Contents

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Introduction

In this thesis, the principal objects of study are Riemann surfaces, and the aim will be to expound the classical Torelli theorem relating Riemann surfaces to their Jacobians, which are central to their study.

Riemann's original definition in his doctoral thesis [Rie51] amounts to saying that a Riemann surface is an n-sheeted branched cover of $\mathbb{P}^1$. At that time, Riemann surfaces were merely a convenient way to represent multi-valued functions. Klein took up the subject after Riemann and studied Riemann surfaces via differential geometry as objects in their own right. Weyl formalised Klein's ideas in his famous monograph *Die Idee der Riemannschen Fläche* [Wey23]. Today we define a Riemann surface as a (compact) connected one dimensional complex manifold. It is interesting to note that the definition of a complex manifold did not appear in the literature until mid 40's. The phrase *komplexe analytische Mannigfaltigkeit* [1] first appeared in Teichmüller's [Tei44], and the English version appears in Chern's [Che46] in 1946. For more on the history of Riemann surfaces, see Remmert's delightful recount in [Rem98].

The Jacobian of a Riemann surface S is a complex torus, and in fact, is an abelian variety. Its definition is intrinsic to S, and captures much of its information. Torelli's theorem states that given a Jacobian of a Riemann surface and an additional piece of data, called the principal polarisation, one can recover the Riemann surface up to isomorphism. The proof which we present follows Andreotti's [And58]. It is interesting to note that Marten published a new proof of the Torelli theorem [Mar63], which uses combinatorial techniques together with Abel's theorem and the Riemann-Roch theorem. Torelli's original publication on Jacobians can be found here [Tor13].

In [Mum75], Mumford speaks of the "amazing synthesis" of algebra, analysis, and geometry that is at the heart of the geometry of algebraic curves. This trichotomy is evident in that complex algebraic curves are in a one to one correspondence with Riemann surfaces, each emphasising different methods used to explore the geometry of these objects. The

[1] This is German for *complex analytic manifold*.

amazing synthesis goes much further; to quote Mumford again [Mum95], algebraic geometry is not an "elementary subject" but draws from, and contributes to, many disparate disciplines in mathematics. So the difficulty for any initiate of algebraic geometry lies in the tremendous amount of background which has to be covered, as well as the depth and breadth of the ideas in algebraic geometry which itself has enjoyed a long and illustrious history. This is the vindication for the long list of topics assumed. For a history of algebraic geometry, we refer the reader to Dieudonné's article, [Die72].

0.1 Assumed Knowledge

We assume knowledge of very basic differential geometry and complex manifolds, an elementary treatment of complex manifolds can be found in chapter 7 of [Che00]. To give an idea of the depth of knowledge assumed, concepts such as Kähler manifolds, Hermitian metrics, differentials, tangent and cotangent bundles will be used without comment.

Also assumed is a basic understanding of algebraic geometry, where the relevant background can be found in the notes of a course on algebraic geometry taught by Daniel Chan at UNSW in 2004. The course was based on [Sha74], and the notes, edited by the author, can be found at [CC04]. We will not list the concepts assumed, and explicit references to these notes will be made in the thesis. Another very good source of information for the subset of algebraic geometry associated to the thesis material is [Mum95].

The basic concepts algebraic topology, homological algebra, and category theory are also assumed. Again to give some idea of what is assumed, the following concepts will be used without digression; categories, functors, cochain complexes, exact sequences, Poincaré duality, the Euler characteristic, the Mayer-Vietoris sequence, and simplicial, de Rham, and Dolbeault cohomology. For an exposition of these concepts we refer the reader [Hat02] for algebraic topology, and [Osb00] for homological algebra.

0.2 Outline

This section provides an outline for the development of the material. The chapters should be read in sequence to maintain coherence.

In chapter 1, we begin with sheaf theory and their cohomology. The reason for beginning with this technical topic is that the chapters which follow employ extensively the language and techniques of sheaves and sheaf cohomology. The definition of coherent sheaves can be found in [Uen01]. Chapters IX and X of [Mir95] contains a very clear exposition on sheaves. Pages 11-18 of [EH00] contain a basic introduction to sheaf theory.

From chapter 2 onwards, the main reference will be [GH78], which is a very comprehensive treatment of algebraic geometry from an analytic perspective.

In chapter 2, we introduce Riemann surfaces, and derive some of their selected properties which will be used in proving the Torelli theorem. The differences between hyperelliptic and non-hyperelliptic Riemann surfaces are discussed. Kirwan's book [Kir92] is elementary in its treatment; [Cle80] contain many interesting examples, but assumes previous knowledge in many areas. The topic of Riemann surfaces are thoroughly developed in both [Mir95] and [FK92].

Chapter 3 and 4 develop more advanced material concerning Riemann surfaces. The Abel and Jacobi theorems are first discussed, then the concepts of divisors, line bundles are introduced. Linear systems are explored in chapter 4, with an emphasis on linear systems on Riemann surfaces, culminating in the Riemann-Roch theorem, which is a formula for the dimension of a linear system on a Riemann surface.

Complex Tori are discussed at length in chapter 5, in anticipation to the discussion of the Jacobian variety in chapter 6. The line bundles on complex tori are classified, and the theta functions are obtained from global sections of such line bundles. The references for this chapter are [Pol03] and [GH78]. The next chapter on the Jacobian variety applies the results for complex tori, culminating in Riemann's theorem.

The penultimate chapter gives the proof of the Torelli theorem, using much of material developed above. Finally, we end with some concluding remarks regarding the Torelli theorem.

Chapter 1

Sheaves and sheaf cohomology

Algebraic geometry was transformed by Serre in the 1950's by his introduction of sheaf theoretic techniques [Ser55]. The main reason for using sheaves in this thesis is to access sheaf cohomology, which is, as we shall see, a powerful and concise technique. We seek to emphasis this approach as much as possible. This chapter contains the elementary definitions and theorems of sheaves and their cohomology. For a full treatment of sheaves in the context of algebraic geometry, see [Uen01].

1.1 Sheaves

The machinery of sheaves allows one to organise local information and extract global properties of a topological space, X. A sheaf associates algebraic data to each open set of X, and does so functorially. Let X denote a topological space; we first define a presheaf over X.

Definition 1.1 *A* ***presheaf*** $\mathcal{F}$ *of abelian groups over* X *is a contravariant functor from the category of open sets of* X*, where the morphisms are given by the inclusion maps, to the category of abelian groups* ***Ab****, where the morphisms are given by the group homomorphisms.*
That is, for every pair of open sets V *and* U *such that* $V \subset U$*, we have the restriction homomorphism* $\rho_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$*. Moreover these restriction homomorphisms satisfy*

1. $\rho_{U,U} = \mathrm{id}_U$ *for all* U*, and*
2. *that the following diagram commutes for all open sets* $W \subset V \subset U$

$$\mathcal{F}(U) \xrightarrow{\rho_{U,V}} \mathcal{F}(V) \xrightarrow{\rho_{V,W}} \mathcal{F}(W), \qquad \mathcal{F}(U) \xrightarrow[\rho_{U,W}]{} \mathcal{F}(W)$$

For concision, we will often write $\rho_{U,V}(\sigma) = \sigma|_V$ *for* $\sigma \in \mathcal{F}(U)$*.*

Presheaves of rings or vector spaces[1] can be analogously defined. When we speak about sheaves in general, we will always refer to sheaves of abelian groups. The presheaf organises local information and stipulates they are consistent. The presheaves form a category with the following

Definition 1.2 *Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves over X. A* ***presheaf morphism*** $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ *is a collection of group morphisms $\alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ for every open set $U \subset X$, such that for every pair $U \subset V$ of open sets in X the following diagram commutes*

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\alpha_U} & \mathcal{G}(U) \\ {\scriptstyle \rho_{U,V}}\downarrow & & \downarrow{\scriptstyle \rho_{U,V}} \\ \mathcal{F}(V) & \xrightarrow{\alpha_V} & \mathcal{G}(V) \end{array}$$

We need an extra patching condition to define a sheaf.

Definition 1.3 *A presheaf $\mathcal{F}$ on X is a* ***sheaf*** *if X satisfies the* ***sheaf condition****: for every open set $U \subset X$, let $\{U_i\}_{i\in I}$ be an open cover of U. If the collection $\sigma_i \in \mathcal{F}(U_i), i \in I$ satisfies $\sigma_i|_{U_{ij}} = \sigma_j|_{U_{ij}}$* [2] *for all $i, j \in I$, then there exists a unique $\sigma \in \mathcal{F}(U)$ such that $\sigma|_{U_i} = \sigma_i$ for all $i \in I$.*
We call the elements $\sigma \in \mathcal{F}(U)$ ***sections*** *of $\mathcal{F}$ over U. If $U = X$ we call σ a* ***global section****.*

Definition 1.4 *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves over X. A* ***sheaf morphism*** $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ *is defined to be the presheaf morphism $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$.*

In particular, let $\{U_i\}_{i\in I}$ be an open cover of X; then to check that $\sigma = \tau$ where $\sigma, \tau \in \mathcal{F}(X)$, it suffices to check that $\sigma|_{U_i} = \tau|_{U_i}$ for all $i \in I$. Note that a presheaf is a priori not a sheaf, so the sheaf condition is not vacuous. We give the following example of a presheaf which is not a sheaf.

Example 1.5 Let $\overline{\mathbb{Z}}$ be the presheaf of *constant* functions on a topological space X, that is, for every open set $U \subset X$,

$$\overline{\mathbb{Z}}(U) \quad = \quad \{f : U \longrightarrow \mathbb{Z} \mid f \text{ is constant}\}.$$

Suppose X has two connected components, X_1 and X_2. Let $\{U_i\}_{i\in I}$ be any open cover of X and we take a refinement such that $U_i \subset X_0$ or $U_i \subset X_1$ for all $i \in I$. Then the

[1] The notion of sheaves of modules requires more explanation, see [Uen01].
[2] Note that $U_{ij} = U_i \cap U_j$. We will keep this notation throughout.

collection of sections $\sigma_i \in \overline{\mathbb{Z}}(U_i)$ where

$$\sigma_i = \begin{cases} 0 & \text{if} \quad U_i \subset X_0 \\ 1 & \text{if} \quad U_i \subset X_1 \end{cases}$$

satisfies $\sigma_i|_{U_{ij}} = \sigma_j|_{U_{ij}}$ for all $i, j \in I$. However, there exists no global section σ such that $\sigma|_{U_i} = \sigma_i$, since if such a σ exists

$$\sigma|_{U_i} = \begin{cases} 0 & \text{if} \quad U_i \subset X_0 \\ 1 & \text{if} \quad U_i \subset X_1 \end{cases} \tag{1.1}$$

contradicting the fact that σ is constant on X. In other words, there is no section which is constant on X and which agrees with the value of σ_i in each U_i.

We see that $\overline{\mathbb{Z}}$ is not a sheaf, and that a possible remedy is the addition of 'extra' sections. This is accomplished by allowing *locally constant* functions. Denote

$$\mathbb{Z}(U) = \{f : U \longrightarrow \mathbb{Z} \mid \forall p \in U, \exists \text{ an open set } U' \subset U \text{ such that } f|_{U'} \text{ is constant}\}$$

for each open $U \subset X$ [3], to be the presheaf of locally constant functions on X. Then we see that in (1.1), $\sigma(x) = \begin{cases} 0 & \text{if} \quad x \in X_0 \\ 1 & \text{if} \quad x \in X_1 \end{cases}$ is locally constant, and hence $\sigma \in \mathbb{Z}(X)$.

Given a presheaf morphism $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ of presheaves over X, define the **kernel** of α, **cokernel** of α, and **image** of α to be the corresponding presheaves,

$$\begin{aligned} \ker(\alpha)(U) &:= \ker(\alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)) \\ \operatorname{im}(\alpha)(U) &:= \operatorname{im}(\alpha_U : \mathcal{F}(U) \longrightarrow \mathcal{G}(U)) \\ \operatorname{coker}(\alpha)(U) &:= \mathcal{G}(U)/\operatorname{im}(\alpha_U) \end{aligned}$$

for all open sets $U \subseteq X$. That the above define presheaves follow from the definition of presheaves.

Proposition 1.6 *Let $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ be sheaves over X and $\alpha : \mathcal{F} \longrightarrow \widetilde{\mathcal{F}}$ be a sheaf morphism. Then the presheaf* $\ker(\alpha)(U)$ *is a sheaf.*

Proof It suffices to check the sheaf condition. Let $\{U_i\}_{i \in I}$ be an open cover of U, and $\sigma_i \in \ker(\alpha)(U_i)$ satisfying $\sigma_i|_{U_{ij}} = \sigma_j|_{U_{ij}}$ for all $i, j \in I$. Now since $\mathcal{F}$ is a sheaf, consider

[3] This is standard notation, so unfortunately the burden is on the reader to remember that this is the sheaf of locally constant functions with values in $\mathbb{Z}$, not the ring of integers, $\mathbb{Z}$. However, the context should eliminate any ambiguity.

σ_i as elements of $\mathcal{F}(U_i)$, so there exists a unique $\sigma \in \mathcal{F}(U)$ such that $\sigma|_{U_i} = \sigma_i$ for all $i \in I$. It remains to show that $\sigma \in \ker(\alpha)(U)$. Consider the following commutative diagram

$$\begin{array}{ccccc} \ker(\alpha_U) & \longrightarrow & \mathcal{F}(U) & \xrightarrow{\alpha_U} & \tilde{\mathcal{F}}(U) \\ {\scriptstyle \rho_{U,U_i}}\downarrow & & {\scriptstyle \rho_{U,U_i}}\downarrow & & {\scriptstyle \tilde{\rho}_{U,U_i}}\downarrow \\ \ker(\alpha_{U_i}) & \longrightarrow & \mathcal{F}(U_i) & \xrightarrow{\alpha_{U_i}} & \tilde{\mathcal{F}}(U_i) \end{array}$$

for all $i \in I$. Hence $\alpha_U(\sigma)|_{U_i} = \alpha_{U_i}(\sigma|_{U_i}) = \alpha_{U_i}(\sigma_i) = 0$ for all $i \in I$, so by the sheaf condition on $\widetilde{\mathcal{F}}$, $\alpha_U(\sigma) = 0$, that is $\sigma \in \ker(\alpha)(U)$. □

However the presheaves $\text{im}(\alpha)$ and $\text{coker}(\alpha)$ need not be sheaves. To define cokernels in the category of sheaves, we need the sheafification construction. First consider an open set $U \subset X$ and an open cover $\{U_i\}_{i \in I}$ of U.

Definition 1.7 *Let $\mathcal{F}$ be a presheaf of abelian groups over X, $U \subset X$ be any open set, and $\{U_i\}_{i \in I}$ be an open cover of U. Define*

$$\mathcal{F}^+(U) \quad = \quad \ker\left(\prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{(j,k)} \mathcal{F}(U_j \cap U_k)\right)$$

for all open sets $U \subset X$ and all ; where

$$\begin{array}{rcl} \prod_i \mathcal{F}(U_i) & \rightrightarrows & \prod_{(j,k)} \mathcal{F}(U_j \cap U_k) \\ (\sigma_j)_{i \in I} & \longmapsto & (\sigma_i|_{U_{ij}} - \sigma_j|_{U_{ij}})_{i,j \in I}. \end{array}$$

*Then the **sheafification** of $\mathcal{F}$, denoted* $\text{sheaf}(\mathcal{F})$*, is defined as the sheaf $\mathcal{F}^{++}$ together with the canonical morphism $\mathcal{F} \xrightarrow{\varphi} \text{sheaf}(\mathcal{F})$.*

Proposition 1.8 *The sheafification $\mathcal{F} \xrightarrow{\varphi} \text{sheaf}(\mathcal{F})$ satisfies the following universal property. Let $\mathcal{F}$ be a presheaf, $\mathcal{G}$ be a sheaf and $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ be a presheaf morphism. Then there exists a unique sheaf morphism $\tilde{\alpha}$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\quad\alpha\quad} & \mathcal{G}\,. \\ & {\scriptstyle\varphi}\searrow \quad \nearrow{\scriptstyle\tilde{\alpha}} & \\ & \text{sheaf}(\mathcal{F}) & \end{array}$$

Example 1.9 Returning to example 1.5, $\mathbb{Z} = \text{sheaf}(\overline{\mathbb{Z}})$. Moreover $\mathbb{Z}(X)$ is a free abelian group with its number of generators equal to the number of connected components of X.

The **kernel**, **cokernel** and **image** of a sheaf morphism $\alpha : \mathcal{F} \longrightarrow \mathcal{G}$ are defined to be the respective sheafifications of the kernel, cokernel, and image of α considered as a presheaf morphism.

Note 1.10 We see that the category of presheaves and the category of sheaves are **abelian categories**, which roughly means a category where kernels and cokernels are well-defined for any of its morphisms. As a result, exact sequences are well defined in abelian categories.

Definition 1.11 *Suppose*

$$\ldots \longrightarrow \mathcal{F}_{n-1} \xrightarrow{\alpha_{n-1}} \mathcal{F}_n \xrightarrow{\alpha_n} \mathcal{F}_{n+1} \longrightarrow \ldots$$

is a sequence of sheaves over X. Then we say that the sequence is ***exact*** *at $\mathcal{F}_n$ if $\alpha_{n-1} \circ \alpha_n = 0$ and $\ker(\alpha_n) = \operatorname{im}(\alpha_{n-1})$. We say the sequence is* ***exact*** *if it is exact at each $\mathcal{F}_k$.*

An important instance of an exact sequence of sheaves is the short exact sequence. For this we need the concept of a zero sheaf. This is simply the assignment $0(U) = 0$ for all open sets U.

Example 1.12 Let

$$0 \longrightarrow \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \longrightarrow 0$$

be an exact sequence of sheaves over X. We call this a **short exact sequence** and we see that $\ker(\psi) = \mathcal{F}$ and $\operatorname{coker}(\varphi) = \mathcal{H}$. In this case, we say $\mathcal{F}$ is a **subsheaf** of $\mathcal{G}$ and $\mathcal{H}$ is the **quotient sheaf** of $\mathcal{G}$ with $\mathcal{F}$, denoted $\mathcal{G}/\mathcal{F}$.

Now given a presheaf $\mathcal{F}$ over X we can define a functor from the category of presheaves to the category of abelian groups by the assignment $\Gamma : \mathcal{F} \longmapsto \mathcal{F}(X)$. This is called the **global sections functor**. The definition for the global sections functor in the category of sheaves is identical.

Definition 1.13 *Suppose $\boldsymbol{A}$ and $\boldsymbol{A}'$ are abelian categories and*

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\phi} C \longrightarrow 0$$

is an exact sequence in $\boldsymbol{A}$. Then a functor $F : \boldsymbol{A} \longrightarrow \boldsymbol{A}'$ is said to be ***exact*** *if the sequence*

$$0 \longrightarrow F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\phi)} F(C) \longrightarrow 0$$

is exact in $\boldsymbol{A}'$*; and* ***left exact*** *if*

$$0 \longrightarrow F(A) \xrightarrow{F(\varphi)} F(B) \xrightarrow{F(\phi)} F(C)$$

is exact in $\boldsymbol{A}'$*.*

Note that right exactness of a functor is defined analogously. The important point here is that the categories of sheaves and presheaves have the same notion of morphisms, but not the same notion of cokernels. A consequence of this is the following

Proposition 1.14

1. *The global sections functor is an exact functor from the category of presheaves to abelian groups.*
2. *The global sections functor is a left exact functor from the category of sheaves to the category of abelian groups. In particular, it is not exact.*

The first part of the above definition is by the definition of a presheaf. To see the second part, we will produce examples to show that the global sections functor in the category of sheaves is not exact. It turns out that this is the reason why there is the need for a cohomology theory for sheaves. The discussion of cohomology of sheaves continue in section 1.3.

1.2 Sheaves associated with functions

There is often a distinguished class of functions over X which we are interested in.

Definition 1.15 *Let* M *be a complex manifold. The assignment* $U \longmapsto \{f : M \longrightarrow \mathbb{C} \mid f \text{ holomorphic}\}$ *for every open set* $U \subseteq M$ *is called the* ***structure sheaf*** *and is denoted* $\mathcal{O}_M$*, or* $\mathcal{O}$ *when there is no ambiguity.*

We have defined the structure sheaf to be the sheaf of holomorphic functions. However, this need not always be the case. For instance, in algebraic geometry over an arbitrary field $\mathbb{K}$ of characteristic 0, one may defines the structure sheaf to be $\mathcal{O}(U) := \{f : U \longrightarrow \mathbb{K} \mid f \text{ rational}\}$ where $U \subset V$ is open and V is a projective variety[4] in $\mathbb{P}^n$. The following example collects some frequently occurring sheaves in complex geometry.

[4]See [CC04] for definition.

Example 1.16 We have met some of these previously. The following are related to $\mathcal{O}_M$ as they depend on the analytic structure of M, these are as follows

$$\begin{array}{ll} \mathcal{O}_M^* & \text{sheaf of nonvanishing holomorphic functions on } M \\ \mathcal{K}_M & \text{sheaf of meromorphic functions on } M \\ \mathcal{K}_M^* & \text{sheaf of meromorphic functions on } M \text{ not identically zero} \\ \Omega^k & \text{sheaf of holomorphic } k\text{-differentials on } M \\ \Omega_{\overline{\partial}}^{p,q} & \text{sheaf of holomorphic differentials of type } (p,q) \\ \mathcal{O}(L) & \text{sheaf of sections of holomorphic line bundle } L \end{array}.$$

We shall adopt the convention that $\Omega^0 = \mathcal{O}$. We also have

$$\begin{array}{ll} C^\infty & \text{sheaf of smooth functions on } M \\ \mathcal{A}_{\overline{\partial}}^{p,q} & \text{sheaf of smooth differentials of type } (p,q) \\ \mathcal{A}^k & \text{sheaf of smooth } k\text{-differentials} \end{array}.$$

Finally we have the locally constant sheaves which are related to the topological structure of M, these are $\mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ for the sheaves of locally constant functions $M \longrightarrow \mathbb{Z}, \mathbb{R}, \mathbb{C}$.

Example 1.17 Let M be a compact complex manifold. A very important short exact sequence is the following

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0,$$

called the **exponential sequence**. The map $\iota_U : \mathbb{Z}(U) \longrightarrow \mathcal{O}(U)$ is simply inclusion, and $\exp_U : \mathcal{O}(U) \longrightarrow \mathcal{O}(U)$ is given by $(\exp_U(f))(z) = e^{2\pi i f(z)}$ for $z \in U$.
Moreover this sequence is exact. Firstly $(\iota_U \circ \exp_U)(f)(z) = \exp(2\pi i f(z)) = 1$ since f is a locally constant function taking integer values. Now

$$\begin{aligned} \ker(\exp)(U) &= \ker(\exp_U : \mathcal{O}(U) \longrightarrow \mathcal{O}^*(U)) \\ &= \mathbb{Z}(U) \end{aligned}$$

so $\ker(\exp) = \mathbb{Z}$ as sheaves. Finally to show exp is surjective as a sheaf map, we show exp has a local inverse. That is, for every $g \in \mathcal{O}^*$, and every $p \in M$, there exists an open neighbourhood of p such that the equation $\exp(2\pi i f)(z) = g(z)$ has a solution: namely $\frac{1}{2\pi i}\log(g(z))$, which is holomorphic on some neighbourhood of p chosen to not contain any branch cuts of $\log(g(z))$.

1.3 Sheaf cohomology

We begin with a sketch of why one studies sheaf cohomology. Firstly, sheaf cohomology replicates important instances of classical cohomology, in particular, we will see that $H^\bullet(X,\mathbb{Z})$, $H^\bullet(X,\mathbb{R})$ and $H^\bullet(X,\Omega^{p,q}_{\bar\partial})$ correspond to simplicial, de Rham and Dolbeault cohomology respectively. As with classical cohomology, one of the aims is to formulate algebraic invariants for topological spaces. Sheaf cohomology, in general, allows one to do so with arbitrary sheaves and in this way generalises classical cohomology theories.

The second reason, as alluded to above, is the fact that the global sections functor from the category of sheaves to the category of abelian groups is not exact. Experience shows that exact sequences are a natural and concise way to express certain facts in mathematics. An exact sequence of sheaves, say $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$, over X generally correspond to some property holding locally, while $0 \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \longrightarrow \mathcal{H}(X) \longrightarrow 0$ correspond to the same property holding globally on X. Hence the obstruction to exactness of the global sections functor correspond to the obstruction to passing from local properties to global properties. We give an example to illustrate this.

Example 1.18 Let $X = \mathbb{C} - \{0\}$. Applying the global sections functor to the exponential sequence

$$0 \longrightarrow \mathbb{Z} \stackrel{\iota}{\longrightarrow} \mathcal{O} \stackrel{\exp}{\longrightarrow} \mathcal{O}^* \longrightarrow 0 \qquad (1.2)$$

over X, we obtain the left exact sequence

$$0 \longrightarrow \mathbb{Z}(X) \longrightarrow \mathcal{O}(X) \stackrel{\exp}{\longrightarrow} \mathcal{O}^*(X)$$

of $\mathbb{C}$-vector spaces. The exponential map in the second sequence is not surjective, since $z \in \mathcal{O}^*(X)$ is not in the image of exp. We can interpret (1.2) as saying that exp is only locally invertible, but does not have a holomorphic inverse on all of X.

We will briefly sketch the derived functor approach to sheaf cohomology, which measures the obstruction to exactness of the global sections functor. No proofs will be given below, see [Uen01] for details.

Definition 1.19 *A sheaf $\mathcal{R}$ over X is said to be **flasque** if the restriction map $\mathcal{R}(X) \longrightarrow \mathcal{R}(U)$ is surjective for all open sets $U \subset X$. A **flasque resolution** of a sheaf is a sequence*

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{R}_1 \longrightarrow \mathcal{R}_2 \longrightarrow \mathcal{R}_3 \longrightarrow \dots \qquad (1.3)$$

such that $\mathcal{R}_1, \mathcal{R}_2, \dots$ are flasque sheaves over X and (1.3) is exact.

A flasque resolution exists for any sheaf $\mathcal{F}$ and in fact the flasque resolution is canonical. Let $\mathcal{G}$ be a sheaf over X, and Γ be the global sections functor. Further let

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{R}_1 \xrightarrow{\delta_1} \mathcal{R}_2 \xrightarrow{\delta_2} \mathcal{R}_3 \xrightarrow{\delta_3} \dots$$

be the canonical choice of flasque resolution for $\mathcal{G}$ and denote $R^\bullet := 0 \longrightarrow \mathcal{R}_1 \longrightarrow \mathcal{R}_2 \longrightarrow \dots$. Apply the functor Γ to obtain

$$0 \longrightarrow \mathcal{G}(X) \longrightarrow \mathcal{R}_1(X) \xrightarrow{\delta_{1_X}} \mathcal{R}_2(X) \xrightarrow{\delta_{2_X}} \mathcal{R}_3(X) \xrightarrow{\delta_{3_X}} \dots$$

which is a cochain complex of abelian groups. We define the **i-th cohomology group** of X **with coefficients in** $\mathcal{G}$ to be

$$H^i(X, \mathcal{G}) \quad := \quad H^i(F(R^\bullet)) \quad = \quad \frac{\ker(\delta_{i_X})}{\operatorname{im}(\delta_{i-1_X})}$$

We have skipped most of the details in the above sketch, the point is to see that sheaf cohomology does in fact measure the obstruction to exactness of Γ. The is called the derived functor approach as it is a special case of such a construction in homological algebra (c.f. [Osb00]). The above constitutes the conceptual scaffold, but it is not a computable theory. We will approach sheaf cohomology via Čech cohomology, which is an alternative, and computable way of doing sheaf cohomology[5].

Definition 1.20 *Let $\mathfrak{U} := \{U_\alpha\}_{\alpha \in A}$ be a locally finite cover of X. For every multi-index $I = \{i_0, \dots, i_k\} \subseteq A$, denote $U_I = \bigcap_{i \in I} I_i$. Define the* ***Čech complex*** *to be*

$$C^\bullet(\mathfrak{U}, \mathcal{F}) \quad := \quad 0 \longrightarrow C^0(\mathfrak{U}, \mathcal{F}) \longrightarrow C^1(\mathfrak{U}, \mathcal{F}) \longrightarrow C^2(\mathfrak{U}, \mathcal{F}) \longrightarrow \dots$$

where $C^k := C^k(\mathfrak{U}, \mathcal{F}) := \prod_{|I|=k+1} \mathcal{F}(U_I)$.

- *We call an element $\sigma \in C^k(\mathfrak{U}, \mathcal{F})$ a* ***k-cochain*** *and write $\sigma = \prod_{|I|=k+1}(\sigma_I) = (\sigma_I)_{|I|=k+1}$ with $\sigma_I \in \mathcal{F}(U_I)$.*
- *The* ***coboundary map****, $\delta : C^k \longrightarrow C^{k+1}$ is given by $\delta\sigma = \prod_{|J|=j+2}(\delta\sigma)_J$ where*

$$(\delta\sigma)_J \quad = \quad \sum_{i=0}^{k+1} (-1)^i \left(\sigma_{J-\{j_i\}}\big|_{U_J} \right)$$

  and we call $\tau \in \delta C^{k+1}$ a ***coboundary****.*

[5]The Čech cohomology groups agree with the derived functor cohomology groups for a quasi-coherent sheaf over a separated Noetherian scheme.

- *An element $\sigma \in \ker(\delta)$ is called a* ***cocycle****.*
- *The* ***p-th Čech cohomology group*** *of $\mathcal{F}$ is the direct limit (see [Osb00] for definition)*

$$H^p(X, \mathcal{F}) \quad = \quad \varinjlim_{\mathfrak{U}} H^p(\mathfrak{U}, \mathcal{F})$$

where $H^p(\mathfrak{U}, \mathcal{F}) = \frac{\ker(\delta : C^p \longrightarrow C^{p+1})}{\operatorname{im}(\delta : C^{p-1} \longrightarrow C^p)}$ is the p-th cohomology group of the Čech complex $C^\bullet(\mathfrak{U}, \mathcal{F})$.

The direct limit which appears in the definition of a Čech cohomology group defies computation. Leray's theorem tells us when the open cover $\mathfrak{U}$ of X is 'good enough' such that $H^p(\mathfrak{U}, \mathcal{F}) = H^p(X, \mathcal{F})$.

Theorem 1.21 *(Leray's theorem) Suppose $\mathcal{F}$ is a sheaf over X and $\mathfrak{U} = \{U_i\}_{i \in I}$ is an open cover of X such that for some integer p, the Čech cohomology groups $H^q(U_{i_1, \ldots, i_p})$ vanish for all $q > 0$ and for all $i_1, \ldots, i_p \in I$. Then*

$$H^\bullet(\mathfrak{U}, \mathcal{F}) \quad = \quad H^\bullet(X, \mathcal{F})$$

Proposition 1.22 *There is a natural isomorphism of vector spaces $\mathcal{F}(X) \simeq H^0(X, \mathcal{F})$.*

Proof Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in A}$ be an open cover for X. Then

$$H^0(\mathfrak{U}, \mathcal{F}) \quad = \quad \ker(\delta : C^0(\mathfrak{U}, \mathcal{F}) \longrightarrow C^1(\mathfrak{U}, \mathcal{F}))/\{0\}$$

For $\sigma \in C^0(\mathfrak{U}, \mathcal{F}) = \prod_{\alpha \in A} \mathcal{F}(U_\alpha)$, $(\delta\sigma)_{\alpha,\beta} = \sigma_\beta|_{U_{\alpha\beta}} - \sigma_\alpha|_{U_{\alpha\beta}}$ for any $\alpha, \beta \in A$. The condition $\sigma \in \ker(\delta : C^0 \longrightarrow C^1)$ holds iff $\sigma_\beta = \sigma_\alpha$ on $U_\alpha \cap U_\beta$ for all $\alpha, \beta \in A$. This is equivalent to $\sigma \in \mathcal{F}(X)$. Taking the limit, $\varinjlim_{\mathfrak{U}} H^0(\mathfrak{U}, \mathcal{F}) = H^0(X, \mathcal{F})$ we have $\mathcal{F}(X) \simeq H^0(X, \mathcal{F})$. □

Note 1.23 The zeroth Čech cohomology groups agree with the zeroth derived functor sheaf cohomology groups.

Note 1.24 Some authors denote the Čech cohomology groups $\check{H}^p(X, \mathcal{F})$, but since most of our discussion will involve Čech cohomology, I will simply denote them $H^p(X, \mathcal{F})$. Classical cohomology groups will be distinguished by the appropriate subscripts. For instance $H^p_{\mathrm{DR}}, H^p_{\overline{\partial}}, H^p_{\mathrm{simplicial}}$, will denote the de Rham, Dolbeault and simplicial homology groups respectively, which appear in proposition 1.27.

The following basic fact from homological algebra produces a most useful corollary. Its name derives from the shape of the accompanying long exact sequence diagram, as shown below.

Proposition 1.25 *(The snake lemma) Let $C^\bullet, D^\bullet, E^\bullet$ be cochain complexes in an abelian category $\boldsymbol{A}$ and suppose the sequence*

$$0 \longrightarrow C^\bullet \xrightarrow{\psi} D^\bullet \xrightarrow{\varphi} E^\bullet \longrightarrow 0$$

is exact. Then this induces the long exact sequence in cohomology

$$\begin{array}{c} \cdots \longrightarrow H^i(C^\bullet) \xrightarrow{\psi^*} H^i(D^\bullet) \xrightarrow{\varphi^*} H^i(E^\bullet) \xrightarrow{c} \\ H^{i+1}(C^\bullet) \xrightarrow{\psi^*} H^{i+1}(D^\bullet) \xrightarrow{\varphi^*} H^{i+1}(E^\bullet) \longrightarrow \cdots \end{array} \tag{1.4}$$

the map c is called a ***connecting homomorphism****.*

Proof Denote $Z^k(A^\bullet) := \ker(\mathrm{d} : A^k \longrightarrow A^{k+1})$ for $k \in \mathbb{N}$ and similarly for B and C. The exact sequence of cochain complexes $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$ is expanded into the two-dimensional complex

$$\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C^{i+1} & \xrightarrow{\psi} & D^{i+1} & \xrightarrow{\varphi} & E^{i+1} & \longrightarrow & 0 \\
 & & \delta\uparrow & & \partial\uparrow & & \mathrm{d}\uparrow & & \\
0 & \longrightarrow & C^{i} & \xrightarrow{\psi} & D^{i} & \xrightarrow{\varphi} & E^{i} & \longrightarrow & 0 \\
 & & \delta\uparrow & & \partial\uparrow & & \mathrm{d}\uparrow & & \\
0 & \longrightarrow & C^{i-1} & \xrightarrow{\psi} & D^{i-1} & \xrightarrow{\varphi} & E^{i-1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & &
\end{array}$$

where the rows are exact. We first define the maps ψ^*, φ^* and c in (1.4).

The induced maps ψ^*, φ^* in cohomology are given by

$$\begin{array}{ccccccc}
H^i(A^\bullet) & \xrightarrow{\psi^*} & H^i(B^\bullet) & \text{and} & H^i(B^\bullet) & \xrightarrow{\varphi^*} & H^i(C^\bullet) \\
[a] & \longmapsto & [\psi a] & & [b] & \longmapsto & [\varphi b]
\end{array}$$

where $[x] \in H^i(X^\bullet)$ denotes the cohomology class of $x \in Z^i(X^\bullet)$, $X = A, B, C$.

Let $a \in Z^i(A^\bullet)$; by commutativity of the top left square, $\partial\psi a = \psi\delta a = \psi 0 = 0$, we obtain $\psi a \in Z^i(B^\bullet)$. Moreover, suppose $a \in \delta A^{i-1}$, that is $a = \delta a'$ for some $a' \in A^{i-1}$. Applying ψ and by commutativity of the bottom left square we obtain $\psi a = \psi\delta a' = \partial\psi a'$, so $\psi a \in \partial B^{i-1}$. Hence ψ^*, and similarly φ^*, are well-defined maps in cohomology.

Let $\gamma \in Z^i(C^\bullet)$; the connecting homomorphism c is given by

$$\begin{array}{ccc} H^i(C^\bullet) & \xrightarrow{c} & H^{i+1}(A^\bullet) \\ [\gamma] & \longmapsto & [a_\gamma] \end{array}$$

where a_γ is defined below. Let us recall that exactness of rows in the two-dimensional complex above means that ψ is injective, $\ker(\varphi) = \mathrm{im}(\psi)$, and φ is surjective. Also recall that along the columns, the maps $\delta^2, \partial^2, \mathrm{d}^2$ are the zero maps. The following diagram will keep track of the various maps and choices in the following paragraphs.

$$\begin{array}{cccccccc} A^{i+2} \ni & 0 & \overset{\psi}{\longmapsto} & 0 & & & & \\ & \uparrow & & \uparrow & & & & \\ & a_\gamma & \overset{\psi}{\longmapsto} & \partial b & \overset{\varphi}{\longmapsto} & 0 & & \\ & & & \uparrow & & \uparrow & & \\ & & & b & \overset{\varphi}{\longmapsto} & \gamma & & \in C^i \end{array} .$$

Since φ is surjective, we can choose $b \in \varphi^{-1}\{\gamma\} \subset B^i$, and since $\ker(\varphi) = \mathrm{im}(\psi)$, $\partial b \in \mathrm{im}(\psi)$. Hence $\psi^{-1}\{\partial b\} \neq \emptyset$, in fact, ψ is injective, so there is a unique choice of $a_\gamma \in \psi^{-1}\{\partial b\}$. Moreover, since $\partial^2 b = 0$, the top right hand corner is zero since ψ is injective, and by commutativity of the top right hand square, $\delta a_\gamma = 0$, that is, $a_\gamma \in Z^{i+1}(A^\bullet)$.

We show that choosing a different $b' \in \varphi^{-1}\{\gamma\}$ in the bottom row, middle position changes a_γ by a coboundary. Now $b - b' \in \ker(\varphi) = \mathrm{im}(\psi)$, so $\psi^{-1}\{b - b'\} \neq \emptyset$. Choose $a \in \psi^{-1}\{b - b'\}$ so a_γ changes by a coboundary, namely δa.

Finally we show that choosing a different representative $\gamma' \in [\gamma]$ does not change a_γ. Let $\gamma' \in Z^i(C^\bullet)$ such that $\gamma - \gamma' \in \mathrm{d}C^{i-1}$. We use the diagram

$$\begin{array}{cccccccc} A^{i+1} \ni & 0 & \overset{\psi}{\longmapsto} & 0 & & & & \\ & \uparrow & & \uparrow & & & & \\ & a''\gamma & \overset{\psi}{\longmapsto} & \partial b'' & \overset{\varphi}{\longmapsto} & \gamma - \gamma' & & \\ & & & \uparrow & & \uparrow & & \\ & & & b'' & \overset{\varphi}{\longmapsto} & \gamma'' & & \in C^{i-1} \end{array}$$

to keep track of the arguments. Choose $\gamma'' \in \mathrm{d}^{-1}\{\gamma - \gamma'\} \subset C^{i-1}$ and $b'' \in \varphi^{-1}\{\gamma''\}$. Following through the rest of the diagram in much the same manner as above, we obtain $a_{\gamma-\gamma'} = 0$. This shows that c is a well-defined map in cohomology.

Exactness at $H^i(B^\bullet)$: Let $[a] \in H^i(A^\bullet)$, then $(\varphi^* \circ \psi^*)[a] = \varphi^*[\psi a] = [(\varphi \circ \psi)(a)] = [0]$, so $\ker(\psi^*) \supset \operatorname{im}(\varphi^*)$. For the converse, we use the following to keep track of arguments.

$$
\begin{array}{cccccccc}
A^{i+1} \ni & & & & 0 & \overset{\psi}{\longmapsto} & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
A^i \ni & b \overset{\varphi}{\longmapsto} \varphi b & & & a & \overset{\psi}{\longmapsto} & \partial b' - b & \overset{\varphi}{\longmapsto} 0 & \in C^i \\
 & \quad\;\; \uparrow & & & & & & & \\
 & b' \overset{\varphi}{\longmapsto} \gamma & \in C^{i-1} & & & & & &
\end{array}
$$

Suppose $[b] \in \ker(\varphi^*)$, that is $\varphi b \in \mathrm{d}C^{i-1}$. So choose $\gamma \in \mathrm{d}^{-1}\{\varphi b\} \subset C^{i-1}$, and since φ is surjective, choose $b' \in \varphi^{-1}\{\gamma\} \subset B^{i-1}$. Now $\varphi\partial b' = \varphi b$ so $\partial b' - b \in \ker(\varphi) = \operatorname{im}(\psi)$, hence choose $a \in \psi^{-1}\{\partial b' - b\}$. Now a is a cocycle, that is $\delta a = 0$, since $\delta a = \delta\psi(\partial b' - b) = \psi\partial(\partial b' - b) = \psi(\partial^2 b' - \partial b) = 0$ since b is a cocycle. Hence $\psi^*[a] = [b]$, so $[b] \in \operatorname{im}(\varphi^*)$ and $\ker(\psi^*) = \operatorname{im}(\varphi^*)$.

Exactness at $H^i(C^\bullet)$: Let $[b] \in H^i(B^\bullet)$, then $(c \circ \varphi^*)[b] = c[\varphi b]$, now $b \in Z^i(B^\bullet)$, so $a_{\varphi b} = 0$ by the definition of c above, and $\ker(c) \supset \operatorname{im}(\varphi^*)$. For the converse, let $[\gamma] \in \ker(c) \subset H^i(C^\bullet)$, that is $a_\gamma \in \delta A^{i-1}$. So let $a \in \delta^{-1}\{a_\gamma\}$, as follows

$$
\begin{array}{ccccc}
A^{i+1} \ni & a_\gamma & \overset{\psi}{\longmapsto} & \partial b & \\
 & \uparrow & & \uparrow & \\
 & a & & b \overset{\varphi}{\longmapsto} \gamma & \in C^i
\end{array}
$$

where $b \in \varphi^{-1}\{\gamma\}$. Now $\partial\psi a = \partial b$ so $\psi a - b \in Z^i(B^\bullet)$. Moreover $\varphi(\psi a - b) = 0 + \gamma$, hence $\varphi^*[\psi a - b] = [\gamma]$, and we have proved $\ker(c) = \operatorname{im}(\varphi^*)$.

Exactness at $H^{i+1}(A^\bullet)$: Let $[\gamma] \in H^i(C^\bullet)$, then $(\psi^* \circ c)[\gamma] = [\psi a_\gamma]$, but ψa_γ is by definition a coboundary, so $(\psi^* \circ c)[\gamma] = 0$, and $\ker(\psi^*) \supset \operatorname{im}(c)$. Conversely, let $[a] \in \ker(\psi^*) \subset H^{i+1}(A^\bullet)$, that is $\psi a \in \partial B^i$. So let $b \in B^i$ such that $\partial b = \psi a$ then $\varphi b \in Z^i(C^\bullet)$ since $\mathrm{d}\varphi b = \varphi\psi a = 0$. So $c[\varphi b] = [a]$. Thus $\ker(\psi^*) = \operatorname{im}(c)$. The following diagram sums up the above paragraph.

$$
\begin{array}{ccccccc}
A^{i+1} \ni & a & \overset{\psi}{\longmapsto} & \psi a & \overset{\varphi}{\longmapsto} & 0 & \\
 & & & \uparrow & & \uparrow & \\
 & & & b & \overset{\varphi}{\longmapsto} & \gamma & \in C^i
\end{array}
$$

This completes the proof of the theorem. □

Corollary 1.26 *The short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

on a topological space X induces the following long exact sequence in cohomology,

$$\cdots \longrightarrow H^i(X,\mathcal{F}) \longrightarrow H^i(X,\mathcal{G}) \longrightarrow H^i(X,\mathcal{H}) \longrightarrow H^{i+1}(X,\mathcal{F}) \longrightarrow H^{i+1}(X,\mathcal{G}) \longrightarrow H^{i+1}(X,\mathcal{H}) \longrightarrow \cdots$$

Proof Simply note that the short exact sequence of sheaves induce an exact sequence of Čech complexes $0 \longrightarrow C^\bullet(X,\mathcal{F}) \longrightarrow C^\bullet(X,\mathcal{G}) \longrightarrow C^\bullet(X,\mathcal{H}) \longrightarrow 0$ and applying the snake lemma gives the corollary. □

The style of the proof of proposition 1.25 is typical in homological algebra, it even has a name: diagram chasing. As one can gather from the above proof, cohomology is an extremely concise language.

We now return to the complex analytic case, a complex manifold M is clearly also a smooth real manifold. Recall that the ordinary Poincaré lemma states that the groups $H^p_{\mathrm{DR}}(U) = 0$ for all $p > 0$ and U an open convex set in M. In sheaf theoretic language, this says that the **de Rham resolution**

$$0 \longrightarrow \mathbb{R} \longrightarrow C^\infty \xrightarrow{\mathrm{d}} \mathcal{A}^1 \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \mathcal{A}^p \xrightarrow{\mathrm{d}} \ldots$$

is exact. Similarly, the $\overline{\partial}$-Poincaré lemma, which states that $H^{p,q}_{\overline{\partial}}(V) = 0$ for $q > 0$ and V a polycylinder in M, is equivalent to the sheaf sequence

$$0 \longrightarrow \Omega^p \longrightarrow \mathcal{A}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1} \xrightarrow{\overline{\partial}} \ldots \xrightarrow{\overline{\partial}} \mathcal{A}^{p,q} \xrightarrow{\overline{\partial}}$$

being exact. This is known as the **Dolbeault resolution**. We can now construct an appropriate 2-dimensional complex and use diagram chasing to verify the claim that Čech cohomology generalise classical cohomology.

Proposition 1.27 *We have the following isomorphisms of cohomology groups*

1. *(de Rham's theorem)* $H^p_{\mathrm{DR}}(M) \simeq H^p(M,\mathbb{R})$
2. *(Dolbeault's thoerem)* $H^{p,q}_{\overline{\partial}}(M) \simeq H^q(M,\Omega^p)$

3. $H^p_{\text{simplicial}}(M) \simeq H^p(M, \mathbb{Z})$

for all $p, q \in \mathbb{Z}$.

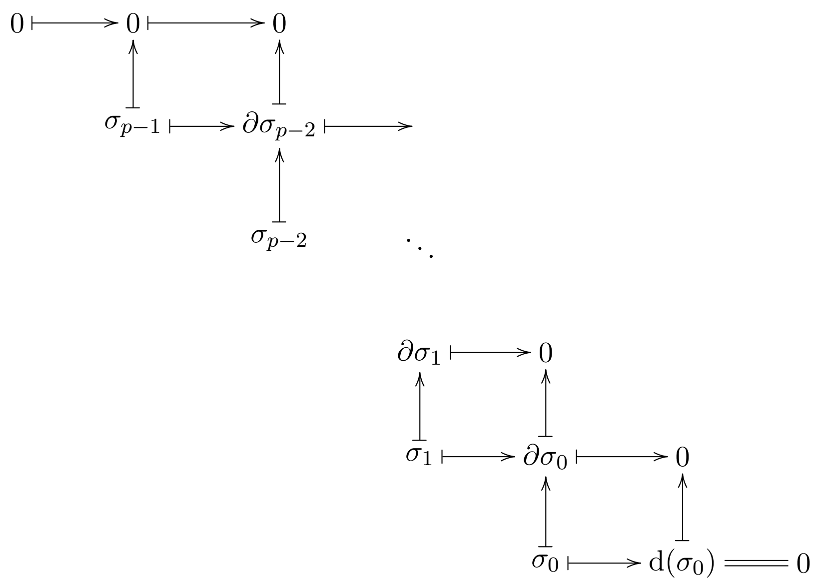
Proof The proof of part 3 can be found in pages 42-43 of [GH78]. Parts 1 and 2 above can be proved by putting the de Rham resolution and the Dolbeault resolution respective in place of $F^\bullet$ (bottom row) in the following two-dimensional complex

$$\begin{array}{cccccccc}
 & & & \vdots & & \vdots & & \\
 & & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \\
 & 0 & \longrightarrow & C^1(\mathcal{F}_1) & \xrightarrow{\mathrm{d}} & C^1(\mathcal{F}_2) & \xrightarrow{\mathrm{d}} & \cdots \\
 & & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \\
 & 0 & \longrightarrow & C^0(\mathcal{F}_1) & \xrightarrow{\mathrm{d}} & C^0(\mathcal{F}_2) & \xrightarrow{\mathrm{d}} & \cdots \\
 & & & \uparrow{\scriptstyle \partial} & & \uparrow{\scriptstyle \partial} & & \\
F^\bullet : & 0 & \longrightarrow & \mathcal{F}_1 & \xrightarrow{\mathrm{d}} & \mathcal{F}_2 & \xrightarrow{\mathrm{d}} & \cdots
\end{array}$$

Since this is similar to the proof of proposition 1.25, we will be more sparing with the details. The relevant part of the double complex is

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & C^{p-1}(\mathcal{F}_1) & \longrightarrow & C^{p-1}(\mathcal{F}_2) & \longrightarrow & & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
0 & \longrightarrow & C^{p-2}(\mathcal{F}_1) & \longrightarrow & C^{p-2}(\mathcal{F}_2) & \longrightarrow & & & & & \\
 & & & & \uparrow & & & & & & \\
 & & & & C^{p-3}(\mathcal{F}_2) & & \ddots & & & & \\
 & & & & & & & & \uparrow & & \\
 & & & & & & C^1(\mathcal{F}_{p-1}) & \longrightarrow & C^1(\mathcal{F}_{p+1}) & & \\
 & & & & & & \uparrow & & \uparrow & & \\
 & & & & & & C^0(\mathcal{F}_{p-1}) & \longrightarrow & C^0(\mathcal{F}_{p+1}) & \longrightarrow & C^0(\mathcal{F}_{p+1}) \\
 & & & & & & & & \uparrow & & \uparrow \\
 & & & & & & & & \mathcal{F}_p & \longrightarrow & \mathcal{F}_{p+1}
\end{array}$$

Let $\sigma_0 \in \ker(\mathrm{d} : \mathcal{F}_p \longrightarrow \mathcal{F}_{p+1})$, and $\sigma_i \in C^{i-1}(\mathcal{F}_{p-i})$ such that $\mathrm{d}\sigma_i = \partial\sigma_{i-1}$. This is possible due to exactness of the rows. We summarise this as follows.



Now $\sigma_{p-1} \in \ker(\partial : C^{p-2}(\mathcal{F}_1) \longrightarrow C^{p-1}(\mathcal{F}_1))$. To show the map $\sigma_0 \longmapsto \sigma_{p-1}$ induces a well defined map in cohomology

$$H^p(F^\bullet) \quad \longrightarrow \quad H^{p-2}(\mathcal{F}_1, X),$$

we check that each choice of σ_i for $0 \leqslant i < p-1$ changes σ_{p-1} by a coboundary. Also we have to show the map is surjective. These are accomplished by tracing through the diagram as in the proof of proposition 1.25, and we will omit these details. □

Note 1.28 Since no properties peculiar to sheaf cohomology were used, this result holds for any double complex in an abelian category with exact rows.

1.4 Vanishing theorems

In the case of left exact functors, we have the long exact sequence in cohomology, and the best one can hope for is the vanishing of some higher cohomology groups. However, there is still a wealth of information encoded in the long exact cohomology sequence. The first theorem identifies a class of sheaves which has trivial Čech cohomology.

Definition 1.29

- *Let $\mathfrak{U} := \{U_i\}_{i\in I}$ be an open cover for M. A family $\{f_i\}_{i\in I}$ where $f_i : \mathcal{F}(U_i) \longrightarrow \mathcal{F}(U)$ is called a* ***partition of unity*** *with respect to the open cover $\mathfrak{U}$, if for all $\sigma \in \mathcal{F}(U)$, $\operatorname{supp}(f_i\sigma) \subset U_i$ and*

$$\sum_{i\in I} f_i(\sigma|_{U_i}) \quad \equiv \quad \sigma.$$

- *A sheaf $\mathcal{F}$ over a topological space X is called a* ***fine*** *sheaf if it admits a partition of unity for any open cover of X.*

Theorem 1.30 *Let $\mathcal{F}$ be a fine sheaf on X. Then $H^p(X,\mathcal{F}) = 0$ for all $p > 0$.*

Proof Let $\mathfrak{U} := \{U_i\}_{i\in I}$ be an open cover for U, and let $\{f_i\}_{i\in I}$ be a partition of unity with respect to the open cover $\mathfrak{U}$. We show that if $\sigma \in C^k(\mathfrak{U},\mathcal{F})$ satisfies $\delta\sigma = 0$, that is,

$$(\delta\sigma)_{i_0,\dots,i_{k+1}} \quad = \quad \sum_{t=0}^{k}(-1)^t \sigma_{i_0,\dots,\widehat{i_t},\dots,i_{k+1}} \quad = \quad 0, \tag{1.5}$$

then $\sigma \in \delta C^{k-1}(\mathfrak{U},\mathcal{F})$. Define $\tau \in C^{k-1}(\mathfrak{U},\mathcal{F})$ by

$$(\tau)_{i_0,\dots,i_{k-1}} \quad = \quad \sum_{\nu\in I} f_\nu \sigma_{\nu,i_0,\dots,i_{k-1}}$$

The claim is that τ satisfies $\delta\tau = \sigma$, we verify this by calculation:

$$\begin{aligned}
(\delta\tau)_{i_0,\dots,i_k} \quad &= \quad \sum_{t=0}^{k-1}(-1)^t \tau_{i_0,\dots,\widehat{i_t},\dots i_k} \\
&= \quad \sum_{t=0}^{k-1}(-1)^t \sum_{\nu\in I} f_\nu \sigma_{\nu,i_0,\dots,\widehat{i_t},\dots,i_k} \\
&= \quad \sum_{\nu\in I}\sum_{t=0}^{k-1}(-1)^t f_\nu \sigma_{\nu,i_0,\dots,\widehat{i_t},\dots,i_k} \\
&= \quad \sum_{\nu\in I}\left(\sum_{t=0}^{k-1}(-1)^t f_\nu \sigma_{\nu,i_0,\dots,\widehat{i_t},\dots,i_k}\right) - f_\nu\sigma_{i_0,\dots,i_k} + f_\nu\sigma_{i_0,\dots,i_k} \\
&\overset{\text{by (1.5)}}{=} \quad \sum_{\nu\in I} -(\delta\sigma)_{\nu,i_0,\dots,i_k} + f_\nu\sigma_{i_0,\dots,i_k} \\
&= \quad \sum_{\nu\in I} f_\nu\sigma_{i_0,\dots,i_k} \\
&= \quad \sigma_{i_0,\dots,i_k}
\end{aligned}$$

This shows $\ker(\delta : C^k \longrightarrow C^{k+1}) = \mathrm{im}(\delta : C^{k-1} \longrightarrow C^k)$, hence the cohomology groups $H^k(\mathfrak{U}, \mathcal{F}) = 0$ for all $k > 0$. Taking the direct limit with respect to $\mathfrak{U}$ we have $H^k(X, \mathcal{F}) = 0$ for all $k > 0$. □

In particular, the sheaf C^∞ over X admits a partition of unity: for any open cover $\{U_i\}_{i\in I}$ of X, there exists functions $f_i : X \longrightarrow \mathbb{R}$ such that $\mathrm{supp}(f_i) \subset U_i$ and $\sum_{i\in I} f_i \equiv 1$. This is simply the ordinary partition of unity construction, so C^∞ has trivial cohomology. Similarly the k-th cohomology of $\mathcal{A}^p, \mathcal{A}^{p,q}_{\overline{\partial}}$ vanish for $k > 0$. We will not prove the next theorem, due to Grothendieck, which deals with the higher cohomology groups of coherent sheaves.

Theorem 1.31 *(Grothendieck vanishing theorem) Let $\mathcal{F}$ be a coherent sheaf and M is a compact submanifold of $\mathbb{P}^n$. Then $H^p(M, \mathcal{F}) = 0$ for all $p > \dim_{\mathbb{C}}(M)$.*

The hypothesis above has been weakened to avoid having to mention schemes. We do not have the space to develop the theory of coherent sheaves, but we simply note that $\mathcal{O}_S$ is a coherent sheaf, where S is a Riemann surface. The vanishing theorem will be applied in the case of $\mathcal{O}_S$ only.

1.5 Cohomology of $\mathbb{C}^n$ and $\mathbb{P}^n$

We conclude the chapter on sheaves and cohomology by determining some cohomology groups, most of which will be used later on. Computations of cohomology by its definition is laborious, which is another reason why the long exact sequence in cohomology is so useful- we can infer the structure of cohomology groups without having to do explicit computations.

In the case[6] of $\mathbb{P}^n$, we have the Hodge decomposition (c.f. theorem 9.3), which states that for a compact Kähler manifold, M, the following holds

$$H^r(M, \mathbb{C}) \simeq \bigoplus_{p+q=r} H^q(M, \Omega^p) \tag{1.6}$$

$$H^q(M, \Omega^p) = \overline{H^q(M, \Omega^p)} \tag{1.7}$$

for all $r, p, q \in \mathbb{N}$. We will denote $h^0(M, \mathbb{C}) = \dim(H^0(M, \mathbb{C}))$ and maintain this convention throughout. This gives the following

Corollary 1.32

[6] $\mathbb{P}^n$ is compact. Moreover $\mathbb{P}^n$ is Kähler via the Study metric, see [Mum95] for more details.

$$H^p(\mathbb{P}^n, \Omega^q) = \begin{cases} \mathbb{C} & \textit{if } p = q \textit{ and } p, q \leqslant n \\ 0 & \textit{otherwise} \end{cases}$$

In particular, $H^k(\mathbb{P}^n, \mathcal{O}) = 0$ *for all* $k > 0$ *and* $n > 0$*.*

Proof First recall that

$$h^k(\mathbb{P}^n, \mathbb{C}) = \begin{cases} 0 & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even} \end{cases}.$$

This can be shown using either by writing $\mathbb{P}^n$ as CW complex or by using the Mayer-Vietoris sequence, but we will omit these details. Hence by (1.6),

$$h^k(\mathbb{P}^n, \mathbb{C}) = \sum_{p+q=k} h^p(\mathbb{P}^n, \Omega^q)$$

This implies $h^p(\mathbb{P}^n, \Omega^q) = 0$ if $p+q$ is odd. Now suppose $p' \neq q' \leqslant k$, then

$$\begin{aligned} 1 = h^{2k}(\mathbb{P}^n, \mathbb{C}) &= \sum_{p+q=2k} h^p(\mathbb{P}^n, \Omega^q) \\ &\geqslant h^{p'}(\mathbb{P}^n, \Omega^{q'}) + h^{q'}(\mathbb{P}^n, \Omega^{p'}) \\ &\overset{\text{by (1.7)}}{=} 2h^{p'}(\mathbb{P}^n, \Omega^{q'}). \end{aligned}$$

So if $p' \neq q'$ then $h^{p'}(\mathbb{P}^n, \Omega^{q'}) = 0$. This leaves $h^p(\mathbb{P}^n, \Omega^p) = 1$ for $p \leqslant k$. so we have the result. □

The cohomology of $\mathbb{C}^n$ is easy: by the $\overline{\partial}$-Poincaré lemma $H^{p,q}_{\overline{\partial}}(\mathbb{C}^n) = 0$ for $q > 0$. Putting $p = 0$ we get

$$0 = H^{0,q}_{\overline{\partial}}(\mathbb{C}^n) = H^q(\mathbb{C}^n, \mathcal{O}).$$

Moreover

$$0 = H^p_{\text{simplicial}}(\mathbb{C}^n) \simeq H^p(\mathbb{C}^n, \mathbb{Z})$$

for $p > 0$. Lastly we finish with an important fact about holomorphic functions on compact complex manifolds.

Proposition 1.33 *Let M be a connected, compact complex manifold, then $H^0(M, \mathcal{O}) = \mathbb{C}$. In other words, the only global holomorphic functions are the constant functions.*

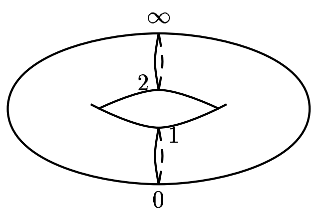
Proof Suppose $f \in H^0(M, \mathcal{O})$ and f obtains a maximum at say $x \in M$. Consider an open set $U \subset M$ containing x. By the maximum principle f is constant on U. Now $f - f(x)$ vanishes on an open set, so by analytic continuation, $f - f(x)$ vanishes on all of M. $\square$

Chapter 2

Riemann surfaces

A distinguishing feature of complex function theory is that there exist natural functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ whose domain of holomorphy is not $\mathbb{C}$. Examples of this include $z \longmapsto \sqrt{z}$ and $z \longmapsto \log(z)$.

Consider the function $f : \mathbb{C} \longrightarrow \mathbb{C}$, $z \longmapsto \sqrt{z(z-1)(z-2)}$. Take two copies of the Riemann sphere, $\mathbb{P}^1$, and make branch cuts along the intervals $[0,1]$ and $[2,\infty]$. Identifying the two copies of $\mathbb{P}^1$ along these cuts, we obtain the following topological picture of the resulting space, T,



The above torus, with the infinity point removed, can be considered as a subset of $\mathbb{C}^2_{x,y}$ satisfying the algebraic equation $p(x,y) = 0$ where $p(x,y) = y^2 - x(x-1)(x-2)$. The map f can be considered as the holomorphic map $\pi_f : T \longrightarrow \mathbb{P}^1$ given by $(x,y) \longmapsto y$, and via the map π_f, T is a two sheeted branched cover of $\mathbb{P}^1$ with ramification points at $0, 1, 2$ and ∞. That is, the fibre of the map π_f is finite with cardinality 2, except for the points $0, 1, 2, \infty$, where it has cardinality 1.

In the above example, a Riemann surface was constructed by analytically continuing the complex valued function f. We see that T is

1. a complex manifold of dimension 1,
2. a complex algebraic variety of dimension 1, that is, it is a complex algebraic curve, and
3. a real manifold of dimension 2, that is, a surface [1].

[1] Here we are using the word **dimension** in three different ways; the dimension of a complex (resp. real) manifold is the complex (resp. real) dimension of the codomain of any local chart, and the dimension of an algebraic variety is the transcendence degree of its coordinate ring.

These three aspects are typical of Riemann surfaces in general and validate what was said in the introduction.

2.1 Properties of Riemann surfaces

We will restrict the definition of an abstract Riemann surface to be compact and connected.

Definition 2.1 *A **Riemann surface** is a one-dimensional, connected, compact complex manifold.*

There are 'non-compact Riemann surfaces,' for instance $\mathbb{C}$, but for the most part of this thesis, we are concerned with the compact case.

Note 2.2 In fact, all Riemann surfaces as defined above can be realised as a n-sheeted branched cover of $\mathbb{P}^1$, so this is equivalent to Riemann's original concept. This amounts to the existence of a nonconstant $f \in \mathcal{K}^*(S)$ with a pole of order n, and the Riemann-Roch theorem (c.f. (4.3)) adequately answers such problems.

Definition 2.3 *Consider $\mathbb{P}^n_{x_0,\ldots,x_n}$ and the set of common zero loci of homogeneous polynomials $p_1, \ldots, p_k \in \mathbb{C}[x_0, \ldots, x_n]$. Denote this set $C := V(p_1, \ldots, p_k) \subseteq \mathbb{P}^n$, then C is called a **complex algebraic curve** if C is a one dimensional submanifold of $\mathbb{P}^n$.*

More generally, we call a subset $X \subseteq \mathbb{P}^n$ **algebraic** if X is the common zero loci of some homogeneous polynomials $q_1, \ldots, q_j \in \mathbb{C}[x_0, \ldots, x_0]$. We first show that any Riemann surface is algebraic.

Proposition 2.4 *Every Riemann surface is a complex algebraic curve.*

Proof By the implicit function theorem, any submanifold of $\mathbb{P}^n$ is an analytic subvariety. Chow's theorem (c.f. page 167 of [GH78]) states that any analytic subvariety of $\mathbb{P}^n$ is an algebraic subvariety of $\mathbb{P}^n$. Hence if there is an embedding of the Riemann surface S into projective space, then it is algebraic. The Kodaira embedding theorem ensures such an embedding exist. To prove this, we will wait until the end of chapter 3. $\square$

Note 2.5 Complex algebraic curves in $\mathbb{P}^n$ are sometimes is referred to as complex projective curves [2].

Invoking the Kodaira embedding theorem is certainly overkill in this case, for a more direct argument, see page 214-215 of [GH78]. The converse to proposition 2.4 is obtained by noting that an embedded complex algebraic curve inherits the complex structure of $\mathbb{P}^n$.

[2] Terminology also used to distinguish between algebraic curves in $\mathbb{A}^n$.

Even though Riemann surfaces and complex algebraic curves are essentially equivalent objects, we will refer to a Riemann surface S in general, and use the terminology of curves in conjunction with a particular embedding- we will see that 'most' Riemann surfaces possess a *canonical* embedding.

The higher dimensional analogue of proposition 2.4 fails for general complex compact manifolds. In chapter 5, we will encounter examples of complex manifolds which are not algebraic. Finally, we give two fundamental properties of Riemann surfaces; the first is topological and the second is analytic.

Proposition 2.6 *Let S be a Riemann surface then*

1. *S is orientable, and*
2. *S is a Kähler manifold.*

Proof Part 1 is simply due to the fact that all complex manifolds have a natural orientation induced by the complex structure. Recall that a complex manifold M with Hermitian metric $\mathrm{d}s^2$ is a Kähler manifold (c.f. page 259 of [Che00]) if the associated (1,1)-form ω of $\mathrm{d}s^2$ satisfies $\mathrm{d}\omega = 0$. Now for $M = S$, $\mathrm{d}\omega \in \mathcal{A}^3(S)$, and since $\dim_{\mathbb{R}}(S) = 2$ we have $\mathrm{d}\omega = 0$. □

Since S is oriented we can assign to S the topological invariant,

$$g \quad = \quad \frac{-\chi(S)+2}{2} \quad = \quad \text{number of 'handles' of underlying real manifold of } S,$$

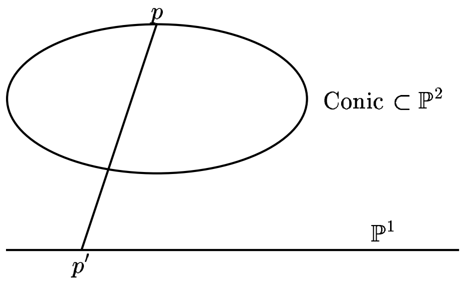
called the **genus** of S. The Kähler condition on S facilitates the use of the Hodge decomposition, which is used to decompose cohomology of S- this is done in section 2.3. The next section contains some examples of Riemann surfaces.

2.2 Examples of Riemann surfaces

The simplest Riemann surface is the Riemann sphere, which we will denote $\mathbb{P}^1$. This is the one point compactification of $\mathbb{C}$ by adding a point at infinity, which we denote ∞. An atlas for $\mathbb{P}^1$ is $\{(U_0, \varphi_0), (U_\infty, \varphi_\infty)\}$ where

$$\begin{array}{rcccc} \{z \in \mathbb{P}^1 \mid z \neq 0\} & = & U_0 & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & 1/z \end{array} \quad \text{and} \quad \begin{array}{rcccc} \{z \in \mathbb{P}^1 \mid z \neq \infty\} & = & U_\infty & \longrightarrow & \mathbb{C} \\ & & z & \longmapsto & z \end{array}$$

It has genus zero and can be realised as a conic in $\mathbb{P}^2$ via the invertible map $p \longmapsto p'$



Diagram

given by projecting from $(0,0)$.

The next simplest example is the elliptic curve, E. The etymology of the name *elliptic* is explained in section 6.1. We start with the complex torus $\mathbb{C}/\Lambda$ where $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, $\Im(\tau) > 0$. We have the classical **Weierstrass $\wp$-function** with respect to Λ,

$$\wp(z) \;=\; \frac{1}{z^2} - \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

which is doubly periodic with periods $1, \tau$. So $\wp$ is naturally a function on $\mathbb{C}/\Lambda$. Define the map

$$\begin{aligned} \varphi : \mathbb{C}/\Lambda &\longrightarrow \mathbb{P}^2 \\ z &\longmapsto \begin{cases} (\wp(z) : \wp'(z) : 1) & \text{if} \quad z \notin \Lambda \\ (0:1:0) & \text{if} \quad z \in \Lambda \end{cases}. \end{aligned}$$

This is an embedding of $\mathbb{C}/\Lambda$ into $\mathbb{P}^2$. Now the function $\wp$ satisfies the important identity,

$$\wp'(z)^2 \;=\; 4\wp(z)^3 - g_2\wp(z) - g_3$$

where $g_2 = 60 \sum_{\omega \in \Lambda - \{0\}} \omega^{-4}$ and $g_3 = 140 \sum_{\omega \in \Lambda - \{0\}} \omega^{-6}$. Hence the image φ is equal to the subset

$$C \;=\; \{(x:y:z) \in \mathbb{P}^2 \mid y^2 z = 4x^3 - g_2 x z^2 - g_3 z\} \;\subset\; \mathbb{P}^2$$

This realises the complex torus as an algebraic curve. We will see more of the $\wp$-function in later chapters, especially its connection with the elliptic integral (c.f. example 6.3).

We will collect some facts about the genus 2 and 3 cases for use later. Let S be a Riemann surface of genus g and let $\omega_1, \ldots, \omega_g$ be the basis of $H^0(S, \Omega^1)$; that $h^0(S, \Omega^1) = g$ will be substantiated in example 4.25. First we make a

Definition 2.7 *Define the* ***canonical map*** *of S to be*

$$\begin{aligned} \iota_K : S &\longrightarrow \mathbb{P}H^0(S, \Omega^1) \simeq \mathbb{P}^{g-1} \\ p &\longmapsto (\omega_1(p) : \ldots : \omega_g(p)) \end{aligned}$$

The image of S in $\mathbb{P}^{g-1}$ is called the **canonical curve** of S. When ι_K is an embedding, this gives a canonical way to study S extrinsically. In example 4.26, we will see that ι_K is an embedding iff S is not hyperelliptic. Now genus 2 Riemann surfaces are hyperelliptic, so ι_K is not an embedding for these Riemann surfaces. In fact, to embed a genus 2 Riemann surface, we need to consider $\mathbb{P}^3$ and use at least three equations [Mum75].
The genus 3 case is the first instance where Riemann surfaces exhibit both hyperelliptic and non-hyperelliptic behaviour. In the non-hyperelliptic case, S can be canonically embedded as a plane curve via $\iota_K : S \longrightarrow \mathbb{P}^2$. Moreover, the degree of ι_K in this case is $2g - 2 = 4$, so $\iota_K(S)$ is a plane quartic. We will need the following fact in the proof of the Torelli theorem.

Proposition 2.8 *Every plane quartic has twenty eight bitangents.*

This is a classical result which can be obtained via the Plücker formulas. We do not have the space to prove this, for more on the Plücker formulas and a proof of proposition 2.8, see pages 277-282 of [GH78]. To prove the Torelli theorem, we only need to know that the plane quartic has a finite number of bitangents.

2.3 Cohomology of Riemann surfaces

As with $\mathbb{P}^n$, any Riemann surface S is a compact Kähler manifold, so we can apply the Hodge decomposition. The decomposition of cohomology can be summarised by the **Hodge diagram**,

$$\begin{array}{ccccc} & & H^1(S, \Omega^1) & & \\ & \diagup & & \diagdown & \\ \overline{H^1(S, \mathcal{O})} & & & & H^1(S, \mathcal{O}) \\ & \diagdown & & \diagup & \\ & & H^0(S, \mathcal{O}) & & \end{array}$$

where $H^n(S, \mathbb{C})$ is isomorphic to direct sum of the entries in the n-th row. In particular, this says $H^0(S, \mathbb{C})$ can be decomposed into holomorphic and anti-holomorphic forms. The bottom row $H^0(S, \mathcal{O}) \simeq H^0(S, \mathbb{C})$ reflects the fact that the only holomorphic functions on S are the constant functions. Moreover $H^k(S, \mathbb{C}) = 0$ for all $k > 2$.

Since S is a two dimensional manifold, Poincaré duality says

$$H^2(S,\mathbb{Z}) \quad \simeq \quad H^0(S,\mathbb{Z})$$

and $H^0(S,\mathbb{Z}) \simeq \mathbb{Z}$.

2.4 The Riemann-Hurwitz formula

Given a holomorphic map $f : S \longrightarrow S'$ of degree d (that is, f is a d to one map), where S and S' are Riemann surfaces with genus g and g' respectively.

Definition 2.9 *Let $f : S \longrightarrow S'$ be a holomorphic map and for $p \in S$, let z be a local coordinate around p and w be a local coordinate around $f(p)$. If f can be given locally at p as $w = z^{\nu(p)}$, for some integer $\nu(p)$, then we say $\nu(p)$ is the **ramification index** of f at p. The point p is a **branch point** if $\nu(p) > 1$. Moreover we define the **branch locus** of p to be the divisor*

$$\sum_{p\in S}(\nu(p) - 1)\cdot p \quad \in \quad \mathrm{Div}(S)$$

or its image

$$\sum_{p\in S}(\nu(p) - 1)\cdot f(p) \quad \in \quad \mathrm{Div}(S')$$

We can see that away from the branch locus, f is a d to 1 covering, and two or more of these sheets come together at the branch locus. The Riemann-Hurwitz formula relates d, g, g' and the numbers $\nu(p)$.

Theorem 2.10 *(Riemann-Hurwitz) Let ν denote the ..., and χ be the Euler characteristic, then we have*

$$\chi(S) \quad = \quad d\chi(S') - \sum_{q\in S}(\nu(q) - 1).$$

Proof (Sketch) A triangulation exists on S' since it is compact. Let $T' = (V', E', F')$ be a triangulation on S' such that all the branch points lie on a vertex. Pull this triangulation back to S via f to obtain a triangulation $T = (V, E, F)$ on S, and we count the numbers of vertices, edges, and faces of T

$$\begin{aligned} |E| &= d|E'| \\ |F| &= d|F'| \\ |V| &= d|V'| - \sum_{q \in S} (\nu(q) - 1) \end{aligned}$$

and we obtain the Riemann-Hurwitz formula. □

2.5 Hyperellipticity

We finish this chapter with a brief discussion of the simplest types of Riemann surfaces. We saw that all Riemann surfaces of genus 2 are hyperelliptic, and in fact, there exists hyperelliptic Riemann surfaces for all genus $g > 2$. Hyperellipticity can be characterised by the existence of meromorphic functions with two poles, this is equivalent to the following

Definition 2.11 *A Riemann surface S is* ***hyperelliptic*** *if it admits a two to one covering map $f : S \longrightarrow \mathbb{P}^1$.*

The function f is essentially unique, as we shall see.

Proposition 2.12 *Let S be a hyperelliptic Riemann surface with genus S, then $f : S \longrightarrow \mathbb{P}^1$ has $2g + 2$ branch points.*

Proof This is a direct application of the Riemann-Hurwitz formula. Since $\chi(\mathbb{P}^1) = 2$,

$$\begin{aligned} 2 - 2g &= 2 \cdot 2 - \sum_{p \in S} (\nu(p) - 1) \\ \sum_{p \in S} (\nu(p) - 1) &= 2g + 2. \end{aligned}$$

Also $1 \leqslant \nu(p) \leqslant 2$, so $\sum_{p \in S} (\nu(p) - 1) =$ number of branch points. □

These branch points actually determine S, we will need this fact when proving the Torelli theorem.

Proposition 2.13 *A hyperelliptic Riemann surface S of genus g with two to one map $f : S \longrightarrow \mathbb{P}^1$. Then S is determined completely by the $2g + 2$ branch points of f.*

Proof Now $S \longrightarrow \mathbb{P}^1_x$ is of degree 2 and $f^* : \mathbb{C}(\mathbb{P}^1_x) = \mathbb{C}(x) \hookrightarrow \mathbb{C}(S)$ is an injective field homomorphism. On identifying $\mathbb{C}(x)$ with its image under f^*, $\mathbb{C}(x)$ is a subfield of $\mathbb{C}(S)$, and moreover $[\mathbb{C}(S) : \mathbb{C}(x)] = 2$. That is, $\mathbb{C}(S)$ is a quadratic extension of $\mathbb{C}(x)$. Given $y \in \mathbb{C}(S)$, $y \notin \mathbb{C}(x)$, it satisfies the quadratic equation

$$y^2 + yf_1(x) + f_2(x) \quad = \quad 0$$

where f_1, f_2 are polynomials in x. Completing the square gives $y^2 = h(x)$, so $\mathbb{C}(S) \simeq \mathbb{C}(x, \sqrt{h(x)})$ where for some polynomial h.

Now $\mathbb{C}(S)$ and $\mathbb{C}(S')$ as isomorphic fields iff to S and S' birationally equivalent. By a theorem in algebraic geometry, birational curves are isomorphic. We have shown that every hyperelliptic Riemann surface S of genus g has $\mathbb{C}(S) \simeq \mathbb{C}(x, \sqrt{h(x)})$, and since the zeroes of h are precisely the $2g + 2$ Weierstrass points of S, these points determine S completely. Moreover S is birationally equivalent to the curve $\{(x, y) \subset \mathbb{C}^2_{x,y} \mid y^2 = h(x)\}$.

□

Hyperelliptic Riemann surfaces often behave differently from their non-hyperelliptic relatives. We will see in example 4.27, that the canonical map $\iota_K : S \longrightarrow \mathbb{P}^{g-1}$ fails to be an embedding iff S is hyperelliptic. This phenomenon will resurface in chapter 7, when we prove the Torelli theorem.

Note that the moduli space of genus g Riemann surfaces has dimension $3g - 3$, this can be determined by counting the parameters which define a Riemann surface. However, the hyperelliptic Riemann surfaces of the same genus has a moduli space of dimension $2g - 1$. This agrees with the fact that in the case of genus $g = 2$, all Riemann surfaces are hyperelliptic; and shows that for genus $g > 2$, 'most' Riemann surfaces are non-hyperelliptic. Chapter 2 of Mumford's book [Mum75], contains a very readable account on moduli spaces of Riemann surfaces.

Chapter 3

The classical theorems of Abel and Jacobi

In the context of the Torelli theorem, Abel's theorem implies Torelli in the genus 1 case; and the Jacobi inversion theorem is needed in section **??** to prove Riemann's theorem. The aim of this chapter is to present Abel's and Jacobi's theorems using the terminology of divisors and line bundles, as well as introducing these essential concepts. Taken together, these theorems give us the following commutative triangle,

$$\begin{array}{ccc} \mathrm{Div}^0(S) & \longrightarrow & \mathcal{J}(S) \\ & \searrow \qquad \nearrow & \\ & \mathrm{Pic}^0(S) & \end{array} \tag{3.1}$$

where S is a Riemann surface, $\mathrm{Div}^0(S)$ is the group of divisors of degree zero, $\mathrm{Pic}^0(S)$ the connected component of $\mathrm{Pic}(S)$ containing the identity, $\mathrm{Pic}(S)$ is the group of isomorphism classes of line bundles on S, and $\mathcal{J}(S)$ is the Jacobian variety of S. This correspondence is remarkable as it relates three seemingly disparate objects. We will use the theorems to describe the geometry of $\mathrm{Pic}(S)$.

3.1 Divisors

The nomenclature in this section have their origins in algebraic number theory, where analogous constructions arose. For an elementary discussion of fractional ideals, the number theoretic analogue of divisors, and the class group, the number theoretic analogue of the Picard group, refer to [Ste79]. Divisors can be thought of as a generalisation of hypersurfaces.

Definition 3.1 *A* ***divisor*** *on a compact complex manifold M is a formal finite sum*

$$D = \sum_{i=1}^{k} n_i H_i$$

where $n_i \in \mathbb{Z}$ *and* $H_i \subset M$ *are irreducible hypersurfaces. Let* $\mathrm{Div}(M)$ *be the free abelian group generated by the divisors on* M*, where the identity element is denoted* 0.

In the case of a Riemann surface S, $D \in \mathrm{Div}(S)$ is simply a formal sum of points $D = \sum_{i=1}^{k} n_i p_i$, where $p_i \in S$. The more general definition above is needed when we discuss divisors on the Jacobian variety.

Let M be a compact complex manifold. To every nonzero meromorphic function $f \in \mathcal{K}^*(M)$ one can associate a divisor (f) as follows. For any hypersurface $H \subset M$ define $\mathrm{ord}_H : \mathcal{K}^*(M) \longrightarrow \mathbb{Z}$ by

$$\mathrm{ord}_H(f) = \begin{cases} n & \text{if } f \text{ has a zero of order } n \text{ along } H \\ -n & \text{if } f \text{ has a pole of order } n \text{ along } H \\ 0 & \text{otherwise} \end{cases}$$

Note that $\mathrm{ord}_H(fg^{-1}) = \mathrm{ord}_H(f) + \mathrm{ord}_H(g^{-1}) = \mathrm{ord}_H(f) - \mathrm{ord}_H(g)$, so ord_H is a group homomorphism. Then define the **divisor of** f to be

$$(f) = \sum_{H \text{ hypersurface in } M} \mathrm{ord}_H(f) H.$$

Of course, not all divisors arise in this way, as the following example shows.

Example 3.2 Let S be a Riemann surface. If $D = p$ where $p \in S$, then D is not the divisor of any nonzero meromorphic function on S. Suppose $f \in \mathcal{K}^*(S)$ such that $p = (f)$. Then f has no poles, $f \in H^0(S, \mathcal{O})$ and since S is compact and connected, proposition 1.33 implies f is constant. Now $f(p) = 0$ implies $f = 0$, contradicting $f \in \mathcal{K}^*(S)$.

Proposition 3.3 *If* $D \in \mathrm{Div}(M)$ *is a divisor of a nonzero meromorphic function, then* D *is called a* ***principal divisor****. The set of principal divisors of* M*, denoted* $\mathrm{PDiv}(M)$*, is a subgroup of* $\mathrm{Div}(M)$.

That $\mathrm{PDiv}(M)$ is a subgroup of $\mathrm{Div}(M)$ follow from the fact that the map $(\cdot) : \mathcal{K}^*(M) \longrightarrow \mathrm{Div}(M)$ is a group homomorphism, since

$$\begin{aligned} (fg^{-1}) &= \sum_{H \text{ hypersurface in } M} (\mathrm{ord}_H(f) + \mathrm{ord}_H(g^{-1}))H \\ &= (f) - (g) \end{aligned}$$

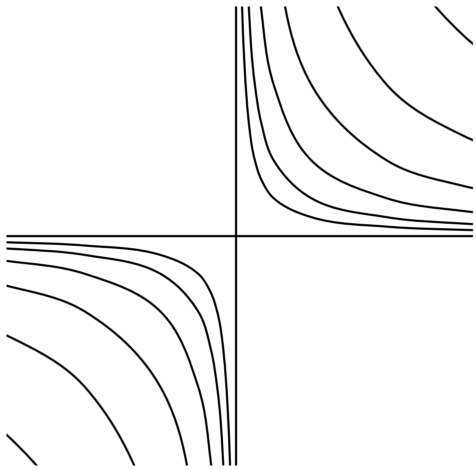
for all $f, g \in \mathcal{K}^*(M)$.

Definition 3.4

- *A divisor* $D = \sum n_i H_i$ ***effective*** *if* $n_i \geqslant 0$ *for all* i*. We use this to define an important partial ordering on* $\operatorname{Div}(M)$*: if* D, D' *be two divisors on* M*, then* $D \geqslant D'$ *iff* $D - D'$ *is effective.*
- *Consider the quotient* $\operatorname{Div}(M)/\operatorname{PDiv}(M)$*, then we say that two divisors are* ***linearly equivalent*** *if they are in the same coset of* $\operatorname{PDiv}(M)$*.*

Define $\operatorname{Pic}(S) = \operatorname{Div}(S)/\operatorname{PDiv}(S)$, we will see the connection between this definition and line bundles in proposition 3.30. Effective divisors reappear in the definition of linear systems in chapter 4.

Example 3.5 One can think of linear equivalent divisors as continuous deformations of each other. Let $M = \mathbb{P}^2_{x_0,x_1,x_2}$ and $f(x_0, x_1, x_2) = \frac{x_2^2 - x_0x_1}{x_0x_1} \in \mathcal{K}^*(M)$, then $D_1 := V(x_2^2 - x_0x_1) \sim V(x_0) + V(x_1) =: D_\infty$. On the open set $\{(x_0, x_1, x_2) \in \mathbb{P}^2 \mid x_2 \neq 0\}$, then for different values of f, we obtain the following



which shows a continuous deformation from D_1 to D_∞.

This point of view is important in intersection theory, under the premise that intersection numbers should be invariant under continuous deformations. Intersection theory is a topic which we do not have the space to develop. The most important theorem therein is Bézout's theorem, c.f. [CC04] Refer to [GH78] or [Sha74] for more details.
We now specialise the discussion to a Riemann surface S. We first define $\operatorname{Div}^0(S)$.

Definition 3.6 *There is a natural group homomorphism,* $\operatorname{Div}(S) \longrightarrow \mathbb{Z}$ *given by*

$$\sum_{p \in S} n_p p \quad \longmapsto \quad \sum_{p \in S} n_p$$

called the ***degree map****.*

Note that $\sum_{p\in S} n_p < \infty$ since it is a finite sum. The kernel of the degree map is the subgroup $\mathrm{Div}^0(M)$.

Proposition 3.7 *A principal divisor on S has degree 0, hence $\mathrm{PDiv}(S)$ is a subgroup of $\mathrm{Div}^0(S)$.*

Proof Note that any $f \in \mathcal{K}^*(S)$ can be considered as an n-sheeted branched covering of $\mathbb{P}^1$, $f : S \longrightarrow \mathbb{P}^1$. Then $\deg\left(\sum_{s\in f^{-1}(r)} \mathrm{ord}_s(f)s\right) = n$ by definition of branched covers. Let $p = (1:0), q = (0:1) \in \mathbb{P}^1$, and we have $\deg((f)) = \deg(f^*(p-q)) = n - n = 0$. $\square$

This means that linearly equivalent divisors have the same degree: if $D, D' \in \mathrm{Div}(S)$ and $D = D' + (f)$, then $\deg(D) = \deg(D') + \deg((f)) = \deg(D')$. The converse to this is false. Moreover proposition 3.7 says that we can define $\mathrm{Pic}^0(S) = \mathrm{Div}^0(S)/\mathrm{PDiv}(S)$.

Note 3.8 The above proposition depends on Riemann surfaces being compact. Consider a non-compact one-dimensional complex manifold, say $\mathbb{C}$, then let

$$f(z) = z^m \prod_{k=1}^{n} \frac{1}{z - a_k}$$

where $m > n$ and $\deg(f) \neq 0$. Now $f \in \mathcal{K}^*(\mathbb{C})$ and $\deg((f)) \neq 0$. However if we consider $\mathbb{P}^1$ with atlas $\{(U_0, \varphi_0), (U_\infty, \varphi_\infty)\}$ and $\tilde{f} \in \mathcal{K}^*(\mathbb{P}^1)$ given by

$$\begin{aligned}\tilde{f}_\infty(z) &= z^m \prod_{k=1}^{n} \frac{1}{z - a_k} && \text{on } U_\infty \\ \tilde{f}_0(z) &= z^{n-m} \prod_{k=1}^{n} \frac{1}{1 - z a_k} && \text{on } U_0\end{aligned}$$

and $(f) = m \cdot \varphi_\infty^{-1}(0) - (\varphi_\infty^{-1}(a_1) + \ldots + \varphi_\infty^{-1}(a_n)) + (n-m) \cdot \varphi_0^{-1}(0)$ has degree 0.

3.2 The Abel-Jacobi map and the Jacobian variety

The Jacobian variety of a Riemann surface S is introduced at this point to state to the Abel and Jacobi theorems. An intrinsic definition will be given in chapter 6, and it will be shown to agree with the following

Definition 3.9 *Let $\omega_1, \ldots, \omega_g$ be a basis for $H^0(S, \Omega^1)$, $\delta_1, \ldots, \delta_{2g}$ be a basis for $H_1(S, \mathbb{Z})$ such that the intersection form on $H_1(S, \mathbb{Z})$ with respect to these basis has the matrix*

$$\begin{pmatrix} & -I \\ I & \end{pmatrix}.$$

*Then define the **Jacobian variety** of S as the quotient*

$$\mathcal{J}(S) \quad = \quad \frac{\mathbb{C}}{\mathbb{Z}\Pi_1 + \ldots + \mathbb{Z}\Pi_{2g}}$$

where $\Pi_i = \left(\int_{\delta_i} \omega_1, \ldots, \int_{\delta_i} \omega_g\right)$ *for all* $i \in [1, g]$.

A basis $\delta_1, \ldots, \delta_{2g}$ for $H_1(S, \mathbb{Z})$ can always be chosen, this will be shown in chapter 6. The implicit claim that $\mathcal{J}(S)$ is a variety will also be verified in chapter 6.

Now we can define the Abel-Jacobi map, this is the prototype for the map $\operatorname{Div}^0(S) \longrightarrow \mathcal{J}(S)$ which appears in (3.1). First pick an arbitrary point, say p_0, on S, called the base point, then we have the

Definition 3.10 *The map* $\mu : S \longrightarrow \mathcal{J}(S)$ *given by*

$$p \quad \longmapsto \quad \left(\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g\right) \tag{3.2}$$

*is called the **Abel-Jacobi map**. Extending via linearity to divisors, we get a map* $\mu : \operatorname{Div}(S) \longrightarrow \mathcal{J}(S)$, *defined by*

$$\sum_{i=1}^{k} p_i \quad \longmapsto \quad \left(\sum_{i=1}^{k} \int_{p_0}^{p_i} \omega_1, \ldots, \sum_{i=1}^{k} \int_{p_0}^{p_i} \omega_g\right) \tag{3.3}$$

One might notice that in (3.2) and (3.3), the right hand side may not be well defined in $\mathbb{C}^g$, owing to the fact that S may not be simply connected. However, we have the

Proposition 3.11 *The Abel-Jacobi map is a well-defined map into the Jacobian of S.*

An argument for this will be provided in section 6.1.

Consider the space of effective degree k divisors, $\operatorname{Div}_+^k(S)$, we can topologise $\operatorname{Div}_+^k(S)$ as follows; denote $\underbrace{S \times \ldots \times S}_{k \text{ times}} / \operatorname{Perm}(k) = S^k / \operatorname{Perm}(k) =: S^{(k)}$ where $\operatorname{Perm}(k)$ acts on S^k by permuting the k coordinates. Then $\operatorname{Div}_+^k(S) = S^{(k)}$, and so inherits the complex structure from S^k. The Abel-Jacobi map restricted to $\operatorname{Div}_+^k(S)$ is

$$\begin{aligned} \mu^{(k)} : S^{(k)} \quad &\longrightarrow \quad \mathcal{J}(S) \\ p_1 + \ldots + p_k \quad &\longmapsto \quad \left(\sum_{i=1}^{g} \int_{p_0}^{p_i} \omega_1, \ldots, \sum_{i=1}^{g} \int_{p_0}^{p_i} \omega_g\right) \end{aligned} \tag{3.4}$$

where $\omega_1, \ldots, \omega_g$ are a basis for $H^0(S, \Omega^1)$. The map μ can be made independent of the base point p_0 by restricting to $\mathrm{Div}^0(S)$. In this case, we obtain

$$\begin{array}{rcl} \mu : \mathrm{Div}^0(S) & \longrightarrow & \mathcal{J}(S) \\ \sum_{i=1}^{k}(p_i - q_i) & = & \left(\sum_{i=1}^{k} \int_{q_i}^{p_i} \omega_1, \ldots, \sum_{i=1}^{k} \int_{q_i}^{p_i} \omega_g \right). \end{array}$$

This is the map μ in (3.1). Recall that proposition 3.7 states that principal divisors have degree zero, and that the converse is false. Abel's theorem give a necessary and sufficient condition for a divisor in $\mathrm{Div}^0(S)$ to be principal; while the Jacobi inversion theorem says that every point in $\mathcal{J}(S)$ corresponds to a linearly equivalent class of divisors of degree 0.

Theorem 3.12 *The sequence* $0 \longrightarrow \mathrm{PDiv}(S) \xrightarrow{\iota} \mathrm{Div}^0(S) \xrightarrow{\mu} \mathcal{J}(S) \longrightarrow 0$ *of abelian groups is exact, where* ι *is the inclusion map and* μ *is the Abel-Jacobi map.*

Note 3.13 Abel's theorem states that $\ker(\mu) = \mathrm{PDiv}(S)$, and the Jacobi inversion theorem states that μ is surjective.

Corollary 3.14 *We have the isomorphism* $\mathrm{Div}^0(S)/\,\mathrm{PDiv}(S) \simeq \mathcal{J}(S)$.

We will see in example 4.22, that in the genus 1 case, there is an isomorphism $S \longrightarrow \mathrm{Div}^0(S)/\,\mathrm{PDiv}(S)$, and hence S is isomorphic with its Jacobian; this is an instance of the Torelli theorem. To prove the Abel and Jacobi theorems we require the following.

An **analytic subvariety** of a complex manifold M is defined as the common zero locus of some subset of $\mathcal{O}(M)$. We will need the proper mapping theorem, which we state without proof

Theorem 3.15 *If* $f : M \longrightarrow N$ *is a holomorphic map between complex manifolds, then if* V *is an analytic subvariety of* M *then* $f(V)$ *is an analytic subvariety of* N.

Now we prove theorem 3.12.

Proof

1. We will first show that $\mathrm{PDiv}(S) \longrightarrow \mathrm{Div}^0(S) \xrightarrow{\mu} \mathcal{J}(S)$ is the zero map. Define

$$\begin{array}{rccccc} \psi_f : & \mathbb{P}^1 & \longrightarrow & \mathrm{Div}(S) & \xrightarrow{\mu} & \mathcal{J}(S) \\ & (x:y) & \longmapsto & (xf - y) & \longmapsto & \mu((xf-y)) \end{array}.$$

Now ψ_f^* is the zero map, since there are no global holomorphic 1 forms on $\mathbb{P}^1$. This is due to Serre duality (c.f. theorem **??**), $H^0(\mathbb{P}^1, \Omega^1) \simeq H^1(\mathbb{P}^1, \mathcal{O})^\vee$, and corollary

1.32, which gives $H^1(\mathbb{P}^1, \mathcal{O}) = 0$. This shows ψ_f is constant, hence $\mu((f)) = \psi_f(1 : 0) - \psi_f(0 : 1) = 0$.

Showing exactness of $0 \longrightarrow \mathrm{PDiv}(S) \longrightarrow \mathrm{Div}^0(S) \xrightarrow{\mu} \mathcal{J}(S)$ will complete the proof of Abel's theorem. The arguments can be found in pages of 232-235 of [GH78].

2. We claim that every $\xi \in \mathcal{J}(S)$ can be written as $\left(\sum_{i=1}^k \int_{p_0}^{p_i} \omega_1, \ldots, \sum_{i=1}^k \int_{p_0}^{p_i} \omega_g\right)$. Hence it suffices to show that $\mu^{(g)} : S^{(g)} \longrightarrow \mathcal{J}(S)$ is surjective. The Jacobian matrix for $\mu^{(g)}$ near $D = p_1 + \ldots + p_g$ with local coordinates $z_1, \ldots, z_g$ is given by

$$D\mu^{(g)} = \begin{pmatrix} \omega_1(p_1)/\mathrm{d}z_1 & \ldots & \omega_1(p_g)/\mathrm{d}z_g \\ \vdots & & \vdots \\ \omega_g(p_1)/\mathrm{d}z_1 & \ldots & \omega_g(p_g)/\mathrm{d}z_g \end{pmatrix}.$$

Note that the n-th column are the coordinates of $\iota_K(p_n)$. This matrix is generically full rank, hence by the inverse function theorem there exists an open set $U \subset S^{(g)}$ such that $\mu^{(g)}$ is a local isomorphism $U \longrightarrow \mu^{(g)}(U)$. Now by the proper mapping theorem $\mathrm{im}(\mu^{(g)})$ is an analytic subvariety of $\mathcal{J}(S)$, but $\mathrm{im}(\mu^{(g)})$ contains the open set $\mu^{(g)}(U)$, so the image of $\mu^{(g)}$ must be equal to its codomain.

□

3.3 Line bundles

A vector bundle formalise the idea of a family of vector spaces parameterised by a smooth manifold, M, and which varies smoothly with respect to points on M. The most common example is that of a tangent bundle of a smooth manifold in differential geometry. In complex differential geometry, we replace the smoothness condition with a holomorphic requirement. A holomorphic line bundle is a holomorphic vector bundle where the vector spaces are one dimensional.

Definition 3.16 *Let E and X be complex manifolds and $\pi : E \longrightarrow X$ a surjective holomorphic map, satisfying the following properties.*

1. *There exists a* ***local trivialisation****. That is, an open cover $\{U_\alpha\}_{\alpha \in A}$ of X together with the biholomorphic maps,*

$$\begin{aligned} \varphi_\alpha : \pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times V \\ p &\longmapsto (\pi(p), \tilde{\varphi}_\alpha(p)) \end{aligned}$$

where V is a complex vector space.

2. *Denote* $\tilde{\varphi}_{\alpha,x} := \tilde{\varphi}_\alpha|_{\pi^{-1}(x)}$. *The functions* $\tilde{\varphi}_\alpha$ *satisfy the following properties.*
   (a) *The restriction* $\tilde{\varphi}_{\alpha,x} : \pi^{-1}(x) \longrightarrow V$ *is biholomorphic for all* $x \in U_\alpha$.
   (b) *The composition*

$$g_{\alpha\beta}(x) \;:=\; \tilde{\varphi}_{\alpha,x} \circ (\tilde{\varphi}_{\beta,x})^{-1} : V \quad\longrightarrow\quad V$$

   is a linear isomorphism for all $x \in U_\alpha \cap U_\beta$, *that is,* $g_{\alpha\beta}(x) \in GL(V)$. *Moreover the assignment* $x \longmapsto g_{\alpha\beta}(x)$ *is holomorphic. We call the family* $\{g_{\alpha\beta}\}_{\alpha,\beta}$ *the* ***transition functions***.

A triple $E \xrightarrow{\pi} X$ *satisfying the above is called a* ***holomorphic vector bundle***. *The* ***rank*** *of the vector bundle is* $\dim_{\mathbb{C}}(V)$. *A holomorphic vector bundle of rank 1 is called a* ***holomorphic line bundle***. *A holomorphic map* $X \xrightarrow{\sigma} E$ *satisfying* $\pi \circ \sigma = \mathrm{id}_X$ *is called a* ***holomorphic section***. *The vector space of all global holomorphic sections is denoted* $\Gamma(E)$ *or* $\mathcal{O}(E)$ [1].

Definition 3.17 *Let* $E \xrightarrow{\pi} X$ *and* $E' \xrightarrow{\pi'} X$ *be holomorphic vector bundles. A* ***morphism of holomorphic vector bundles***, *is a holomorphic map*

$$\varphi : E \quad\longrightarrow\quad E'$$

such that $\varphi|_X : X \longrightarrow X$ *is the identity map and the diagram*

$$\begin{array}{ccc} E & \xrightarrow{\quad\varphi\quad} & E' \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle\pi'} & \\ & B & \end{array}$$

commutes. We say that E *and* E' *are* ***isomorphic*** *if there exists morphisms* $\varphi : E \longrightarrow E'$ *and* $\psi : E' \longrightarrow E$ *such that* $\varphi \circ \psi = \mathrm{id}_E$ *and* $\psi \circ \varphi = \mathrm{id}_{E'}$.

Definition 3.18 *Suppose* f *is a holomorphic map* $X \longrightarrow Y$ *and suppose* $E \longrightarrow Y$ *is a holomorphic vector bundle on* Y. *Define the* ***pullback*** *of* E, $f^*E \longrightarrow X$, *with* $f^*E := \{(x,e) \in X \times E \mid f(x) = \pi(e)\}$ *and projection* $f^*\pi : f^*E \longrightarrow Y$ *given by* $f^*\pi(x,e) = x$.

Now line bundles enjoy the property of being specified completely by their transition functions. Let M be a complex manifold with open cover $\mathfrak{U} := \{U_\alpha\}_{\alpha \in A}$ and a family

[1] More generally, $\mathcal{O}(E)(U)$ denotes the sections of E over U.

$\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$ where $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$, satisfying

$$g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} = \mathrm{id}|_{U_{\alpha\beta\gamma}} \quad (3.5)$$

$$g_{\alpha\beta} \cdot g_{\beta\alpha} = \mathrm{id}|_{U_{\alpha\beta}} \quad (3.6)$$

for all $\alpha, \beta, \gamma \in A$. Note that these are precisely the identities satisfied by the transition functions defined above.

We can construct a line bundle with the family $\{g_{\alpha\beta}\}$ as transition functions. Consider the union $L := \bigcup_{\alpha\in A} U_\alpha \times \mathbb{C}$ and projections $\pi_\alpha : U_\alpha \times \mathbb{C} \longrightarrow U_\alpha$ for each α, we identify the fibre over $z \in U_{\alpha\beta}$ via the bijection $\pi_\alpha^{-1}(z) \longrightarrow \pi_\beta^{-1}(z)$ given by $p \longmapsto g_{\alpha\beta} \cdot p$. Then $L \xrightarrow{\pi} X$ is a holomorphic line bundle with $\pi|_{U_\alpha} = \pi_\alpha$ for all $\alpha \in A$.

This also tells us how to 'glue' sections together, given $\sigma_\alpha : U_\alpha \longrightarrow L$ and $\sigma_\beta : U_\beta \longrightarrow L$, then on $U_{\alpha\beta}$, we have $\sigma_\alpha = g_{\alpha\beta} \cdot \sigma_\beta$. In the following, let M be a complex manifold.

Definition 3.19 *Define* Pic(M) *to be the group of isomorphism classes of holomorphic line bundles on* M*.*

Definition 3.20 *Let* $L \longrightarrow M$ *and* $L' \longrightarrow M$ *be two holomorphic line bundles over* X *with transition functions* $\{g_{\alpha\beta}\}$ *and* $\{h_{\alpha\beta}\}$ *given on the same open cover* [2] *of* X *respectively.*

Define the ***dual*** *of* L*, denoted* L^**, to be the line bundle given by the transition functions* $\{g_{\alpha\beta}^{-1}\}$*. Also define the* ***tensor product*** *of* L *and* L'*, denoted* $L \otimes L'$ *to be the line bundle given by the transition functions* $\{g_{\alpha\beta} \cdot h_{\alpha\beta}\}$*.*

Note that the tensor product makes Pic(M) a group. We can characterise Pic(M) as a Čech cohomology group.

Proposition 3.21 *There is an isomorphism of groups* $H^1(M, \mathcal{O}^*) \simeq \mathrm{Pic}(M)$*.*

Proof Let $\mathfrak{U} := \{U_\alpha\}_{\alpha\in A}$ be an open cover of M, we have established above that a family $g := \{g_{\alpha\beta}\}_{\alpha,\beta}$, with $g_{\alpha\beta} \in \mathcal{O}^*(U_{\alpha\beta})$ satisfying (3.5) and (3.6) determines a line bundle L. We check that the map $g \longmapsto L$ is well defined and is an isomorphism.

Firstly $g \in Z^1(M, \mathcal{O}^*)$, since

$$\begin{aligned}
(\delta g)_{\alpha\beta\gamma} &= g_{\beta\gamma} \cdot g_{\alpha\gamma}^{-1} \cdot g_{\beta\gamma} \\
&\overset{\text{by}(3.6)}{=} g_{\beta\gamma} \cdot g_{\gamma\alpha} \cdot g_{\beta\gamma} \\
&\overset{\text{by}(3.5)}{=} \mathrm{id}\,|_{U_{\alpha\beta\gamma}}
\end{aligned}$$

[2]One can always take a refinement of the two open covers of X if they are different.

If we pick a different representative of $[g]$, say $g' = \{f_\beta \cdot f_\alpha^{-1} \cdot g_{\alpha\beta}\}_{\alpha,\beta}$, then this amounts to picking a different trivialisation, and defines the same line bundle L. The definition of tensor product in $\mathrm{Pic}(M)$ coincide with the group operation on $H^1(M, \mathcal{O}^*)$, so the map $g \longmapsto L$ is a group homomorphism. The existence of the inverse to $g \longmapsto L$ is clear, since L is simply mapped to its transition functions, and choosing a different trivialisation changes the image by a coboundary. □

We give some examples of holomorphic line bundles.

Example 3.22 Let M be an n-dimensional complex manifold, and $T^*(M)$ be the cotangent bundle. Then $K := \bigwedge^n T^*(M)$ is a line bundle, called the **canonical bundle** of M.

Example 3.23 Consider a hyperplane $H \subset \mathbb{P}^n_{x_0,\ldots,x_n}$. This is a codimension one subvariety as it is given by a linear form $\alpha_0 x_0 + \ldots + \alpha_n x_n$ for some $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$. The line bundle corresponding to the divisor class $[H] \in \mathrm{Pic}(\mathbb{P}^n)$ is called the **hyperplane bundle**. The dual of the hyperplane bundle is called the **universal bundle**.

Example 3.24 The trivial line bundle $L := X \times \mathbb{C} \xrightarrow{\pi} X$ over a complex manifold X corresponds to the structure sheaf $\mathcal{O}_X$ via the identification $\mathcal{O}_X(U) = \mathcal{O}(L)(U)$.

Example 3.25 We determine $\mathrm{Pic}(\mathbb{P}^n)$. First consider

$$\begin{aligned} \deg : \mathrm{Div}(\mathbb{P}^n) &\longrightarrow \mathbb{Z} \\ V(f) &\longmapsto \deg(f) \end{aligned}$$

where f is a irreducible homogeneous polynomial, and extend deg to all of $\mathrm{Div}(\mathbb{P}^n)$ via linearity. This is actually the explicit form of the **chern class map** for $\mathrm{Pic}(\mathbb{P}^n)$, whose general definition will be given in the next section. We see that if $\deg(D) = \sum_{i=1}^m n_i V(f_i) = 0$, then

$$\begin{aligned} \sum_{i=1}^m n_i \deg(f_i) &= 0 \\ \sum_{i=1}^m \deg(f_i^{n_i}) &= 0 \end{aligned}$$

and after a suitable renumbering of the f_j's,

$$\begin{aligned} g &= \frac{f_1^{n_1} \cdots f_j^{n_j}}{f_{j+1}^{n_{j+1}} \cdots f_n^{n_m}} \in \mathcal{K}^*(\mathbb{P}^n) \\ (g) &= D. \end{aligned}$$

So the kernel is $\mathrm{PDiv}(\mathbb{P}^n)$. Now deg is surjective, so we have $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$. We will show $\mathrm{Pic}(\mathbb{P}^n) = \mathbb{Z}$ another way using cohomology.
Note that in the above proof, we have implicitly assumed that all meromorphic functions in $\mathbb{P}^n$ are rational functions. For proof of this, we refer the reader to page 168 of [GH78].

Example 3.26 The above proposition shows that $\mathrm{Pic}(\mathbb{P}^n)$ is generated by one element, $[H]$, since $\deg(H) = 1$. So given any divisor, $D \in \mathrm{Pic}(\mathbb{P}^n)$, there exists some $d \in \mathbb{Z}$ such that $D \sim dH$. This shows that there is no ambiguity in writing

$$\mathcal{O}(D) \quad = \quad \mathcal{O}([D]) \quad = \quad \mathcal{O}([dH]) \quad = \quad \mathcal{O}(dH) \quad = \quad \mathcal{O}(d)$$

so all the line bundles on $\mathbb{P}^n$ are in the form $\mathcal{O}(d)$ for $d \in \mathbb{Z}$. In fact, we can be even more explicit,

Proposition 3.27 *We have the following isomorphism of vector spaces*

$$H^0(\mathbb{P}^n, \mathcal{O}(D)) \quad \simeq \quad \mathbb{C}[x_0, \ldots, x_n]_{\deg(D)}$$

A proof of the above proposition can be found in pages 164-166 of [GH78].

3.4 Pic(S)

We specialise the discussion to holomorphic line bundles over a Riemann surface S. Henceforth, holomorphic line bundles will be referred to as simply line bundles, and will be denoted $L \xrightarrow{\pi} S$. In this section, we will examine the structure of $\mathrm{Pic}(S)$.
The relationship between line bundles and divisors is best expressed in sheaf theoretic language. In definition **??**, $\mathrm{Pic}(S)$ is defined as a Čech cohomology group. We now express $\mathrm{Div}(S)$ in terms of such a group.

Proposition 3.28 *There is an isomorphism of groups* $\varphi : H^0(S, \mathcal{K}^*/\mathcal{O}^*) \longrightarrow \mathrm{Div}(S)$.

Proof Let $\sigma \in H^0(S, \mathcal{K}^*/\mathcal{O}^*)$ given by an open cover $\{U_i\}_{i \in I}$ of S, and $\sigma|_{U_i} = \sigma_i \in H^0(U_i, \mathcal{K}^*/\mathcal{O}^*)$, satisfying

$$\sigma_i|_{U_{ij}} \mathcal{O}^*(U_{ij}) \quad = \quad \sigma_j|_{U_{ij}} \mathcal{O}^*(U_{ij}) \tag{3.7}$$

as cosets for all $i, j \in I$. We can associate to σ the divisor

$$D_\sigma \quad = \quad \sum_{p \in S} \mathrm{ord}_p(\sigma_i) p$$

where i is chosen such that $p \in U_i$. The value of $\text{ord}_p(\sigma_i)$ does not depend on such a choice, since by (3.7), σ_i and σ_j has the same poles and zeroes in U_{ij}. So for $p \in U_{ij}$, σ_i has a zero (or pole) at p iff σ_j has a zero (or pole) at p.

Conversely, given any divisor $D \in \text{Div}(S)$, choose an open cover $\{V_j\}_{j \in J}$ such that for each V_j there exist $f_j \in H^0(V_j, \mathcal{K}^*)$ such that f_j has poles or zeroes at the corresponding points in D. This gives, $\frac{f_i|_{U_{ij}}}{f_j|_{U_{ij}}} \in \mathcal{O}^*(U_{ij})$, hence the by the sheaf condition there exists an $f \in H^0(S, \mathcal{K}^*/\mathcal{O}^*)$ such that $D_f = D$. We call f the **local defining function** for the divisor D. Finally, it is clear that φ is a homomorphism. $\square$

Note 3.29 We can define divisors as elements of $H^0(S, \mathcal{K}^*/\mathcal{O}^*)$, in which case they are called **Cartier divisors**. Divisors defined as a formal sum of irreducible codimension 1 subvarieties are called **Weil divisors**. When S is smooth, these definitions are equivalent, as the above isomorphism shows. When singularities are present in S, this is not true.

Consider the exact sequence of sheaves on S,

$$0 \longrightarrow \mathcal{O}^* \longrightarrow \mathcal{K}^* \longrightarrow \mathcal{K}^*/\mathcal{O}^* \longrightarrow 0$$

This induces a long exact sequence in cohomology, from which we extract the following,

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^0(S,\mathcal{K}^*) & \longrightarrow & H^0(S,\mathcal{K}^*/\mathcal{O}^*) & \longrightarrow & H^1(S,\mathcal{O}^*) & \longrightarrow \cdots \\ & \| & & \downarrow \varphi & & \| & \\ & H^0(S,\mathcal{K}^*) & \longrightarrow & \text{Div}(S) & \longrightarrow & \text{Pic}(S) & \end{array}$$

where φ is the isomorphism of proposition 3.28. The kernel of the map $\text{Div}(S) \longrightarrow \text{Pic}(S)$ is $H^0(S, \mathcal{K}^*)$ by exactness. We first determine the map $\text{Div}(S) \longrightarrow \text{Pic}(S)$ above explicitly. Let $D \in \text{Div}(S)$ with local defining equations $f_j \in \mathcal{K}^*(V_j)$ with respect to some open cover $\{V_j\}_{j \in J}$ of S. Then let $g_{ij} = f_i/f_j$, and the family $\{g_{ij}\}_{i,j \in J}$ satisfy the conditions for transition functions:

$$\begin{aligned} g_{ij} \cdot g_{ji} &= \tfrac{f_i}{f_j} \cdot \tfrac{f_j}{f_i} = \text{id}_{U_{ij}} \\ g_{ij} \cdot g_{jk} \cdot g_{ki} &= \tfrac{f_i}{f_j} \cdot \tfrac{f_j}{f_k} \cdot \tfrac{f_k}{f_i} = \text{id}_{U_{ijk}} \end{aligned}$$

for all $i, j, k \in J$. Denote $[D]$ to be the line bundle defined by $\{g_{ij}\}_{i,j \in J}$. Now if we choose a different set of local defining equations with respect to the same open cover,[3] say

[3] We can always take a refinement of two different open covers, so we can assume without loss of generality that the local defining functions are on the same open cover.

$h_j \in \mathcal{K}^*(V_j)$, for D. Then for each $j \in J$, $f_j = \varphi_j h_j$ for some nonvanishing holomorphic function φ_j. Now let $g'_{ij} = h_i/h_j$, then

$$g_{ij} = \frac{f_i}{f_j} = \frac{\varphi_i h_i}{\varphi_j h_j} = \frac{\varphi_i}{\varphi_j} g'_{ij}$$

and since $\varphi_i/\varphi_j \in \mathcal{O}^*(U_{ij})$, $\{g_{ij}\}$ and $\{g'_{ij}\}$ define the same line bundle. Now the map $H^0(S, \mathcal{K}^*) \longrightarrow \text{Div}(S)$ is given by $f \longmapsto (f)$ by proposition 3.28. This gives the following

Proposition 3.30 *The sequence* $0 \longrightarrow \text{PDiv}(S) \longrightarrow \text{Div}(S) \xrightarrow{[\cdot]} \text{Pic}(S) \longrightarrow 0$ *is exact.*

The above proposition shows that linearly equivalent divisors give rise to the same line bundles. Moreover for $D, D' \in \text{Div}(S)$, $[D + D'] = [D] \otimes [D']$. This follows from the fact that the transition functions for $[D + D']$ are obtained by multiplying the transition functions of $[D]$ and $[D']$.

Definition 3.31 *We say a section σ of the line bundle $[D]$ over S is* ***holomorphic*** *iff* $(\sigma) + D' \geqslant 0$ *for some* $D' \sim D$.

Next we introduce the chern class map. Let S be a Riemann surface, and consider the exponential sheaf sequence on S

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0$$

Again this sequence is exact, so it induces a long exact sequence in cohomology,

$$\begin{array}{l} 0 \longrightarrow H^0(S,\mathbb{Z}) \longrightarrow H^0(S,\mathcal{O}) \xrightarrow{\exp} H^0(S,\mathcal{O}^*) \xrightarrow{\delta} \\ \longrightarrow H^1(S,\mathbb{Z}) \longrightarrow H^1(S,\mathcal{O}) \longrightarrow H^1(S,\mathcal{O}^*) \xrightarrow{c_1} \\ \longrightarrow H^2(S,\mathbb{Z}) \longrightarrow H^2(S,\mathcal{O}) \longrightarrow H^2(S,\mathcal{O}^*) \longrightarrow \cdots \end{array}$$

Since $\mathcal{O}$ is a coherent sheaf and $\dim_{\mathbb{C}}(S) = 1$, $H^2(S, \mathcal{O})$ is trivial by Grothendieck's vanishing theorem (c.f. theorem 1.31). Moreover $\text{im}(\exp) = H^0(S, \mathcal{O}^*) = \ker(\delta)$, hence we can extract the following exact sequence,

$$0 \longrightarrow H^1(S,\mathbb{Z}) \longrightarrow H^1(S,\mathcal{O}) \longrightarrow H^1(S,\mathcal{O}^*) \xrightarrow{c_1} H^2(S,\mathbb{Z}) \longrightarrow 0$$

Definition 3.32 *The connecting homomorphism* $H^1(S, \mathcal{O}^*) \xrightarrow{c_1} H^2(S, \mathbb{Z})$ *is called the* ***chern class map****.*

Note 3.33 More correctly, this is called the *first* chern class map, which explains the subscript in c_1. But since the higher chern class maps will not be used, we will simply stick to *chern class map.*

Example 3.34 We can use the above long exact sequence to determine $\mathrm{Pic}(\mathbb{P}^n)$.

$$\dots \longrightarrow H^1(\mathbb{P}^n, \mathcal{O}) \longrightarrow H^1(\mathbb{P}^n, \mathcal{O}^*) \longrightarrow H^2(\mathbb{P}^n, \mathbb{Z}) \longrightarrow H^2(\mathbb{P}^n, \mathcal{O}) \longrightarrow \dots$$

The groups $H^1(\mathbb{P}^n, \mathcal{O})$ and $H^2(\mathbb{P}^n, \mathcal{O})$ were determined to be both zero in corollary 1.32, and also $H^2(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$. This gives the exact sequence

$$0 \longrightarrow \mathrm{Pic}(\mathbb{P}^n) \longrightarrow \mathbb{Z} \longrightarrow 0$$

and hence the isomorphism $\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$.

The above long exact sequence in cohomology is needed to finish off the proof of proposition 2.4; every Riemann surface is algebraic.

Proof (of proposition 2.4) Let S be a Riemann surface and $\mathrm{d}s^2$ be a metric on S with associated $(1,1)$-form ω, normalised such that $\int_S \omega = 1$. Then $[\omega] \in H^2(S, \mathbb{Z})$ under the identification $H^2(S, \mathbb{R}) \simeq H^2_{\mathrm{DR}}(S)$ and the injection $H^2(S, \mathbb{Z}) \hookrightarrow H^2(S, \mathbb{R})$. By the exact sequence

$$\dots \longrightarrow H^1(S, \mathcal{O}^*) \longrightarrow H^2(S, \mathbb{Z}) \longrightarrow 0$$

$[\omega]$ is a positive form under the identification $H^2(S, \mathbb{Z}) \simeq \mathbb{Z}$, hence there exists a positive line bundle L with $c_1(L) = [\omega]$. By the Kodaira embedding theorem, S can be embedded into projective space. □

Theorem 3.35 *The chern class map* $\mathrm{Pic}(S) \xrightarrow{c_1} H^2(S, \mathbb{Z})$ *for a Riemann surface* S *coincides with the degree map* $\deg : \mathrm{Div}(S) \longrightarrow \mathbb{Z}$.

For a proof of theorem 3.35, see pages 141-144 of [GH78]. To determine $\mathrm{Pic}(S)$ for S a Riemann surface, we need to examine $\mathrm{Pic}^0(S)$, which is defined as

$$\mathrm{Pic}^0(S) \quad = \quad \ker(c_1)$$

Proposition 3.36 *We have the following isomorphism of groups,*

$$\mathrm{Pic}^0(S) \quad \simeq \quad \frac{H^1(S, \mathcal{O})}{H^1(S, \mathbb{Z})}$$

Proof Denote $\varphi : H^1(S, \mathcal{O}) \longrightarrow H^1(S, \mathcal{O}^*)$ to be the map in the long exact sequence above. Exactness implies the $\ker(c_1)$ is isomorphic to $\operatorname{im}(\varphi)$. By the first isomorphism theorem

$$\operatorname{im}(\varphi) \;\simeq\; \frac{H^1(S,\mathcal{O})}{\ker(\varphi)}$$

But $\ker(\varphi) = \operatorname{im}(H^1(S,\mathbb{Z}) \longrightarrow H^1(S,\mathcal{O})) \simeq H^1(S,\mathbb{Z})$ since exactness implies the map, $H^1(S,\mathbb{Z}) \longrightarrow H^1(S,\mathcal{O})$, is injective. So we have the required isomorphism. □

Note 3.37 The above characterisation of $\operatorname{Pic}^0(S)$ together with the Jacobi inversion theorem says that every point on $\mathcal{J}(S)$ corresponds to some line bundle with trivial chern class.

Note 3.38 We can use the above characterisation (c.f. proposition 3.36)

$$\operatorname{Pic}^0(S) \;\simeq\; \frac{H^1(S,\mathcal{O})}{H^1(S,\mathbb{Z})}$$

to approach Abel's theorem another way. The right hand side of the above is actually isomorphic to $\mathcal{J}(S)$ via Serre duality (c.f. definition 6.4). So we can form the following sequence

$$\varphi : \operatorname{Div}^0(S) \longrightarrow \frac{\operatorname{Div}^0(S)}{\operatorname{PDiv}(S)} \xrightarrow{\sim} \operatorname{Pic}^0(S) \xrightarrow{\sim} \mathcal{J}(S)$$

and it can be shown that φ agrees with the Abel-Jacobi map.

The map $\pi : \operatorname{Div}(S) \longrightarrow \operatorname{Pic}(S)$ from theorem 3.28 restricts to a map $\pi : \operatorname{Div}^0(S) \longrightarrow \operatorname{Pic}^0(S)$. The Hodge theorem (c.f. theorem 9.3) gives the isomorphism, $H^1(S,\mathcal{O}) \simeq \overline{H^0(S,\Omega^1)}$, so $H^1(S,\mathcal{O})$ is naturally a g-dimensional complex vector space. Proposition 3.36 tells us that $\operatorname{Pic}^0(S)$ for a Riemann surface S is a complex torus and we can pull back the geometry of $\operatorname{Pic}^0(S)$ to $\operatorname{Div}^0(S)/\operatorname{PDiv}(S)$ via the map $\pi : \operatorname{Div}^0(S) \longrightarrow \operatorname{Pic}^0(S)$. The theorems of Abel and Jacobi can now be summarised by the following commutative diagram

$$\begin{array}{ccc} \operatorname{Div}^0(S) & \xrightarrow{\quad\mu\quad} & \mathcal{J}(S) \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle\tilde{\mu}} & \\ & \operatorname{Pic}^0(S) & \end{array}$$

Proposition 3.39 *There is a non-canonical isomorphism of* $\operatorname{Pic}^0(S)$*-sets* $\operatorname{Pic}^j(S) \simeq \operatorname{Pic}^{j+1}(S)$. *Hence* $\operatorname{Pic}^j(S) \simeq \operatorname{Pic}^0(S)$.

Proof We can map $\mathrm{Pic}^j(S) \longrightarrow \mathrm{Pic}^{j+1}(S)$ by $D \longmapsto D + p$ for some point $p \in S$. The inverse map is given by $D' \longmapsto D' - p$ for $D' \in \mathrm{Pic}^{j+1}(S)$. $\square$

This leads to the following non-canonical characterisation of $\mathrm{Pic}(S)$,

$$\mathrm{Pic}(S) \quad = \quad \bigcup_{n \in \mathbb{Z}} \mathrm{Pic}^n(S) \;\simeq\; \bigcup_{n \in \mathbb{Z}} \mathrm{Pic}^0(S).$$

This shows that the moduli space of isomorphism classes of line bundles is indexed by a discrete parameter, as well as a 'continuous' parameter. As is typical in problems of moduli, the discrete parameter is usually easier to determine, and as we have seen, a lot more work was required to work out $\mathrm{Pic}^0(S)$. In section 4.2, example 4.22 we will see Abel's theorem applied to the classical case of an elliptic curve.

Chapter 4

Linear systems and the Riemann-Roch theorem

Linear systems are closely related to divisors and line bundles. Given an effective divisor D on a Riemann surface S, consider the $\mathbb{C}$-vector space [1]

$$\mathcal{L}(D) \quad = \quad \{f \in \mathcal{K}^*(S) \mid (f) + D \geqslant 0\}$$

and one might surmise that we can use the functions in $\mathcal{L}(D)$ to map S into projective space, which allows the extrinsic [2] study of S. Denote $\ell(D) = \dim(\mathcal{L}(D))$. The reason why we only consider effective divisors $D \in \mathrm{Div}(S)$ is due to the following

Proposition 4.1 *If* $\deg(D) < 0$*, then* $\mathcal{L}(D) = 0$.

Proof Suppose $f \in \mathcal{K}^*(S)$ such that $(f) + D \geqslant 0$, then $0 + \deg(D) \geqslant 0$ contradicting $\deg(D) < 0$. □

We actually have met the vector space $\mathcal{L}(D)$ before. Recall that holomorphic sections σ of $\mathcal{O}(D)$ satisfy $(\sigma) + D \geqslant 0$, hence we obtain the

Proposition 4.2 *There is a natural isomorphism of* $\mathbb{C}$*-vector spaces* $\mathcal{L}(D) \simeq H^0(S, \mathcal{O}(D))$.

We can characterise this in terms of divisors. Define, set-theoretically,

$$|D| \quad = \quad \{D' \in \mathrm{Div}(S) \mid D' \sim D, D' \geqslant 0\}$$

then we have the following

Proposition 4.3 *There is a bijection* $|D| \longrightarrow \mathbb{P}(\mathcal{L}(D))$.

Proof Let $D' \in |D|$, then $D' = D + (f)$ for some $f \in \mathcal{K}^*(S)$. The function f is unique up to scalar multiplication, and satisfies $(f) + D \geqslant 0$ since D' is effective. So define a map $|D| \longrightarrow \mathbb{P}(\mathcal{L}(D))$ by $D' \longmapsto [f]$ where $[f] = \{\lambda f \in \mathbb{P}(\mathcal{L}(D)) \mid \lambda \in \mathbb{C}\}$. The inverse is

[1] This is known as the **Riemann-Roch space** associated to the divisor D.
[2] That is, we study S via its embedding in $\mathbb{P}^N$.

given by $[f] \longmapsto D + (f) \in |D|$ and clearly does not depend on choice of representative of $[f]$. □

We can now define linear systems for a general compact complex manifold M.

Definition 4.4 *Let $|D|$ be the* ***complete linear system*** *associated to the divisor D. The effective divisors corresponding to subspaces of $\mathbb{P}H^0(M, \mathcal{O}(D))$ are called* ***linear systems****.*

The space $|D|$ contains all effective divisors linearly equivalent to D, hence the name *complete* linear system. The same constructions are valid for any compact complex manifold M, in which case, maps to projective space are even more important- they are candidates for embedding M into $\mathbb{P}^N$. If this occurs, then M is algebraic by Chow's theorem, so we can study M using algebro-geometric techniques.

Example 4.5 Let M be a compact complex manifold, D an effective divisor on M, and $f_0, \ldots, f_N$ be a basis for $H^0(M, \mathcal{O}(D))$. Consider [3]

$$\begin{aligned} \iota_D : M & \dashrightarrow \mathbb{P}H^0(M, \mathcal{O}(D)) \simeq \mathbb{P}^N \\ p & \longmapsto (f_0(p) : \ldots : f_N(p)). \end{aligned}$$

The map φ is well-defined provided that $f_0, \ldots, f_N$ do not simultaneously vanish at some point $p \in M$. Let $X := \{p \in M \mid f_0(p) = \ldots = f_N(p) = 0\}$, then $\varphi : M - X \longrightarrow \mathbb{P}^N$ is a well-defined map.

Definition 4.6 *The* ***dimension*** *of a complete linear system is defined by*

$$\dim |D| = \dim(\mathbb{P}H^0(M, \mathcal{O}(D))) = h^0(M, \mathcal{O}(D)) - 1.$$

Linear systems of dimensions $1, 2,$ and 3 are respectively known as ***pencils, nets,*** *and* ***webs****.*

In general, for ι_D to be an embedding $M \longrightarrow \mathbb{P}^N$, the dimension N of $|D|$ must to be greater than M. Recall in example 4.5, the map ι_D is not defined for points where elements of $H^0(M, \mathcal{O}(D))$ all vanish. This leads to the following

Definition 4.7 *A point $p \in M$ is a* ***base point*** *of a linear system $W \subset |D|$ if every element of W contains p, that is, for all $D \in W$, $D \geqslant p$. Call the set of all base points of $|D|$ its* ***base locus****. A linear system is* ***base point free*** *if its base locus is empty.*

We can view this in terms of the sections in $H^0(M, \mathcal{O}(D))$.

[3] The notation $\dashrightarrow$ means a map which is not everywhere defined.

Example 4.8 Let $f_1, \ldots, f_r$ span a linear subspace W of $\mathbb{P}H^0(M, \mathcal{O}(D))$, then the base locus of the linear system corresponding to W is the set $B = \{p \in M \mid f_0(p) = \ldots = f_r(p) = 0\}$.

As example 4.5 indicates, the map $M - \to \mathbb{P}W$ is well defined and holomorphic away from its base locus B.

Given any holomorphic map $\varphi : M \longrightarrow \mathbb{P}^N$, we can pullback the hyperplane divisors $H \in \mathrm{Div}(\mathbb{P}^N)$ to obtain divisors $\varphi^* H \in \mathrm{Div}(M)$, provided that $\varphi(M)$ is not contained in H. Then the linear system corresponding to the map φ is given by $\{\varphi^* H\}_{H \in (\mathbb{P}^N)^\vee, \varphi(M) \not\subseteq H}$. Let φ be given by $m \longmapsto (f_0(m) : \ldots : f_N(m))$, and

$$D \;=\; - \sum_{H \text{ a hypersurface in } M} \left(\min_{1 \leqslant i \leqslant N} \mathrm{ord}_H(f_i) \right) H \;\in\; \mathrm{Div}(M).$$

Then if $H \in \left(\mathbb{P}^N\right)^\vee$ is given by the linear form $\alpha_0 x_0 + \ldots + \alpha_N x_N$, we have

$$\varphi^*(H) \;=\; (\alpha_0 f_0 + \ldots + \alpha_N f_N) + D.$$

So the basic correspondence

$$\left\{ \begin{array}{c} \text{linear systems } |D| \\ \text{with base locus } B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{holomorphic maps} \\ \iota_D : M - B \longrightarrow \mathbb{P}(\mathcal{L}(D)) \end{array} \right\} \tag{4.1}$$

restricted to base point free linear systems give holomorphic maps $M \longrightarrow \mathbb{P}(\mathcal{L}(D))$. Note that this is not a priori an embedding. The following proposition gives a condition for $\iota_D : S \longrightarrow \mathbb{P}(\mathcal{L}(D))$ to be an embedding in the case of Riemann surface

Proposition 4.9 *Let S be a Riemann surface, and $|D|$ be a base point free linear system. Then ι_D is an embedding iff $\ell(D - p - q) = \ell(D) - 2$ for all $p, q \in S$.*

Proof Let $f_0, \ldots, f_N$ be a basis for $\mathcal{L}(D)$, so that $\iota_D : S \longrightarrow \mathbb{P}^N$. Suppose for some distinct points $p, q \in S$, $\iota_D(p) = \iota_D(q)$. Choose coordinates on $\mathbb{P}^N$ such that $\iota_D(p) = (1 : 0 : \ldots : 0)$, then $f_1(p) = \ldots = f_N(p) = 0$. This implies $f_1, \ldots, f_N$ is a basis for $\mathcal{L}(D - p)$. The same argument show that $f_1, \ldots, f_N$ are a basis for $\mathcal{L}(D - q)$. So

$$\mathcal{L}(D - p) \;=\; \mathcal{L}(D - q) \;=\; \mathcal{L}(D - p - q) \tag{4.2}$$

The arguments are reversible, so ι_D is not one to one iff there exists distinct points $p, q \in S$ such that (4.2) holds.

Now suppose ι_K is one to one. Note that $\ell(D) \geqslant \ell(D-p)$, with equality iff p is a base point of $|D|$. Since $|D|$ is base point free, we have $\ell(D-p) = \ell(D) - 1$. Moreover $\ell(D-p-q) \leqslant \ell(D-p)-1$, with equality implying (4.2) holds. So $\ell(D-p-q) = \ell(D)-2$. Conversely, if $\ell(D-p-q) = \ell(D)-2$ holds, we must have $\mathcal{L}(D-p-q) \subset \mathcal{L}(D-p) \subset \mathcal{L}(D)$. Hence ι_D is one to one.

Finally ι_D is embedding at p iff there exists $f \in \mathcal{L}(D)$ vanishing exactly to order 1 at p. That is, there exists $f \in \mathcal{L}(D-p)$ but $f \notin \mathcal{L}(D-2p)$. Now

$$\mathcal{L}(D-2p) \quad \subseteq \quad \mathcal{L}(D-p)$$

so we have $\ell(D-2p) = \ell(D-p) - 1 = \ell(D) - 2$. □

Concerning the generic elements of a linear system away from the base locus, we have Bertini's theorem. The term generic has a precise meaning in algebraic geometry: a property holds **generically** on a variety X if it fails to hold on some subvariety of X of strictly lower dimension. An example of this appears in an instance of Bezout's theorem; the generic hyperplane H intersects a curve C of degree n in $\mathbb{P}^m$ in n distinct points. This is because H is tangent to C at only a finite number of points, hence this set of points of tangency has dimension 0.

Theorem 4.10 *(Bertini) Let $W \subset |D|$ be a linear system and B its base locus. Then a generic element $D' \in W$, $D' \notin B$ is smooth.*

Bertini's theorem is the analogue to Sard's theorem in differential geometry, which states that for a smooth map f between smooth manifolds X and Y, the set $\{f : X \longrightarrow Y \mid \text{rank}(f_*) < \dim(Y)\}$ has Lebesgue measure 0. We will not prove Bertini's theorem, but give an example of how it applies.

Example 4.11 Let S be a Riemann surface with projective embedding $\varphi : S \longrightarrow \mathbb{P}^n$ and consider the pullback divisors $\varphi^* H$ where $H \in (\mathbb{P}^n)^\vee$. All such divisors $\varphi^* H$ have the same degree, and are in fact linearly equivalent, since

$$\varphi^* H \quad = \quad \varphi^* H' + (f)$$

where f is the quotient of the linear forms defining H and H' respectively. The set $\{\varphi^* H \mid H \in (\mathbb{P}^n)^\vee\}$ is a complete linear system, since the divisors there correspond to $H^0(S, \mathcal{O}(1))$. It is clearly base point free. The divisor $\varphi^* H$ is generic iff H is a generic hyperplane. Since the points of intersection of a generic hyperplane with S are distinct, so too are the points in φ^*H. *In this case, the condition* $D = p_1 + \ldots + p_d$ *where* $p_1, \ldots, p_d$ *are distinct characterises smoothness.*

The author admits that the last statement is a bit mysterious, but the notion of smoothness cannot be formalised without schemes, which we definitely do not have the space to develop. We will use a similar argument in the proof of the Torelli theorem, so the above italicised statement will be taken as the definition of smoothness in this case. Below are some examples of linear systems arising in algebraic geometry.

Example 4.12

- Suppose $\deg(D) = d$, then there is a natural isomorphism of vector spaces

$$H^0(\mathbb{P}^n_{z_0,\ldots,z_n}, \mathcal{O}(D)) \longrightarrow \mathbb{C}[z_0, \ldots, z_n]_d$$

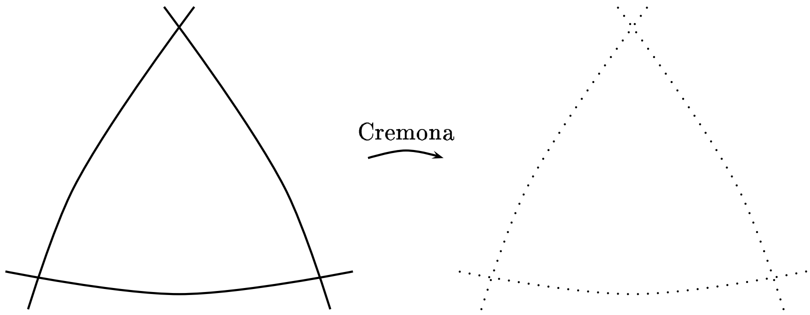
  This is proposition 3.27, and using the above, we obtain the d-**Veronese embedding**

$$\begin{aligned} V : \mathbb{P}^n &\longrightarrow \mathbb{P}^N \\ (x_0 : \ldots : x_n) &\longmapsto (m_0 : \ldots : m_N) \end{aligned}$$

  where $m_0, \ldots, m_N$ is the monomial basis of $\mathbb{C}[z_0, \ldots, z_n]_d$. This is base point free, since the equations $m_0(\boldsymbol{x}) = \ldots = m_N(\boldsymbol{x}) = 0$ implies $x_0 = \ldots = x_n = 0$.
- The **Cremona transformation**

$$\begin{aligned} C : \mathbb{P}^2 &\dashrightarrow \mathbb{P}^2 \\ (x : y : z) &\longmapsto (xy : yz : zx) \end{aligned}$$

  is the map into the subspace of $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(2))$ spanned by xy, yz, zx. The associated linear system has base points where any two of x, y, z are zero, that is, $(0 : 0 : 1), (0 : 1 : 0), (1 : 0 : 0)$.



Example 4.13 Denote $H_{\lambda x+\mu y+\nu z} \subset \mathbb{P}^2$ to be the divisor corresponding to the line defined by $\lambda x + \mu y + \nu z = 0$. Consider the linear system $\mathcal{L}$ corresponding to $W = \text{span}\{x, y\} \leqslant \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(1))$. The base locus of $\mathcal{L}$ is the point $p := H_x \cap H_y = (0 : 0 : 1)$. The map $\mathbb{P}^2 \longrightarrow \mathbb{P}W \simeq \mathbb{P}^1$ corresponding to the linear system $\mathcal{L}$ is simply projection away from p, and so is not defined at p. Bertini's theorem is trivial in this case, since the generic divisor $H_{\lambda x+\mu y} \in \mathcal{L}$ is nonsingular, unless $\lambda = \mu = 0$, that is, at the base point p.

Example 4.14 Consider the complete linear system corresponding to $\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(2H_x))$, and let

$$\left\{1, \frac{yx}{x^2}, \frac{zx}{x^2}, \frac{yz}{x^2}, \frac{y^2}{x^2}, \frac{z^2}{x^2}\right\}$$

be a basis for $H^0(\mathbb{P}^2, \mathcal{O}(2H_x))$. The linear system $|2H_x|$ is base point free, and any $D \in |2H_x|$ is given by $D = 2H_x + (f)$ for some $f \in H^0(\mathbb{P}^2, \mathcal{O}(2H_x))$, that is

$$\begin{aligned} D &= 2H_x + \left(\frac{\lambda_0 + \lambda_1 yx + \lambda_2 zx + \lambda_3 yz + \lambda_4 y^2 + \lambda_5 z^2}{x^2}\right) \\ &= (\lambda_0 + \lambda_1 yx + \lambda_2 zx + \lambda_3 yz + \lambda_4 y^2 + \lambda_5 z^2) \\ &= \text{a conic in } \{(x : y : z) \in \mathbb{P}^2 \mid x \neq 0\} \end{aligned}$$

In this case, Bertini's theorem says that the generic conic is nonsingular.

Example 4.15 Recall that the Grassmannian is defined as

$$\mathcal{G}(r, n) = \{V \leqslant \mathbb{C}^n \mid \dim(V) = r\}.$$

It is naturally embedded into projective space as follows

$$\begin{aligned} \mathcal{G}(r, n) &\longrightarrow \mathbb{P}\left(\bigwedge^r \mathbb{C}^n\right) \\ V &\longmapsto (v_1 \wedge \ldots \wedge v_r) \end{aligned}$$

where $v_1, \ldots, v_r$ is a basis of the r-dimensional subspace V. Again we can pullback hyperplanes in $\mathbb{P}\left(\bigwedge^r \mathbb{C}^n\right)$ to obtain a linear system. The Grassmannian will make its appearance again in the proof of the Torelli theorem.

4.1 The Riemann-Roch theorem

This is the principal tool in the study of complete linear systems on a Riemann surface S of genus g. Determining the dimension of $H^0(S, \mathcal{O}(D))$ is equivalent to finding the number of linearly independent meromorphic functions satisfying $D + (f) \geqslant 0$, which is not always an easy task. The Riemann-Roch theorem gives a formula for $h^0(S, \mathcal{O}(D))$, the caveat is that $h^0(S, \mathcal{O}(K - D))$ must also be known.

$$h^0(S, \mathcal{O}(D)) - h^0(S, \mathcal{O}(K - D)) \quad = \quad \deg(D) - g + 1 \tag{4.3}$$

We will see in section 4.2 when one can extract useful information from the above formula. In this section, we will examine some interpretations and a proof of the Riemann-Roch theorem.

If we restrict to the case where D is an effective divisor, a geometric interpretation of (4.3) can be given. First we make a

Definition 4.16 *If $p_1, \ldots, p_k$ are points in $\mathbb{P}^n$, then the **linear span** of the points $p_1, \ldots, p_k$ is the intersection of all hyperplanes containing $p_1, \ldots, p_k$. If no hyperplanes contain all of $p_1, \ldots, p_k$, then we say the linear span of $p_1, \ldots, p_k$ is $\mathbb{P}^n$.*

We can generalise this to the linear span of an effective divisor $D \in \mathrm{Div}(S)$ where $\iota : S \hookrightarrow \mathbb{P}^n$. A hyperplane $H \in (\mathbb{P}^n)^\vee$ **contains** D if $\iota^* H \geqslant D$, then we define the **linear span**, $\overline{D}$, of D to be $\left\{H \in (\mathbb{P}^n)^\vee \mid \iota^* H \geqslant D\right\}$. If $D = p_1 + \ldots + p_k$ where $p_1, \ldots, p_k$ are distinct points, then the linear span of D is precisely the linear span of $p_1, \ldots, p_k$.

Proposition 4.17 *Let $D \in \mathrm{Div}(S)$ be an effective divisor. Then there is a one to one correspondence between the space of hyperplanes containing D, and $|K - D|$*

Proof Let $H \in \left(\mathbb{P}^{g-1}\right)^\vee$ containing D and consider $\iota_K^* H$. Then $\iota_K^* H \in |K - D|$, since $\iota_K^* H$ contains D and by definition $\iota_K^* H \in |K|$. Conversely, given any $D' \in |K - D|$, $D' + D \sim K$ so there exists a hyperplane H' in $\mathbb{P}^{g-1}$ such that $\iota_K^* H' = D' + D$, hence H' contains D. This gives the one to one correspondence. □

Moreover the set of hyperplanes containing D is a linear subspace of $\left(\mathbb{P}^{g-1}\right)^\vee$, and its dimension is equal to $g - 2 - \dim(\overline{D})$ [4]. Now applying the Riemann-Roch theorem, we

[4] Think of lines in $\mathbb{P}^2$ which contain a point p. These lines span a one dimensional space, and the span of p is zero dimensional. Adding these we get 1, which is 1 less than 2.

get

$$\begin{aligned} h^0(K-D)-1 &= g-2-\dim(\overline{D}) \\ h^0(D)-(\deg(D)-g+1)-1 &= g-2-\dim(\overline{D}) \\ \dim|D| &= \deg(D)-1-\dim(\overline{D}) \end{aligned}$$

The Riemann-Roch theorem in this form relates the dimension of the complete linear system $|D|$ to the geometry of the canonical curve $\iota_K(S)$. The theorem can also be proven, for effective divisors, purely in its geometric form. For details see pages 248-249 of [GH78].

We can also interpret the Riemann-Roch theorem in the context of sheaves and cohomology. While not as geometric, this formulation paves the way for a generalisation to higher dimensional varieties [5] and leads to a simple, concise proof using Serre duality.

Definition 4.18 *Define the **holomorphic Euler characteristic** of the line bundle L and M to be the alternating sum*

$$\chi(L) = \sum_{p\in\mathbb{N}}(-1)^p h^p(M,\mathcal{O}(L)) \tag{4.4}$$

Proposition 4.19 *Let L be a line bundle and S be a Riemann surface, then*

$$\chi(L) = \chi(\mathcal{O}_S)+c_1(L) \tag{4.5}$$

holds and is equivalent to (4.3).

Proof Specialising (4.4) to $M=S$, we have

$$\begin{aligned} \chi(L) &= h^0(S,\mathcal{O}(L))-h^1(S,\mathcal{O}(L)) \\ &= h^0(S,\mathcal{O}(L))-h^0(S,\mathcal{O}(K-L)) & (4.6) \\ \chi(\mathcal{O}_S) &= h^0(S,\mathcal{O})-h^1(S,\mathcal{O}) \\ &= 1-g & (4.7) \end{aligned}$$

Equation (4.6) is due to Serre duality (c.f. theorem 9.5); equation (4.7) holds since the only holomorphic functions on S are the constant ones (c.f. proposition 1.33), and

[5] This is the Hirzebruch-Riemann-Roch theorem (c.f. page 437 of [GH78])

$h^1(S,\mathcal{O}) = h^0(S,\Omega^1) = g$ (c.f. example 4.25). By proposition 3.35, $c_1([D]) = \deg(D)$. Putting these facts together gives equivalence of (4.3) and (4.5).

To prove (4.5), we proceed by induction. First note that if $L = \mathcal{O}_S$, then $c_1(\mathcal{O}_S) = 0$ and (4.5) holds. Now suppose (4.5) is true for $L = [D]$, we show that this implies it is true for $L = [D+p]$ and $L = [D-p]$, hence for all $L \in \mathrm{Pic}(S)$. The following exact sequence of sheaves over S

$$0 \longrightarrow \mathcal{O}(D) \xrightarrow{\iota} \mathcal{O}(D+p) \xrightarrow{\mathrm{Res}_p} \mathbb{C}_p \longrightarrow 0$$

gives

$$0 \longrightarrow H^0(S,\mathcal{O}(D)) \longrightarrow H^0(S,\mathcal{O}(D+p)) \longrightarrow \mathbb{C}_p \longrightarrow H^1(S,\mathcal{O}(D)) \longrightarrow H^1(S,\mathcal{O}(D+p)) \longrightarrow 0$$

The alternating sum of dimensions of summands in an exact sequence is zero, so

$$\begin{aligned} 0 &= h^0(S,\mathcal{O}(D)) - h^0(S,\mathcal{O}(D+p)) + \dim(\mathbb{C}_p) - h^1(S,\mathcal{O}(D)) + h^1(S,\mathcal{O}(D+p)) \\ &= (h^0(S,\mathcal{O}(D)) - h^1(S,\mathcal{O}(D))) + 1 - (h^0(S,\mathcal{O}(D+p)) - h^1(S,\mathcal{O}(D+p))) \end{aligned}$$

giving $\chi([D]) + 1 - \chi([D+p]) = 0$. By assumption $\chi([D]) = \chi(\mathcal{O}_S) + c_1(D)$, so

$$\chi([D+p]) = \chi([D]) + 1 = \chi(\mathcal{O}_S) + c_1(D) + 1 = \chi(\mathcal{O}_S) + c_1([D+p])$$

as required. The proof for $L = [D-p]$ is identical, and we have proved the Riemann-Roch theorem. □

4.2 Application and Examples

The Riemann-Roch theorem does not always give $h^0(S,\mathcal{O}(D))$ easily, due to the appearance of the term $h^0(S,\mathcal{O}(K-D))$, whose value may not be obvious. However, it turns out that for a generic divisor $D = p_1 + \ldots + p_d$, that is $p_1,\ldots,p_d$ is in general position, the subspace spanned will have dimension $g-d$. This gives the following

$$h^0(S,\mathcal{O}(D)) = d - g + 1 - (g-d) = 1$$

Also if $h^0(S,\mathcal{O}(K-D)) = 0$, which occurs if $\deg(K-D) < 0$, (c.f. proposition 4.1), then

$$h^0(S,\mathcal{O}(D)) = d - g + 1$$

So for a generic divisor D,

$$h^0(S,\mathcal{O}(D)) = \begin{cases} 1 & \text{if} \quad \deg(D) < 2g-2 \\ d-g+1 & \text{if} \quad \deg(D) \geqslant 2g-2 \end{cases}$$

We emphasise that $D \neq K$ in the above formula, whence $h^0(S,\mathcal{O}(K)) = g$ as shown below in example 4.25.

Applying the Riemann-Roch theorem is somewhat harder when $h^0(S,\mathcal{O}(K-D)) \neq 0$, and we call such linear systems $|D|$ **special linear systems**. Regarding special linear systems, we have Clifford's theorem,

Theorem 4.20 *(Clifford) Suppose $D \in \mathrm{Div}(S)$ is any special effective divisor such that $D \neq 0$, $D \neq K$. Then we have*

$$\dim|D| \leqslant \frac{\deg(D)}{2}$$

with equality holding only if S is hyperelliptic.

Proof Let D_1 and D_2 be two effective divisors on S. If $D_1' = D_1 + (f_1)$ and $D_2' = D_2 + (f_2)$ for some $f_1, f_2 \in \mathcal{K}^*(S)$, then $D_1' + D_2' = D_1 + D_2 + (f_1 f_2)$. This gives

$$\dim|D_1| + \dim|D_2| \leqslant \dim|D_1 + D_2| \tag{4.8}$$

If $|D|$ is a special linear system, then $h^0(S,\mathcal{O}(K-D)) \neq 0$, and we can substitute D and $K-D$ into (4.8). By Riemann-Roch, $\dim|K-D| = \dim|D| - (\deg(D) - g + 1)$ so

$$\begin{aligned} \dim|D| + (\dim|D| - (\deg(D) - g + 1)) &\leqslant g-1 \\ 2\dim|D| &\leqslant \deg(D) \end{aligned} \tag{4.9}$$

Equality holds in (4.8) iff $|K| = |D_1| + |D_2|$ where $D_1 \in |K - D_2|$. Clearly, then, equality holds if $D_1 = 0$ and $D_1 = K$. Also, let S be hyperelliptic let $f : S \longrightarrow \mathbb{P}^1$ be the double cover to $\mathbb{P}^1$. Consider the pullback divisor f^*H for some $H \in \mathrm{Div}(\mathbb{P}^1)$, then since $2\deg(H) = \deg(f^*H)$

$$\begin{aligned} h^0(K - \pi^*H) &= g - \deg(H) \\ h^0(\pi^*H) &= \deg(\pi^*H) - g + 1 + g - \deg(H) \\ \dim|\pi^*H| &= \deg(H) = \frac{\deg(f^*H)}{2} \end{aligned}$$

In fact, these are the only instances where equality in (4.8) holds. Let S be non-hyperelliptic, and consider the canonical embedding $\iota_K : S \longrightarrow \mathbb{P}^{g-1}$. Recall that every element in $|K|$ is the pullback $\iota_K^* H$ of some hyperplane $H \in \left(\mathbb{P}^{g-1}\right)^\vee$. Suppose $|K| = |D_1| + |D_2|$, and D_1 is not 0 or K, so assume $\deg(D_2) \leqslant g-1$. Then $H \in \left(\mathbb{P}^{g-1}\right)^\vee$ contains the points of D_2, which are linearly dependent, since $h^0(S, \mathcal{O}(K - D_2)) > g - \deg(D_2)$. This contradicts the fact any $g-1$ points of a generic hyperplane section are in general position. Hence if S is non-hyperelliptic equality in (4.9) only holds if $D = 0$ or $D = K$.

□

We conclude this chapter with some important applications of the Riemann-Roch theorem. The letter g will always denote the genus of S.

Example 4.21 Suppose S has genus 0, then S is isomorphic to the Riemann sphere. Using the Riemann-Roch theorem, we get for any $p \in \mathrm{Div}(S)$

$$\begin{aligned} h^0(S, \mathcal{O}(p)) &= \deg(p) - 0 + 1 + h^0(K - p) \\ &= 2 \end{aligned}$$

since $\deg(K - p) = 2 \cdot 0 - 2 - 1 = -3$ implies $h^0(K - p) = 1$ by proposition 4.1. Hence there must be a nonconstant $f \in H^0(S, \mathcal{O}(p))$, giving the isomorphism $f : S \longrightarrow \mathbb{P}^1$.

Example 4.22 Another application of the Riemann-Roch theorem is to prove the addition law on the elliptic curve. This is a Riemann surface, E, of genus 1. Recall that Abel's theorem gives the following isomorphism

$$\mathrm{Pic}^0(S) \simeq \mathcal{J}(S)$$

where S is any Riemann surface. On picking an arbitrary base point $p_0 \in S$, we aim to show the following isomorphism in the genus 1 case,

$$\begin{aligned} \varphi : E &\longrightarrow \mathrm{Pic}^0(E) \\ p &\longmapsto [p - p_0]. \end{aligned}$$

Then by Abel's theorem $E \simeq \mathrm{Pic}^0(E) \simeq \mathcal{J}(E)$ and the additive group structure on $\mathcal{J}(E)$ pulls back to E via φ.

Let $p, q \in E$ such that $p \neq q$. If $p - q = (f)$ for some $f \in \mathcal{K}^*(E)$, then $h^0(E, \mathcal{O}(p-q)) > 0$. Now by Riemann-Roch,

$$h^0(E, \mathcal{O}(p)) \quad = \quad 1 - 1 + 1 + h^0(E, \mathcal{O}(K - p)) \quad = \quad 1 \tag{4.10}$$

and by proposition 4.1, $\deg(K - p) = -1$ implies $h^0(E, \mathcal{O}(K - p)) = 0$. This means $H^0(E, \mathcal{O}(p))$ consists of only the constant meromorphic functions. If $\sigma \in H^0(E, \mathcal{O}(p-q))$ then $\sigma \in H^0(E, \mathcal{O}(p))$ and σ satisfies $\sigma(q) = 0$, so we obtain $h^0(E, \mathcal{O}(p - q)) = 0$. Hence $p \nsim q$.

Now suppose $D \in \mathrm{Div}^1(E)$. By similar reasoning as (4.10), we have

$$h^0(E, \mathcal{O}(D)) \quad = \quad 1$$

so $|D| = \mathbb{P}H^0(E, \mathcal{O}(D)) \simeq$(effective divisors linearly equivalent to D) contains one point, p_D. Hence any divisor class, $[D]$, of degree 1 has a unique effective representative, $p_D \in E$; and we have an inverse to $[\cdot]$, given by $[D] \longmapsto p_D$.

Note 4.23 The argument above using Riemann-Roch fails for higher genus. If S has genus $g > 1$, then (4.10) becomes

$$h^0(S, \mathcal{O}(p)) \quad = \quad 1 - g + 1 + h^0(E, \mathcal{O}(K - p)) \ \leqslant \ 0$$

In fact, a simple dimension count shows, $S \not\simeq \mathrm{Pic}^0(S)$ for $g > 1$.

Example 4.24 The third application will be to verify the claim in chapter 2 that all genus 2 Riemann surfaces are hyperelliptic. Since $\deg(K) = 2$, and the canonical map $\iota_K : S \longrightarrow \mathbb{P}^1$. Pick any point z on $\mathbb{P}^1$ and consider the pullback divisor $\iota_K^* z$. Since ι_K is two to one, $\iota_K z = 2p$ where $p \in \iota_K^{-1} z$ and $2p \sim K$. So

$$\begin{aligned} h^0(S, \mathcal{O}(2p)) \quad &= \quad \deg(2p) - g + 1 + h^0(S, \mathcal{O}(K - 2p)) \\ &= \quad 1 + 1. \end{aligned}$$

Hence there exists a nonconstant meromorphic function on S with a double pole at p.

Example 4.25 We can determine $\deg(K)$ and $h^0(S, \Omega^1) = h^0(S, \mathcal{O}(K))$ (these are equal due to Serre duality and the Hodge theorem $\overline{H^0(S, \Omega^1)} \simeq H^1(S, \mathcal{O}) \simeq H^0(S, \mathcal{O}(K))$). by putting $D = K$ and $D = 0$ in (4.3). This gives

$$\begin{aligned} h^0(S, \mathcal{O}(K)) &= \deg(K) - g + 1 + h^0(S, \mathcal{O}) \\ h^0(S, \mathcal{O}) &= -g + 1 + h^0(S, \mathcal{O}(K)) \end{aligned}$$

Solving for $\deg(K)$ and $h^0(S, \mathcal{O}(K))$ simultaneously we obtain

$$\begin{aligned} \deg(K) &= h^0(S, \mathcal{O}(K)) + g - 1 - h^0(S, \mathcal{O}) \\ &= h^0(S, \mathcal{O}) + g - 1 + g - 1 - h^0(S, \mathcal{O}) \\ &= 2g - 2 \\ h^0(S, \mathcal{O}(K)) &= (2g - 2) - g + 1 + h^0(S, \mathcal{O}) \\ &= g \end{aligned}$$

since $h^0(S, \mathcal{O}) = 1$ (c.f. proposition 1.33).

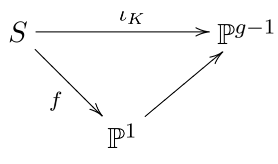
Example 4.26 Recall that a Riemann surface S is hyperelliptic if there exists a meromorphic function f with a double pole, that is $h^0(S, \mathcal{O}(p + q)) > 1$ for all $p, q \in S$. Applying the Riemann-Roch theorem,

$$\begin{aligned} h^0(S, \mathcal{O}(p + q)) &= 2 - g + 1 + h^0(S, \mathcal{O}(K - p - q)) \\ g - 2 &< h^0(S, \mathcal{O}(K - p - q)). \end{aligned}$$

By proposition 4.9, $\iota_K : S \longrightarrow \mathbb{P}^{g-1}$ is not an embedding, note that the converse holds as well.

Example 4.27 In this example, we examine the differences between the canonical curve of a hyperelliptic and non-hyperelliptic Riemann surface. Consider pullback divisors of hyperplanes $\iota_K^* H \in \text{Div}(S)$ where $H \in \left(\mathbb{P}^{g-1}\right)^{g-1}$, then $\iota_K^* H \sim K$ so $\deg(\iota_K H) = 2g - 2$. In the non-hyperelliptic case ι_K is an embedding, so H intersects $\iota_K(S)$ in $2g - 2$ points, counting multiplicity.

However in the hyperelliptic case, there is a two to one map $f : S \longrightarrow \mathbb{P}^1$. We claim that the canonical map factors through f, that is, the following commutes



First write the hyperelliptic Riemann surface S as the completion of $\{(x,y) \in \mathbb{C}^2_{x,y} \mid y^2 = h(x)\}$ for some $h \in \mathbb{C}[x]_{2g+2}$, with the two to one covering of $\mathbb{P}^1$ given by π, the projection onto the first coordinate. We can then work out a basis for $H^0(S, \Omega^1)$, firstly $\mathrm{d}y/x \in H^0(S, \Omega^1)$, and if ω is any holomorphic 1-form on S, we can write $\omega = p(x)\frac{\mathrm{d}y}{x}$ where $p \in \bigoplus_{k=0}^{g-1} \mathbb{C}[x]_k$ Hence the following,

$$\left\{ \frac{\mathrm{d}y}{x}, x\frac{\mathrm{d}y}{x}, \ldots, x^{g-1}\frac{\mathrm{d}y}{x} \right\}$$

is a basis for $H^0(S, \Omega^1)$. With respect to this basis, the canonical map is given by

$$\begin{aligned} \iota_K : S &\longrightarrow \mathbb{P}^{g-1} \\ (x,y) &\longmapsto (1 : x : x^2 : \ldots : x^{g-1}). \end{aligned}$$

It is then clear that $\iota_K = \varphi \circ \pi$ where $\varphi : \mathbb{P}^1_x \longrightarrow \mathbb{P}^{g-1}$ is given by $x \longmapsto (1 : x : \ldots : x^{g-1})$. For any $H \in \left(\mathbb{P}^{g-1}\right)^\vee$, H intersects $\iota_K(S)$ at $g-1$ points, counting multiplicities; while the pullback $\iota_K^*(H) \in \mathrm{Div}(S)$ will have degree $2g-2$, since ι_K is two to one.

The proof of the Torelli theorem in chapter 7 will make use of both the Riemann-Roch theorem and Clifford's theorem.

Chapter 5

Complex tori

This chapter on complex tori is motivated by the fact that Jacobians are complex tori. A complex torus is defined to be the quotient space V/Λ where V is an n-dimensional complex vector space, and Λ is a lattice spanned by $2n$ linearly independent vectors in V. It is a Kähler manifold with the induced Euclidean metric on V. Complex tori are the simplest examples of compact higher dimensional varieties, although not all complex tori are algebraic.

Definition 5.1 *A positive definite Hermitian form H on V such that $E := \Im(H)$ takes integer values on Λ is called a* ***polarisation*** *of V/Λ. We sometimes also refer to E as the polarisation, since this determines H uniquely.*

A consequence of the Kodaira embedding theorem is that any complex torus V/Λ admitting a polarisation can be embedded into projective space, since the polarisation guarantees the existence of a positive line bundle on V/Λ.

Definition 5.2 *A complex torus which admits a polarisation is called an* ***abelian variety****.*

Note 5.3 The definition of an abelian variety given above is analytic. An algebraic definition can be given, see page 100 of [Pol03]. This makes the definition possible over finite characteristic, and hence is an important construction in number theory.

A polarisation is **principal** if $\det(E) = 1$ and an abelian variety admitting a principal polarisation is called **principally polarised abelian variety**. We will write $(V/\Lambda, H)$ if the polarisation is explicitly given. These are important for two reasons. Firstly the Jacobian is, in fact, principally polarised. We will see why in chapter 6. Moreover there exists a sublattice Λ' of Λ such that $\mathbb{C}/\Lambda'$ together with $\frac{1}{n}H$ for some positive integer n is a principally polarised abelian variety. The existence of a polarisation on V/Λ can be expressed in the following coordinate form, known as the **Riemann conditions**. First we make a

Definition 5.4 *Let $\lambda_1, \ldots, \lambda_{2n}$ be a basis for Λ over $\mathbb{Z}$ and $e_1, \ldots, e_n$ be a basis for V over $\mathbb{C}$. Define the* ***period matrix*** *of V/Λ to be the change of basis matrix $\Omega \in M_{n,2n}(\mathbb{C})$, that is, Ω satisfies*

$$\Omega \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{2n} \end{pmatrix} = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

Then the Riemann conditions are given by

Theorem 5.5 *The complex torus V/Λ is an abelian variety iff there exists bases given in definition 5.4 such that the period matrix satisfies the following conditions*

$$\Omega = (\Delta_\delta, Z) \qquad Z = Z^T \qquad \Im(Z) \text{ is positive definite}$$

where $\Delta_\delta = \begin{pmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{pmatrix}$ *and $\delta_1, \ldots, \delta_n$ are integers satisfying $\delta_1 | \delta_2 | \ldots | \delta_n$.*

The calculation involves finding conditions relating two bases of $H^\bullet(M, \mathbb{Z})$ and $H^\bullet(M, \mathbb{C})$. We will prove this explicitly in the case of the Jacobian variety.

5.1 Cohomology of complex tori

As usual, we will compute the some useful cohomology groups. The first proposition is a simple application of the Künneth formula, which relates the cohomology groups of product spaces

$$H^m(X \times Y, \mathbb{Z}) \simeq \bigoplus_{p+q=m} H^p(X, \mathbb{Z}) \otimes H^q(Y, \mathbb{Z})$$

c.f. chapter 3 of [Hat02].

Proposition 5.6 *We have the canonical isomorphism of cohomology rings $H^\bullet(V/\Lambda, \mathbb{Z}) \simeq \bigwedge^\bullet H^1(V/\Lambda, \mathbb{Z})$.*

Proof Consider the natural map

$$\begin{aligned} \alpha : \bigwedge\nolimits^\bullet H^1(V/\Lambda, \mathbb{Z}) &\longrightarrow H^\bullet(V/\Lambda, \mathbb{Z}) \\ \xi_1 \wedge \ldots \wedge \xi_n &\longmapsto \xi_1 \smile \ldots \smile \xi_n \end{aligned}$$

for all $n \in \mathbb{N}$, and extended to other elements via linearity. This is an homomorphism since the cup product is skew-symmetric. The complex torus V/Λ is homeomorphic to $(S^1)^{2n}$, so we apply Künneth on the m-th graded piece to obtain

$$\begin{aligned} H^m(V/\Lambda, \mathbb{Z}) \simeq H^m((S^1)^{2n}, \mathbb{Z}) &\simeq \bigoplus_{i_1+i_2=m} H^{i_1}((S^1)^{2n-1}, \mathbb{Z}) \otimes H^{i_2}(S^1, \mathbb{Z}) \\ &\simeq \bigoplus_{\substack{i_1+\ldots+i_{2n}=m \\ 0\leqslant i_1,\ldots,i_{2n}\leqslant 1}} H^{i_1}(S^1, \mathbb{Z}) \otimes \ldots \otimes H^{i_{2n}}(S^1, \mathbb{Z}) \\ &:= \bigoplus_{\substack{i_1+\ldots+i_{2n}=m \\ 0\leqslant i_1,\ldots,i_{2n}\leqslant 1}} H^{i_1,\ldots,i_{2n}} \end{aligned}$$

since $H^k(S^1, \mathbb{Z}) = 0$ for all $k > 1$, and where the last line is a definition. Similarly

$$\bigwedge^m H^1(V/\Lambda, \mathbb{Z}) \simeq \bigwedge^m \bigoplus_{\substack{i_1+\ldots+i_{2n}=1 \\ 0\leqslant i_1,\ldots,i_{2n}\leqslant 1}} H^{i_1,\ldots,i_{2n}}$$

so $h^m(V/\Lambda, \mathbb{Z}) = \binom{2n}{m} = \dim\left(\bigwedge^m H^1(V/\Lambda, \mathbb{Z})\right)$. Finally α is surjective since the all cohomology of $2n$-tori are cup products of 1-dimensional classes. □

Now there is a canonical isomorphism $H^1(V/\Lambda, \mathbb{Z}) \simeq \mathrm{Hom}(\Lambda, \mathbb{Z}) := \Lambda^\vee$, so for all $m \in \mathbb{N}$ we have $H^m(V/\Lambda, \mathbb{Z}) \simeq \bigwedge^m \Lambda^\vee$. The next task is to determine $H^\bullet(V/\Lambda, \mathcal{O})$

Proposition 5.7 *We have the isomorphism of cohomology rings $H^\bullet(V/\Lambda, \mathcal{O}) \simeq \bigwedge^\bullet \overline{V}^\vee$, where $\overline{V}$ denotes complex conjugation.*

Proof See pages 4-5 of [Pol03] for proof. □

5.2 Line bundles on complex tori

In this section, we classify all line bundles on complex tori. In keeping with the cohomological language, we will show $\mathrm{Pic}(V/\Lambda) \xrightarrow{\phi} H^1(\Lambda, \mathcal{O}^*(V))$ is an isomorphism. Here $H^1(\Lambda, \mathcal{O}^*(V))$ refers to *group cohomology*, not sheaf cohomology, and is the first cohomology group of Λ with coefficients in $\mathcal{O}^*(V)$. The necessary definitions for group cohomology are given in section 9.1.

Let $L \xrightarrow{\rho} V/\Lambda$ be any holomorphic line bundle, we recall the pullback bundle $\pi^*L \longrightarrow V$ defined in definition **??**,

$$\begin{array}{ccc} \pi^*L & \longrightarrow & L \\ {\scriptstyle \pi^*\rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ V & \xrightarrow{\pi} & V/\Lambda \end{array}$$

Recall that $H^1(V,\mathcal{O}) = H^1(V,\mathbb{Z}) = 0$ (c.f. section 1.5), by the long exact sequence in cohomology $H^1(V,\mathcal{O}^*) = 0$, every holomorphic line bundle $L \longrightarrow V$ is trivial. So choose a global trivialisation $\varphi : \pi^*L \longrightarrow V \times \mathbb{C}$ of the pullback π^*L. We adopt the following conventions: for $z \in V$, $L_z := \rho^{-1}(z+\Lambda)$, $(\pi^*L)_z := (\pi^*\rho)^{-1}(z)$, and $\varphi_z := \varphi|_{\{z\}}$. The action of Λ on V permutes fibres, for $\lambda \in \Lambda$, $\lambda \cdot (\pi^*L)_z = (\pi^*L)_{z+\lambda}$, which induces a map $e_\lambda(z) : \mathbb{C} \longrightarrow \mathbb{C}$ via the commutative square,

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\;e_\lambda\;} & \mathbb{C} \\ \varphi_z \downarrow & & \downarrow \varphi_{z+\lambda} \\ (\pi^*L)_z & \xrightarrow{\;\lambda\;} & (\pi^*L)_{z+\lambda} \end{array}$$

Since φ_z is a linear isomorphism for all $z \in V$, $e_\lambda(z) \in \mathrm{Aut}(\mathbb{C}) \simeq \mathbb{C}^*$ for all $\lambda \in V$. That is $e : \Lambda \longrightarrow \mathcal{O}^*(V)$, and $\{e_\lambda\}_{\lambda\in\Lambda}$ are known as **multipliers** of the line bundle $L \longrightarrow V/\Lambda$. Note that the definition of e depends on the choice of $\varphi : \pi^*L \longrightarrow V \times \mathbb{C}$. For any $\lambda, \lambda' \in \Lambda$, the following diagrams commute

$$\mathbb{C} \xrightarrow{e_\lambda(z)} \mathbb{C} \xrightarrow{e_{\lambda'}(z+\lambda)} \mathbb{C}, \quad \mathbb{C} \xrightarrow{e_{\lambda+\lambda'}(z)} \mathbb{C} \qquad \text{and} \qquad \mathbb{C} \xrightarrow{e_{\lambda'}(z)} \mathbb{C} \xrightarrow{e_\lambda(z+\lambda')} \mathbb{C}, \quad \mathbb{C} \xrightarrow{e_{\lambda+\lambda'}(z)} \mathbb{C}$$

Given a representative ε of a class in $H^1(\Lambda, \mathcal{O}^*(V))$, we see that by the 1-cocycle equations $\mathrm{d}e = 1$, ε satisfies the conditions above. We can obtain a line bundle on V/Λ by defining the following equivalence relation on $V \times \mathbb{C}$, identified with the image of a global trivialisation of $\tilde{L} \longrightarrow V$. Define $(z,\ell) \sim (z+\lambda, \varepsilon_\lambda(z)\cdot\ell)$, then $(V\times\mathbb{C})/\sim$ with projection onto the first factor is a line bundle over V/Λ with $\{\varepsilon_\lambda\}_{\lambda\in\Lambda}$ as multipliers.

Theorem 5.8 *We have the isomorphism of groups* $\mathrm{Pic}(V/\Lambda) \xrightarrow{\phi} H^1(\Lambda, \mathcal{O}^*(V))$.

Proof Note that Λ acts on $\mathcal{O}^*(V)$ by $\lambda \cdot f(z) = f(z+\lambda)$. For notations of group cohomology, refer to section 9.1. The map $\phi : L \longrightarrow [e]$, where $\{e_\lambda\}_{\lambda\in\Lambda}$ are the multipliers of L (with respect to φ) and $[e]$ is the equivalence class of e in $H^1(\Lambda, \mathcal{O}^*(V))$, is well-defined. This is because $e_{\lambda+\lambda'}(z) = e_{\lambda'}(z+\lambda)e_\lambda(z)$ is precisely the condition for $e \in \ker(\mathrm{d} : C^1 \longrightarrow C^2)$ by note 9.2.
Also, suppose $\varphi : \pi^*L \longrightarrow V \times \mathbb{C}$ and $\varphi' : \pi^*L \longrightarrow V \times \mathbb{C}$ are two different global trivialisations for L over V. Then $\varphi' = f \cdot \varphi$ for some $f \in \mathcal{O}^*(V)$. Hence the multipliers of π^*L with respect to the two different trivialisations are e_λ and $\alpha_\lambda(z) := f(z+\lambda)f^{-1}(z)e_\lambda(z)$. We conclude $\alpha \in [e]$ since $\alpha_\lambda e_\lambda^{-1} = f(z+\lambda)f^{-1}(z) \in \mathrm{d}C^0(\Lambda, \mathcal{O}^*(V))$ again by note 9.2, so ϕ is well defined.

Moreover if $L \overset{\phi}{\longmapsto} [e]$ and $L' \overset{\phi}{\longmapsto} [e']$, then $L \otimes L' \in \mathrm{Pic}(V/\Lambda)$ has multipliers $\{e_\lambda e'_\lambda\}_{\lambda \in \Lambda}$, which gives $L_1 \otimes L_2 \overset{\phi}{\longmapsto} [e_1 \cdot e_2]$. Hence ϕ is a group homomorphism.

That ϕ has an inverse is evident from the discussion immediately preceding the theorem. Denote this inverse β. We check that β is a well defined map in cohomology, that is changing e by a coboundary does not change the line bundle defined by $\varphi(e)$. Let $e \cdot \varepsilon$, where $\varepsilon \in \mathrm{d}C^0(\Lambda, \mathcal{O}^*(V))$. So write $\varepsilon_\lambda(z) = f(z+\lambda)f^{-1}(z)$ for some $f \in \mathcal{O}^*(V)$. Then the line bundle defined is $(V \times \mathbb{C})/\sim$ where $V \times \mathbb{C}$ is identified with the image of some trivialisation of a line bundle $\tilde{L} \longrightarrow V$ and

$$(z, \ell) \quad \sim \quad (z+\lambda, e_\lambda(z) f(z+\lambda) f^{-1}(z) \cdot \ell).$$

By discussion above, this amounts to choosing a different global trivialisation $\tilde{L} \longrightarrow V \times \mathbb{C}$. □

The next step is the determination of the chern class of a line bundle specified by $\{e_\lambda\}$. First we have

Proposition 5.9 *For any non-degenerate, skew-symmetric $\mathbb{R}$-bilinear form E, there exists a basis $\lambda_1, \ldots, \lambda_{2g}$ for Λ such that with respect to this basis, E has the matrix*

$$\begin{pmatrix} 0 & \Delta_{\boldsymbol{\delta}} \\ -\Delta_{\boldsymbol{\delta}} & 0 \end{pmatrix} \quad \textit{where} \quad \Delta_{\boldsymbol{\delta}} \quad = \quad \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_n \end{pmatrix}$$

Proof This is an easy application of classification theorem of PID modules together with a Gram-Schmidt type argument. Refer to page 304-305 of [GH78] for details. □

This gives a decomposition, called an **isotropic decomposition** of the lattice $\Lambda = \Lambda_1 \oplus \Lambda_2$ such that $E(\lambda, \lambda') = 0$ if $\lambda, \lambda' \in \Lambda_1$ or $\lambda, \lambda' \in \Lambda_2$. Let $e_1 = \delta_1^{-1}\lambda_1, \ldots, e_g = \delta_g^{-1}\lambda_g$ be a basis for V. Then we have the following,

$$E(e_i, \lambda_j) \quad = \quad 0 \quad \text{and} \quad E(e_i, \lambda_{g+j}) \quad = \quad \delta_{ij}$$

Proposition 5.10 *Define the multipliers $e_\lambda(z) = \exp(-2\pi i E(\lambda, z))$ for all $\lambda \in \Lambda$, and denote the line bundle defined by these multipliers L_E. Then we have*

$$c_1(L_E) \quad = \quad E$$

under the identification $H^2(V/\Lambda, \mathbb{Z}) \simeq \bigwedge^2 \Lambda^\vee$.

Proof From the proof of lemma 9.1 the coboundary map $\mathrm{d} : C^1(M, G) \longrightarrow C^2(M, G)$ where $M \in G$-**Mod**,

$$\mathrm{d}\varphi(g_1, g_2) = g_1 \cdot \varphi(g_2) - \varphi(g_1 + g_2) + \varphi(g_1)$$

for $\varphi \in C^1(M, G)$ and $g_1, g_2 \in G$. Now for $M = \mathcal{O}^*(V), G = \Lambda$, the abelian group operation in $\mathcal{O}^*(V)$ is written multiplicatively. So we check that e satisfies the 1-cocycle equation $\mathrm{d}e = 1$,

$$\begin{aligned}
(\mathrm{d}e)(\lambda_1, \lambda_2) &= \frac{\lambda \cdot e_{\lambda_2} + e_{\lambda_1}}{e_{\lambda_1+\lambda_2}} \\
&= \frac{\exp(-2\pi i E(\lambda_2, \lambda_1 + z)) \exp(-2\pi i E(\lambda_1, z))}{\exp(-2\pi i E(\lambda_1 + \lambda_2, z))} \\
&= \frac{\exp(-2\pi i E((\lambda_2, z) + E(\lambda_1, z))}{\exp(-2\pi i E(\lambda_1 + \lambda_2, z))} \\
&= 1.
\end{aligned}$$

Hence e represents a class in $H^1(\Lambda, \mathcal{O}^*(V))$. From the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}(V) \longrightarrow \mathcal{O}^*(V) \longrightarrow 0$$

of Λ modules, we extract the following from the long exact sequence

$$H^1(\Lambda, \mathcal{O}^*(V)) \xrightarrow{c} H^2(\Lambda, \mathbb{Z}).$$

(recall that the snake lemma holds in any abelian category, in this case Λ-**Mod**), where c is the connecting homomorphism. Construct the diagram

$$\begin{array}{ccccc}
H^1(V/\Lambda, \mathcal{O}^*) & \xrightarrow{c_1} & H^2(V/\Lambda, \mathbb{Z}) & \longrightarrow & \bigwedge^2 \Lambda^\vee \\
\downarrow{\scriptstyle \phi} & & & \nearrow & \\
H^1(\Lambda, \mathcal{O}^*(V)) & \xrightarrow{c} & H^2(\Lambda, \mathbb{Z}) & &
\end{array}$$

where the two diagonal arrows are natural identifications, and it can be checked that the left hand square commutes. The image of the class represented by e under c is

$$\begin{aligned} c(e)(\lambda_1, \lambda_2) &= -\lambda \cdot E(\lambda_2, v) + E(\lambda_1 + \lambda_2, v) - E(\lambda_1, v) \\ &= -E(\lambda_2, v + \lambda_1) + E(\lambda_2, v) \\ &= E(\lambda_1, \lambda_2) \in \bigwedge^2 \Lambda^\vee \end{aligned}$$

Hence by commutativity of the above diagram, we have $c_1(L_E) = E$. $\square$

Note 5.11 With the basis given in proposition 5.9, the multipliers become

$$e_{\lambda_1}(z) = \ldots = e_{\lambda_g}(z) = 1$$
and
$$e_{\lambda_{g+1}}(z) = e^{-2\pi i z_1}, \ldots, e_{\lambda_{2g}}(z) = e^{-2\pi i z_g}$$

where $z = z_1 e_1 + \ldots + z_n e_n$. This is the form of the multipliers given in [GH78], and they prove proposition 5.10 by calculations involving the metric and curvature of the line bundle. I have spared the reader from reading a mess of calculations by simplifying the group cohomological argument given in pages 6-7 of [Pol03].

We have constructed line bundles on abelian varieties as quotients of trivial line bundles $L \longrightarrow V$. Furthermore, we have shown how to construct line bundles of any given chern class. To finish this section, we have the

Proposition 5.12 *Let $\tau_\alpha : V/\Lambda \longrightarrow V/\Lambda$ be the translation map $\tau_\alpha : [\mu] \longmapsto [\mu + \alpha]$.*

1. *Let L be a line bundle over V/Λ, then $c_1(\tau_\alpha^* L) = c_1(L)$.*
2. *If L has multipliers $\{e : \lambda \longmapsto e_\lambda\}_{\lambda \in \Lambda}$, then for $\alpha \in V/\Lambda$, $\tau_\alpha^* L$ has multipliers $\{\varepsilon : \lambda \longmapsto \varepsilon_\lambda\}$, where $\varepsilon_\lambda(z) = e_\lambda(z + \alpha)$.*
3. *Let L, L' be line bundles over V/Λ, then $c_1 L = c_1 L'$ implies $L = \tau_\lambda^* L'$.*

Proof Recall that if two continuous maps between topological spaces are homotopic, then they induce the same map in cohomology. The translation map $\tau_\alpha : V/\Lambda \longrightarrow V/\Lambda$ for any $\alpha \in V/\Lambda$ is homotopic to the identity, hence the following diagram commutes

$$\begin{array}{ccc} \mathrm{Pic}(V/\Lambda) & \xrightarrow{\tau_\lambda^*} & \mathrm{Pic}(V/\Lambda) \\ \downarrow c_1 & & \downarrow c_1 \\ H^2(V/\Lambda, \mathbb{Z}) & \xrightarrow{\mathrm{id}} & H^2(V/\Lambda, \mathbb{Z}) \end{array}$$

giving part 1. Part 2 is clear. For part 3, we show that any line bundle L with $c_1(L) = 0$ has constant multipliers. The following maps between exact sheaf sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}^* & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \iota_1} & & \uparrow{\scriptstyle \iota_2} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}^* & \longrightarrow & 0
\end{array}$$

induce the following commutative diagram in cohomology

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & H^1(V/\Lambda, \mathcal{O}) & \xrightarrow{\exp} & H^1(V/\Lambda, \mathcal{O}^*) & \xrightarrow{c_1} & H^2(V/\Lambda, \mathbb{Z}) & \longrightarrow & \cdots . \\
 & & \uparrow{\scriptstyle \iota_1^*} & & \uparrow{\scriptstyle \iota_2^*} & & \uparrow{\scriptstyle \mathrm{id}} & & \\
\cdots & \longrightarrow & H^1(V/\Lambda, \mathbb{C}) & \longrightarrow & H^1(V/\Lambda, \mathbb{C}^*) & \longrightarrow & H^2(V/\Lambda, \mathbb{Z}) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \mathrm{Hodge}} & & & & & & \\
 & & H^1(V/\Lambda, \mathcal{O}) & & & & & & \\
 & & \oplus & & & & & & \\
 & & H^0(V/\Lambda, \Omega^1) & & & & & &
\end{array}$$

Under the Hodge decomposition isomorphism, ι_1^* is projection onto the first factor, hence it is surjective. If $e \in \ker(c_1)$, then $e \in \mathrm{im}(\exp)$. So there is some $\xi \in H^1(V/\Lambda, \mathbb{C})$ such that $\exp(\iota_1^* \xi) = e$, and by commutative of the leftmost square, we have $e \in \mathrm{im}(\iota_2^*)$. Hence e_λ are constant functions. □

The last proposition says that the chern class determines a line bundle up to translation.

5.3 Theta functions

Given any positive line bundle $L \xrightarrow{\pi} \mathbb{C}^g/\Lambda$, the pullback of any global section of of L via $\mathbb{C}^g \longrightarrow \mathbb{C}^g/\Lambda$ is a holomorphic function on $\mathbb{C}^g$. We call these functions **theta functions**. Using the multipliers constructed in the previous section, we see that any such function $\theta : \mathbb{C}^g \longrightarrow \mathbb{C}$ must satisfy the functional equations

$$\theta(z + \lambda_j) \quad = \quad \exp(-2\pi i E(\lambda_j, z))\theta(z) \qquad j \in [1, g]$$

The aim is to determine $h^0(V/\Lambda, \mathcal{O}(L))$. So let $L \longrightarrow \mathbb{C}^g/\Lambda$ be a line bundle, with multipliers $e_{\lambda_i} \equiv 1$, $e_{\lambda_{i+g}}(z) = e^{-2\pi i z_i}$ normalised with respect to some given $E \in \bigwedge^2 \Lambda^\vee$. We will translate L by $\mu := \frac{1}{2}(Z_{11}, \ldots, Z_{gg}) \in \mathbb{C}^g/\Lambda$ and consider $\tau_\mu^* L$. the functional equations become

$$\begin{aligned} \theta(z+\lambda_j) &= \theta(z) && (5.1) \\ \theta(z+\lambda_{g+j}) &= e^{-2\pi i z_j - \pi i Z_{ii}}\theta(z) && (5.2) \end{aligned}$$

for $j \in [1, g]$. Note that $h^0(V/\Lambda, \mathcal{O}(L)) = h^0(V/\Lambda, \mathcal{O}(\tau_\mu^* L))$. The translation by μ has the effect of simplifying following proof. We will solve these equations and derive a closed form for the theta functions. The equations (5.1) are periodicity conditions, hence by Fourier analysis, we can write

$$\begin{aligned} \theta(z) &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^g} \alpha_{\boldsymbol{\ell}} \cdot \exp\left(2\pi i \ell_1 z_1 \delta_1^{-1}\right) \ldots \exp\left(2\pi i \ell_1 z_1 \delta_1^{-1}\right) \\ &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^g} \alpha_{\boldsymbol{\ell}} \cdot \exp\left(2\pi i \left\langle \boldsymbol{\ell}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) \end{aligned}$$

The equations (5.2) will give recurrence relations for the $\alpha_{\boldsymbol{\ell}}$'s,

$$\begin{aligned} \theta(z+\lambda_{g+j}) &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^g} \alpha_{\boldsymbol{\ell}} \cdot \exp\left(2\pi i \left\langle \boldsymbol{\ell}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) \exp\left(2\pi i \left\langle \boldsymbol{\ell}, \Delta_{\boldsymbol{\delta}}^{-1} \lambda_{g+j}\right\rangle\right) \\ &\overset{\text{by } 5.2}{=} \sum_{\boldsymbol{\ell}\in\mathbb{Z}^g} \alpha_{\boldsymbol{\ell}} \cdot \exp\left(2\pi i \left\langle \boldsymbol{\ell}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) \exp\left(-2\pi i z_j\right) \exp\left(-\pi i Z_{ii}\right) \\ &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^g} \alpha_{\boldsymbol{\ell}} \cdot \exp\left(2\pi i \left\langle \boldsymbol{\ell}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) \exp\left(-2\pi i \left\langle \Delta_{\boldsymbol{\delta}}\xi_j, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) \exp\left(-\pi i Z_{ii}\right) \\ &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^g} \alpha_{\boldsymbol{\ell}} \cdot \exp\left(2\pi i \left\langle \boldsymbol{\ell} - \Delta_{\boldsymbol{\delta}}\xi_j, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) \exp\left(-\pi i Z_{ii}\right) \\ &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^g} \left(\alpha_{\boldsymbol{\ell}+\Delta_{\boldsymbol{\delta}}\xi_j} \exp\left(-\pi i Z_{ii}\right)\right) \cdot \exp\left(2\pi i \left\langle \boldsymbol{\ell}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) \end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the dual pairing on V, and $\langle \xi_j, z_i\rangle = \delta_{ij}$. Comparing the first and last lines give

$$\alpha_{\boldsymbol{\ell}} \cdot \exp\left(2\pi i \left\langle \boldsymbol{\ell}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) = \alpha_{\boldsymbol{\ell}+\Delta_{\boldsymbol{\delta}}\xi_j} \exp\left(-\pi i Z_{ii}\right) \quad (5.3)$$

Let $\Lambda' := \delta_1\mathbb{Z} + \ldots + \delta_g\mathbb{Z}$ be a sublattice of Λ, and identify Λ/Λ' with $\Gamma := \{\boldsymbol{\gamma} \in \Lambda \mid 0 \leqslant \gamma_i < \delta_i \text{ for all } i \in [1, g]\} \subset \Lambda$, then the coefficients $\{\alpha_{\boldsymbol{\gamma}}\}_{\boldsymbol{\gamma}\in\Gamma}$ determine θ.

Proposition 5.13 *The series*

$$\theta(z) = \sum_{\boldsymbol{\ell}\in\mathbb{Z}^g} \alpha_{\boldsymbol{\ell}} \cdot \exp\left(2\pi i \left\langle \boldsymbol{\ell}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) \quad (5.4)$$

converges uniformly on any compact subset of $\mathbb{C}$ *for any choice of* $\alpha_{\boldsymbol{\ell}}$*'s, hence gives a well defined holomorphic function* $\theta : V \longrightarrow \mathbb{C}$.

Proof Let $K \subseteq V$ be compact. We first reorder the summation in (5.4),

$$\begin{aligned}\theta(z) &= \sum_{\boldsymbol{\gamma}\in\Gamma}\sum_{\boldsymbol{\ell}\in\Lambda'} \alpha_{\boldsymbol{\gamma}+\boldsymbol{\ell}} \cdot \exp\left(2\pi i \left\langle \boldsymbol{\gamma}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) \exp\left(2\pi i \left\langle \boldsymbol{\ell}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) \\ &= \sum_{\boldsymbol{\gamma}\in\Gamma} \exp\left(2\pi i \left\langle \boldsymbol{\gamma}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right) \left(\sum_{\boldsymbol{\ell}\in\Lambda'} \alpha_{\boldsymbol{\gamma}+\boldsymbol{\ell}} \cdot \exp\left(2\pi i \left\langle \boldsymbol{\ell}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right)\right)\end{aligned}$$

Let $M = \sup_{z\in K, \boldsymbol{\gamma}\in\Gamma} \left|\exp\left(2\pi i \left\langle \boldsymbol{\gamma}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right)\right|$, this exists since K is compact and Γ is finite, then

$$|\theta(z)| \leqslant M|\Gamma| \sum_{\boldsymbol{\ell}\in\Lambda'} |\alpha_{\boldsymbol{\gamma}+\boldsymbol{\ell}}| \cdot \left|\exp\left(2\pi i \left\langle \boldsymbol{\ell}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right)\right|$$

We can solve (5.3), and, omitting the details, we obtain

$$\alpha_{\boldsymbol{\gamma}+\boldsymbol{\ell}} = \exp\left(\pi i \left\langle \Delta_{\delta}^{-1}\boldsymbol{\ell}, Z\Delta_{\delta}^{-1}\boldsymbol{\ell}\right\rangle + 2\pi i \left\langle \Delta_{\delta}^{-1}\boldsymbol{\gamma}, Z\Delta_{\delta}^{-1}\boldsymbol{\ell}\right\rangle\right)$$

or more neatly, let $\boldsymbol{\ell}' \in \mathbb{Z}^g$ such that $\boldsymbol{\ell} = \Delta_{\boldsymbol{\delta}}\boldsymbol{\ell}'$,

$$\alpha_{\boldsymbol{\gamma}+\Delta_{\boldsymbol{\delta}}\boldsymbol{\ell}'} = \exp\left(\pi i \left\langle \boldsymbol{\ell}, Z\boldsymbol{\ell}\right\rangle + 2\pi i \left\langle \boldsymbol{\gamma}, Z\boldsymbol{\ell}\right\rangle\right)$$

Now we can put a bound on the norm of $\alpha_{\boldsymbol{\gamma}+\boldsymbol{\ell}}$,

$$|\alpha_{\boldsymbol{\gamma}+\boldsymbol{\ell}}| = \left|\alpha_{\boldsymbol{\gamma}+\Delta_{\boldsymbol{\delta}}\boldsymbol{\ell}'}\right| = \exp\left(-\pi \left\langle \boldsymbol{\ell}', \Im(Z)\boldsymbol{\ell}'\right\rangle - 2\pi \left\langle \Delta_{\delta}^{-1}\boldsymbol{\gamma}, \Im(Z)\boldsymbol{\ell}'\right\rangle\right)$$

Since $\Im(Z)$ is positive definite, so all its eigenvalues are real and positive. Let ρ be the smallest eigenvalue of $\Im(Z)$, then

$$\left\langle \boldsymbol{\ell}', \Im(Z)\boldsymbol{\ell}'\right\rangle \geqslant \left\langle \boldsymbol{\ell}', \rho\boldsymbol{\ell}'\right\rangle = \rho\|\boldsymbol{\ell}'\|^2$$

Now let P be the largest eigenvalue of $\Im(Z)$, then

$$\begin{aligned}\left\langle \Delta_{\delta}^{-1}\boldsymbol{\gamma}, \Im(Z)\boldsymbol{\ell}'\right\rangle &\leqslant \mathrm{P}\left\langle \Delta_{\delta}^{-1}\boldsymbol{\gamma}, \boldsymbol{\ell}'\right\rangle \\ &\leqslant \mathrm{P}'\|\ell'\|\end{aligned}$$

for some constant $\mathrm{P}' > 0$ since $\boldsymbol{\gamma}$ is bounded. So for some constant $\mathrm{P}'' > 0$, we have

$$|\alpha_{\boldsymbol{\gamma}+\boldsymbol{\ell}}| \quad \leqslant \quad \exp\left(-\mathrm{P}''\|\Delta_{\delta}^{-1}\boldsymbol{\ell}\|^2\right)$$

for all $\boldsymbol{\ell} \in \Gamma \cap \{z \in V \mid |z_i| > R$ for all $i \in [1, n]\}$ for some $R > 0$. Hence the series (5.4) converges on all compact subsets of V. □

Now since $\theta(z)$ is well defined and holomorphic, independently of the coefficients $\{\alpha_{\boldsymbol{\gamma}}\}_{\boldsymbol{\gamma}\in\Gamma}$, so we can easily count the dimension of $H^0(S, \mathcal{O}(L))$. Since $|\Gamma| = \delta_1 \ldots \delta_g$, the sections

$$\tilde{\theta}_{\gamma}(z) \quad = \quad \sum_{\boldsymbol{\ell}\in\Gamma'} \exp\left(2\pi i \left\langle \boldsymbol{\ell} + \boldsymbol{\gamma}, \Delta_{\boldsymbol{\delta}}^{-1} z\right\rangle\right)$$

span $H^0(S, \mathcal{O}(L))$. Hence we have the following

Corollary 5.14 *Let L be a positive line bundle, then*

$$h^0(V/\Lambda, \mathcal{O}(L)) \quad = \quad \delta_1 \ldots \delta_g$$

In the special case where $c_1(L)$ is a principal polarisation, $h^0(V/\Lambda, \mathcal{O}(L)) = 1$ and the vector space $H^0(V/\Lambda, \mathcal{O}(L))$ is spanned by one element, say, $\tilde{\theta}$. In the case of a Jacobian, the pullback $\theta : V \longrightarrow \mathbb{C}$ is called the **Riemann theta function**

$$\theta(z) \quad = \quad \sum_{\boldsymbol{\ell}\in\mathbb{Z}^n} \exp\left(\pi i \langle \boldsymbol{\ell}, Z\boldsymbol{\ell}\rangle + 2\pi i \langle \boldsymbol{\ell}, z\rangle\right)$$

and its associated divisor $\Theta = (\tilde{\theta})$ is called the **Riemann theta divisor.**
It is clear then that θ is an even function

$$\begin{aligned}
\theta(-z) &= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^n} \exp\left(\pi i \langle \boldsymbol{\ell}, Z\boldsymbol{\ell}\rangle + 2\pi i \langle \boldsymbol{\ell}, -z\rangle\right) \\
&= \sum_{\boldsymbol{\ell}\in\mathbb{Z}^n} \exp\left(\pi i \langle \boldsymbol{\ell}, Z\boldsymbol{\ell}\rangle + 2\pi i \langle -\boldsymbol{\ell}, z\rangle\right) \\
&= \sum_{\boldsymbol{\ell}'\in\mathbb{Z}^n} \exp\left(\pi i \left\langle \boldsymbol{\ell}', Z\boldsymbol{\ell}'\right\rangle + 2\pi i \left\langle \boldsymbol{\ell}', z\right\rangle\right) \\
&= \theta(z) \qquad (5.5)
\end{aligned}$$

we shall make use of this fact in the proof of Riemann's theorem (c.f. theorem 6.14). In fact, the Weierstrass $\wp$-function in chapters 2 and 6 can be obtained naturally from theta functions (see pages 85-89 of [Cle80]). There are many more applications of theta functions. We will describe one more, recall that the Kodaira embedding theorem gives a bound k_0 such that for all $k \geqslant k_0$, the positive line bundle $L^k \longrightarrow M$ gives an embedding of M into projective space. The following Lefschetz embedding theorem uses theta functions to make k_0 explicit in the case of complex tori.

Theorem 5.15 *(Lefschetz) Let $L \longrightarrow V/\Lambda$ be a positive line bundle and $\sigma_1, \ldots, \sigma_N$ be a basis of $H^0(V/\Lambda, \mathcal{O}(L^k))$. Then for $k \geqslant 3$, the map $\varphi : V/\Lambda \longrightarrow \mathbb{P}^N$ given by $p \longmapsto (\sigma_1(p) : \ldots : \sigma_N(p))$ is an embedding of V/Λ into $\mathbb{P}^N$.*

Proof See pages 321-324 of [GH78] or pages 32-35 (section 3.4) of [Pol03]. □

We mention the Lefschetz theorem in the spirit of specificity; the Kodaira embedding theorem is a very general existence result, and it is pleasing to know that it can be sharpened in this instance using the intrinsic features of complex tori. As mentioned previously, the theory of theta functions is rich and has wide applications in the study of Riemann surfaces, for example to the Schottky problem, which we will mention in chapter 7, and Riemann's theorem. A major omission from this section are the theta characteristics, ...References: [Mumford- Tata lectures on Theta]

Chapter 6

The Jacobian Variety

The Jacobian variety $\mathcal{J}(S)$ is the cornerstone in the study of the Riemann surface S. Torelli's theorem says that all of the information in S is captured by $\mathcal{J}(S)$ and its principal polarisation. Since $\mathcal{J}(S)$ is a complex torus, it is 'linear' and hence much more accessible to study than S. The construction we give via complex tori is analytic, and a purely algebraic definition can be given. In fact, this approach is the basis of the proof of the Riemann Hypothesis in characteristic p.

The Jacobian variety is also directly related to abelian integrals, as we shall see in the following.

6.1 Motivation: Abelian integrals

Historically, such integrals first arose as **elliptic integrals**, so named for their connection with the arc length of an ellipse. Let $\mathcal{E} := \left\{(v,w) \in \mathbb{R}^2 \mid \frac{v^2}{a^2} + \frac{w^2}{b^2} = 1\right\}$ be an ellipse, then the arc length of $\mathcal{E}$ from p to q is given by,

$$\begin{aligned} L := \int_p^q \sqrt{\left(\frac{\mathrm{d}v}{\mathrm{d}w}\right)^2 + 1}\ \mathrm{d}v &= \int_p^q \sqrt{\frac{a^4 + (b^2 - a^2)v^2}{a^2(a^2 - v^2)}}\ \mathrm{d}v \\ &= \int_p^q \frac{a^4 + (b^2 - a^2)v^2}{\sqrt{a^2(a^2 - v^2)(a^4 + (b^2 - a^2)v^2)}}\ \mathrm{d}v \end{aligned}$$

This can be transformed into a nicer looking integral, let $x = a^4 + (b^2 - a^2)v^2$, then

$$\begin{aligned} L &= \frac{1}{a}\int_{\tilde{p}}^{\tilde{q}} \frac{x}{\sqrt{x\left(a^2 - \frac{x-a^4}{b^2-a^2}\right)}} \cdot \frac{\mathrm{d}x}{2\sqrt{(b^2 - a^2)(x - a^4)}} \\ &= \frac{1}{2a}\int_{\tilde{p}}^{\tilde{q}} \frac{x\mathrm{d}x}{\sqrt{x\left(a^2b^2 - x\right)\left(x - a^4\right)}} := \int_{\tilde{p}}^{\tilde{q}} x\frac{\mathrm{d}x}{y} \end{aligned}$$

on setting $y^2 = 4a^2x\left(a^2b^2 - x\right)(x - a^4)$. A natural generalisation of this is to consider

$$I := \int_p^q R(x,y)\mathrm{d}\,x \qquad (6.1)$$

where R is a rational function and x, y satisfies a polynomial equation $\rho(x, y) = 0$. For $\deg(\rho) \leqslant 2$, I in (6.1) can be expressed in terms of elementary functions. But for $\deg(\rho) > 2$, this is not the case; and such integrals are known as **abelian integrals** after the Norwegian mathematician Niels Hendrik Abel, who first studied them. Elliptic integrals are then abelian integrals with $\deg(\rho) = 3$ or 4.

The following example demonstrates why an elliptic integral is not expressible as elementary functions.

Example 6.1 Consider the elliptic integral

$$E \quad := \quad \int_p^q \frac{\mathrm{d}x}{\sqrt{x(x-1)(x-2)}}$$

Let $y^2 = x(x-1)(x-2)$ and $\rho(x, y) = y^2 - x(x-1)(x-2)$ then $E = \int_p^q \frac{\mathrm{d}x}{y}$ can be treated as a line integral on the Riemann surface $S := \{(x, y) \in \mathbb{C}^2 \mid \rho(x, y) = 0\} \subseteq \mathbb{C}^2$. We saw in the example at the beginning of chapter 2 that S is has topological genus 1. Note that the surface S is not simply connected. So let α, β be a symplectic basis for S and $\gamma, \gamma' : [0, 1] \longrightarrow S$ be two paths such that $\gamma(0) = \gamma'(0) = p$, $\gamma(1) = \gamma'(1) = q$ and $\gamma - \gamma' \sim n\alpha + m\beta$ for some $n, m \in \mathbb{Z}$. Then the difference in evaluating E along γ and γ' is

$$\int_\gamma \frac{\mathrm{d}x}{y} - \int_{\gamma'} \frac{\mathrm{d}x}{y} \quad = \quad \int_{\gamma-\gamma'} \frac{\mathrm{d}x}{y} \quad = \quad \int_{n\alpha+m\beta} \frac{\mathrm{d}x}{y}$$

We show $\omega := \mathrm{d}x/y$ is a holomorphic differential. Implicitly differentiating ρ, we have

$$\omega \quad = \quad \frac{\mathrm{d}x}{y} \quad = \quad \frac{2\mathrm{d}y}{3x^2 - 6x + 2}$$

and since $3x^2 - 6x + 2$ and y are not simultaneously zero [1], $\omega \in H^0(S, \Omega^1)$. This shows $\int_{n\alpha+m\beta} \omega$ is an element of $\Lambda := \mathbb{Z}\Pi_1 + \mathbb{Z}\Pi_2$, where Π_1 and Π_2 are the periods defined below.

The difficulty with E is that it is not a well-defined number- the natural range E is $\mathbb{C}/\Lambda$, the Jacobian of S. With essentially the same argument, we can generalise this phenomenon to

Proposition 6.2 *Let* $I := \int_p^q R(x, y)\mathrm{d}x$ *be an abelian integral with* x *and* y *satisfying* $\rho(x, y) = 0$ *for some* $\rho \in \mathbb{C}[x, y]$. *Then the natural range of* I *is the Jacobian,* $\mathcal{J}(S)$, *of the Riemann surface,* $S = V(\rho) \subseteq \mathbb{C}^2$.

[1] In general, this depends on ρ having distinct roots.

Example 6.3 Consider the elliptic integral $I := \int_{q_0}^{q} \frac{\mathrm{d}x}{4x^3 - g_2 x - g_3}$. Then we see that

$$\begin{aligned} \mu : S &\longrightarrow \mathcal{J}(S) \\ q &\longmapsto \int_{q_0}^{q} \frac{\mathrm{d}x}{4x^3 - g_2 x - g_3} \end{aligned}$$

is the Abel-Jacobi map. Moreover let $\wp$ be the Weierstrass $\wp$-function, then the map

$$\begin{aligned} \mathbb{C}/\Lambda \simeq \mathcal{J}(S) &\longrightarrow \mathbb{P}^2 \\ z &\longmapsto (\wp(z) : \wp'(z) : 1) \end{aligned}$$

is the explicit inverse to μ.

6.2 Properties of the Jacobian variety

Much of this section consists of working out some of the constructions of the last chapter in the case of $\mathcal{J}(S)$.

Definition 6.4 *The* ***Jacobian variety*** *of a Riemann surface S is defined as*

$$\mathcal{J}(S) = \frac{H^0(S, \Omega^1)^\vee}{H_1(S, \mathbb{Z})}$$

This definition is intrinsic to S, and note that we clearly have $\mathcal{J}(S) \simeq \frac{H^1(S,\mathcal{O})}{H_1(S,\mathbb{Z})} \simeq \mathrm{Pic}^0(S)$ via Serre duality. Choosing bases $\omega_1, \ldots, \omega_g \in H^0(S, \Omega^1)$, and $\delta_1, \ldots, \delta_{2g} \in H_1(S, \mathbb{Z})$ we have the following map

$$\begin{aligned} H^0(S, \Omega^1)^\vee &\longrightarrow \mathbb{C}^g/\Lambda \\ \alpha &\longmapsto (\alpha(\omega_1), \ldots, \alpha(\omega_g)) \end{aligned}$$

which realises $\mathcal{J}(S)$ explicitly as a complex torus $\mathbb{C}^g/\Lambda$, where

$$\begin{aligned} \Lambda &= \mathbb{Z}\Pi_1 + \ldots + \mathbb{Z}\Pi_{2g} \\ \Pi_i &= \left(\int_{\delta_i} \omega_1, \ldots, \int_{\delta_i} \omega_g \right) \in \mathbb{C}^g. \end{aligned}$$

This agrees with definition 3.9. We proceed to show $\mathcal{J}(S)$ is an abelian variety. First we choose a nice basis for Λ and $\mathbb{C}^g$. Proposition 5.9 implies the following,

Proposition 6.5 *There exists a basis for $H_1(S, \mathbb{Z})$, $\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$, such that with respect to this basis, the matrix of the intersection form, E, is given by*

$$\begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$$

The basis obtained above is called a **symplectic basis**. By Poincaré duality, we can choose a basis $\omega_1, \ldots, \omega_g$ for $H^0(S, \Omega^1)$ dual to $\alpha_1, \ldots, \alpha_g$, that is, $\int_{\alpha_i} \omega_j = \delta_{ij}$. Then clearly we have,

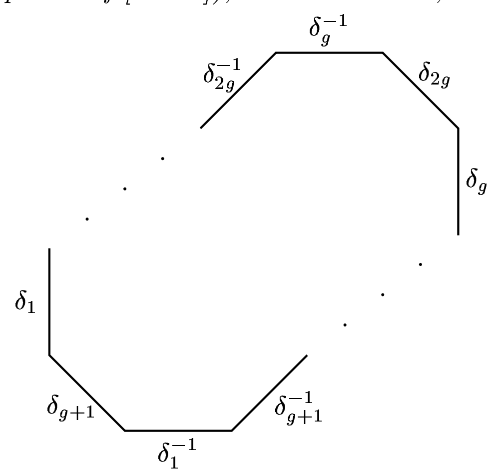
Proposition 6.6 *With the bases above, the period matrix of $\mathcal{J}(S)$ has the form*

$$\Omega \;:=\; \begin{pmatrix} I & Z \end{pmatrix}$$

where $Z = \left(\int_{\beta_i} \omega_j\right)_{i,j}$ and I is the $g \times g$ identity matrix. The columns of Ω, denoted Π_i for $i \in [1, 2g]$ are called the ***periods*** *of Ω.*

As promised, we will verify the Riemann conditions (c.f. theorem 5.5) in the case of $\mathcal{J}(S)$, thus showing it is algebraic. First we prove

Proposition 6.7 *Let S be a Riemann surface and $\delta_1\delta_1^{-1}\delta_2\delta_2^{-1}\ldots\delta_{2g}\delta_{2g}^{-1}$ be its $4g$-polygon representation (c.f. chapter 3 of [FG01]), as shown below,*



Let η be a meromorphic 1-form on S with simple poles at $p_1, \ldots, p_k$, and $\omega \in H^0(S, \Omega^1)$. Denote $\Pi_i := \int_{\delta_i} \omega$ and $\mathrm{N}_i := \int_{\delta_i} \eta$ for $i \in [1, 2g]$, then we have the

$$\sum_{i=1}^{g} (\Pi_i \mathrm{N}_{g+i} - \Pi_{g+i}\mathrm{N}_i) \;=\; 2\pi\, i \sum_{i=1}^{k} \mathrm{res}_{p_i}(\eta) \int_{p_0}^{p_i} \omega \tag{6.2}$$

This is classically known as the ***reciprocity formula****.*

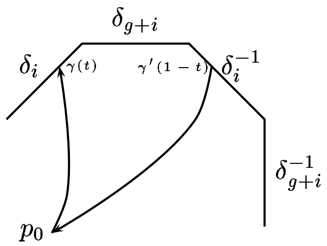
Proof By choice of the cycle representatives $\delta_1, \ldots, \delta_{2g}$, we can assume without loss of generality that none of the poles of η lie on any δ_i. Denote the interior of the above polygon P and ∂P be its boundary. Pick a point p_0 in P, not a pole of η, and we stipulate that all integrals are taken over paths lying entirely in P, or entirely in ∂P. Consider $I(p) := \int_{p_0}^{p} \omega$, this is a holomorphic function on P satisfying $\mathrm{d}I = \omega$. We evaluate $\int_{\partial P} \eta I$ in two different ways to derive (6.2). First we use the residue theorem to obtain

$$\int_{\partial P} \eta I \quad = \quad 2\pi i \sum_{p \in P} \mathrm{res}_p(\eta I) \quad = \quad 2\pi i \sum_{i=1}^{k} \mathrm{res}_{p_i}(\eta) \int_{p_0}^{p_i} \omega$$

Let $\gamma, \gamma' : [0,1] \longrightarrow S$ parameterise the cycles δ_i and δ_i^{-1} respectively, then $\eta(\gamma(t)) = \eta(\gamma'(1-t))$ for $t \in [0,1]$ so

$$\begin{aligned} \int_{\delta_i} \eta I + \int_{\delta_i^{-1}} \eta I \quad &= \quad \int_0^1 (\eta \cdot I)(\gamma(t))\mathrm{d}t + \int_0^1 (\eta \cdot I)(\gamma'(t))\mathrm{d}t \\ &= \quad \int_0^1 \eta(\gamma(t)) \Big(I(\gamma(t)) - I(\gamma'(1-t)) \Big) \mathrm{d}t \end{aligned}$$

Now for any $t \in [0,1]$,



$$\begin{aligned} I(\gamma(t)) - I(\gamma'(1-t)) \quad &= \quad \int_{p_0}^{\gamma(t)} \omega - \int_{p_0}^{\gamma'(1-t)} \omega \quad = \quad \int_{\gamma'(1-t)}^{\gamma(t)} \omega \\ &= \quad \int_{\gamma'(1-t)}^{\gamma'(0)} \omega - \int_{\delta_{g+i}} \omega + \int_{\gamma(1)}^{\gamma(t)} \omega \\ &= \quad -\Pi_{g+i} \end{aligned}$$

since $\int_{\gamma'(1-t)}^{\gamma'(0)} \omega = -\int_{\gamma(1)}^{\gamma(t)} \omega$, giving $\int_{\delta_i} \eta I + \int_{\delta_i^{-1}} \eta I = -\Pi_{g+i}\mathrm{N}_i$. We can similarly derive $\int_{\delta_{g+i}} \eta I + \int_{\delta_{g+i}^{-1}} \eta I = \Pi_i \mathrm{N}_{g+i}$, thus we obtain

$$\int_{\partial P} \eta I \quad = \quad \sum_{i=1}^{g} (\Pi_i \mathrm{N}_{g+i} - \Pi_{g+i}\mathrm{N}_i)$$

as required. □

In the genus 1 case, the above reduces to Legendre's relation $\Pi_1 \mathrm{N}_2 - \Pi_2 \mathrm{N}_1 = 2\pi i$, with $\eta(z) = \zeta(z)\mathrm{d}z$ where ζ is the Weierstrass ζ-function. We obtain the proof of the Riemann conditions as a corollary.

Corollary 6.8 *The period matrix, Ω, of $\mathcal{J}(S)$ satisfy the Riemann conditions,*

$$\Omega = (I, Z) \qquad Z = Z^T \qquad \Im(Z) \quad \text{positive definite.}$$

Proof We have seen in proposition 6.6 how to write Ω in the form (I, Z). Let $\delta_1, \ldots, \delta_{2g}$ be a basis of $H_1(S, \mathbb{Z})$ and $\omega_1, \ldots, \omega_g$ be a basis of $H^0(S, \Omega^1)$ giving the period matrix in the form $\Omega = (I, Z)$. Denote $\Pi_{i,j} = \int_{\delta_i} \omega_j$ for $i \in [1, 2g]$ and $j \in [1, g]$, then $\Pi_{i,j} = \delta_{i,j}$ for $i, j \in [1, g]$, and substituting this into (6.2), we obtain

$$0 \quad = \quad \sum_{i=1}^{g} (\delta_{i,j}\Pi_{g+i,k} - \Pi_{g+i,j}\delta_{i,k}) \quad = \quad \Pi_{g+j,k} - \Pi_{g+k,j}$$

Hence $Z = Z^T$. Let $\mathcal{I}_j(p) = \int_{p_0}^{p} \omega_j$ and consider the positive definite form $(\cdot, \cdot) = i \int_P \omega_j \wedge \overline{\omega}_k$ on $H^0(S, \Omega^1)$,

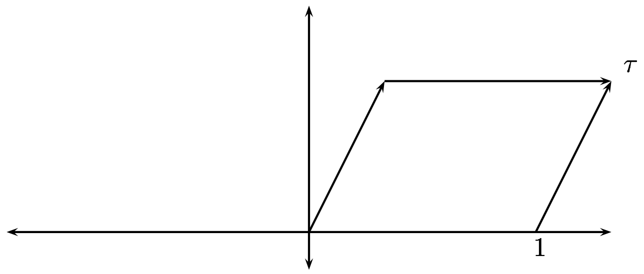
$$\begin{aligned} i \int_P \omega_j \wedge \overline{\omega}_k &= i \int_P \mathrm{d}(\mathcal{I}_j \overline{\omega}_k) \\ &= i \int_{\partial P} \mathcal{I}_j \overline{\omega}_k \\ &= i \sum_{i=1}^{g} \left(\Pi_{i,j} \overline{\Pi}_{g+i,k} - \Pi_{g+i,j} \overline{\Pi}_{i,k} \right) \\ &= i \left(\overline{\Pi}_{g+j,k} - \Pi_{g+k,j} \right) \\ &= 2\Im(\Pi_{g+k,j}) \end{aligned}$$

so $\Im(Z)$ is the matrix of $\frac{1}{2}(\cdot, \cdot)$ with respect to the chosen basis, hence $\Im(Z)$ is positive definite. □

Example 6.9 In the genus 1 case, let E be an elliptic curve. This means that we can write the period matrix as

$$\Omega \quad = \quad (1, \tau)$$

where $\tau = \int_\beta \omega$, α, β symplectic basis for $H_1(S, \mathbb{Z})$ and ω a basis of $H^0(S, \Omega^1)$ dual to α, and $\Im(\tau) > 0$. So $\mathcal{J}(E)$ is a complex torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$



$\frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$

We claimed that $\mathcal{J}(S)$ has a principal polarisation, this is a consequence of the intersection form on S and the natural isomorphism $H_1(S,\mathbb{Z}) \simeq \Lambda$.

Proposition 6.10 *The intersection form $E : \Lambda \times \Lambda \longrightarrow \mathbb{Z}$ induces a unique positive definite Hermitian form H on $H^0(S,\Omega^1)$. Moreover these are related by $\Im(H) = E$, and H is a principal polarisation of $\mathcal{J}(S)$.*

Proof The unique positive definite form on $H^0(S,\Omega^1)^\vee$ is given by

$$H(u,v) \quad = \quad E(iu,v) + iE(u,v).$$

Also with respect to a symplectic basis,

$$\det(E) \quad = \quad (-1)^g \begin{vmatrix} -I & 0 \\ 0 & I \end{vmatrix} \quad = \quad (-1)^{2g} \quad = \quad 1$$

and since det is invariant under basis change, H is a principal polarisation. □

6.3 Riemann's theorem

Since $\mathcal{J}(S)$ has a principal polarisation given by the intersection form $E \in \bigwedge^2 \Lambda^\vee$ (c.f. proposition 6.10), we have by corollary 5.14 $h^0(S, L_E) = 1$ where $L_E \in \mathrm{Pic}(\mathcal{J}(S))$ has chern class E under the identification $H^2(\mathcal{J}(S),\mathbb{Z}) \simeq \bigwedge^2 \Lambda^\vee$. Note that E specifies L_E up to translation by proposition **??**, so we can associate, up to translation, the divisor $\Theta = (\theta)$ to E. Conversely, if a divisor D satisfies $c_1([D]) = E$, then $[D]$ must be a translate of L_E. Hence E and Θ up to translation are equivalent data [2]. The divisor Θ is called the **Riemann theta divisor**.

Denote $\Theta_\lambda = (\theta(z-\lambda))$ to be the translated theta divisor, and $A \cdot B$ =intersection number of divisors A and B. First we prove the following lemma

[2] Some references go as far as saying Θ is the principal polarisation as does [FK92] and [Mum75], we shall refrain from this

Lemma 6.11 *Suppose $\mu(S) \nsubseteq \Theta_\lambda$, then $\mu(S)\cdot\Theta_\lambda = g$, that is, $\mu(S)$ and Θ_λ intersect at g points. Denote these points of intersection $z_1(\lambda),\ldots,z_g(\lambda) \in \mathcal{J}(S)$, then there exists a constant $\kappa \in \mathcal{J}(S)$ such that*

$$(z_1(\lambda) + \ldots + z_g(\lambda)) + \kappa = \lambda \tag{6.3}$$

Proof First we establish $\mu(S)\cdot\Theta_\lambda = g$. Let $\alpha_1,\ldots,\alpha_g,\beta_1,\ldots,\beta_g$ be a symplectic basis for $H_1(S,\mathbb{Z})$, and $\omega_1,\ldots,\omega_g$ be a basis for $H^0(S,\Omega^1)$ such that the period matrix of $\mathcal{J}(S)$ has the form (I, Z). We will denote Z_k to be the k-th column of Z and Z_{kj} to be the k,j-th element of Z. Let $P := \alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\ldots\alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}$ be the associated $4g$-polygon representation, as in proposition 6.7.
The Abel-Jacobi map with respect to a base point z_0 lifts to a map $\tilde{\mu} : P \longrightarrow \mathbb{C}^g$ by

$$\tilde{\mu}(z) = \left(\int_{z_0}^{z}\omega_1,\ldots,\int_{z_0}^{z}\omega_g\right)$$

This is summed by the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\mu}} & \mathbb{C}^g \\ \downarrow & & \downarrow \\ S & \xrightarrow{\mu} & \mathcal{J}(S) \end{array}.$$

The translated Riemann theta function θ_λ can be pulled back to P via $\tilde{\mu}$. Then, after adjusting $\alpha_1,\beta_1,\ldots,\alpha_g,\beta_g$ such that no zeroes of $\tilde{\mu}^*\theta_\lambda$ lie on the boundary of P, the number of zeroes of $\tilde{\mu}^*\theta_\lambda$ is equal to the number of points of intersection of $\mu(S)$ and Θ_λ. Now by continuity of the translatio map $\theta \longmapsto \theta_\lambda$ and the residue theorem,

$$\begin{aligned}\deg((\tilde{\mu}^*\theta_\lambda)) = \deg((\tilde{\mu}^*\theta)) &= \frac{1}{2\pi i}\int_{\partial P} \mathrm{d}\,\log(\theta(\tilde{\mu}(z)) \\ &= \frac{1}{2\pi i}\left(\sum_{j=1}^{g}\int_{\alpha_j} + \int_{\alpha_j^{-1}} + \sum_{j=1}^{g}\int_{\beta_j} + \int_{\beta_j^{-1}}\right).\end{aligned}$$

- Case $\int_{\alpha_j} + \int_{\alpha_j^{-1}}$. Let z and z^* be points on α_j such that they are identified on S. Then we have the following identities

$$\begin{aligned}\tilde{\mu}(z^*) &= \tilde{\mu}(z) + Z_j \\ \theta(\tilde{\mu}(z^*)) &= \exp\left(-2\pi i\left(\tilde{\mu}_j(z) + \frac{Z_{jj}}{2}\right)\right)\theta(\tilde{\mu}(z^*))\end{aligned}$$

where the second line is due to the quasiperiodic condition of θ (c.f. (**??**)). This gives

$$\begin{aligned}
\int_{\alpha_j} \mathrm{d}\ \log(\theta(\tilde{\mu}(z)) + \int_{\alpha_j^{-1}} \mathrm{d}\ \log(\theta(\tilde{\mu}(z)) &= \int_{\alpha_j} \mathrm{d}\ \log(\theta(\tilde{\mu}(z)) - \mathrm{d}\ \log(\theta(\tilde{\mu}(z^*)) \\
&= 2\pi i \int_{\alpha_j} \mathrm{d} \left(\tilde{\mu}_j(z) + \frac{Z_{jj}}{2} \right) \\
&= 2\pi i \int_{\alpha_j} \mathrm{d} \int_{z_0}^{z} \omega_j \\
&= 2\pi i
\end{aligned}$$

- Case $\int_{\beta_j} + \int_{\beta_j^{-1}}$. In this case the identities become

$$\begin{aligned}
\tilde{\mu}(z^*) &= \tilde{\mu}(z) - e_j \\
\theta(\tilde{\mu}(z^*)) &= \theta(\tilde{\mu}(z))
\end{aligned}$$

  which gives

$$\begin{aligned}
\int_{\beta_j} \mathrm{d}\ \log(\theta(\tilde{\mu}(z)) + \int_{\beta_j^{-1}} \mathrm{d}\ \log(\theta(\tilde{\mu}(z)) &= \int_{\beta_j} \mathrm{d}\ \log(\theta(\tilde{\mu}(z)) - \mathrm{d}\ \log(\theta(\tilde{\mu}(z)) \\
&= 0.
\end{aligned}$$

Adding everything up we have

$$\begin{aligned}
\deg((\tilde{\mu}^* \theta_\lambda)) &= \frac{1}{2\pi i} \left(\sum_{j=1}^{g} 2\pi i + \sum_{j=1}^{g} 0 \right) \\
&= g.
\end{aligned}$$

This gives the first assertion.

To prove (6.3), we use a similar argument. If f has zeroes of orders $n_1, \ldots, n_m$ at $z_1, \ldots, z_m$, respectively, then, then by the residue theorem

$$\frac{1}{2\pi i} \int_\gamma z \mathrm{d} \log(f(z)) = n_1 z_1 + \ldots + n_m z_m. \tag{6.4}$$

By above, $\mu(S) \cap \Theta_\lambda$ are the only points where $\mu^*\theta_\lambda$ is zero. We consider the i-th component of $\mu(S) \cap \Theta_\lambda$, then by (6.4),

$$\begin{aligned} \tilde{\mu}_i(z_1(\lambda)) + \ldots + \tilde{\mu}_i(z_g(\lambda)) &= \frac{1}{2\pi i}\int_{\partial P} \tilde{\mu}_i(z) \mathrm{d} \log(\theta_\lambda(\tilde{\mu}(z)) \\ &= \frac{1}{2\pi i}\left(\sum_{j=1}^{g}\int_{\alpha_j} + \int_{\alpha_j^{-1}} + \sum_{j=1}^{g}\int_{\beta_j} + \int_{\beta_j^{-1}}\right) \end{aligned} \tag{6.5}$$

where in the last line we have omitted the integrand. The $\int_{\alpha_j} + \int_{\alpha_j^{-1}}$ and $\int_{\beta_j} + \int_{\beta_j^{-1}}$ cases are dealt with separately. We will denote $\xi(z) = \log(\theta_\lambda(\tilde{\mu}(z))$ for concision.

- For the $\int_{\alpha_j} + \int_{\alpha_j^{-1}}$ case. Let z and z^* be points on α_j such that they are identified on S. As before

$$\begin{aligned} \tilde{\mu}_i(z^*) &= \tilde{\mu}_i(z) + Z_{ij} \\ \mathrm{d}\,\xi(z^*) &= \mathrm{d}\xi(z) - 2\pi i \omega_j(z). \end{aligned}$$

  Now

$$\begin{aligned} \int_{\alpha_j} \tilde{\mu}_i(z)\mathrm{d}\xi(z) + \int_{\alpha_j} \tilde{\mu}_i(z)\mathrm{d}\xi(z) &= \int_{\alpha_j} \tilde{\mu}_i(z)\mathrm{d}\xi(z) - \tilde{\mu}_i(z^*)\mathrm{d}\,\xi(z^*) \\ &= \int_{\alpha_j} \tilde{\mu}_i(z)\mathrm{d}\xi(z) - (\tilde{\mu}_i(z) + Z_{ij})\,(\mathrm{d}\,\xi(z) - 2\pi i\omega_j(z)) \\ &= \int_{\alpha_j} 2\pi i\omega_j(z)(\tilde{\mu}_i(z) + Z_{ij}) + Z_{ij}\mathrm{d}\,\xi(z) \\ &:= A_{ij} + Z_{ij}\int_s^t \mathrm{d}\ \log(\theta_\lambda(\tilde{\mu}(z)) \\ &= A_{ij} + Z_{ij}\log\left(\frac{\theta_\lambda(\tilde{\mu}(t))}{\theta_\lambda(\tilde{\mu}(s))}\right) \end{aligned}$$

  where $A_{ij} = \int_{\alpha_j} 2\pi i\omega_j(z)(\tilde{\mu}_i(z) + Z_{ij})$ and s, t are the endpoints of α_j. Now $\tilde{\mu}(t) = \tilde{\mu}(s) + e_j$, so $\theta_\lambda(\tilde{\mu}(t)) = \theta_\lambda(\tilde{\mu}(s))$ and the above becomes

$$\begin{aligned} A_{i*j} + Z_{ij}\mathrm{log}\left(\frac{\theta_\lambda(\tilde{\mu}(t))}{\theta_\lambda(\tilde{\mu}(s))}\right) &= A_{i*j} + Z_{i*j}\,(\mathrm{Log}(1) + 2\pi i * M) \\ &= A_{i*j} + Z_{i*j}2\pi i * M \end{aligned} \tag{6.6}$$

  for some $M \in \mathbb{Z}$. Note that none of these depend on λ.

- For the $\int_{\beta_j} + \int_{\beta_j^{-1}}$ case, we have

$$\begin{aligned}\tilde{\mu}_i(z^*) &= \tilde{\mu}_i(z) - \delta_{ij} \\ \mathrm{d}\xi(z^*) &= \mathrm{d}\,\xi(z)\end{aligned}$$

where z and z^* are identified on S. Following the same trail as before,

$$\begin{aligned}\int_{\beta_j} \tilde{\mu}_i(z)\mathrm{d}\xi(z) + \int_{\beta_j} \tilde{\mu}_i(z)\mathrm{d}\xi(z) &= \int_{\beta_j} \tilde{\mu}_i(z)\mathrm{d}\xi(z) - \tilde{\mu}_i(z^*)\mathrm{d}\,\xi(z^*) \\ &= \int_{\beta_j} \tilde{\mu}_i(z)\mathrm{d}\xi(z) - (\tilde{\mu}_i(z) - \delta_{ij})\,\mathrm{d}\,\xi(z) \\ &:= \int_u^v \delta_{ij}\mathrm{d}\,\log(\theta_\lambda(\tilde{\mu}(z))) \\ &= \delta_{ij}\log\left(\frac{\theta_\lambda(\tilde{\mu}(v))}{\theta_\lambda(\tilde{\mu}(u))}\right)\end{aligned}$$

where $u^{(j)}$ and $v^{(j)}$ are the endpoints of β_j. Now $\tilde{\mu}(v^{(j)}) = \tilde{\mu}(u^{(j)}) + Z_j$, so

$$\theta_\lambda(\tilde{\mu}(v^{(j)})) = \exp\left(-2\pi i\left(\tilde{\mu}_i(u^{(j)}) + \frac{Z_{ii}}{2} - \lambda_i\right)\right)\theta_\lambda(\tilde{\mu}(u^{(j)}))$$

and the above becomes

$$\delta_{ij}\mathrm{log}\left(\frac{\theta_\lambda(\tilde{\mu}(v))}{\theta_\lambda(\tilde{\mu}(u))}\right) = \delta_{i*j}\left(-2\pi i\left(\tilde{\mu}_i(u^{(j)}) + \frac{Z_{i*i}}{2} - \lambda_i\right) + 2\pi i * N\right) \quad (6.7)$$

for some $N \in \mathbb{Z}$.

Now substitute (6.6) and (6.7) into (6.5),

$$\begin{aligned}\sum_{k=1}^{g} \tilde{\mu}_i(z_k(\lambda)) &= \frac{1}{2\pi i}\left(\sum_{j=1}^{g} A_{ij} + Z_{ij}2\pi i M + \sum_{j=1}^{g}\delta_{ij}\left(-2\pi i\left(\tilde{\mu}_i(u^{(j)}) + \frac{Z_{ii}}{2} - \lambda_i\right) + 2\pi i M\right)\right) \\ &= \left(-\tilde{\mu}_i(u^{(i)}) - \frac{Z_{ii}}{2} + \lambda_i + N + \sum_{j=1}^{g}\frac{A_{ij}}{2\pi i} + Z_{ij}M\right) \\ &\in \lambda_i + \kappa_i + \mathbb{Z}e_1 + \ldots \mathbb{Z}e_g + \mathbb{Z}Z_1 + \ldots + \mathbb{Z}Z_g\end{aligned}$$

where κ_i includes all the constant terms not depending on λ_i. This gives (6.3). □

Note 6.12 This gives the explicit solution to the Jacobi inversion theorem.

To state Riemann's theorem, we need to recall the Abel-Jacobi map from (3.4), $\mu^{(k)}$: $\mathrm{Div}_+^k(S) \longrightarrow \mathcal{J}(S)$. Denote the image of $\mu^{(k)}$, by W_k.

Note 6.13 Using this notation, the Jacobi inversion theorem says $W_g = \mathcal{J}(S)$.

We can identify W_{g-1} with a translate of Θ; this is the content of Riemann's theorem,

Theorem 6.14 *The equation $W_{g-1} = \Theta_{-\kappa}$ holds, where κ is defined in lemma 6.11.*

Proof First we use the properties of theta functions to show $W_{g-1} \subset \Theta_{-\kappa}$. Let D be a generic effective divisor of degree g such that $\mu(S) \not\subseteq \mu^{(g)}(D) + \kappa =: \lambda$. Now D is generic means we can write $D = p_1 + \ldots + p_g$, where $p_1, \ldots, p_g \in S$ are distinct. Applying lemma 6.11, we see that

$$\mu(S) \cap \Theta_\lambda \quad = \quad \mu(D) \quad = \quad \mu(p_1) + \ldots + \mu(p_g)$$

Recall that $\theta_\lambda(\mu(p_1)) = \ldots = \theta_\lambda(\mu(p_g)) = 0$, and θ is even (c.f. equation (5.5)), so

$$\theta(\mu(p_1) + \ldots \mu(p_{g-1}) + \kappa) \quad = \quad \theta(\lambda - \mu(p_g)) \quad = \quad \theta(\mu(p_g) - \lambda) \quad = \quad \theta_\lambda(\mu(p_g)) \quad = \quad 0.$$

This gives $\theta_{-\kappa}(\mu(p_1) + \ldots \mu(p_{g-1})) = 0$ for all generic effective divisors $D = p_1 + \ldots + p_{g-1}$. Thus $\mu^*\theta_{-\kappa}$ vanishes on an *open* set in $S^{(g-1)}$, so is identically zero by analytic continuation. Hence $W_{g-1} \subset \Theta_{-\kappa}$.
Now W_{g-1} is irreducible since it is the image of an irreducible algebraic variety[3], hence we can write $\Theta_{-\kappa} = nW_{g-1} + \Xi$ for some $\Xi \in \mathcal{J}(S)$. The next two lemmas will show that $n = 1$ and $\Xi = 0$, thus giving $\Theta_{-\kappa} = W_{g-1}$. □

Lemma 6.15 *We have $\mu(S) \cdot W_{g-1} \geqslant g$ and $\mu(S) \cdot \Xi \geqslant 0$. Then since*

$$g \quad = \quad \mu(S) \cdot \Theta_{-\kappa} \quad = \quad n(\mu(S) \cdot W_{g-1}) + (\mu(S) \cdot \Xi)$$

we conclude $\mu(S) \cdot W_{g-1} = g, \mu(S) \cdot \Xi = 0$ hence $n = 1$.

Proof Pick a generic point $\chi := \mu(q_1) + \ldots + \mu(q_g) \in \mathcal{J}(S)$ such that $\mu(S) \not\subseteq \Xi + \chi$ and $-\mu(S) \not\subseteq W_{g-1} - \chi$.
Recall that the intersection number of two cycles only depends on their respective homology classes. Firstly $\mu(S)$ is homologous to $-\mu(S)$ since the involution $\mathcal{J}(S) \ni \xi \longmapsto -\xi \in \mathcal{J}(S)$ on $H_2(\mathcal{J}(S), \mathbb{Z})$ acts as the identity. Moreover, $W_{g-1} - \zeta$ is homologous to W_{g-1} for a generic choice of $\zeta \in \mathcal{J}(S)$.

[3] S is irreducible so S^{g-1} is irreducible. Hence the image of $S^{g-1} \longrightarrow \mathrm{Div}_+^{(g-1)}(S) \longrightarrow W_{g-1}$ is irreducible. A more detailed argument on why the holomorphic image of an irreducible subvariety is irreducible is given in the proof of the Torelli theorem.

The following is the important step

$$-\mu(p_j) \;=\; \sum_{i\neq j}\mu(p_i)-\chi\in W_{g-1}-\chi$$

for all $j\in[1,g]$, hence $-\mu(S)$ intersects $W_{g-1}-\chi$ at (at least)$-\mu(p_1),\ldots,-\mu(p_g)$, so $(-\mu(S))\cdot(W_{g-1}-\chi)\geqslant g$. Now the intersection number only depends on homology class, and by the discussion in the previous paragraph,

$$\begin{aligned}-\mu(S)\cdot(W_{g-1}-\chi) &= \mu(S)\cdot(W_{g-1}-\chi)\\ &= \mu(S)\cdot W_{g-1}\;\geqslant\; g.\end{aligned}$$

Similarly $\mu(S)\cdot\Xi=\mu(S)\cdot(\Xi+\chi)$. Since $\mu(S)\not\subseteq\Xi+\chi$, $\mu(S)\cdot(\Xi+\chi)\geqslant 0$ and the lemma is proved. □

Lemma 6.16 *The divisor* Ξ *is zero.*

Proof First assume $\Xi\neq 0$. Suppose $\mu(p)\in\Xi_\lambda$ for some $p\in S$. Then $\mu(S)\subset\Xi_\lambda$, since otherwise $\mu(S)\cdot\Xi_\lambda\geqslant 1$ contradicting $\mu(S)\cdot\Xi_\lambda=\mu(S)\cdot\Xi=0$. This is true for any $\lambda\in\mathcal{J}(S)$.

Now if $\mu(p_0)+\mu(q_0)\in\Xi_\lambda$ for some $p_0,q_0\in S$, then

$$\mu(q_0)\;\in\;\Xi_{\lambda+\mu(p_0)}$$

so by above, $\mu(S)\subset\Xi_{\lambda+\mu(p_0)}$. That is $\mu(q)\in\Xi_{\lambda+\mu(p_0)}$ for all $q\in S$. Now

$$\mu(p_0)\;\in\;\Xi_{\lambda+\mu(q)}$$

so $\mu(S)\subset\Xi_{\lambda+\mu(q)}$ for all $q\in S$. These statements imply that for any $\lambda\in\mathcal{J}(S)$ if $\mu(p_0)+\mu(q_0)\in\Xi_\lambda$ for some $p_0,q_0\in S$ then $\mu(p)+\mu(q)\in\Xi_\lambda$ for all $p,q\in S$. That is, $W_2\subset\Xi_\lambda$.

Repeating the above argument we have the following,
for any $\lambda\in\mathcal{J}(S)$
if $\mu(a_1)+\ldots+\mu(a_n)\in\Xi_\lambda$ for some $a_1,\ldots,a_n\in S$,
then $\mu(b_1)+\ldots+\mu(b_n)\in\Xi_\lambda$ for all $b_1,\ldots,b_n\in S$
that is $W_2\subset\Xi_\lambda$.

Now for $n=g$, the Jacobi inversion theorem states that $W_g=\mathcal{J}(S)$, so there exists $a_1,\ldots,a_g\in S$ such that $\mu(a_1)+\ldots+\mu(a_n)\in\Xi_\lambda$ for any $\lambda\in\mathcal{J}(S)$. By the above argument, this implies $\mathcal{J}(S) = W_g \subset \Xi_\lambda$, which is a contradiction since Ξ_λ is codimension 1. Hence $\Xi = \Xi_\lambda = 0$. □

This completes the proof of theorem 6.14. Riemann's theorem is significant as it relates two seemingly different objects, $\Theta_{-\kappa}$ which is defined in terms of $\mathcal{J}(S)$ and E, and W_{g-1} which is defined by the geometry of the Abel-Jacobi map μ and S. This provides the essential link leading to the Torelli theorem.

Chapter 7

The Torelli theorem

The Torelli theorem is the theoretical justification of the comment made at the beginning of chapter 6, that studying a Riemann surface via its Jacobian incurs no loss of information. The proof relies heavily on Riemann's theorem (c.f. theorem 6.14). By the discussion in section 6.3, the Riemann theta divisor Θ of a Riemann surface S is specified up to translation by the principal polarisation E; the following proof is a geometric recipe for reconstructing S from Θ. First we introduce some notation. Let X be an analytic variety; we define the **singular locus** of X, X^{sing}, to be the union of all singular points of X. Denote $X^{\mathrm{sm}} = X - X^{\mathrm{sing}}$ to be the **smooth locus** of X.

Theorem 7.1 *(Torelli) Let $(\mathcal{J}(S), H)$ and $(\mathcal{J}(S'), H')$ be principally polarised Jacobians of S and S' respectively. If $(\mathcal{J}(S), H)$ and $(\mathcal{J}(S'), H')$ are isomorphic as principal polarised abelian varieties, then S and S' are isomorphic.*

7.1 Proof of the Torelli theorem

This section will be devoted to proving the Torelli theorem.

Definition 7.2 *Let M be a complex manifold of dimension n and X be a dimension k analytic subvariety. Define the **Gauss map** of X to be*

$$\begin{aligned} \mathcal{G}_X : X^{\mathrm{sm}} &\longrightarrow G(k,n) \\ x &\longmapsto T'_x(X) \subset T'_x(M) \end{aligned}$$

where $T'_x(X)$ is the holomorphic tangent space of X at $x \in X$, identified with $\mathbb{C}^n$, and $G(k,n)$ is the Grassmannian of k-planes in $\mathbb{C}^n$ (c.f. example 4.15).

Example 7.3 Consider the Abel-Jacobi map $\mu : S \longrightarrow \mathcal{J}(S)$ defined by $p \longmapsto \left(\int_{p_0}^{p} \omega_1, \ldots, \int_{p_0}^{p} \omega_g\right)$. On identification $T_p(\mathcal{J}(S))$ with $\mathbb{C}^g$, the Gauss map

$$\begin{aligned} \mathcal{G}_S : S &\longrightarrow G(1,g) \\ p &\longmapsto T_p'(S) \subset T_p'(\mathcal{J}(S)) \end{aligned}$$

is simply the canonical map $S \longrightarrow G(1,g) = \mathbb{P}^{g-1}$.

Lemma 7.4 *Consider the Gauss map* $\mathcal{G} : \Theta^{\mathrm{sm}}_{-\kappa} \longrightarrow G(g-1,g) = \left(\mathbb{P}^{g-1}\right)^{\vee}$ *of the Riemann theta divisor. Moreover the generic fibres contains* $\binom{2g-2}{g-1}$ *elements.*[1]

Proof Under the identification $\Theta^{\mathrm{sm}}_{-\kappa} = W^{\mathrm{sm}}_{g-1}$, let $\mu(D) = \mu(p_1) + \ldots + \mu(p_{g-1}) \in W_{g-1}$. The tangent hyperplane at $\mu(D)$, $\mathcal{G}(\mu(D)) \in \left(\mathbb{P}^{g-1}\right)^{\vee}$, is simply the hyperplane spanned by the points $\iota_K(p_1), \ldots, \iota_K(p_{g-1})$. We verify this by computation. Let $z_0, z_1, \ldots, z_{g-1}$ be local coordinates around the points $p_0, p_1, \ldots, p_{g-1}$, then

$$\begin{aligned} \mu(D) \quad = \quad \mu(p_1 + \ldots + p_{g-1}) \quad &= \quad \left(\sum_{k=1}^{g-1} \int_{z_0}^{z_k} \omega_1, \ldots, \sum_{k=1}^{g-1} \int_{z_0}^{z_i} \omega_g \right) \\ \frac{\partial}{\partial z_i}\mu(D)\bigg|_{z_i=0} \quad &= \quad \left(\frac{\partial}{\partial z_i} \sum_{k=1}^{g-1} \int_{z_0}^{z_k} \omega_1 \bigg|_{z_i=0}, \ldots, \frac{\partial}{\partial z_i} \sum_{k=1}^{g-1} \int_{z_0}^{z_i} \omega_g \bigg|_{z_i=0} \right) \\ &= \quad (\omega_1(p_i)/\mathrm{d}z_i, \ldots, \omega_g(p_i)/\mathrm{d}z_i) \ =: \ \boldsymbol{v}_i \end{aligned}$$

Under the identification $G(g-1,g) \simeq \bigwedge^{g-1} \mathbb{C}^g$, the tangent hyperplane at $\mu(D)$ is $\boldsymbol{v}_1 \wedge \ldots \wedge \boldsymbol{v}_{g-1} \in G(g-1,g)$. Upon projectivising, we see that $\boldsymbol{v}_i$ corresponds to $\iota_K(p_i) \in \mathbb{P}^{g-1}$, and the tangent hyperplane at $\mu(D)$ is simply the hyperplane in $\mathbb{P}^{g-1}$ spanned by the points $\iota_K(p_1), \ldots, \iota_K(p_{g-1})$.

From the discussion in chapter 4, these points span a unique hyperplane iff D is **regular**, that is, if $h^0(D) = 1$ iff $h^0(K - D) = 1$ by Riemann-Roch.

Recall that a generic hyperplane intersects $\mu(S)$ in $2g-2$ points (c.f. example 4.27). Then for a generic $H \in \left(\mathbb{P}^{g-1}\right)^{\vee}$ the generic fibre contains $\binom{2g-2}{g-1}$ elements. Moreover, each hyperplane intersects $\mu(S)$ in a finite number of points, so all fibres of $\mathcal{G}$ are finite.

□

We arrive at the proof of Torelli's theorem. As hinted in section 2.5, hyperelliptic and non-hyperelliptic Riemann surfaces often exhibit different behaviour. Accordingly, the proof will be given in two parts, with the first part covering the non-hyperelliptic case. Let $C := \iota_K(S), C' := \iota_K(S')$ be the canonical curves of S and S' respectively; points in

[1] This is another way of saying $\Theta^{\mathrm{sm}}_{-\kappa}$ is a $\binom{2g-2}{g-1}$-sheeted branched cover of $(\mathbb{P}^{g-1})^{\vee}$.

S will be denoted $p_1, \ldots, p_k$ and their images under ι_K in C, $\xi_1, \ldots, \xi_k$. We follow the arguments of [And58], also found in pages 359-362 of [GH78].

Proof of theorem 7.1.

Elliptic curve case, $g = 1$. This is a direct consequence of Abel's theorem (c.f. example 4.22).

Non-hyperelliptic case, $g \geqslant 3$. Let $B \subset \left(\mathbb{P}^{g-1}\right)^\vee$ be the branch locus of $\mathcal{G}$, that is B is the union of the images of the singular points of $\mathcal{G}$. Define,

$$C^\vee \quad = \quad \{H \in (\mathbb{P}^{g-1})^\vee \mid H \text{ is a tangent hyperplane of } C\} \;\subset\; \left(\mathbb{P}^{g-1}\right)^\vee$$

and call this the **hyperplane envelop** of C. We will show that $C^\vee$ is determined by $\Theta_{-\kappa}$ and $\mathcal{J}(S)$, explicitly, $\overline{B} = C^\vee$ where B is the branch locus of $\mathcal{G}$. That is, if two curves, C, C', have isomorphic Jacobians with the same principal polarisations, then $C^\vee \simeq C'^{\,\vee}$; this is the content of lemma **??**. The converse is proved in lemma 7.6; in the non-hyperelliptic case, the canonical map $\iota_K : S \longrightarrow \mathbb{P}^{g-1}$ is an embedding, so this completes the proof of the theorem.

Hyperelliptic case, $g \geqslant 2$. The difference here is that the canonical map $\iota_K : S \longrightarrow \mathbb{P}^1$ is not an embedding (c.f. example 4.26), we claim that in this case

$$\overline{B} \quad = \quad C^\vee \cup \{p^\vee\}_{p \text{ is a branch point of } \iota_K}$$

where $p^\vee := \left\{H \in \left(\mathbb{P}^{g-1}\right)^\vee \mid p \in H\right\} \subset \left(\mathbb{P}^{g-1}\right)^\vee$, the dual of p. This is substantiated in lemma 7.7. Now $\overline{B}$ determines $C^\vee$ as well as the ramification points of the two to one map $f : S \longrightarrow \mathbb{P}^1$. By the discussion in section 2.5, this determines S completely. □

Lemma 7.5 *Suppose S is non-hyperelliptic and denote $C = \iota_K(S)$. The hyperplane envelop, $C^\vee$ of C, is equal to the closure of the branch locus B of $\mathcal{G}$. Then since $\mathcal{G}$ is intrinsically defined by $\mathcal{J}(S)$ and $\Theta_{-\kappa}$, so is $C^\vee$.*

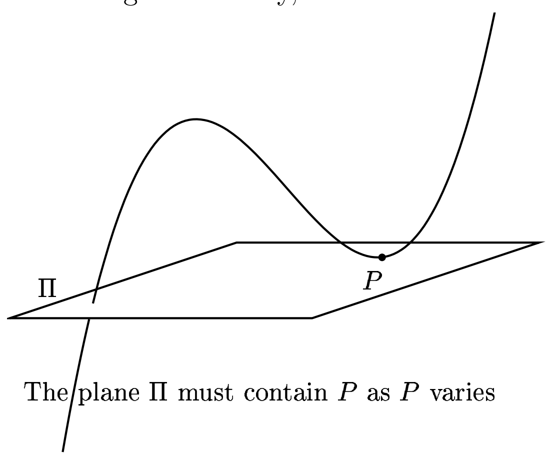
Proof First define, set theoretically,

$$V \quad = \quad \left\{H \in \left(\mathbb{P}^{g-1}\right)^\vee \mid \text{ the intersection of } H \cap C \text{ are in not general position}\right\}$$

The condition on V is equivalent to the following. There exists $g-1$ points out of the $2g-2$ points in $H \cap C$ which are linearly dependent, that is, whose linear span has dimension less than $g-2$. This is a clearly a proper subvariety of $\left(\mathbb{P}^{g-1}\right)^\vee$. We wish to show $B = C^\vee \cap V^c$. Consider the maps

$$\begin{array}{ccccc} S^{g-1}/\sim & \xrightarrow{\mu^{(g-1)}} & W_{g-1} & \xrightarrow{\mathcal{G}} & \left(\mathbb{P}^{g-1}\right)^\vee \\ p_1+\ldots+p_{g-1} & \longmapsto & \mu(p_1)+\ldots+\mu(p_{g-1}) & \longmapsto & \overline{\iota_K(p_1),\ldots,\iota_K(p_{g-1})} \end{array}.$$

If $H \in \left(\mathbb{P}^{g-1}\right)^\vee$ is tangent to C at some point, then the pullback divisor $\iota_K^* H$ contains multiple points. Hence by the proof of lemma 7.4, $|\mathcal{G}^{-1}(H)| < \binom{2g-2}{g-1}$, hence H is a branch point of $\mathcal{G}$. This fact is obvious geometrically, consider the following diagram



The plane Π must contain P as P varies

so that an infinitesimal change in the position of P will leave Π stationary, so if z is a local coordinate for p then $\frac{\partial}{\partial z}\mathcal{G}(z)\big|_{z=0} = 0$. Conversely, suppose H is a hyperplane not tangent to C, so it intersects C at $2g-2$ *distinct* points, $\iota_K(q_1),\ldots,\iota_K(q_{2g-2})$, in fact, this is true for any H' in some neighbourhood of H in $\left(\mathbb{P}^{g-1}\right)^\vee$, hence H is not a singular point of $\mathcal{G}$. Hence

$$B \quad = \quad C^\vee \cap V^c.$$

We now show $C^\vee$ is irreducible. Define [2]

$$I \quad := \quad \{(p,H) \mid H \text{ is tangent to } C \text{ at } p\} \;\subset\; C \times \left(\mathbb{P}^{g-1}\right)^\vee$$

and consider

$$\begin{array}{ccc} & I & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ C & & C^\vee \end{array}$$

[2] I is called the **incidence correspondence**.

where π_1 and π_2 are the projections onto the first and second factors respectively. First note that π_1 and π_2 are both surjective and continuous. Now C is irreducible, and the fibres of π_1 are all irreducible, since they are hyperplanes with dimension $g-1$, we conclude that I is irreducible (c.f. [CC04]). Now if $C^\vee$ is reducible, write $C^\vee = \Delta_1 \cup \Delta_2$ where Δ_1, Δ_2 are nonempty closed sets. Then since π_2 is surjective and defined on all of I, $I = \pi_2^{-1}(\Delta_1) \cup \pi_2^{-1}(\Delta_2)$ is a nontrivial decomposition of I into a union of two closed sets, contradicting irreducibility of I. So $C^\vee$ is irreducible.

It follows that $\overline{B} = \overline{C^\vee \cap V^c} = C^\vee$, otherwise we have the decomposition $C^\vee = \left(\overline{C^\vee \cap V^c}\right) \cup (C^\vee \cap V)$ into two nonempty closed subsets, contradicting irreducibility of $C^\vee$. This gives $\overline{B} = C^\vee$ as required. □

Lemma 7.6 *Suppose C is non-hyperelliptic. Then the hyperplane envelop, $C^\vee$, of C determine C up to isomorphism.*

Proof Suppose C and C' are two curves with $C^\vee = C'^\vee$. We claim that there is a well-defined regular bijection

$$\begin{aligned} \rho : C &\longrightarrow C' \\ p &\longmapsto T_p(C) \cap C' \end{aligned} \qquad (7.1)$$

where $T_p(C)$ is the tangent line to C at p. Define the set,

$$X_p := \left\{ H \in \left(\mathbb{P}^{g-1}\right)^\vee \mid H \text{ contains the tangent line to } C \text{ at } p \right\}$$

and consider the linear system, $\mathcal{L}$, obtained by

$$\mathcal{L} := \{H \cap C'\}_{H \in X_p}.$$

The base locus of $\mathcal{L}$ is

$$\beta_p := \bigcap_{H \in X_p} H \cap C' = T_p(C) \cap C'$$

since all $H \in X_p$ contains $T_p(C)$. Now any $H \in X_p$ is tangent to C at p, so $H \in C^\vee$. Since $C^\vee = C'^\vee$, H must be tangent to $C'^\vee$ also. We claim that H is tangent to C' at β_p.

To prove this, recall that Bertini's theorem (c.f. theorem 4.10) states that the generic element of a linear system away from its base locus is smooth. Applying this to $\mathcal{L}$, we see that for any $H \in X_p$, H cannot be tangent to C' at the points $(H \cap C') - \beta_p$; hence H must be tangent to C' at β_p.

For $g > 3$, we claim that C' has no bitangents [3], so β_p is the unique point of tangency of $T_p(C)$ to C'. To see this, suppose ℓ is a bitangent of C' at the points $p, q \in C'$. By the geometric version of Riemann-Roch, $\dim |D| = (\deg(D) - 1) - \dim(\overline{D})$, where

$$\dim |2p + 2q| \quad = \quad (4-1) - 1 \quad = \quad 2 \quad = \quad \frac{\deg(2p+2q)}{2}$$

and Clifford's theorem (c.f. theorem 4.20) implies S is hyperelliptic. Hence the map in (7.1) is well-defined and is a regular bijection.

For $g = 3$, we see that $X_p = T_p(C)$, and by proposition 2.8, C' only has a finite number of bitangents. So we obtain a rational map

$$\begin{aligned} \rho : C \quad & \dashrightarrow \quad C' \\ p \quad & \longmapsto \quad T_p(C) \cap C' \end{aligned}$$

defined on the open set consisting of points p, where $T_p(C')$ is not a bitangent. Now by a theorem in algebraic geometry (c.f. [CC04]), birationally equivalent smooth projective curves are isomorphic, so $C \simeq C'$. □

Lemma 7.7 *Suppose S is hyperelliptic, and recall $C = \iota_K(S)$. Let $B_{\iota_K} \subset C$ be the set of branch points of $\iota_K : S \longrightarrow \mathbb{P}^{g-1}$, then*

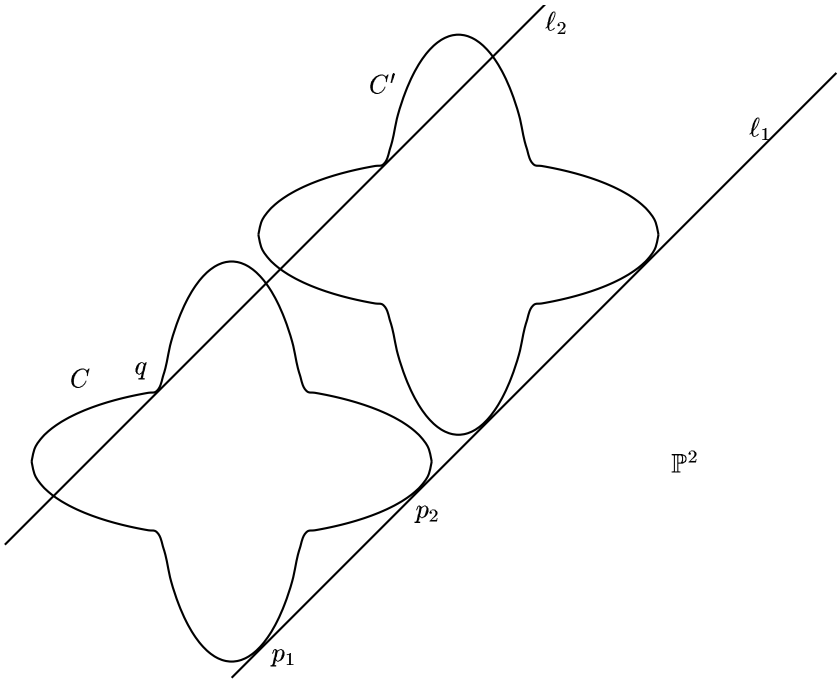
$$\overline{B} \quad = \quad C^\vee \cup \{p^\vee\}_{p \in B_{\iota_K}}$$

where B is the branch locus of $\mathcal{G}$.

Proof In the hyperelliptic case, a hyperplane $H \in \left(\mathbb{P}^{g-1}\right)^\vee$ intersects the canonical curve of S at $g-1$ points. So if $\iota_K^* H$ contains multiple points, then either H is tangent to S or H passes through a branch point of ι_K. This determines the branch points of ι_K and by section 2.5 determines S. □

A neat picture demonstrating the proof of the Torelli theorem can be drawn for the non-hyperelliptic genus 3 Riemann surface, S. In this case, the canonical map is an embedding, hence we can embed S into $\mathbb{P}^2$. As with all diagrammatic representations of complex curves, we can only draw a "real" cross section.

[3]A bitangent to a curve $C \subset \mathbb{P}^n$ is a line in $\mathbb{P}^n$ which is tangent to C at two distinct points.



The genus 3 Riemann surface embedded as a plane quartic

We have shown a common bitangent and a common tangent of C and C'. [4]

[4]The equation of the bottom left curve is given by $\left(\frac{x^2}{7} + y^2 - 1\right)\left(x^2 + \frac{y^2}{7} - 1\right) - \frac{1}{100} = 0$.

Chapter 8

Concluding remarks

In the proof of the Torelli theorem, we recovered the Riemann surface via its canonical curve. The Jacobian of the Riemann surface S stores its analytic structure, whereas the principal polarisation specifies, up to translation, a divisor in $\mathcal{J}(S)$, such that the Riemann surface can be reconstructed from this information.

To study Riemann surfaces directly is difficult. Mumford describes in detail in [Mum75] that as the genus of S grows, it becomes increasingly difficult to find explicit descriptions for it. The Weierstrass $\wp$-function which so neatly does the job in genus 1 has no analogues in higher genera, and as we saw in chapter 2, the number of equations increase as well - three equations are needed already to cut out a genus 2 curve in $\mathbb{P}^3$. The upshot of Torelli's theorem is that it is enough to study the Jacobian, with its theta function, in order to study the Riemann surface. For instance, in order to count the number of bitangents of a plane quartic, it is enough to count the so called *odd theta characteristics*, c.f. pages 150-155 (section 5.2) of [Cle80].

This classification of Riemann surfaces is however not completely satisfactory; for we do not have a description of all the Jacobians of a given dimension. This will be explained in the following section.

8.1 The Schottky problem

From corollary 6.8, we see that the period matrix Ω of $\mathcal{J}(S)$ can be given in the form $\Omega = (I, Z)$. Define

$$\mathfrak{S}_g \quad := \quad \{X \in M_g(\mathbb{C}) \mid X = X^T, \Im(X) \text{ positive definite}\}$$

then we see that $Z \in \mathfrak{S}_g$. The space $\mathfrak{S}_g \subseteq M_g(\mathbb{C})$ is known as the **Siegel upper half space**. In the case of $g = 1$, this is simply the upper half plane of $\mathbb{C}$. The information in Ω determines $\mathcal{J}(S)$ completely, so determining which $Z \in \mathfrak{S}_g$ such that (I, Z) is a period matrix is equivalent to identifying all the Jacobians of a given dimension. This is known as the **Schottky problem**.

In the language of moduli, let $\mathcal{M}_g$ be the moduli space of all Riemann surfaces of genus g. Consider the map

$$\begin{array}{rcl} \mathcal{M}_g & \longrightarrow & \mathfrak{S}_g \\ S & \longmapsto & Z \end{array}$$

associating each Riemann surface of genus g in $S \in \mathcal{M}_g$ to its period matrix. Torelli's theorem states that this map is injective, and the Schottky problem is the problem of determining its image in $\mathfrak{S}_g$. This is still an open problem; Mumford discusses several approaches to the Schottky problem in chapter 4 of [Mum75].

Chapter 9

Background material

The appendix contains the basic definitions of group cohomology and some major theorems referred to in the thesis.

9.1 Group cohomology

The point of view of studying G-modules using the fixed point functor $\cdot^G$, which assigns to a G-module M the abelian group $M^G := \{m \in M \mid gm = m\}$, leads to group cohomology. As with the case with sheaf cohomology, exact sequences of G-modules $0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$ is carried, under $\cdot^G$, to the left exact $0 \longrightarrow M_1^G \longrightarrow M_2^G \longrightarrow M_3^G$. A cohomology theory with $H^0(G, M) = M^G$ will allow us to apply the snake lemma (proposition 1.25), giving the long exact sequence

$$0 \longrightarrow M_1^G \longrightarrow M_2^G \longrightarrow M_3^G \longrightarrow H^1(M_1, G) \longrightarrow H^1(M_2, G) \longrightarrow H^1(M_3, G) \longrightarrow \cdots$$

and the cohomology groups are the obstructions for $\cdot^G$ from exactness.

Let G be a group and $M \in G$-**Mod**[1], define $C^0 := C^0(G, M) = M$, the abelian group $C^k := C^k(G, M) = \{\varphi : G^m \longrightarrow M\}$, and form the cochain complex $C^\bullet : 0 \longrightarrow C^0 \xrightarrow{d_0} C^1 \longrightarrow \ldots$ with the coboundary map $\mathrm{d} : C_n \longrightarrow C_{n+1}$,

$$\begin{aligned}\mathrm{d}(\varphi)(g_1, \ldots, g_{n+1}) &= g_1\varphi(g_2, \ldots, g_{n+1}) + \sum_{j=1}^{n} (-1)^j \varphi(g_1, \ldots, g_{j-1}, g_j g_{j+1}, \ldots, g_{n+1}) \\ &\quad + (-1)^{n+1}\varphi(g_1, \ldots, g_n)\end{aligned}$$

Call $H^k(M, G)$ of the complex $C^\bullet$ the **n-th cohomology group of G with coefficients on M**. With this definition, $H^0(G, M) = M^G$ since $\mathrm{d}m(g) = gm - m = 0$ so $m \in \ker(\mathrm{d} : C^0 \longrightarrow C^1)$ iff $m \in M^G$. We will need the following lemma for the proof of theorem 5.8.

[1] G-**Mod** denotes the category of (left) G-modules.

Lemma 9.1 *The kernel of* $\mathrm{d} : C^1 \longrightarrow C^2$ *is the set of all* $\varphi : G \longrightarrow M$ *satisfying* $g_1\varphi(g_2) + \varphi(g_1) - \varphi(g_1 g_2) = 0$ *for all* $g_1, g_2 \in G$. *The image of* $\mathrm{d} : C^0 \longrightarrow C^1$ *is the set of all* $\psi : G \longrightarrow M$ *satisfying* $\psi(g) = gm - m$ *for some* $m \in M$.

Proof Let $\varphi \in C^1(G, M)$, the condition $\mathrm{d}\varphi = 0$ implies $\mathrm{d}\varphi(g_1, g_2) = g_1\varphi(g_2) - \varphi(g_1 g_2) + \varphi(g_1) = 0$, that is, φ satisfies $\varphi(g_1 g_2) = g_1\varphi(g_2) + \varphi(g_1)$ for all $g_1, g_2 \in G$. We have seen above that $\psi \in \mathrm{d}C^0(M, G)$ iff $\psi(g) = gm - m$ for some $m \in M$. □

Note 9.2 The group operation of M is traditionally written as addition, and the operation of G as multiplication, but in the case of the Λ-module $\mathcal{O}^*(V)$ in theorem 5.8, this is reversed! The above equations become $e(\lambda + \lambda') = \lambda \cdot e(\lambda')e(\lambda)$ and $\varepsilon(\lambda) = (\lambda \cdot f)f^{-1}$ for $\lambda, \lambda' \in \Lambda$, and $e, \varepsilon \in C^1(\Lambda, \mathcal{O}^*(V))$, $f \in \mathcal{O}^*(V)$.

9.2 Major theorems

We give the statements of the Hodge decomposition theorem for compact Kähler manifolds, the Serre duality theorem, and the Kodaira embedding theorem.

Theorem 9.3 *(Hodge) Let* M *be a compact Kahler manifold, then we have the following*

$$\begin{aligned} H^r(M, \mathbb{C}) &\simeq \bigoplus_{p+q=r} H^q(M, \Omega^p) \\ H^q(M, \Omega^p) &= \overline{H^q(M, \Omega^p)}. \end{aligned}$$

Example 9.4 The most basic case of the Hodge decomposition states the following,

$$\begin{aligned} H^1(M, \mathbb{C}) &\simeq H^0(M, \Omega^1) \oplus H^1(M, \mathcal{O}) \\ &\simeq H^{1,0}_{\overline{\partial}}(M) \oplus H^{0,1}_{\overline{\partial}}(M) \end{aligned}$$

which is the decomposition of forms into their holomorphic and anti-holomorphic components.

Theorem 9.5 *(Serre) Let* M *be a connected, compact complex manifold of dimension* n. *Then the following holds*

1. $H^n(M, \Omega^n) \simeq \mathbb{C}$ *and*
2. *the pairing* $H^q(M, \Omega^p) \otimes H^{n-q}(M, \Omega^{n-p}) \longrightarrow H^n(M, \Omega^n)$ *is nondegenerate.*

In particular we have the isomorphism $H^1(S, \mathcal{O}) \simeq H^0(S, \Omega^1)^\vee$.

Definition 9.6 *Let* M *be a complex manifold. A line bundle is called* ***positive*** *if its chern class can be represented by a positive form in* $H^2_{\mathrm{DR}}(M)$.

Theorem 9.7 *(Kodaira) Let M be a compact complex manifold and $L \longrightarrow M$ be a positive line bundle. Then there exists $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$*

$$\iota_{L^k} : M \longrightarrow \mathbb{P}H^0(M, \mathcal{O}(L^k)) \simeq \mathbb{P}^N$$

is an embedding of M into $\mathbb{P}^N$.

References

[And58] A. Andreotti. On torelli's theorem. *Am. J. Math*, 80:801–821, 1958.

[CC04] Daniel Chan and Kenneth Chan. Algebraic geometry course notes. 2004. http://web.maths.unsw.edu.au/ danielch/. Direct link: http://web.maths.unsw.edu.au/ danielch/aglec.pdf.

[Che46] S.S. Chern. Characteristic classes of hermitian manifolds. *Ann. of Math.*, 46:85–121, 1946.

[Che00] S.S. Chern. *Lectures on Differential Geometry.* World Scientific, 2000.

[Cle80] C. Herbert Clemens. *A Scrapbook of Complex Curve Theory.* Plenum Press, New York, 1980.

[Die72] Jean Dieudonné. The historical development of algebraic geometry. *The American Mathematical Monthly*, 79(8):827–866, 1972.

[EH00] David Eisenbud and Joe Harris. *The Geometry of Schemes.* Springer Verlag, 2000.

[FG01] Peter Firby and Cyril Gardner. *Surface Topology, 3rd Edition.* Horwood Publishing Ltd, 2001.

[FK92] Hershel M. Farkas and Irwin Kra. *Riemann Surfaces, 2nd edition.* Springer-Verlag, 1992.

[GH78] Phillips Griffiths and Joe Harris. *Principles of Algebraic Geometry.* John Wiley and Sons, Inc., 1978.

[Hat02] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002.

[Kir92] Frances Kirwan. *Complex Algebraic Curves.* Cambridge University Press, 1992.

[Mar63] Henrik H. Martens. A new proof of torelli's theorem. *Ann. of Math*, 78:107–111, 1963.

[Mir95] Rick Miranda. *Algebraic Curves and Riemann Surfaces.* American Mathematical Society, 1995.

[Mum75] David Mumford. *Curves and their Jacobians.* University of Michigan, 1975.

[Mum95] David Mumford. *Algebraic Geometry 1, Complex Projective Varieties.* Springer-Verlag, 1995.

[Osb00] M. Scott Osborne. *Basic Homological Algebra*. Springer-Verlag, 2000.

[Pol03] Alexander Polishchuk. *Abelian Varieties, Theta Functions and the Fourier Transform*. Cambridge University Press, 2003.

[Rem98] Reinhold Remmert. From Riemann Surfaces to Complex Spaces. *Séminaires et Congrés*, 3:203–241, 1998.

[Rie51] Bernhard Riemann. Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse. *Inaugural dissertation, Göttingen*, 1851.

[Ser55] Jean-Pierre Serre. Faisceaux algébriques cohérents. *Ann. of Math.*, 61:197–278, 1955.

[Sha74] Igor Shafarevich. *Basic Algebraic Geometry*. Springer-Verlag, 1974.

[Ste79] Ian Stewart. *Algebraic Number Theory*. Chapman and Hall, 1979.

[Tei44] Oswald Teichmüller. Veränderliche Riemannsche Flächen. *Deutche Math.*, 7:344–359, 1944.

[Tor13] R. Torelli. Sulle varietà di jacobi. *Rend. Acc. Lincei*, 22:98, 1913.

[Uen01] Kenji Ueno. *Algebraic Geometry 2: Sheaves and Cohomology*. American Mathematical Society, 2001.

[Wey23] Hermann Weyl. *Die Idee der Riemannschen Fläche*. Berlin, 1923.