. Recall that $A_1(\mathbb{C})$ has a basis $B = \{t^a\delta^b \mid a, b \geq 0\}$ by Proposition 1.10. Hence, we can assume that,

$$f(t, \delta) = \sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{ij} t^i \delta^j$$

with at least one $\alpha_{mj} \neq 0$ for $j = 1, \ldots, n$ and at least one $\alpha_{in} \neq 0$ for $i = 1, \ldots, m$. This guarantees that the values we use for m and n are unique.

We need to define a degree function on elements of R^λ_t, as we will use it in an inductive proof shortly. To define a degree that works for us, we first consider terms with the greatest power of δ and then amongst these terms we consider the term with the greatest power of t.

Definition 3.20. *Define the* **degree** *of* $f(t,\delta) = \sum_{i=0}^{m}\sum_{j=0}^{n} \alpha_{ij} t^i \delta^j \in D(\mathbb{C}[t])$ *to be, deg* $f = (l,n) \in \mathbb{N}\times\mathbb{N}$, *where* n *is the greatest power of* δ *in all the terms of* f, f *contains a term of the form* $t^l\delta^n$, *and* f *contains no terms of the form* $t^i\delta^n$ *with* $i > l$. *Define the following* **order** *on elements of* $D(\mathbb{C}[t])$.

$$(m,n) \leq (p,q) \Leftrightarrow n < q \text{ or } (m \leq p \text{ if } n = q)$$

This order is a reverse lexicographic order, and since $\mathbb{N}$ is totally ordered, we can see that this is actually a total order of elements of $D(\mathbb{C}[t])$.

We haven't used the fact that if $f \in R_t^\lambda$, then $\tilde{\theta}_\lambda(f(t,\delta)) \in D(\mathbb{C}[t^{-1}])$. In the next proposition, we see how this defining property affects f.

Proposition 3.21. *If* $f(t,\delta) = \sum_{i=0}^{m}\sum_{j=0}^{n} \alpha_{ij} t^i \delta^j \in R_t^\lambda$ *then* $m \leq 2n$.

Proof Evidently,

$$\begin{aligned} \tilde{\theta}_\lambda(f(t,\delta)) &= \tilde{\theta}_\lambda(\sum_{i=0}^{m}\sum_{j=0}^{n} \alpha_{ij} t^i \delta^j) \\ &= \sum_{i=0}^{m}\sum_{j=0}^{n} \alpha_{ij} t^i (\delta + \lambda t^{-1})^j \end{aligned}$$

Consider all the non-zero elements of the form α_{mj} with $j = 1,\ldots,n$. Suppose that the term such that j is maximal is where $j = k$. That is, $\alpha_{mk} \neq 0$ and $\alpha_{mj} = 0$ for all $j = 1,\ldots,n$ with $j > k$. This produces a term, $\alpha_{mk}t^m(\delta + \lambda t^{-1})^k$, upon which after expanding we obtain the summand $\alpha_{mk}t^m\delta^k$. Recall that $D(\mathbb{C}[t^{-1}])$ is generated by $\{t^{-1}, -t^2\delta\}$ over $\mathbb{C}$. We use $\delta t = t\delta + 1$ to move all the t's to the left of the δ's. Furthermore, using $\delta t = t\delta + 1$ only adds terms of less than or equal degree. Thus we can see that the term containing $t^i\delta^k$ with i maximal in $D(\mathbb{C}[t])$ is

$$(t^2\delta)^k = t^{2k}\delta^k + \text{terms of lower degree}$$

This implies that $m \leq 2k$. Thus $m \leq 2n$. □

We are now ready to define the set of generators for R_t^λ. In the following Lemma, we prove that R_t^λ does in fact contain these generators.

Lemma 3.22. *The subring of $D(\mathbb{C}[t])$ generated by,*

$$X := \{-2t\delta + \lambda, -\delta, t^2\delta - \lambda t\}$$

is contained in R_t^λ.

Proof We explicitly compute $\tilde{\theta}_\lambda(x)$ for all $x \in X$.

$$\begin{aligned}
\tilde{\theta}_\lambda(-2t\delta + \lambda) &= -2t(\delta + \lambda t^{-1}) + \lambda \\
&= -2t\delta - \lambda \in D(\mathbb{C}[t^{-1}]) \\
\tilde{\theta}_\lambda(-\delta) &= -\delta - \lambda t^{-1} \in D(\mathbb{C}[t^{-1}]) \\
\tilde{\theta}_\lambda(t^2\delta - \lambda t) &= t^2(\delta + \lambda t^{-1}) - \lambda t \\
&= t^2\delta \in D(\mathbb{C}[t^{-1}])
\end{aligned}$$

Hence $X \subseteq R_t^\lambda$. □

Theorem 3.23. *R_t^λ is generated by the set X.*

Proof We prove the result by using induction on the order of f. For the base case suppose $deg\ f = (0,0)$. Then f is a constant function that is trivially generated by X (we only listed the non-trivial generators in X). For the induction step, suppose $deg\ f = (l, n)$ with $0 \leq l \leq m \leq 2n$ and all elements of R_t^λ of lower degree are generated by X. So we can assume that $f(t,\delta) = \sum_{i=0}^m \sum_{j=0}^n \alpha_{ij} t^i \delta^j$ and there are no terms of the form $t^i\delta^n$ with $l < i \leq m$. This means that

$$\tilde{\theta}_\lambda(f(t,\delta)) = \sum_{i=0}^m \sum_{j=0}^n \alpha_{ij} t^i (\delta + \lambda t^{-1})^j = \alpha_{ln} t^l \delta^n + \text{terms of lower degree}$$

Now, we know that $l \leq m \leq 2n$. Thus consider the following element of $D(\mathbb{C}[t])$ generated by X in each case,

$$f_1(t,\delta) := \begin{cases} \alpha_{ln}(-2t\delta + \lambda)^l(-\delta)^{n-l}(-1)^{n-l}(\frac{-1}{2})^l & \text{if } l \leq n \\ \alpha_{ln}(t^2\delta - \lambda t)^{\frac{l}{2}}(-\delta)^{n-\frac{l}{2}}(-1)^{n-\frac{l}{2}} & \text{if } n < l \leq 2n \text{ and } l \text{ even} \\ \alpha_{ln}(-2t\delta + \lambda)(t^2\delta - \lambda t)^{\frac{l-1}{2}}(\delta)^{n-\frac{l+1}{2}}(\frac{-1}{2}) & \text{if } n < l \leq 2n \text{ and } l \text{ odd} \end{cases}$$

In all cases, upon expansion and simplification using $t^{-1}\delta - \delta t^{-1} = t^{-2}$,

$$f_1(t,\delta) = \alpha_{ln} t^l \delta^n + \text{terms of lower degree}$$

and also by Lemma 3.22, $\tilde{\theta}_\lambda(f_1(t,\delta)) \in D(\mathbb{C}[t^{-1}])$. Furthermore, $deg(f - f_1) < deg\ f$ so $f - f_1$ is generated by elements of X by the inductive hypothesis. Thus $f = (f - f_1) + f_1$ is generated by elements of X. □

In this section we have used Swan's technique to analyse the global sections for sheaves of twisted rings of differential operators. We have discovered that the global sections are intimately associated to the kernel of the Cech map, U_λ, that the global sections formed a ring, and that this ring was generated by the following set of elements.

$$\left\{(-2t\delta + \lambda, -2td - \lambda), (-\delta, -\delta - \lambda t^{-1}), (t^2\delta - \lambda t, t^2\delta)\right\}$$

What remains, is to show that the only relations between these elements are the ones defined by $D(\mathbb{C}[t])$ and $D(\mathbb{C}[t^{-1}])$. That is, the relations $\delta t - t\delta = 1$ and $t^{-1}\delta - \delta t^{-1} = t^{-2}$. To show this, we prove a remarkable result, which connects our global sections, U_λ, to the *special linear Lie algebra*, $\mathfrak{sl}_2(\mathbb{C})$. This result is one of the most fascinating results of this thesis, forming a bridge between two areas of mathematics that didn't seem to have any obvious connections.

3.4 Universal Enveloping Algebras

Sophus Lie, a Norwegian mathematician from the 19th-century, essentially single-handedly discovered two classes of objects in modern mathematics and found connections between them. These are Lie groups, and Lie algebras. Lie groups are most commonly found in differential geometry, whereas Lie algebras seem to be purely algebraic. In this section we will describe the *special linear Lie algebra*, $\mathfrak{sl}_2(\mathbb{C})$, and also its *universal enveloping algebra*. The universal enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$, is the Lie algebra equivalent of the group algebra for a group. The universal enveloping algebra will help us to explicitly describe the global sections of $\mathscr{D}^\lambda$, U_λ. As a reference for this section, see [13].

Proposition 3.24. *If x, y, z are elements of an associative algebra, and $[\cdot,\cdot]$ is the commutator bracket map defined in chapter 1, then,*

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

Proof The result follows from simply expanding the left hand side. □

Definition 3.25. *A* ***Lie algebra over*** $\mathbb{C}$ *is a vector space* $\mathfrak{g}$ *over* $\mathbb{C}$ *with a* ***Lie bracket*** *map* $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, *written* $(x,y) \mapsto [x,y]$ *satisfying the following properties.*

1. *The Lie bracket is bilinear, that is, for all $a, b \in \mathbb{C}, x, y, z \in \mathfrak{g}$,*

$$[ax + by, z] = a[x, z] + b[y, z]$$
$$[x, ay + bz] = a[x, y] + b[x, z]$$

2. *The Lie bracket is skew-symmetric, that is* $[y, x] = -[x, y]$ *for all* $x, y \in \mathfrak{g}$
3. *The Lie bracket satisfies the Jacobi identity, that is,*

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0, \text{ for all } x, y, z \in \mathfrak{g}$$

Note here the Lie bracket map is not necessarily the commutator bracket map. Proposition 3.24 tells us that the commutator bracket satisfies the requirements of a Lie bracket. This tells us that every associative $\mathbb{C}$-algebra can be made into a Lie-algebra by using the commutator bracket.

Proposition 3.26. *An associative* $\mathbb{C}$*-algebra* R *becomes a Lie algebra if we use the commutator* $[x, y] = xy - yx$ *as the Lie Bracket.*

Proof We've checked the Jacobi identity in Proposition 3.24. Bilinearity follows from bilinearity of multiplication in R, and skew-symmetry is obvious from the definition. □

Definition 3.27. *A* ***linear Lie algebra*** *is a subspace of* $M_n(\mathbb{C})$*, the set of* $n \times n$ *matrices with complex entries, for some* $n > 0$*, closed under the commutator bracket map.*

Example 3.28. Corresponding to various subgroups of $GL_n(\mathbb{C})$, the set of invertible matrices in $M_n(\mathbb{C})$, we have the ***general linear Lie algebra***

$$\mathfrak{gl}_n(\mathbb{C}) = M_n(\mathbb{C}) \text{ (considered as a Lie algebra)}$$

the ***special linear Lie algebra***

$$\mathfrak{sl}_n(\mathbb{C}) = \{x \in \mathfrak{gl}_n(\mathbb{C}) \mid tr(x) = 0\}$$

and the ***special orthogonal Lie algebra***

$$\mathfrak{so}_n(\mathbb{C}) = \{x \in \mathfrak{gl}_n(\mathbb{C}) \mid x^t = -x\}$$

To prove that these are in fact Lie algebras amounts to a few simple calculations. Since for us the field will always be $\mathbb{C}$, we will simply write $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, and $\mathfrak{so}_n$. We are most interested in $\mathfrak{sl}_2$, so we only show that $\mathfrak{sl}_2$ is closed under commutator bracket. Suppose $x, y \in \mathfrak{sl}_2$, then we know that $tr(xy) = tr(yx)$. Hence $tr([x, y]) = tr(xy - yx) = tr(xy) - tr(yx) = 0$. Since $\mathfrak{sl}_2$ is a subspace of the general linear

Lie algebra $\mathfrak{gl}_2$ (which is a Lie algebra by Proposition 3.26), it inherits the three properties of the Lie bracket on $\mathfrak{gl}_2$.

Definition 3.29. *The* ***standard basis*** *of* $\mathfrak{sl}_2$ *as a vector space over* $\mathbb{C}$ *is* $\{e, h, f\}$ *where*

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Proposition 3.30. *The following commutator relations hold in* $\mathfrak{sl}_2$.

$$\begin{aligned} [e, f] &= h \\ [h, e] &= 2e \\ [h, f] &= -2f \end{aligned}$$

The proof of this proposition amounts to a few simple matrix calculations, so we omit it here. We are now ready to define the universal enveloping algebra of $\mathfrak{sl}_2$. We can define the universal enveloping algebra for an arbitrary Lie algebra, $\mathfrak{g}$, but this is unnecessary since we only need that of $\mathfrak{sl}_2$.

Definition 3.31. *The* ***universal enveloping algebra*** *of* $\mathfrak{sl}_2$, *denoted by* $U(\mathfrak{sl}_2)$, *is the* $\mathbb{C}$*-algebra generated by the three symbols* $\{e, h, f\}$ *subject to the relations,*

$$ef - fe = h$$
$$he - eh = 2e$$
$$hf - fh = -2f$$

It is vital to realise that in this definition, we are no longer thinking of e, h, f as matrices. In $U(\mathfrak{sl}_2)$, these are just formal symbols, and the only information that they carry into the universal enveloping algebra are the Lie bracket relations. Another interesting fact is that elements like ef, h^2, fe are elements of $U(\mathfrak{sl}_2)$ but not elements of $\mathfrak{sl}_2$ as they are not of trace 0. Universal enveloping algebras are to Lie algebras as the group algebra (which allows one to form a module from a group representation) is to a group, that is, representations of a Lie algebra $\mathfrak{g}$ are the same as modules of the universal enveloping algebra $U(\mathfrak{g})$. The next proposition will allow us to be more familiar with elements of $U(\mathfrak{sl}_2)$, as we show that arbitrary elements of the algebra can be simplified into a neat structural form.

Proposition 3.32. *Suppose $x \in U(\mathfrak{sl}_2)$, then x can be written in the form*

$$\sum_{i=j=k=0}^{i+j+k=n} \alpha_{ijk} e^i h^j f^k \text{ for } \alpha_{ijk} \in \mathbb{C} \tag{3.1}$$

Proof We begin by defining an order on elements of $U(\mathfrak{sl}_2)$. Let $ord\ x =$ maximum total degree over all the terms in x. We will prove the result via induction on $ord\ x$. Suppose $ord\ x = 0$, then x is trivially in the form (3.1). Suppose all elements of order less than n are in the form (3.1), and take an element $x \in U(\mathfrak{sl}_2)$ with $ord\ x = n$. Then there exists terms with e, h, f in no particular order such that the total degree of the term is n. We can use $he = eh + 2e$ to move all the e's to the left of the $h's$ and introduce terms of lower total degree. Thus at this stage, all terms of order n can be put in the form where all e's are to the left of the h's with f's dispersed randomly in the term. We can then use $fe = ef + h$ to move all the e's to the left of the f's. Although we introduce h's with this relation, they are will be in terms of lower total order, so they don't concern us. Lastly, we can use $fh = hf + 2f$ to move all the h's to the left of the f's. This means that we can use the defining relations of $U(\mathfrak{sl}_2)$ to write x in the form,

$$x = \sum_{i+j+k=n} \alpha_{ijk} e^i h^j f^k + \text{terms of lower total order}$$

The inductive assumption implies that we can write x in the form (3.1). □

Understanding how to neatly describe each element of $U(\mathfrak{sl}_2)$ is a fundamental tool that we will need in the next section. The last concept related to $U(\mathfrak{sl}_2)$ that we need to cover is a special element called the *casimir* element of $U(\mathfrak{sl}_2)$. It is special because it is contained by the centre of $U(\mathfrak{sl}_2)$.

Proposition 3.33. *The casimir element of $U(\mathfrak{sl}_2)$, $\Delta = (h-1)^2 + 4ef$ commutes with every element of $U(\mathfrak{sl}_2)$.*

Proof It suffices to show that Δ commutes with only e, h, f. We only show that Δ commutes with e here as the other two follow from similar calculations.

$$\begin{aligned}
[\Delta, e] &= \Delta e - e\Delta \\
&= (h-1)^2 e + 4efe - e(h-1)^2 - 4e^2 f \\
&= h^2 e - 2he + 4efe - eh^2 + 2eh - 4e^2 f \\
&= heh - 2eh - eh^2 \\
&= (eh + 2e)h - 2eh - eh^2 \\
&= 0
\end{aligned}$$

□

In this section we have defined the universal enveloping algebra of the special linear Lie algebra, $U(\mathfrak{sl}_2)$. It will appear in one of the major results for this thesis, and it will allow us to explicitly compute the global sections of $\mathscr{D}^\lambda$ on $\mathbb{P}^1_{\mathbb{C}}$, U_λ.

3.5 Global Sections of $\mathscr{D}^\lambda$ on $\mathbb{P}^1_{\mathbb{C}}$

We are finally ready to explicitly compute the global sections of $\mathscr{D}^\lambda$ on $\mathbb{P}^1_{\mathbb{C}}$. This section provides a remarkable connection between universal enveloping algebras and the global sections of sheaves of twisted rings of differential operators. In section 3.3, we found that we could define a generating set for the global sections, U_λ, for each twist $\lambda \in \mathbb{C}$. We still need to prove that there are no other relations in U_λ. This is done by finding a mapping from $U(\mathfrak{sl}_2)$ to U_λ.

Since $U(\mathfrak{sl}_2)$ is a $\mathbb{C}$-algebra, we can think of it as a ring.

Proposition 3.34. *There is a surjective ring homomorphism over* $\mathbb{C}$,

$$\Psi_\lambda : U(\mathfrak{sl}_2) \to U_\lambda$$

Proof We need a mapping into U_λ. As we saw in section 3.3, it suffices to find a mapping into R_t^λ. This is because if we have a surjective ring homomorphism $\tilde{\Psi}_\lambda : U(\mathfrak{sl}_2) \to R_t^\lambda$ over $\mathbb{C}$, then we can define Ψ_λ by,

$$\Psi_\lambda(x) := (\tilde{\Psi}_\lambda(x), (\tilde{\theta}_\lambda \circ \tilde{\Psi}_\lambda)(x))$$

We define $\tilde{\Psi}_\lambda$ by giving the image of the generators e, h, f. Define $\tilde{\Psi}_\lambda$ by sending,

$$\begin{aligned} h &\mapsto -2t\delta + \lambda \\ e &\mapsto -\delta \\ f &\mapsto t^2\delta - \lambda t \end{aligned}$$

For $\tilde{\Psi}_\lambda$ to be well defined, it must be compatible with the defining relations of $U(\mathfrak{sl}_2)$. We carry out the calculations here.

$$\begin{aligned} \tilde{\Psi}_\lambda(ef - fe - h) &= (-\delta)(t^2\delta - \lambda t) - (t^2\delta - \lambda t)(-\delta) - (-2t\delta + \lambda) \\ &= -\delta t^2\delta + \lambda\delta t + t^2\delta^2 - \lambda t\delta + 2t\delta - \lambda \\ &= -(1 + t\delta)t\delta + t^2\delta^2 + 2t\delta \\ &= t^2\delta^2 + t\delta - t(1 + t\delta)\delta \\ &= 0 \end{aligned}$$

$$\begin{aligned} \tilde{\Psi}_\lambda(he - eh - 2e) &= (-2t\delta + \lambda)(-\delta) - (-\delta)(-2t\delta + \lambda) - 2(-\delta) \\ &= 2t\delta^2 - \lambda\delta - 2\delta t\delta + \lambda\delta + 2\delta \\ &= 2t\delta^2 - 2(1 + t\delta)\delta + 2\delta \\ &= 0 \end{aligned}$$

$$\begin{aligned} \tilde{\Psi}_\lambda(fh - hf - 2f) &= (t^2\delta - \lambda t)(-2t\delta + \lambda) - (-2t\delta + \lambda)(t^2\delta - \lambda t) - 2(t^2\delta - \lambda t) \\ &= -2t^2\delta t\delta + \lambda t^2\delta + 2\lambda t^2\delta - \lambda^2 t + 2t\delta t^2\delta \\ &\qquad - 2\lambda t\delta t - \lambda t^2\delta + \lambda^2 t - 2t^2\delta + 2\lambda t \\ &= -2t^2\delta t\delta + 2\lambda t^2\delta + 2t\delta t^2\delta - 2\lambda t\delta t - 2t^2\delta + 2\lambda t \\ &= -2t^2\delta t\delta + 2\lambda t(t\delta - \delta t) + 2t\delta t^2\delta - 2t^2\delta + 2\lambda t \\ &= 2t\delta t^2\delta - 2t^2\delta - 2t^2\delta t\delta \\ &= 2t\delta t^2\delta - 2t^2\delta(1 + t\delta) \\ &= 2t\delta t^2\delta - 2t^2\delta^2 t \\ &= 2t(\delta t^2\delta - t\delta^2 t) \\ &= 2t((1 + t\delta)(\delta t - 1) - t\delta^2 t) \\ &= 2t(\delta t - t\delta - 1) \\ &= 0 \end{aligned}$$

By Theorem 3.23, R_t^λ is generated by $\{-2t\delta + \lambda, -\delta, t^2\delta - \lambda t\}$, which are precisely the images of $\{h, e, f\}$ under $\tilde{\Psi}_\lambda$. Hence Ψ_λ defined as above is a surjective ring homomorphism over $\mathbb{C}$ from $U(\mathfrak{sl}_2) \to U_\lambda$. □

Our aim is to find the kernel of Ψ_λ, and then use the first isomorphism theorem for rings. This will give us an isomorphic copy of U_λ, in the form of a quotient of $U(\mathfrak{sl}_2)$, and this will allow us to deduce that there are in fact no extra relations in U_λ except for the defining relations in $D(\mathbb{C}[t])$ and $D(\mathbb{C}[t^{-1}])$.

Theorem 3.35. *Consider the sheaf of twisted rings of differential operators on $\mathbb{P}^1_{\mathbb{C}}$, $\mathscr{D}^\lambda$, with twist $\lambda \in \mathbb{C}$. Then the global sections of $\mathscr{D}^\lambda$ are given by the following isomorphism of rings,*

$$\mathscr{D}^\lambda(\mathbb{P}^1_{\mathbb{C}}) = U_\lambda \simeq \frac{U(\mathfrak{sl}_2)}{(\Delta - (\lambda+1)^2)}$$

Where Δ is the casimir element of $U(\mathfrak{sl}_2)$, that is, $\Delta = (h-1)^2 + 4ef$.

Proof As in the previous proposition, we focus our attention on the ring homomorphism, $\tilde{\Psi}_\lambda : U(\mathfrak{sl}_2) \to R_t^\lambda$. We first show that $\Delta - (\lambda+1)^2$ is in the kernel of $\tilde{\Psi}_\lambda$.

$$\begin{aligned}
\tilde{\Psi}_\lambda\left(\Delta - (\lambda+1)^2\right) &= \tilde{\Psi}_\lambda\left((h-1)^2 + 4ef - (\lambda+1)^2\right) \\
&= (-2t\delta + \lambda - 1)^2 + 4(-\delta)(t^2\delta - \lambda t) - (\lambda+1)^2 \\
&= 4t\delta t\delta - 4(\lambda-1)t\delta + (\lambda-1)^2 - 4\delta t^2\delta + 4\lambda\delta t - (\lambda+1)^2 \\
&= 4t\delta t\delta + 4t\delta - 4\delta t^2\delta + 4\lambda(\delta t - t\delta) + (\lambda-1)^2 - (\lambda+1)^2 \\
&= -4(\delta t - t\delta - 1)t\delta + 4\lambda + (\lambda-1)^2 - (\lambda+1)^2 \\
&= 0
\end{aligned}$$

We now show that $\Delta - (\lambda+1)^2$ is in fact the only generator of the kernel of $\tilde{\Psi}_\lambda$. We prove this by showing that if $x \in ker\ \tilde{\Psi}_\lambda$, then $x = 0$ in $\frac{U(\mathfrak{sl}_2)}{(\Delta-(\lambda+1)^2)}$. Suppose $x \in ker\ \tilde{\Psi}_\lambda$, then by Proposition 3.32 we can write x as

$$x = \sum_{i=j=k=0}^{i+j+k=n} \alpha_{ijk} e^i h^j f^k$$

with $\alpha_{ijk} \in \mathbb{C}$ for all i, j, k. Furthermore, we can use $\Delta - (\lambda+1)^2 = 0$ as we are working in $\frac{U(\mathfrak{sl}_2)}{(\Delta-(\lambda+1)^2)}$. This gives us a fourth relation to work with, and we see that $\Delta - (\lambda+1)^2 = 0$ simplifies to $h^2 = 2h - 4ef + \lambda^2 + 2\lambda$. This means that we can assume that $j = 0, 1$ in x (after some rearranging).

We now prove that if $x \in ker\ \tilde{\Psi}_\lambda$ then x must be zero in $\frac{U(\mathfrak{sl}_2)}{(\Delta-(\lambda+1)^2)}$. Recall the degree and total order defined on elements of $D(\mathbb{C}[t])$ in Definition 3.20. We will prove this by induction on $deg\ \tilde{\Psi}_\lambda(x)$. For the base case, suppose $deg\ \tilde{\Psi}_\lambda(x) = (0,0)$. Then $\tilde{\Psi}_\lambda(x)$ is a constant polynomial, so $\tilde{\Psi}_\lambda(x) = x = 0$, so x is trivially zero in $\frac{U(\mathfrak{sl}_2)}{(\Delta-(\lambda+1)^2)}$. For the induction step, suppose $deg\ \tilde{\Psi}_\lambda(x) = (l,n)$, then

$$\tilde{\Psi}_\lambda(x) = \sum_{i=k=0,j=0,1}^{i+j+k=n} \alpha_{ijk}(-\delta)^i(-2t\delta+\lambda)^j(t^2\delta-\lambda t)^k = 0 \tag{3.2}$$

with $\alpha_{ijk} \in \mathbb{C}$ for all i,j,k. We know that we can rewrite $\tilde{\Psi}_\lambda(x)$ in the form

$$\tilde{\Psi}_\lambda(x) = \sum_{r=0}^{m}\sum_{s=0}^{n} \beta_{rs} t^r \delta^s = 0$$

where each β_{rs} is a polynomial in the α_{ijk}'s. Thus we can denote each $\beta_{rs} = \beta_{rs}(\alpha_{ijk})$. Since $\tilde{\Psi}_\lambda(x) = 0$, each coefficient, $\beta_{rs}(\alpha_{ijk}) = 0$ as terms of the form $t^r\delta^s$ are a basis for $D(\mathbb{C}[t])$ as a vector space over $\mathbb{C}$ (by Proposition 1.10 and Theorem 1.14). This means that each coefficient of $t^r\delta^s$, forms a relation on α_{ijk}. We focus on the relation produced by $\beta_{ln}(\alpha_{ijk}) = 0$.

Since $deg\ \tilde{\Psi}_\lambda(x) = (l,n)$, $\tilde{\Psi}_\lambda(x)$ contains a term in $t^l\delta^n$ but no terms of the form $t^r\delta^n$ for $r > l$. Consider terms of maximal degree in (3.2), that is, terms containing $t^l\delta^n$ after simplification by $\delta t - t\delta = 1$. Since $j = 0,1$ we see there is only one way to create a term in $t^l\delta^n$. The reason is, if l were odd then we have no choice but to use $j = 1$ as we will only obtain a term with an even power of t with $j = 0$, thus we are forced to use $k = \frac{l-1}{2}$ because there is no other way to obtain a term in t^l. This implies that we must use $i = n - \frac{l+1}{2}$ so that we obtain a term in $t^l\delta^n$. We also see that i must be non-negative since $l \leq 2n$ by Proposition 3.21 and the fact that $l \leq m$. The argument for l even is similar. Thus, the following element of $\frac{U(\mathfrak{sl}_2)}{(\Delta-(\lambda+1)^2)}$ must be a summand in x,

$$x_1 := \begin{cases} \alpha_{(n-\frac{l+1}{2})(1)(\frac{l-1}{2})} e^{n-\frac{l+1}{2}} h f^{\frac{l-1}{2}} & \text{if } l \text{ is odd} \\ \alpha_{(n-\frac{l}{2})(0)(\frac{l}{2})} e^{n-\frac{l}{2}} f^{\frac{l}{2}} & \text{if } l \text{ is even} \end{cases}$$

This means that,

$$\begin{aligned}\tilde{\Psi}_\lambda(x_1) &:= \begin{cases} \alpha_{(n-\frac{l+1}{2})(1)(\frac{l-1}{2})}(-\delta)^{n-\frac{l+1}{2}}(-2t\delta+\lambda)(t^2\delta-\lambda t)^{\frac{l-1}{2}} & \text{if } l \text{ is odd} \\ \alpha_{(n-\frac{l}{2})(0)(\frac{l}{2})}(-\delta)^{n-\frac{l}{2}}(t^2\delta-\lambda t)^{\frac{l}{2}} & \text{if } l \text{ is even} \end{cases} \\ &= \beta_{ln}t^l\delta^n + \text{terms of lower degree}\end{aligned}$$

What this tells us is that we can now compute β_{ln} as a polynomial in the α_{ijk}'s. Thus we can see that

$$\beta_{ln} = \begin{cases} \alpha_{(n-\frac{l+1}{2})(1)(\frac{l-1}{2})}(-1)^{n-\frac{l+1}{2}}(-2) & \text{if } l \text{ is odd} \\ \alpha_{(n-\frac{l}{2})(0)(\frac{l}{2})}(-1)^{n-\frac{l}{2}} & \text{if } l \text{ is even} \end{cases}$$

Since $\beta_{ln} = 0$ we see that,

$$0 = \begin{cases} \alpha_{(n-\frac{l+1}{2})(1)(\frac{l-1}{2})}(-1)^{n-\frac{l+1}{2}}(-2) & \text{if } l \text{ is odd} \\ \alpha_{(n-\frac{l}{2})(0)(\frac{l}{2})}(-1)^{n-\frac{l}{2}} & \text{if } l \text{ is even} \end{cases}$$

This tells us that $x_1 = 0$ in $\frac{U(\mathfrak{sl}_2)}{(\Delta-(\lambda+1)^2)}$. Hence, as $x \in ker\ \tilde{\Psi}_\lambda$,

$$0 = \tilde{\Psi}_\lambda(x) = \tilde{\Psi}_\lambda(x - x_1) + \tilde{\Psi}_\lambda(x_1) = \tilde{\Psi}_\lambda(x - x_1)$$

So $x - x_1 \in ker\ \tilde{\Psi}_\lambda$, but $\tilde{\Psi}_\lambda(x - x_1)$ is of degree less than (l, n), so by the inductive assumption, $x - x_1 = 0$ in $\frac{U(\mathfrak{sl}_2)}{(\Delta-(\lambda+1)^2)}$ as well. Thus, in $\frac{U(\mathfrak{sl}_2)}{(\Delta-(\lambda+1)^2)}$,

$$x = (x - x_1) + x_1 = 0 + 0 = 0$$

This proves that if $x \in ker\ \tilde{\Psi}_\lambda$ then $x = 0$ in $\frac{U(\mathfrak{sl}_2)}{(\Delta-(\lambda+1)^2)}$, that is, then x is in the ideal generated by $\Delta - (\lambda+1)^2$ in $U(\mathfrak{sl}_2)$. This shows us that $ker\ \tilde{\Psi}_\lambda = (\Delta - (\lambda+1)^2)$.

The first isomorphism theorem for rings then implies that,

$$U_\lambda = R_t^\lambda \oplus R_{t^{-1}}^\lambda \simeq R_t^\lambda \simeq \frac{U(\mathfrak{sl}_2)}{(\Delta - (\lambda+1)^2)}$$

□

In this chapter we have described a fascinating property of differential operators on $\mathbb{P}^1_{\mathbb{C}}$. In section 3.2, we found out that every sheaf of rings of differential operators on $\mathbb{P}^1_{\mathbb{C}}$ could be viewed as a sheaf of twisted rings of differential operators. In section 3.3 we described how obtaining global sections amounted to finding U_λ, the kernel of the Cech map on $\mathscr{D}^\lambda$. In this section, we discovered that the kernel of the Cech map is in fact isomorphic to a quotient of $U(\mathfrak{sl}_2)$, generated by the casimir element and a constant depending on the twist, λ. Specifically we see that the isomorphism is $\Psi_\lambda : U(\mathfrak{sl}_2) \to U_\lambda$, defined by,

$$\begin{aligned} h &\mapsto (-2t\delta + \lambda, -2t\delta - \lambda) \\ e &\mapsto (-\delta, -\delta - \lambda t^{-1}) \\ f &\mapsto (t^2\delta - \lambda t, t^2\delta) \end{aligned}$$

Chapter 4

The Beilinson-Bernstein Theorem and the Translation Principle

The much celebrated Beilinson-Bernstein Theorem was proved in 1981. It advanced research in the area of algebraic geometry, in particular the study of the geometry of flag varieties with representation theory and $\mathscr{D}$-modules (see [12]). In the first section of this chapter we will cover the necessary category theory to understand the Beilinson-Bernstein Theorem. We will see that the *global sections functor* plays an important role in this theorem. After understanding the theorem, we will demonstrate the usefulness of the global sections functor, as it will give us a geometric construction of finite dimensional irreducible $U(\mathfrak{sl}_2)$-modules. We will then end the chapter by demonstrating the usefulness of the Beilinson-Bernstein Theorem, as we prove the *translation principle* for U_λ-modules. This will give us a greater appreciation for the module categories produced by the global sections of $\mathscr{D}^\lambda_{\mathbb{P}^1_\mathbb{C}}$.

4.1 Equivalent Categories and the Beilinson-Bernstein Theorem

Recall that in the definition of a category $\mathscr{C}$, there is an identity existence condition that states that for each $\mathscr{C}$-object A, there exists a $\mathscr{C}$-morphism e such that $dom(e) = A = cod(e)$ and

1. $f \circ e = f$ whenever $f \circ e$ is defined
2. $e \circ g = g$ whenever $e \circ g$ is defined

We will now show that this morphism e is unique. For more details on any of the category theory in this chapter see [16].

Proposition 4.1. *Let $\mathscr{C}$ be a category and A be a $\mathscr{C}$-object. Then there exists exactly one $\mathscr{C}$-morphism $e : A \to A$ such that*

1. $f \circ e = f$ whenever $f \circ e$ is defined

2. *$e \circ g = g$ whenever $e \circ g$ is defined*

Proof Suppose that e and $\tilde{e}$ are such morphisms. Then by condition (1), $\tilde{e} \circ e = \tilde{e}$ and by condition (2), $\tilde{e} \circ e = e$. Thus $\tilde{e} = e$. □

Definition 4.2. *For each object A of a category $\mathscr{C}$, the unique $\mathscr{C}$-morphism $e : A \to A$ from Proposition 4.1 is denoted by 1_A and is called the* ***$\mathscr{C}$-identity*** *of A.*

Proposition 4.1 implies that there is a one-to-one correspondence between $\mathscr{C}$-objects and $\mathscr{C}$-identities, namely, $A \leftrightarrow 1_A$.

Definition 4.3. *Let $\mathscr{B}$ and $\mathscr{C}$ be categories. A* ***functor from $\mathscr{B}$ to $\mathscr{C}$*** *is a triple $(\mathscr{B}, F, \mathscr{C})$ where F is a function $F : Mor(\mathscr{B}) \to Mor(\mathscr{C})$ satisfying the following conditions,*

1. *F* ***preserves identities****; if e is a $\mathscr{B}$-identity, then $F(e)$ is a $\mathscr{C}$-identity*
2. *F* ***preserves composition****; $F(f \circ g) = F(f) \circ F(g)$ for $f, g \in Mor(\mathscr{B})$*

Instead of writing $(\mathscr{B}, F, \mathscr{C})$ is a functor, it is more common to write $F : \mathscr{B} \to \mathscr{C}$ or to simply say that F is a functor from $\mathscr{B}$ to $\mathscr{C}$.

Proposition 4.4. *Suppose F is a functor from $\mathscr{B}$ to $\mathscr{C}$. Then F induces a unique function $Ob(\mathscr{B}) \to Ob(\mathscr{C})$ which we also denote by F (we are abusing notation) such that $F(1_A) = 1_{F(A)}$ for all $A \in Ob(\mathscr{B})$. Furthermore, $F[hom_{\mathscr{B}}(A, B)] \subseteq hom_{\mathscr{C}}(F(A), F(B))$.*

Proof Because functors preserve identities and there is a one-to-one correspondence between objects and identities ($A \leftrightarrow 1_A$), we see that a functor $F : \mathscr{B} \to \mathscr{C}$ induces a unique mapping $Ob(\mathscr{B}) \to Ob(\mathscr{C})$ that makes the following diagram commute,

$$\begin{array}{ccc} 1_A & \xrightarrow{F} & 1_{F(A)} \\ \updownarrow & & \updownarrow \\ A & \overset{F}{\dashrightarrow} & F(A) \end{array}$$

Suppose $\sigma \in hom_{\mathscr{B}}(A, B)$ then $F(\sigma) \in Mor_{\mathscr{C}}(C, D)$ for some $C, D \in Ob(\mathscr{C})$. Since F preserves composition, $F(1_B \circ \sigma) = 1_{F(B)} \circ F(\sigma)$ so $D = F(B)$. Similarly, $F(\sigma \circ 1_A) = F(\sigma) \circ 1_{F(A)}$ so $C = F(A)$. □

Just as morphisms are injective or surjective, we want a similar concept of injectivity and surjectivity for a functor.

Definition 4.5. *Suppose $F : \mathscr{B} \to \mathscr{C}$ is a functor. Then F is,*

1. ***full*** *if* $F : hom_{\mathscr{B}}(A, B) \to hom_{\mathscr{C}}(F(A), F(B))$ *is surjective*
2. ***faithfull*** *if* $F : hom_{\mathscr{B}}(A, B) \to hom_{\mathscr{C}}(F(A), F(B))$ *is injective*
3. ***dense*** *if for each* $C \in ob(\mathscr{C})$ *there exists some* $B \in ob(\mathscr{B})$ *such that* $F(B)$ *is isomorphic to* C

We will now give some important examples of functors. The first is a basic example, the identity functor. After this we will see the localisation functor, and also the global sections functor, these two functors in particular will help us to understand the Beilinson-Bernstein Theorem.

Example 4.6. For any category $\mathscr{C}$, define the functor $(\mathscr{C}, 1_{Mor\mathscr{C}}, \mathscr{C})$ that takes $1_A \mapsto 1_A$ for each $A \in Ob(\mathscr{C})$. This functor is called the ***identity functor on*** $\mathscr{C}$ and is denoted by $1_{\mathscr{C}}$.

Definition 4.7. *Let* R *be a ring,* $X = Spec\ R$*, and recall,* $\mathscr{O}_X$*, the structure sheaf of* X *with restriction map* r_{fg} *for open sets* $E(g) \subseteq E(f)$ *given be the universal property of localisation. Suppose* M *is an* R*-module. Then define a sheaf* $\mathscr{O}_X \otimes M$ *on* X *by the folowing data,*

1. *For a basis open set* $E(f) \subseteq X$*, define,*

$$(\mathscr{O}_X \otimes M)(E(f)) = \mathscr{O}_X(E(f)) \otimes_R M = R\left[\frac{1}{f}\right] \otimes_R M$$

2. *For an inclusion of basis open sets* $E(g) \subseteq E(f)$*, define the restriction map* $\rho_{fg} : R\left[\frac{1}{f}\right] \otimes_R M \to R\left[\frac{1}{g}\right] \otimes_R M$ *by,*

$$\rho_{fg}(a \otimes m) = r_{fg}(a) \otimes m$$

This construction is a generalisation of the object $\mathscr{O}(n) \otimes N$ of $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$ given in Proposition 3.6. We need to make sure this is infact a sheaf before we can describe how this sheaf induces a functor. Thankfully it follows from the fact that $\mathscr{O}_X$ is a sheaf.

Proposition 4.8. *Suppose we have the set up of the preceding definition. Then* $\mathscr{O}_X \otimes_R M$ *is a sheaf on* X*.*

By definition $\mathscr{O}_X \otimes_R M$ is in fact a quasi-coherent sheaf of $\mathscr{O}_X$-modules. Thus if we are given an R-module, we now have a method of obtaining a quasi-coherent sheaf of $\mathscr{O}_X$-modules. This describes a functor from the category of R-modules (R-**mod** with R-module homomorphisms) to the category of quasi-coherent sheaves of $\mathscr{O}_X$-modules ($\mathscr{O}_X$-**Qcoh** with morphisms of quasi-coherent sheaves). We call this functor the *localisation functor*.

Definition 4.9. *Consider the following functor* $\otimes : R\text{-}\textbf{mod} \to \mathscr{O}_X\text{-}\textbf{Qcoh}$ *given by* $M \mapsto \mathscr{O}_X \otimes M$ *for an* R*-module,* M*. Call this functor the* ***localisation functor****.*

The localisation functor is quite non-trivial as an example of a functor, there are much simpler examples of functors of categories, but the localisation functor will be valuable to us in understanding the Beilinson-Bernstein theorem. Another functor that will be of interest to us is the *global sections functor* and thankfully it is a little simpler to understand.

Definition 4.10. *Consider the following functor* $\Gamma(\mathbb{P}^1_{\mathbb{C}}, -) : \mathscr{O}_X\text{-}\textbf{Qcoh} \to R\text{-}\textbf{mod}$ *defined by* $\Gamma(\mathbb{P}^1_{\mathbb{C}}, \mathscr{F}) := \mathscr{F}(\mathbb{P}^1_{\mathbb{C}})$*, the global sections of the sheaf* $\mathscr{F}$ *on* $\mathbb{P}^1_{\mathbb{C}}$*.*

Note here that we are claiming that the global sections for a quasi-coherent sheaf form an R-module. This result comes from Proposition 3.17 where we showed that the kernel of the Cech map was a ring, and hence a module over itself (our ring was $R = \mathbb{C}[t] \oplus \mathbb{C}[t^{-1}]$).

The Beilinson-Berstein Theorem essentially an equivalence of categories. The equivalence of categories, in some sense, describes when two categories are the same. We begin by describing an isomorphism of categories, which seems appropriate at first glance, but is actually too strong. This will lead us to the idea of the equivalence of categories which is a weaker concept. All the category theory here is covered in more detail in [16].

Definition 4.11. *A functor* $F : \mathscr{A} \to \mathscr{B}$ *is said to be an* ***isomorphism from*** $\mathscr{A}$ ***to*** $\mathscr{B}$ *provided that there exists a functor* $G : \mathscr{B} \to \mathscr{A}$ *such that* $G \circ F = 1_{\mathscr{A}}$ *and* $F \circ G = 1_{\mathscr{B}}$*. In this case we say that the categories* $\mathscr{A}$ *and* $\mathscr{B}$ *are* ***isomorphic*** *and they are denoted by* $\mathscr{A} \simeq \mathscr{B}$*.*

Example 4.12. Suppose $\mathscr{C}$ is a category. Then the identity functor is trivially an isomorphism. This suggests to us that there is a bijection between the objects of isomorphic categories. As it turns out this is the case.

Proposition 4.13. *Suppose* $F : \mathscr{A} \to \mathscr{B}$ *is a functor. Then the following are equivalent,*

1. *F is an isomorphism*
2. *The function* $F : Mor(\mathscr{A}) \to Mor(\mathscr{B})$ *is a bijection*
3. *F is full and faithful and the associated object function* $F : Ob(\mathscr{A}) \to Ob(\mathscr{B})$ *is a bijection*

Consider the category of all vector spaces over a field K and the category of all vector spaces over K of the form K^n. The first category clearly has many more objects than the second, so these categories are not isomorphic. The vital point here is that if we view objects of each category up to isomorpism, then in some sense these categories are the same. Roughly speaking, this situation describes an equivalence of categories. An arbitrary category may have isomorphic objects, hence we need to incorporate these objects as a single object when we think of categories being equivalent. This leads us to the definition of a skeleton of a category.

Definition 4.14. *A category $\mathscr{C}$ is called **skeletal** provided that isomorphic $\mathbb{C}$-objects are identical. A **skeleton** of a category $\mathscr{C}$ is a maximal full skeletal subcategory of $\mathscr{C}$.*

It is important to note that a full subcategory $\mathscr{B}$ of category $\mathscr{C}$ is a subcategory that has the property that $hom_{\mathscr{B}}(A, B) = hom_{\mathscr{C}}(A, B)$ for all objects $A, B \in ob(\mathscr{B})$.

Example 4.15. For any field K, the full subcategory of all finite powers K^n is a skeleton for the category of finite dimensional vector spaces over K.

Proposition 4.16. *Every category $\mathscr{C}$ has a skeleton.*

Proof If we let $A \simeq B$ mean that there is a $\mathscr{C}$-isomorphism from A to B, then $\simeq$ is an equivalence relation on $Ob(\mathscr{C})$. Hence, by the Axiom of Choice, it has a system of representatives $\mathscr{S}$. Let $\mathscr{B}$ be the full subcategory of $\mathscr{C}$ that is generated by $\mathscr{S}$. It follows that $\mathscr{B}$ is skeletal and maximal. □

This proof also shows us that if we found two skeletons of a category, than they must be isomorphic.

Definition 4.17. *A category $\mathscr{A}$ is **equivalent** to a category $\mathscr{B}$ (denoted by $\mathscr{A} \sim \mathscr{B}$ if $\mathscr{A}$ and $\mathscr{B}$ have isomorphic skeletons.*

Naturally there is a better way to show two categories are equivalent than to simply check whether or not their skeletons are isomorphic. We can use functors to prove equivalence of categories.

Theorem 4.18. *Two categories $\mathscr{A}$ and $\mathscr{B}$ are equivalent if and only if there exists a functor $F : \mathscr{A} \to \mathscr{B}$ that is full, faithful and dense.*

The last concept we will need to cover before arriving at the Beilinson-Bernstein theorem is the idea of a $\mathscr{D}$-affine scheme.

Definition 4.19. *Let X be an algebraic variety over a field K which is algebraically closed and has characteristic zero. Let $\mathscr{O}_X = \mathscr{O}$ be the structure sheaf of X, $\mathscr{D}_X = \mathscr{D}$ be a sheaf of differential operators on X, and $\mathscr{D}(X)$ be the global sections of the sheaf (which is a ring). Also let $\mathscr{D}_X$-**mod**, be the category of $\mathscr{D}_X$-modules which* *are quasi-coherent as $\mathscr{O}_X$-modules, and let $\mathscr{D}(X)$**-mod**, be the category of $\mathscr{D}(X)$-modules. Then X is $\mathscr{D}$**-affine** if each sheaf $\mathscr{F} \in ob(\mathscr{D}_X$**-mod**$)$ is generated over $\mathscr{D}$ by global sections and $H^i(X, \mathscr{F}) = 0$ for $i > 0$.*

We have not defined algebraic varieties yet, but they are similar to projective varieties, except that the space X is affine. It means we do not need homogenous polynomials to describe the closed sets, but rather just arbitrary polynomials. Similarly the structure sheaf can be defined as the sheaf of regular functons on the affine variety. See [6] for more details. Also the condition, $H^i(X, \mathscr{F}) = 0$ for $i > 0$, on the cohomology groups may seem unfamiliar to us, but this condition is exactly the same as the one given by Swan when he defined the Cech complex for a quasi-coherent sheaf. Namely that $ker\ d$ gave the global sections and no other information could be extracted from the complex. The condition of the cohomology groups being zero is equivalent to obtaining no extra information from the sequence, and also that the cohomology group $H^0(X, \mathscr{F}) = ker\ d$. We now give an example of a $\mathscr{D}$-affine scheme which was proved so by Beilinson and Bernstein in [17].

Theorem 4.20. *Let K be an algebraically closed field with charactersic zero. Then $\mathbb{P}^n_K$ is $\mathscr{D}$-affine.*

The proof of this result is highly non-trivial and we omit it here. Beilinson and Bernstein showed that the property of being $\mathscr{D}$-affine was a vital premise for a remarkable equivalence of two categories. We are now ready for the Beilinson-Bernstein Theorem. Released in 1981, it was a highly celebrated theorem in relation to the study of the geometry of flag varieties, and the Kazhdan-Lusztig multiplicity conjecture.

Theorem 4.21. *(Beilinson-Bernstein) If X is $\mathscr{D}$-affine, the global sections functor, $\Gamma(X, -) : \mathscr{F} \to \Gamma(X, \mathscr{F})$ induces an equivalence of categories between $\mathscr{D}_X$**-mod** and $\mathscr{D}(X)$**-mod**.*

It is interesting to note that the inverse functor is given by the localisation functor, $\mathscr{D}_X \otimes_{\mathscr{D}_X} - : \mathscr{D}(X)$**-mod** $\to \mathscr{D}_X$**-mod**. Kashiwara gives an equivalent statement in [12]. He uses the terms anti-dominant and regular, which for our case of $\mathfrak{sl}_2$ amounts to saying that $\lambda \neq -1, 0, 1, \ldots$

Theorem 4.22. *If $\lambda \neq -1, 0, 1, \ldots$ (ie if λ is anti-dominant and regular), the category of finitely generated $U_\lambda(\mathfrak{g})$-modules are equivalent to the category of coherent $\mathscr{D}^\lambda_{\mathbb{P}^1_{\mathbb{C}}}$-modules.*

In this thesis, $\mathfrak{g} = \mathfrak{sl}_2$ and $U_\lambda(\mathfrak{g}) = U_\lambda$. This last theorem is telling us that there exists a U_λ-module for each quasi-coherent sheaf $\mathscr{D}^\lambda$ (the global sections are the desired module), which is precisely what we showed in the third chapter, as we can think of $\mathscr{D}^\lambda$ as a $\mathscr{D}^\lambda$-module.

In this section we have set out the necessary foundations in category theory to understand the Beilinson-Bernstein Theorem. We have used the global sections functor and the localisation functor which are commonly found in algebraic geometry. In the next section we will demonstrate the usefulness of the global sections functor by providing explicit examples of all the finite dimensional irreducible $U(\mathfrak{sl}_2)$-modules.

4.2 Irreducible $U(\mathfrak{sl}_2)$ Modules

In this section we will investigate a case where the Beilinson-Bernstein Theorem does not apply. For this section, we will keep the following notation fixed. Let $X = \mathbb{P}^1_{\mathbb{C}}$, $\mathscr{O} = \mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}$ and $\mathscr{D}^\lambda = \mathscr{D}^\lambda_{\mathbb{P}^1_{\mathbb{C}}}$ for $\lambda \in \mathbb{C}$. We will describe a $\mathscr{D}^n$-module for $n \in \mathbb{Z}, n \geq -1$. We will see that the global sections of this sheaf will be a $U(\mathfrak{sl}_2)$-module. These $U(\mathfrak{sl}_2)$-modules will be concrete examples of finite-dimensional irreducible $U(\mathfrak{sl}_2)$-modules.

Consider $\mathscr{O}(n)$, the sheaf of regular functions on $\mathbb{P}^1_{\mathbb{C}}$ with the Serre twist $n \geq -1$. It is described by the object $(\mathbb{C}[t], \mathbb{C}[t^{-1}], t^{-n})$ in the category $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$ defined in Chapter 3. To prove that $\mathscr{O}(n)$ is a $\mathscr{D}^n$-module, we require the module action to be compatible with the restriction maps of the two sheaves. The details of this condition are covered in the following Lemma.

Lemma 4.23. *Consider the differential operators, $t^{-n}, \delta, \delta + nt^{-1} \in D(\mathbb{C}[t^{-1}])$. Then $t^{-n}\delta = (\delta + nt^{-1})t^{-n}$.*

Proof We proceed via induction on n. For the base case, $(\delta+t^{-1})t^{-1} = \delta t^{-1}+t^{-2} = t^{-1}\delta$. For the induction step, the right hand side is

$$\begin{aligned}
(\delta + nt^{-1})t^{-n} &= (\delta + (n-1)t^{-1} + t^{-1})t^{-n+1}t^{-1} \\
&= (\delta + (n-1)t^{-1})t^{-n+1}t^{-1} + t^{-n-1} \\
&= t^{-n+1}\delta t^{-1} + t^{-n-1} \text{ (by the inductive assumption)} \\
&= t^{-n+1}(t^{-1}\delta - t^{-2}) + t^{-n-1} \\
&= t^{-n}\delta
\end{aligned}$$

□

Now we are ready to prove that $\mathscr{O}(n)$ is in fact a $\mathscr{D}^n$-module. Once we accomplish this, we can proceed to show that the global sections of this sheaf will be a $U(\mathfrak{sl}_2)$-module.

Proposition 4.24. *The sheaf $\mathscr{O}(n)$, is a $\mathscr{D}^n$-module.*

Proof Firstly, we must describe the module action of $\mathscr{D}^n$ on the sections of $\mathscr{O}(n)$. On U_0, $\mathscr{O}(n)(U_0) = \mathbb{C}[t]$ which is a $D(\mathbb{C}[t])$-module via the action of the differential operators on elements of $\mathbb{C}[t]$. Similary, $\mathscr{O}(n)(U_\infty)$ is a $D(\mathbb{C}[t^{-1}])$-module.

Secondly, we need to show that the module action is compatible with the restriction maps of the two sheaves. We show this by considering the inclusion $U_{0\infty} \subseteq U_0 \subseteq \mathbb{P}^1_{\mathbb{C}}$. We need to show that the following diagram commutes,

$$\begin{array}{ccc}
\mathscr{D}^n(U_0) \times \mathscr{O}(n)(U_0) & \xrightarrow{M_{U_0}} & \mathscr{O}(n)(U_0) \\
\Big\downarrow {\scriptstyle \rho^{\mathscr{D}^n} \times \rho^{\mathscr{O}(n)}} & & \Big\downarrow {\scriptstyle \rho^{\mathscr{O}(n)}} \\
\mathscr{D}^n(U_{0\infty}) \times \mathscr{O}(n)(U_{0\infty}) & \xrightarrow{M_{U_{0\infty}}} & \mathscr{O}(n)(U_{0\infty})
\end{array}$$

We can demonstrate the commutativity of this diagram with the following observation. Suppose we have a polynomial $f(t) \in \mathbb{C}[t]$. By the previous Lemma, we can see that the diagram commutes if we only consider elements of the form $(\delta, f(t)) \in \mathscr{D}^n(U_0) \times \mathscr{O}(n)(U_0)$. Thus by composition, the diagram commutes for any $(\delta^k, f(t))$. Since t commutes with the map t^{-n}, and elements of the form $t^i\delta^j$ form a basis for $D(\mathbb{C}[t])$, we see that the diagram commutes. Hence, $\mathscr{O}(n)$ is a $\mathscr{D}^n$-module. □

Proposition 4.25. *The global sections of $\mathscr{O}(n)$ form a $U(\mathfrak{sl}_2)$-module.*

Proof We can compute the global sections of this sheaf by writing down its Cech complex and computing the kernel of its Cech map,

$$0 \longrightarrow \mathbb{C}[t] \oplus \mathbb{C}[t^{-1}] \xrightarrow{d_n} \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} \mathbb{C}[t^{-1}] \longrightarrow 0$$

Where $d_n(x, y) = t^{-n}(1 \otimes x) - 1 \otimes y = t^{-n} \otimes x - 1 \otimes y$. Without too much difficulty we can see that $ker\ d_n$ is the $\mathscr{D}^n(\mathbb{P}^1_{\mathbb{C}})$-module generated by,

$$\{(1, t^{-n}), (t, t^{1-n}), \ldots, (t^n, 1)\}$$

Recall that the ring, $\mathscr{D}^n(\mathbb{P}^1_{\mathbb{C}})$, is the subring of $D(\mathbb{C}[t]) \oplus D(\mathbb{C}[t^{-1}])$ generated by the following 3 elements,

$$\big\{(-2t\delta + \lambda, -2td - \lambda), (-\delta, -\delta - \lambda t^{-1}), (t^2\delta - \lambda t, t^2\delta)\big\}$$

Seeing as $\mathscr{D}^n(\mathbb{P}^1_{\mathbb{C}}) \simeq \frac{U(\mathfrak{sl}_2)}{(\Delta-(n+1)^2)}$, there is a natural action of $U(\mathfrak{sl}_2)$ on the global sections of $\mathscr{O}(n)$. □

Thus, we see that $\Gamma(\mathbb{P}^1_{\mathbb{C}}, \mathscr{O}(n))$ is a $U(\mathfrak{sl}_2)$-module given by the isomorphism proved in Chapter 3. We will show that this $U(\mathfrak{sl}_2)$-module is in fact an irreducible module. To do so, we will breifly describe the classification of irreducible $U(\mathfrak{sl}_2)$-modules. For more details on the classification of irreducible $U(\mathfrak{sl}_2)$-modules see Chapter 5 in [13].

Definition 4.26. *Let V be an $U(\mathfrak{sl}_2)$-module. For $a \in \mathbb{C}$, the a-eigenspace of h_V (this is the image of the representation of h) is written V_a and called the* ***weight space of weight*** *a, and any a-eigenvector is called a* ***weight vector of weight*** *a. The eigenvalues of h_V are called the* ***weights*** *of V, and dim V_a is called the* ***multiplicity*** *of the weight a. A weight vector v such that $ev = 0$ is called a* ***highest-weight vector***.

We have defined a seemingly interesting vector, namely the highest weight vector. Its interesting nature comes from the submodule that it generates.

Proposition 4.27. *Let V be a non-zero finite dimensional $U(\mathfrak{sl}_2)$-module.*

1. *V contains a highest-weight vector of some weight*
2. *If $v_0 \in V$ is a highest weight vector of weight n, then $n \in \mathbb{N}$ and the submodule generated by v_0 has basis $\{v_0, v_1, \ldots, v_n\}$ satisfying*

$$\begin{aligned} ev_i &= \begin{cases} (n-i+1)v_{i-1} & \text{if } 1 \le i \le n \\ 0 & \text{if } i = 0 \end{cases} \\ fv_i &= \begin{cases} (i+1)v_{i+1} & \text{if } 0 \le i \le n-1 \\ 0 & \text{if } i = n \end{cases} \\ hv_i &= (n-2i)v_i \end{aligned}$$

Modules that satisfy the above relations turn out to be the only irreducible $U(\mathfrak{sl}_2)$-modules up to isomorphism, which allows us to classify all the finite dimensional irreducible $U(\mathfrak{sl}_2)$-modules. Recall that an irreducible module is a module with no non-trivial submodules.

Theorem 4.28.

1. *For any $n \in \mathbb{N}$, we can define an irreducible $U(\mathfrak{sl}_2)$-module,*

$$V(n) := \mathbb{C}\{v_0, v_1, \ldots, v_n\}$$

by the above formulae

2. *Any irreducible finite-dimensional* $U(\mathfrak{sl}_2)$*-module* V *is isomorphic to* $V(n)$ *where* $n = dim\ V - 1$

The proofs of all these results can be found in [13]. Now that we understand the structure of irreducible $U(\mathfrak{sl}_2)$-modules, we are ready to show that the global sections of our twisted sheaf, $\Gamma(\mathbb{P}^1_{\mathbb{C}}, \mathscr{O}(n))$, are indeed such modules.

Proposition 4.29. *The global sections of the twisted sheaf of regular functions,* $\Gamma(\mathbb{P}^1_{\mathbb{C}}, \mathscr{O}(n))$*, is isomorphic to the irreducible* $U(\mathfrak{sl}_2)$*-module of dimension* $n + 1$*,* $V(n)$*. More explicitly this isomorphism is given by setting,*

$$v_i = {}^nC_i(-1)^i(t^i, t^{i-n})$$

Proof Proving this statement reduces to carrying out the calculations for the action on e, f, h on $v_0, v_1, \ldots, v_n$ given by $V(n)$. We will carry out the calculations for e here. We first calculate ev_0,

$$\begin{aligned} ev_0 &= (-\delta, -\delta - nt^{-1})(1, t^{-n}) \\ &= (0, nt^{-n-1} - nt^{-n-1}) \\ &= (0, 0) \end{aligned}$$

Now we will calculate ev_i for $1 \leq i \leq n$,

$$\begin{aligned} ev_i &= (-\delta, -\delta - nt^{-1})^nC_i(-1)^i(t^i, t^{i-n}) \\ &= {}^nC_i(-1)^i(-it^{i-1}, -(i-n)t^{i-n-1} - nt^{i-n-1}) \\ &= {}^nC_i(-1)^i(-1)(i)(t^{i-1}, t^{i-n-1}) \\ &= (n-i+1)^nC_{i-1}(-1)^{i-1}(t^{i-1}, t^{i-n-1}) \\ &= (n-i+1)v_{i-1} \end{aligned}$$

Note that in the third line we used the identity, ${}^nC_i i = {}^nC_{i-1}(n-i+1)$. Thus, we can see that $\Gamma(\mathbb{P}^1_{\mathbb{C}}, \mathscr{O}(n)) = V(n)$ with $v_i = {}^nC_i(-1)^i(t^i, t^{i-n})$. □

In this section we have explored a use of the global sections functor, by using the twisted sheaf of regular functions $\mathscr{O}(n)$. We showed that $\mathscr{O}(n)$ was a $\mathscr{D}^n$-module. Lastly we saw how these global sections were in fact irreducible $U(\mathfrak{sl}_2)$-modules, which is why the casimir element Δ acts as a scalar on it. This explains why really these modules are U_n-modules. In the next section we will use the Beilinson-Bernstein Theorem to prove the translation principle for U_λ-modules. This allows us to connect sheaves of twisted differential operators with different twists.

4.3 The Translation Principle

In this section we will be displaying an example of the power of the Beilinson-Bernstein Theorem. We will show an equivalence of module categories, namely that U_λ**-mod** $\simeq U_{\lambda+1}$**-mod** for $\lambda \neq -1, 0, \ldots$ This shows us that many of the global sections of the twisted sheaves of differential operatores have equivalent module categories. In doing so we will prove the translation principle for the sheaves of differential operators, that is we will find a connection between the two sheaves $\mathscr{D}^{\lambda+1}$ and $\mathscr{D}^{\lambda}$ for any $\lambda \in \mathbb{C}$. For this section, we will keep the following notation fixed. Let $X = \mathbb{P}^1_{\mathbb{C}}$, $\mathscr{O} = \mathscr{O}_{\mathbb{P}^1_{\mathbb{C}}}$ and $\mathscr{D}^{\lambda} = \mathscr{D}^{\lambda}_{\mathbb{P}^1_{\mathbb{C}}}$ for $\lambda \in \mathbb{C}$.

Proposition 4.30. *There is an isomorphism of sheaves*

$$\mathscr{D}^{\lambda+1} \simeq \mathscr{O}(1) \otimes_{\mathscr{O}} \mathscr{D}^{\lambda} \otimes_{\mathscr{O}} \mathscr{O}(-1)$$

Proof To prove this result we will show that these two objects are isomorphic in the category $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$, given in Swan's article, [9]. Thus we need an object in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$ that corresponds to the sheaf $\tilde{\mathscr{D}}^{\lambda} := \mathscr{O}(1) \otimes_{\mathscr{O}} \mathscr{D}^{\lambda} \otimes_{\mathscr{O}} \mathscr{O}(-1)$. Recall the following objects in $\mathscr{M}od(\mathbb{P}^1_{\mathbb{C}})$,

$$\begin{aligned} \mathscr{O}(1) &= (\mathbb{C}[t], \mathbb{C}[t^{-1}], t^{-1}) \\ \mathscr{D}^{\lambda} &= (D(\mathbb{C}[t]), D(\mathbb{C}[t^{-1}]), \theta_{\lambda}) \\ \mathscr{O}(-1) &= (\mathbb{C}[t], \mathbb{C}[t^{-1}], t) \end{aligned}$$

Thus, for the sheaf $\tilde{\mathscr{D}}^{\lambda}$, on U_0 and U_∞ respectively,

$$\begin{aligned} \tilde{\mathscr{D}}^{\lambda}(U_0) &= \mathbb{C}[t] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t]) \otimes_{\mathbb{C}[t]} \mathbb{C}[t] \\ \tilde{\mathscr{D}}^{\lambda}(U_\infty) &= \mathbb{C}[t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}]) \otimes_{\mathbb{C}[t^{-1}]} \mathbb{C}[t^{-1}] \end{aligned}$$

Lastly we have the gluing isomorphism,

$$\psi_\lambda : \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t]} \tilde{\mathscr{D}}^{\lambda}(U_0) \to \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} \tilde{\mathscr{D}}^{\lambda}(U_\infty)$$

By definition ψ_λ works independently on each tensor product so that, $\psi_\lambda = id \otimes t^{-1} \otimes \theta_\lambda \otimes t$. We want $\psi_\lambda(t \otimes 1 \otimes 1 \otimes 1) = t \otimes 1 \otimes 1 \otimes 1$. It is evident that according to the definition of ψ_λ, $(id \otimes t^{-1} \otimes \theta_\lambda \otimes t)(t \otimes 1 \otimes 1 \otimes 1) = t \otimes t^{-1} \otimes 1 \otimes t$, which is not well defined since, in the last position, $t \notin \mathbb{C}[t^{-1}]$. Since the image of ψ_λ is contained in a tensor product over $\mathbb{C}[t^{-1}]$ we can move the t^{-1} from the second position to the last position giving the desired result, that is $\psi_\lambda(t \otimes 1 \otimes 1 \otimes 1) = t \otimes 1 \otimes 1 \otimes 1$ is well defined.

For the action of ψ_λ on $1 \otimes 1 \otimes \delta \otimes 1$, we see from the definition that $(id \otimes t^{-1} \otimes \theta_\lambda \otimes t)(1 \otimes 1 \otimes \delta \otimes 1) = 1 \otimes t^{-1} \otimes (\delta + \lambda t^{-1}) \otimes t$. Again, we see that the t in the fourth position is not well defined as an element of $\mathbb{C}[t^{-1}]$. Thus as before we move the t^{-1} from the second position through the tensor product to gain a well defined image. Thus we see that,

$$\begin{aligned} 1 \otimes t^{-1} \otimes \left(\delta + \lambda t^{-1}\right) \otimes t &= 1 \otimes 1 \otimes t^{-1}\left(\delta + \lambda t^{-1}\right) \otimes t \\ &= 1 \otimes 1 \otimes \left(t^{-1}\delta + \lambda t^{-2}\right) \otimes t \\ &= 1 \otimes 1 \otimes \left(\delta t^{-1} + t^{-2} + \lambda t^{-2}\right) \otimes t \\ &= 1 \otimes 1 \otimes \left(\delta + (\lambda+1)t^{-1}\right) t^{-1} \otimes t \\ &= 1 \otimes 1 \otimes \left(\delta + (\lambda+1)t^{-1}\right) \otimes 1 \end{aligned}$$

Thus we see that ψ_λ is a well defined isomorphism that sends,

$$\begin{aligned} t \otimes 1 \otimes 1 \otimes 1 &\mapsto t \otimes 1 \otimes 1 \otimes 1 \\ 1 \otimes 1 \otimes (\delta + \lambda t^{-1}) \otimes 1 &\mapsto 1 \otimes 1 \otimes \left(\delta + (\lambda+1)t^{-1}\right) \otimes 1 \end{aligned}$$

To show that $\tilde{\mathscr{D}}^\lambda \simeq \mathscr{D}^{\lambda+1}$, it suffices to find a ring isomorphisms $f^+ : \mathscr{D}^{\lambda+1}(U_0) \to \tilde{\mathscr{D}}^\lambda(U_0)$ and $f^- : \mathscr{D}^{\lambda+1}(U_\infty) \to \tilde{\mathscr{D}}^\lambda(U_\infty)$ such that the following diagram commutes,

$$\begin{array}{ccc} \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} D(\mathbb{C}[t]) & \xrightarrow[\simeq]{\theta_{\lambda+1}} & \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} D(\mathbb{C}[t^{-1}]) \\ \big\downarrow {\scriptstyle 1\otimes f^+} & & \big\downarrow {\scriptstyle 1\otimes f^-} \\ \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} \tilde{\mathscr{D}}^\lambda(U_0) & \xrightarrow[\simeq]{\psi_\lambda} & \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} \tilde{\mathscr{D}}^\lambda(U_\infty) \end{array}$$

It is clear that f^+ and f^- are just injections, that is,

$$\begin{aligned} f^+(\beta^+(t,\delta)) &= 1 \otimes \beta^+(t,\delta) \otimes 1 \\ f^-(\beta^-(t^{-1}, t^2\delta)) &= 1 \otimes \beta^-(t^{-1}, t^2\delta)) \otimes 1 \end{aligned}$$

The inverse morphisms are even simpler, all they do is collapse upon the two tensor products. This shows that f^+, f^- are in fact ring isomorphisms and hence that $\mathscr{D}^{\lambda+1} \simeq \tilde{\mathscr{D}}^\lambda := \mathscr{O}(1) \otimes_{\mathscr{O}} \mathscr{D}^\lambda \otimes_{\mathscr{O}} \mathscr{O}(-1)$ □

The translation principle not only gives us a useful way to connect $\mathscr{D}^{\lambda+1}$ with $\mathscr{D}^\lambda$ but we can go the other way too by swapping the tensor products $\mathscr{O}(1)$ and $\mathscr{O}(-1)$. We omit the proof as it is identical to the proof above with $\mathscr{O}(1)$ and $\mathscr{O}(-1)$ and their respective sections swapped.

Corollary 4.31. *There is another isomorphism of sheaves,*

$$\mathscr{D}^{\lambda-1} \simeq \mathscr{O}(-1) \otimes \mathscr{D}^{\lambda} \otimes \mathscr{O}(1)$$

Next we will define the functors that will give us the equivalence of module categories. Define the functor $F : \mathscr{D}^{\lambda}\textbf{-Mod} \to \mathscr{D}^{\lambda+1}\textbf{-Mod}$ which maps $M \mapsto \mathscr{O}(1) \otimes_{\mathscr{O}} M$ for the sheaf $M \in \mathscr{D}^{\lambda}\textbf{-Mod}$. Also, we define the functor $G : \mathscr{D}^{\lambda}\textbf{-Mod} \to \mathscr{D}^{\lambda-1}\textbf{-Mod}$ wich maps $M \mapsto \mathscr{O}(-1) \otimes_{\mathscr{O}} M$ for the sheaf $M \in \mathscr{D}^{\lambda}\textbf{-Mod}$. We need to make sure these functors are well defined, that is that $\mathscr{O}(1) \otimes M$ is a $\mathscr{D}^{\lambda+1}$-module and that $\mathscr{O}(-1) \otimes M$ is a $\mathscr{D}^{\lambda-1}$-module. To do so we need the following result.

Proposition 4.32. *There is an isomorphism of sheaves,*

$$\mathscr{O}(1) \otimes_{\mathscr{O}} \mathscr{O}(-1) \simeq \mathscr{O} \simeq \mathscr{O}(-1) \otimes_{\mathscr{O}} \mathscr{O}(1)$$

Proof We define ring isomorphisms

$$f^+ : \mathbb{C}[t] \otimes_{\mathbb{C}[t]} \mathbb{C}[t] \to \mathbb{C}[t] : a \otimes b \mapsto ab$$
$$f^- : \mathbb{C}[t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} \mathbb{C}[t^{-1}] \to \mathbb{C}[t^{-1}] : a \otimes b \mapsto ab$$

These have inverses,

$$g^+ : \mathbb{C}[t] \to \mathbb{C}[t] \otimes_{\mathbb{C}[t]} \mathbb{C}[t] : a \mapsto a \otimes 1$$
$$g^- : \mathbb{C}[t^{-1}] \to \mathbb{C}[t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} \mathbb{C}[t^{-1}] : a \mapsto a \otimes 1$$

Furthermore, they allow the following diagram to commute,

$$\begin{array}{ccc} \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} \mathbb{C}[t] \otimes_{\mathbb{C}[t]} \mathbb{C}[t] & \xrightarrow[\simeq]{1\otimes t^{-1}\otimes t} & \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} \mathbb{C}[t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} \mathbb{C}[t^{-1}] \\ \Big\downarrow {\scriptstyle 1\otimes f^+} & & \Big\downarrow {\scriptstyle 1\otimes f^-} \\ \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t]} \mathbb{C}[t] & \xrightarrow[\simeq]{1\otimes 1} & \mathbb{C}[t,t^{-1}] \otimes_{\mathbb{C}[t^{-1}]} C[t^{-1}] \end{array}$$

This shows that $\mathscr{O}(1) \otimes_{\mathscr{O}} \mathscr{O}(-1) \simeq \mathscr{O}$, and similarly with $\mathscr{O}(1)$ and $\mathscr{O}(-1)$ swapped, $\mathscr{O} \simeq \mathscr{O}(-1) \otimes_{\mathscr{O}} \mathscr{O}(1)$. □

Proposition 4.33. *The functors F and G are well defined.*

Proof Suppose M is a $\mathscr{D}^{\lambda}$-module. Then for each open set $U \subseteq \mathbb{P}^1_{\mathbb{C}}$, $M(U)$ is a $\mathscr{D}^{\lambda}(U)$-module. This means that there is morphism of sheaves from $\mathscr{D}^{\lambda} \otimes_{\mathscr{O}} M \to M$ which describes the module action of $\mathscr{D}^{\lambda}(U)$ on $M(U)$ for each open set $U \subseteq \mathbb{P}^1_{\mathbb{C}}$.

Thus we need a morphism of sheaves $\mathscr{D}^{\lambda+1} \otimes_{\mathscr{O}} \mathscr{O}(1) \otimes_{\mathscr{O}} M \to \mathscr{O}(1) \otimes_{\mathscr{O}} M$. Using the translation principle, we see that there is a morphism of sheaves,

$$\mathscr{D}^{\lambda+1} \otimes_{\mathscr{O}} \mathscr{O}(1) \otimes_{\mathscr{O}} M \to \mathscr{O}(1) \otimes_{\mathscr{O}} \mathscr{D}^{\lambda} \otimes_{\mathscr{O}} \mathscr{O}(-1) \otimes_{\mathscr{O}} \mathscr{O}(1) \otimes_{\mathscr{O}} M$$

Without too much difficulty, we can see that $\mathscr{O}(-1) \otimes_{\mathscr{O}} \mathscr{O}(1) \simeq \mathscr{O}$ are isomorphic as sheaves. Since we are tensoring over $\mathscr{O}$ this tensor product collapses. Thus we have a morphism of sheaves

$$\mathscr{D}^{\lambda+1} \otimes_{\mathscr{O}} \mathscr{O}(1) \otimes_{\mathscr{O}} M \to \mathscr{O}(1) \otimes_{\mathscr{O}} \mathscr{D}^{\lambda} \otimes_{\mathscr{O}} M$$

But since M is a $\mathscr{D}^{\lambda}$-module, we see that the last tensor product collapses which gives us the desired module action. More explicity this shows there is a morphism of sheaves,

$$\mathscr{D}^{\lambda+1} \otimes_{\mathscr{O}} \mathscr{O}(1) \otimes_{\mathscr{O}} M \to \mathscr{O}(1) \otimes_{\mathscr{O}} M$$

Similarly G is also well-defined. □

These two functors are in fact inverses of each other, the vital ingredient of proof of this fact is that $\mathscr{O}(1)\otimes_{\mathscr{O}}\mathscr{O}(-1) \simeq \mathscr{O} \simeq \mathscr{O}(-1)\otimes_{\mathscr{O}}\mathscr{O}(1)$. This means that we have an equivalence of categories, $\mathscr{D}^{\lambda}$-**Mod** $\simeq \mathscr{D}^{\lambda+1}$-**Mod**. Thus, the Beilinson-Bernstein Theorem gives us the desired result.

Theorem 4.34. *(Translation Principle) There is an equivalence of categories,*

$$U_{\lambda}\textbf{-}\boldsymbol{mod} \simeq U_{\lambda+1}\textbf{-}\boldsymbol{mod}$$

for $\lambda \in \mathbb{C}, \lambda \neq -1, 0, \ldots$

In this section we have discovered that many of the module categories of global sections have isomorphic skeletal categories. This allows us to make a connection between the global sections of sheaves of differential operators with different twists.

In the next, and last, chapter we will expand our view to the general setting. We will see that so far, the whole thesis has been a special case of a much more general setting. The general setting is related to the study of the geometry of flag varieties through representation theory and $\mathscr{D}$-modules.

Chapter 5

The Geometry of Flag Varieties

In this chapter we will see that the results given so far in this thesis can be completely generalised, following M. Kashiwara's work, [12]. We will apply this general setting to our special case to obtain a greater appreciation, in some sense, of the complexities of the general case.

5.1 The General Setting

The general setting is considerably more abstract than what has been studied in this thesis, and as a result we will not be defining all the terms used. The terms are, thankfully, quite standard, hence they can easily be found in sources like [12] and [14]. In [12], Kashiwara studies the geometry of flag manifolds via $\mathscr{D}$-modules and representation theory. We will define a flag and understand how $\mathbb{P}^1_{\mathbb{C}}$ is a flag.

Definition 5.1. *Suppose K is a field and $V = K^n$ is a vector space over K. A* ***complete flag*** *in V is a sequence of subspaces*

$$\mathfrak{f} : 0 = V_0 < V_1 < \ldots < V_v = V$$

such that dim $V_i = i$ for each i. Let the set of all complete flags in V be denoted by F_V.

Taking the case where $V = \mathbb{C}^2$ we see that a complete flag of V is a sequence of subspaces

$$\mathfrak{f} : 0 < \mathbb{C}\begin{pmatrix} x \\ y \end{pmatrix} < \mathbb{C}^2$$

Thus to each flag, $\mathfrak{f}$ we can associate a line through the origin in $\mathbb{C}^2$, $\mathbb{C}\begin{pmatrix} x \\ y \end{pmatrix}$. This is explicitly an element of $\mathbb{P}^1_{\mathbb{C}}$. We can let elements of $SL_2(\mathbb{C})$ act on flags $\mathfrak{f} \in F_{\mathbb{C}^2}$ by matrix multiplication. That is, $A \in SL_2(\mathbb{C})$ acts on $\mathfrak{f} = 0 < \mathbb{C}\begin{pmatrix} x \\ y \end{pmatrix} < \mathbb{C}^2 \in F_{\mathbb{C}^2}$ by,

$$A\mathfrak{f} : 0 < \mathbb{C}A\begin{pmatrix} x \\ y \end{pmatrix} < \mathbb{C}^2$$

This action induces an action of $SL_2(\mathbb{C})$ on $\mathbb{P}^1_{\mathbb{C}}$ and more interestingly on U_0 and U_∞. From the definition of U_0 and U_∞ we see that $U_0 = \left\{\mathbb{C}\begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{P}^1_{\mathbb{C}}\right\}$ and similarly $U_\infty = \left\{\begin{pmatrix} 1 \\ y \end{pmatrix} \in \mathbb{P}^1_{\mathbb{C}}\right\}$. This means we can coordinatise U_0 by x and U_∞ by y. Then we see that the action on $SL_2(\mathbb{C})$ on $\mathbb{P}^1_{\mathbb{C}}$ is given by the following proposition.

Proposition 5.2. *Suppose* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$. *The action of* $SL_2(C)$ *on* $\mathbb{P}^1_{\mathbb{C}}$ *is given by,*

$$\begin{array}{rcl} U_0 : x & \mapsto & \dfrac{ax+b}{cx+d} \\ U_\infty : y & \mapsto & \dfrac{dy+c}{by+a} \end{array}$$

Proof Suppose $x \in U_0$. Then x represents the element $\begin{pmatrix} x \\ 1 \end{pmatrix} \in \mathbb{P}^1_{\mathbb{C}}$. Thus, $A\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} ax+b \\ cx+d \end{pmatrix} = \begin{pmatrix} \frac{ax+b}{cx+d} \\ 1 \end{pmatrix}$ which we identify with $\frac{ax+b}{cx+d} \in U_0$. Also, suppose $y \in U_\infty$. Then y represents the element $\begin{pmatrix} 1 \\ y \end{pmatrix} \in \mathbb{P}^1_{\mathbb{C}}$. Thus, $A\begin{pmatrix} 1 \\ y \end{pmatrix} = \begin{pmatrix} a+by \\ c+dy \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{dy+c}{by+a} \end{pmatrix}$ which we identify with $\frac{dy+c}{by+a} \in U_\infty$. □

We have used our identification of $F_{\mathbb{C}^2}$ with $\mathbb{P}^1_{\mathbb{C}}$ to define how $\mathfrak{sl}_2$ acts, not only on $\mathbb{P}^1_{\mathbb{C}}$, but on our open patches U_0 and U_∞. Thus, as we have a topology on $\mathbb{P}^1_{\mathbb{C}}$, we can see the set of all flags in $\mathbb{C}^2$, $F_{\mathbb{C}^2}$, forms a projective variety, called the *flag variety* of $SL_2(\mathbb{C})$. This is in fact part of a much more general statement, which can be found in [12].

Proposition 5.3. *Let* G *be a connected algebraic reductive group defined over* $\mathbb{C}$. *The set of Borel subgroups of* G *forms an algebraic variety called the* ***flag variety*** *of* G. *We denote the flag variety by* X. *Then* G *acts on* X *transitively.*

Furthermore, for $x \in X$, the stabiliser of x, $B(x)$, coincides with the Borel subgroup corresponding to $x \in X$ and $G/B(x) \to X : g \mapsto gx$ is an isomorphism.

When we are given a set of flags F_V and a connected algebraic reductive group, G, which acts on it, we see that there is a connection between the flag variety and the set of flags. In general, if we define the standard basis of V as $\{e_1, e_2, \ldots, e_n\}$, it turns out that the stabiliser subgroup of the flag, $\mathfrak{f}_0 := 0 < Ke_1 < Ke_1 \oplus Ke_2 < \ldots < V$, under the group action is a Borel subgroup.

In our case, $G = SL_2$, and we can see that B is the subgroup of all upper triangular matrices in G. Since we know that SL_2 acts on $\mathbb{P}^1_{\mathbb{C}}$, let $A \in \mathfrak{sl}_2 = T_1 SL_2$. Then $\{e^{-rA} \mid r \in \mathbb{R}\}$ is a 1-parameter subgroup of SL_2 which has a velocity of A at $r = 0$. That is, $\frac{d}{dr}\left(e^{-rA}\right)|_{r=0} = A$. This means that there is a natural infinitesimal action from $\mathfrak{sl}_2$ to the sheaf of tangent vector fields on $\mathbb{P}^1_{\mathbb{C}}$, $\theta_{\mathbb{P}^1_{\mathbb{C}}}$. We will see that the image of e, h, f through this action will be the global sections of $\mathscr{D}^0$.

Proposition 5.4. *Define the infinitesimal action $\mathfrak{sl}_2 \to \theta_{\mathbb{P}^1_{\mathbb{C}}}$ by,*

$$A \mapsto \frac{d}{dr}\left(f\left(e^{-rA}x\right)\right)|_{r=0}$$

Then in this case on U_0,

$$\begin{aligned} h &\mapsto -2x\delta_x \\ e &\mapsto -\delta_x \\ f &\mapsto t^2\delta_x \end{aligned}$$

where $\delta_x = \frac{d}{dx}$. Furthermore, on U_∞,

$$\begin{aligned} h &\mapsto 2y\delta_y \\ e &\mapsto y^2\delta_y \\ f &\mapsto -\delta_y \end{aligned}$$

where $\delta_y = \frac{d}{dy}$.

Proof We show that this result is true for just h, as the calculation is similar for e and f. Using this map, on U_0,

$$\begin{aligned} h(f(x)) &= \frac{d}{dr}\left(f\left(e^{-rh}x\right)\right)|_{r=0} \\ &= \frac{d}{dr}\left(f\left(\left(\begin{array}{cc} e^{-r} & 0 \\ 0 & e^{r} \end{array}\right)x\right)\right)|_{r=0} \\ &= \frac{d}{dr}\left(f\left(\frac{e^{-r}x}{e^{r}}\right)\right)|_{r=0} \\ &= \frac{d}{dr}\left(f\left(e^{-2r}x\right)\right)|_{r=0} \\ &= f'\left(e^{-2r}x\right)\frac{\partial}{\partial r}\left(e^{-2r}x\right)|_{r=0} \\ &= f'\left(e^{-2r}x\right)(-2)e^{-2r}x|_{r=0} \\ &= -2xf'(x) \end{aligned}$$

Also, on U_∞,

$$\begin{aligned} h(f(y)) &= \frac{d}{dr}\left(f\left(e^{-rh}y\right)\right)|_{r=0} \\ &= \frac{d}{dr}\left(f\left(\left(\begin{array}{cc} e^{-r} & 0 \\ 0 & e^{r} \end{array}\right)y\right)\right)|_{r=0} \\ &= \frac{d}{dr}\left(f\left(\frac{e^{r}y}{e^{-r}}\right)\right)|_{r=0} \\ &= \frac{d}{dr}\left(f\left(e^{2r}y\right)\right)|_{r=0} \\ &= f'\left(e^{2r}y\right)\frac{\partial}{\partial r}\left(e^{2r}y\right)|_{r=0} \\ &= f'\left(e^{2r}y\right)2e^{2r}y|_{r=0} \\ &= 2yf'(y) \end{aligned}$$

□

This proposition shows us that our choice for the generators for U_0 was indeed, quite a natural choice. More so, if we use the relationship, $xy = 1$, to rewrite everything in terms of x and x^{-1} we will obtain the global sections of the untwisted sheaf of differential operators, $\mathscr{D}^0$, on $\mathbb{P}^1_{\mathbb{C}}$.

Kashiwara then continues to show a generalised version of the result concerning the global sections we saw in the last chapter, $U_\lambda \simeq \frac{U(\mathfrak{sl}_2)}{(\Delta-(\lambda+1)^2)}$.

Definition 5.5. *Let $Z(\mathfrak{g})$ be the center of the universal enveloping algebra, $U(\mathfrak{g})$, of $\mathfrak{g}$. Suppose M is a $U(\mathfrak{g})$-module, and $\chi : U(\mathfrak{g}) \to \mathbb{C}$ is a character on $U(\mathfrak{g})$. Then*

χ *is an* ***infinitesimal character*** *on* M *if* $Pu = \chi(P)u$ *for all* $u \in M$ *and* $P \in Z(\mathfrak{g})$.

In [15], Kashiwara neatly shows that the global sections of $\mathscr{D}^\lambda$ on an arbitrary flag variety, X, are isomorphic to a quotient of the universal enveloping algebra, $U(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G.

Theorem 5.6. *Let* $X \simeq G/B$ *be a flag variety. Then the global sections of* $\mathscr{D}^\lambda$ *on* X, $\Gamma(X; \mathscr{D}^\lambda)$, *is isomorphic to a quotient of the universal enveloping algebra of* $\mathfrak{g}$, *the Lie algebra corresponding to* G. *That is,*

$$\Gamma(X; \mathscr{D}^\lambda) \simeq \frac{U(\mathfrak{g})}{U(\mathfrak{g})ker(\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C})}$$

In this chapter, we have seen that the results of this thesis so far are indeed a special case of a deep and wonderful area of mathematics. Kashiwara thoroughly generalises these results in his study of flag varieties using representation theory and $\mathscr{D}$-modules.

Chapter 6

Concluding Remarks

Algebraic Geometry has given us the vital tools to study differential operatoring on the projective line. That is, the theory of varieties, sheaves and schemes. These are some of the basic building blocks of Algebraic Geometry, and we saw that they comfortably allowed us to describe how differential operators could be understood on $\mathbb{P}^1_{\mathbb{C}}$.

The theory developed in this thesis, paints a simplified picture of a more general setting. Kashiwara, in [12], describes differential operatoring on an arbitrary flag variety. We used similar techniques to that used by Kashiwara, that is, that of sheaves, $\mathscr{D}$-modules and representation theory. Kashiwara, in [12], uses these techniques to describe the geometry of these flag varieties, which is something we saw in the last chapter where we saw that there was an infinitesimal action from $U(\mathfrak{sl}_2)$ which created a vector field on $\mathbb{P}^1_{\mathbb{C}}$.

Although we were unable to prove the Beilinson-Bernstein Theorem for the case of $\mathfrak{sl}_2$, we demonstrated its uses by proving the translation principle. The proof of the Beilinson-Bernstein Theorem in the general setting is *highly non-trivial.* Beilinson and Bernstein, in their article *Localisation de* $\mathfrak{g}$*-modules*, [17], gave a sketch of a new classification of irreducible Harish-Chandra modules. More importantly, they proved the Kazhdan-Lusztig multiplicity conjecture which made their work so significant and influential for future research.

This thesis provides the first few baby steps in the direction of Algebraic Geometry. Algebraic Geometry is such a diverse and deep field, that it can be approached from many different angles, we followed Hartshorne in [6] by beginning with varieties, sheaves and schemes. These were effectivley the only tools we needed, but we should remark that the usefulness of Algebraic Geometry, truly is, *limitless.*

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