Resolving Singularities of Orders on Surfaces

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by

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Originality Statement

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Abstract

The Mori program for orders on surfaces provides a framework for studying noncommutative birational geometry. Many results from the birational geometry of surfaces have their counterparts in this theory, for example, the existence of resolutions of singularities, the Castelnuovo contraction theorem, and of course the Enriques classification of surfaces. Singularities of the Mori program for varieties, namely, the terminal, canonical, log terminal and log canonical singularities, also have noncommutative incarnations. Terminal orders play the role of smooth models; the canonical and log terminal orders are noncommutative analogues of quotient singularities. We use the resolution of singularities for orders to study these "singular" orders, and in particular, we investigate the question of what is a good notion of rational singularities for orders.

To this end, we introduce the notion of numerical rationality for orders, which generalises rational singularities for varieties. We show that numerical rationality is a property independent of the choice of resolution of singularities, and that log terminal orders are numerically rational. Both these results generalise well known facts about rational singularities of varieties.

We next study the noncommutative blowup of orders, which is the building block for birational morphisms of terminal orders and the main ingredient for resolving singularities of orders. Van den Bergh shows in [VdB01] that one can construct the blowup of smooth noncommutative surfaces in the framework of category theory. We show that these two procedures of noncommutative blowing up, performed on terminal orders, yield equivalent objects. We also give a complete description of the noncommutative blowups for local canonical orders.

Singularities of orders over surfaces

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Chapter 1

Introduction

The aim of birational geometry is to classify algebraic varieties by their function fields. Varieties with the same function field are said to be in the same birational equivalence class. The basic questions are then to classify the types of such equivalence classes, say of a given dimension, and to study the relationships between varieties in the same birational equivalence class. We can extend this to noncommutative algebras by replacing a function field K with a division K-algebra, or more generally, a central simple K-algebra B. This gives rise to noncommutative birational geometry. In this setting, the noncommutative generalisation of a variety, or more accurately, its structure sheaf, is an order contained in B. In the following discussion all objects will be defined over an algebraically closed field k with characteristic zero.

When the transcendence degree of $K = Z(B)$ over k is equal to 2, orders contained in B give examples of noncommutative surfaces. Artin's conjecture (c.f. [Art97], Conjecture 4.1) roughly classifies noncommutative surfaces into two (non-disjoint) classes, those which are finite over their centres, and those which are birationally ruled. This motivates the study of orders by illuminating their central role from the point of view of classification, since orders provide examples of noncommutative surfaces which are finite over their centres. Besides their prominent place in noncommutative birational geometry, the study of orders have applications in algebra and algebraic geometry. In [AM72], the authors exhibit examples of unirational varieties which are not rational by studying the Brauer-Severi varieties of certain maximal orders. In [vdB04], certain orders having good homological properties are thought of as noncommutative crepant resolutions of commutative singularities. Azumaya algebras are probably the most well-known examples of orders outside noncommutative geometry, and they were used in the proof of the period-index theorem for surfaces (c.f.

[dJ04]).

The majority of results to date regarding the birational geometry of orders are in dimension 2, that is, when the transcendence degree of K over k is equal to 2. The highlight of this theory is the minimal model program for orders over surfaces (c.f. [CI05]), where the authors adapt the techniques of Mori theory for orders. As in the Mori program for varieties, singularities appear as an essential feature of the theory. Noncommutative singularities have been studied previously, for example, in [CBH98] and [Cha00]. The distinguishing feature of our approach is the use of resolutions of singularities for a class of orders containing the maximal orders. The study of noncommutative singularities via their resolutions raises new questions, and some of these will be explored in this thesis.

Our main motivating question is the following, does there exist a theory of rational singularities for maximal orders? The definition of rational singularities makes essential use of the existence of resolutions[1]; a variety Z has rational singularities if for some resolution $\sigma : \tilde{Z} \longrightarrow Z$, the natural morphism $\mathcal{O}_Z \longrightarrow \boldsymbol{R}\sigma_* \mathcal{O}_{\tilde{Z}}$ of complexes is a quasi-isomorphism. In particular, rational singularities are normal and Cohen-Macaulay. For surfaces, rational singularities can be characterised using numerical invariants of its minimal resolution (c.f. [Art66]). They can also be characterised as the normal surface singularities having finite Picard group (c.f. [Lip69]). Rational singularities occur throughout algebraic geometry. We will simply mention a few well known examples: quotient singularities, singularities of toric varieties, and log terminal singularities of the minimal model program. The richness of the theory of rational singularities in algebraic geometry inspires us to hope that the corresponding noncommutative theory will be just as interesting.

In pursuing the above question, we were led to study the resolutions of singularities of orders in detail. As for singularities of varieties, blowing up is the main method for resolving singularities of orders. The blowup of orders first appeared in [CI05], Definition 4.4, in the context of the noncommutative Mori program. As with algebraic surfaces, there is a Zariski factorisation for orders which states that blowups are the building blocks of birational maps between "smooth" orders.

The blowup of orders involve taking a kind of noncommutative integral closure inside a central simple algebra, which produces phenomena not found in the commutative situation. For simplicity, we will restrict the following discussion to maximal orders. Let A be a

[1] There do exist characterisations of rational singularities which do not use resolutions, for example, pseudorationality (c.f. [LT81]) and the notion of F-rationality via tight closure theory (c.f. [Smi97]).

maximal order on a surface Z. The blowup of A at a closed point $p \in Z$ consists of a blowup $\sigma : \tilde{Z} \longrightarrow Z$ at p and a maximal order on $\tilde{Z}$ containing $\sigma^* A$. Let $\mathrm{Ex}(\sigma) = \bigcup_i E_i$ denote the decomposition of $\mathrm{Ex}(\sigma)$ into its irreducible components. Since σ is an isomorphism away from p, the order $\sigma^* A$ is maximal away from E, however, the localisations $(\sigma^* A)_{E_i}$ are not necessarily maximal orders. Hence there may be different maximal orders on $\tilde{Z}$, owing to the choices of maximal orders containing $(\sigma^* A)_{E_i}$, which are blowups of A at p.

To study the question of what it means for an order A to have rational singularities, it is necessary to investigate the cohomological behaviour of its resolutions. The preceding discussion shows that for a maximal order A there may exist different resolutions over the same centre. While different resolutions have the same ramification data (c.f. [CI05], Lemma 3.4), they may have different cohomologies. This motivates our study of blowups of terminal and canonical orders in Chapters 5 and 6.

We would like to have a definition of rational singularities for orders which, like rational singularities of varieties, includes many interesting examples. In the commutative theory, rational singularities include the canonical singularities; which, in dimension 2, are quotient singularities of the form $\mathbb{A}^2/G$ where G is a finite subgroup of $SL_2(k)$. Moreover, canonical singularities are Gorenstein, and Gorenstein rational singularities are canonical, so canonical singularities are in this sense the mildest rational singularities. In light of these facts, the litmus test we chose for any definition of rational singularities for orders is that it must include the canonical orders, which are a type of noncommutative quotient singularity.

Our examples indicated that Artin's topological characterisation of rational singularities, generalised appropriately for orders, seems to be immune from the above objections. We call this numerical rationality (c.f Definition 3.1.5), and with this definition, we were able to prove some theorems which generalise the commutative theory. Our main result is that log terminal orders (c.f. Section 2.2.3) are numerically rational.

We conclude this introduction by describing a connection between blowups of orders and a noncommutative blowup which appeared in a different context. A more thorough discussion of this point will be done in Chapter 5. In [VdB01], Van den Bergh defined blowing up for noncommutative smooth surfaces in the general framework of quasi-schemes. A quasi-scheme is just a Grothendieck category, so a maximal order A can be considered as a quasi-scheme by taking the category $\mathrm{Mod}\, A$ of quasi-coherent left A modules (c.f. introduction to Chapter 5). A natural question is then whether we can blowup orders as quasi-schemes, and whether such blowups coincide with our blowups of orders. We find that for terminal orders, we can indeed carry out Van den Bergh's construction and interpret blowing up of orders as blowups of quasi-schemes (c.f. Theorem 5.2.6). This is interesting since it provides geometric meaning to the choices of maximal orders made in constructing order-theoretic blowups. It also hints at the possibility of understanding the blowup of orders in a more general framework. However, the hypotheses in [VdB01] are too restrictive to blowup even canonical orders, which is perhaps not surprising since the stated aim is to blowup smooth noncommutative surfaces. We expect the development of a theory for blowing up non-terminal orders to be an interesting future direction.

1.1 Summary

We begin in Chapter 2 by reviewing the necessary background on singularities and orders over surfaces. In Chapter 3, we define numerically rational orders which generalise rational singularities of varieties. We show that numerical rationality is independent of all choices of resolution (c.f. Proposition 3.1.3); this parallels the analogous fact for rational singularities for varieties. The main result in this chapter states that log terminal orders are numerically rational (c.f. Theorem 3.2.1). The proof of this uses a formula of Artin and de Jong, which computes the Euler characteristic of an order restricted to divisors. This formula along with a proof outline appears in the unpublished manuscript, and we provide a proof in Section 3.3 which fills in some details missing from [AdJ].

The next part of the thesis concerns the construction of blowups of orders. As mentioned above, this requires an understanding of the maximal orders containing a given order over a discrete valuation ring. In Chapter 4, we explain a method of "reduction to artin rings" which we apply in the subsequent chapters to classify blowups of terminal and canonical orders. In Chapter 5, we determine all the blowups of terminal orders and describe explicitly a bijective correspondence between the these blowups and blowups of terminal orders obtained by Van den Bergh's method (c.f. [VdB01]). Having settled the case of terminal orders, we move on to canonical orders in Chapter 6, where we describe the blowups of all the local canonical orders. We include in Appendix A generators and relations for canonical orders which are used extensively in the last chapter.

Chapter 2

Background

In this chapter we give an overview of rational singularities of varieties and the birational geometry of orders. We set up some conventions which will be in force throughout. Let k be an algebraically closed field of characteristic zero. A variety is an integral separated k-scheme of finite type. A curve (resp. surface) is a k-variety of dimension 1 (resp. 2). We often work in the étale or complete neighbourhood of a point on a surface, so strictly speaking we should refer to integral normal k-schemes of dimension 2 instead of surfaces. However, for the sake of brevity, we will not enforce this distinction in non-technical discussions. For instance, we will speak of orders over surfaces, even if their centres are not of finite type over k.

Let Z be a integral normal k-scheme of dimension 2. We denote by Z^1 its codimension 1 skeleton, that is, the set consisting of all irreducible codimension one subvarieties on Z. Given a birational morphism $\sigma : \tilde{Z} \longrightarrow Z$, we define its *exceptional locus* $\mathrm{Ex}(\sigma) \subset \tilde{Z}$ to be the subset of $\tilde{Z}$ where σ is not an isomorphism. We give $\mathrm{Ex}(\sigma)$ the reduced induced scheme structure, so $\mathrm{Ex}(\sigma)$ is a reduced connected divisor on $\tilde{Z}$. A $\mathbb{Q}$-divisor on Z is a finite $\mathbb{Q}$-linear combination $\sum_i a_i D_i$ where $D_i \in Z^1$. We say that a $\mathbb{Q}$-divisor D on $\tilde{Z}$ is σ-nef if $DE_i \geqslant 0$ for all components E_i of $\mathrm{Ex}(\sigma)$.

The necessary background material for orders over discrete valuation rings and quasi-schemes will be reviewed in Chapters 4 and 5.

2.1 Singularities

Let $\sigma : \tilde{Z} \longrightarrow Z$ be a resolution of a normal variety with $\mathrm{Ex}(\sigma) = \bigcup_{i=1}^{r} E_i$ the irreducible decomposition of $\mathrm{Ex}(\sigma)$. We can write the canonical divisor $K_{\tilde{Z}}$ of $\tilde{Z}$ as follows

$$K_{\tilde{Z}} \equiv \sigma^* K_Z + \sum_i a(E_i, Z) E_i$$

The rational numbers $a(E_i, Z)$ is called the *discrepancy* of Z at E_i. The *discrepancy* of Z, denoted by $\mathrm{discrep}(Z)$, is defined as the number $\inf\{a(E_i, Z)\}$ where the infimum ranges over all exceptional curves E_i of all resolutions of Z. The *index* m of Z is defined as the smallest integer such that mK_Z is a Cartier divisor.

We associate a weighted graph, called the *dual resolution graph*, to σ as follows. Let Γ denote the incidence graph of $\{E_i\}$ and for each vertex i define its weight by $b_i = -E_i^2$. The *numerical cycle* Z_{num} is the minimal effective exceptional divisor such that $-Z_{\mathrm{num}}$ is σ-nef. We denote by

$$I = (E_i E_j)_{i,j}$$

the intersection matrix of $\{E_i\}$. Note that I is negative definite and the numerical cycle Z_{num} can be computed directly from I.

2.1.1 Rational singularities

Let Z be a normal variety of dimension 2 and $\sigma : \tilde{Z} \longrightarrow Z$ be a resolution of singularities. We say that σ is a *rational resolution* if the natural morphism $\mathcal{O}_Z \longrightarrow \boldsymbol{R}\sigma\mathcal{O}_{\tilde{Z}}$ is a quasi-isomorphism, and Z has *rational singularities* if it has a rational resolution. If $\sigma : \tilde{Z} \longrightarrow Z$ is a rational resolution, then any resolution of Z is also rational (c.f. [KM98], Theorem 5.10). The Grauert-Riemenschneider vanishing theorem and Grothendieck duality allow us to reformulate this in terms of the canonical sheaf: the resolution σ is rational if and only if $\sigma_* \omega_{\tilde{Z}} \subset \omega_Z$ and Z is Cohen-Macaulay (c.f. [Kol97], Corollary 11.9). Rational singularities can also be characterised by numerical invariants of the resolution (c.f. [Art66], Proposition 1): a resolution $\sigma : \tilde{Z} \longrightarrow Z$ of a surface Z is rational if and only if for every effective divisor E contracted by σ, we have $\chi(\mathcal{O}_E) \geqslant 1$. In particular, all the irreducible components of $\mathrm{Ex}(\sigma)$ are smooth rational curves and the dual resolution graph of σ contains no cycles.

Unlike arbitrary normal surface singularities, rational singularities can be resolved by a sequence of blowups at points (c.f. [Lip69], Proposition 1.2), without the need for normalisation. For a rational surface singularity, its multiplicity is equal to $-Z_{\mathrm{num}}^2$ (c.f.

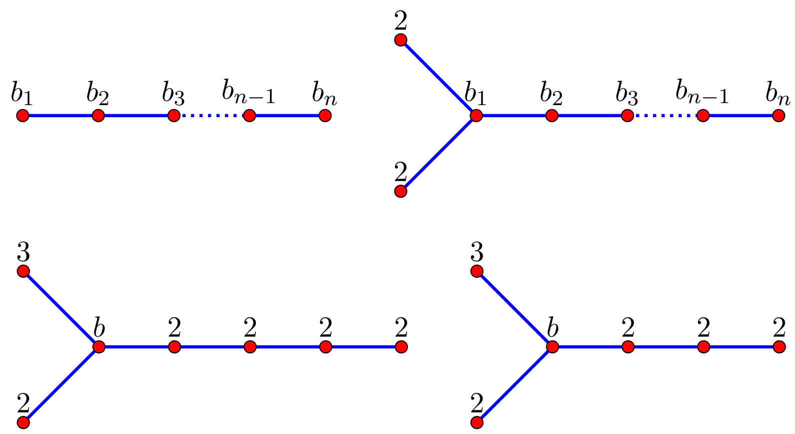
[Rei97], Section 4.17 and [Art66], Corollary 6); and the discrepancies a_i can be computed from the intersection matrix I of the exceptional curves $\{E_i\}_{i=1}^r$ (c.f. [Nik89] (1.6))

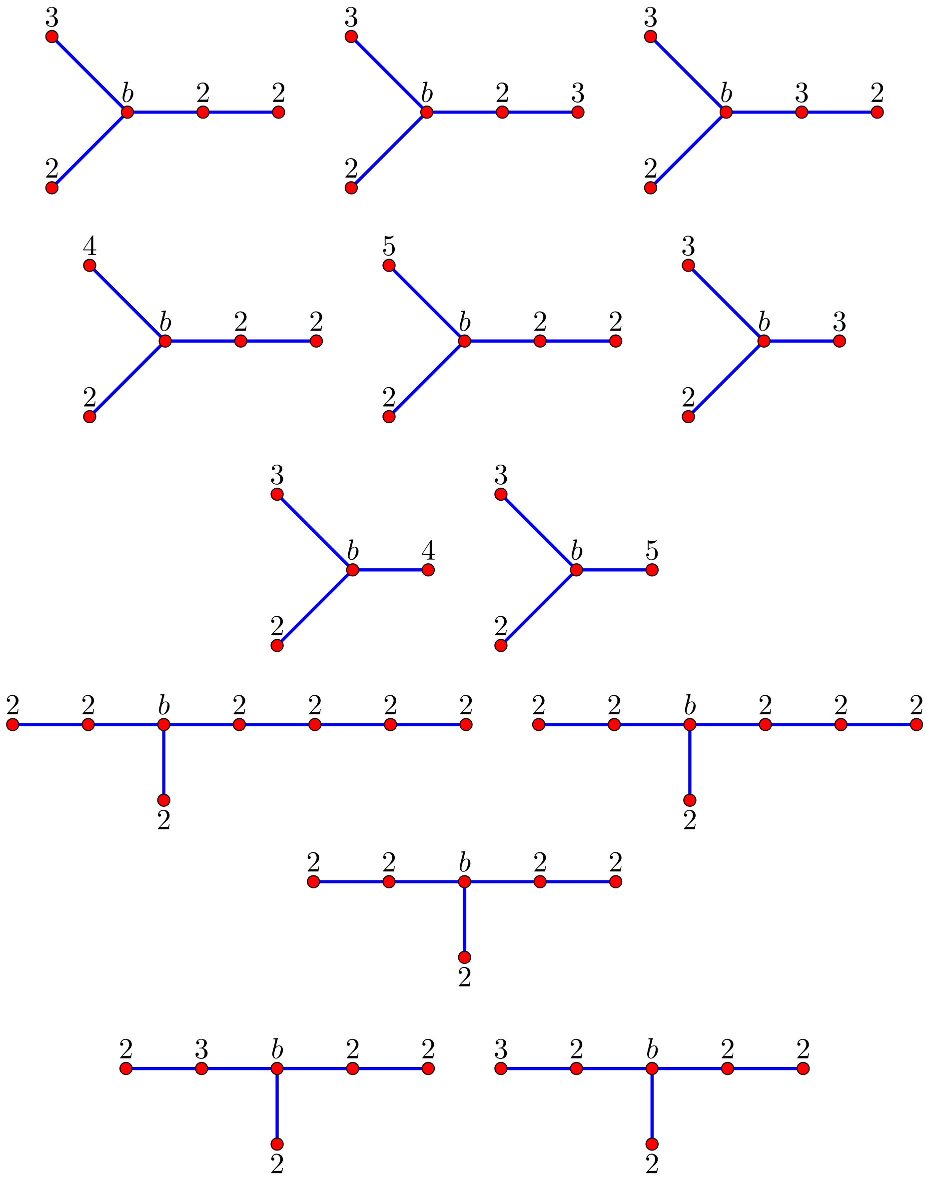
$$\begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} = -I^{-1} \begin{pmatrix} 2-b_1 \\ \vdots \\ 2-b_r \end{pmatrix}.$$

This also gives a convenient way to compute the index, which is equal to the greatest common divisor of the denominators of the a_i's. Classification of the dual resolution graphs of rational singularities has been carried out for low index: $m = 3$ in [Art66] and $m = 4$ in [Ste91]. Of course, if $m = 1$ then Z is Gorenstein so Z has canonical singularities. While we do not have good noncommutative analogues for the multiplicity and embedding dimension, the index is well defined for orders.

2.1.2 Log terminal singularities

Let Z be a normal integral scheme of dimension 2. We say that Z has *log terminal* (resp. *canonical*, *terminal*) singularities if $\mathrm{discrep}(Z) > -1$ (resp. $\mathrm{discrep}(Z) \geqslant 0$, $\mathrm{discrep}(Z) > 0$). If Z has dimension 2, then it has terminal singularities if and only if it is smooth. The classification of canonical singularities of surfaces is well known: locally they are isomorphic to a quotient of $\mathbb{A}^2$ by a finite subgroup of $SL_2(k)$, and their dual resolution graphs are the ADE Dynkin diagrams. Perhaps not so well known is the classification of the dual resolution graphs of log terminal surface singularities. This can be found, for example, in [Nik89] and we reproduce the classification here for the reader's convenience. As usual, each node i corresponds to an exceptional curve E_i, the integer b_i above node i denotes $-E_i^2$, and there are E_iE_j edges between the vertices i and j. Note that $b_i \geqslant 2$ for each vertex i in the diagram below.





We will refer to the above list in the proof of Proposition 3.2.9.

2.2 Orders over surfaces

Let Z be an integral normal k-scheme of dimension 2. An $\mathcal{O}_Z$ *-order* A is a coherent, torsion-free sheaf of $\mathcal{O}_Z$-algebras which is generically a central simple $k(Z)$-algebra; the scheme Z is called the *centre* of A, and we denote it by $Z(A)$. We sometimes refer to an $\mathcal{O}_Z$-order as an order over Z or simply an order if there is no ambiguity. For this notion of orders to be well-behaved, we need to impose some geometric hypotheses. An order A is *maximal* if it is maximal with respect to inclusions of orders in $A \otimes k(Z)$. Every order A can be embedded in a maximal order A'. This is the noncommutative analogue of taking an integral closure, although, unlike in the commutative case, the maximal order A' is in general not unique. Maximality is a Zariski local property, but it is not preserved under étale localisations or completions. For this reason, we introduce a slight weakening of maximality to cover such cases. An $\mathcal{O}_Z$-order A is called *normal* if it satisfies the following conditions

N1 A is reflexive as an $\mathcal{O}_Z$-module

N2 for all $C \in Z^1$, the $\mathcal{O}_{Z,C}$-order A_C is hereditary

N3 for all $C \in Z^1$, $A_C \simeq \omega_{A,C}$ as left and right A_C-modules (but not necessarily as bimodules)

N4 if the transcendence degree of $k(C)$ over k is finite, then A_C is maximal.

The conditions N1 and N2 imply that normal orders are tame orders in the sense of [RVdB89] and [Sil68]. In the situation of a blowup, we have a normal $\mathcal{O}_{\tilde{Z}}$-order $\tilde{A}$ on $\tilde{Z}$ together with a birational map $\tilde{Z} \longrightarrow Z$ where Z is some formal surface germ. The restriction imposed by N4 means that $\tilde{A}_E$ is maximal for exceptional curves E, but $\tilde{A}_C$ is allowed to be non-maximal if C is not an exceptional curve on $\tilde{Z}$.

2.2.1 Ramification data

Let A be a normal order over Z and C be an irreducible curve on Z. We denote by A_C the localisation $A \otimes \mathcal{O}_{Z,C}$ of A at C and $J(A_C)$ the Jacobson radical of A_C. The $k(C)$-algebra $A_C/J(A_C)$ is semi-simple (but not necessarily $k(C)$-central) so we can write $A_C/J(A_C)$ as a product of matrix algebras over some field extension of $k(C)$. The normality condition ensures that the simple decomposition of $A_C/J(A_C)$ is isotypic (c.f. [CI05], Proposition 2.4), that is,

$$A_C/J(A_C) \simeq \underbrace{M_m(L) \times \cdots \times M_m(L)}_{n \text{ times}}$$

where L is a cyclic field extension of $k(C)$.

We define the *ramification index* e_C of A at C to be the integer $\dim_{k(C)} L^n$. If $e_C > 1$, we say that A is *ramified* at C and *unramified* otherwise. If A is ramified at C, we denote $\pi_C : \tilde{C} \longrightarrow C$ to be the degree e_C cyclic cover determined by the integral closure of $\mathcal{O}_C$ in L^n. If π_C is ramified at $p \in C$, then we say that A has *secondary ramification* at p. The *ramification data* of A is defined as the cover $\pi : \tilde{D} \longrightarrow D$ where D is the union (as subsets of Z) $\bigcup_{e_C>1} C$, $\tilde{D}$ is the disjoint union $\dot{\bigcup}_{e_C>1}\tilde{C}$ and the map π is induced by the cyclic covers π_C. The *ramification divisor* of A is defined as the $\mathbb{Q}$-divisor

$$\Delta \;=\; \sum_{e_C>1}\left(1-\frac{1}{e_C}\right)C,$$

and the *canonical divisor* of A is defined to be

$$K_A \;=\; K_Z+\Delta.$$

The ramification data retain important information about A. For instance, the order A_C is maximal if and only if $\tilde{C}$ is connected. In Chapter 6, Proposition 6.3.3, we use the ramification data to compute the discriminant ideal of certain canonical orders. We can also define terminal orders in terms of the ramification data. An order A on Z is *terminal* if Z is smooth and its ramification data $\tilde{D} \longrightarrow D$ satisfies the following conditions, $D=\sum_i D_i$ is a simple normal crossing divisor and at any node $p\in D_i\cap D_j$

T π_{D_i} is a degree e cyclic cover, totally ramified at p; and π_{D_j} is a degree ne cyclic cover, ramified at p with index e.

The étale local structure of a terminal order A is determined completely by its ramification data. We denote by $\mathcal{O}^{sh}_{Z,p}$ the strict henselisation of $\mathcal{O}_{Z,p}$ at its maximal ideal $\mathfrak{m}_p$ and let u,v be generators of $\mathfrak{m}_p$. If A is a terminal order, then by [CI05], Section 2.3, we have

$$A^{sh}_p := A\otimes_Z \mathcal{O}^{sh}_{Z,p} \;\simeq\; \begin{pmatrix} S & \cdots & \cdots & S\\ xS & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & \vdots\\ xS & \cdots & xS & S\end{pmatrix}^{m\times m}$$

where

$$S \;\simeq\; \frac{\mathcal{O}^{sh}_{Z,p}\langle x,y\rangle}{(x^e-u,\,y^e-v,\,xy-\zeta yx)} \tag{2.2.1}$$

with ζ a primitive e-th root of unity. We will use the local structure for terminal orders in Section 3.3 for the proof of the adjunction formula (c.f. Theorem 3.1.2).

2.2.2 Blowing up and resolutions of singularities

Terminal orders are the smooth models in the noncommutative Mori program in that we can always perform a noncommutative resolution of singularities to obtain a terminal order. We can therefore measure how singular an order is by considering at how far its ramification data is from being terminal. Resolving singularities of an order is accomplished by a procedure called blowing up, which we define below.

Let Z be an integral normal k-scheme of dimension 2 and A be an order on Z. A *blowup* of A at $p \in Z$ is given by a pair $(\sigma : \tilde{Z} \longrightarrow Z, \tilde{A})$ where σ is the blowup of Z at p and $\tilde{A}$ is an order on $\tilde{Z}$ such that $\tilde{A}|_{\sigma^{-1}(Z-p)} = (\sigma^* A)|_{\sigma^{-1}(Z-p)}$, and $\tilde{A}_E$ is a maximal order containing $(\sigma^* A)_E$ for every component E of $\mathrm{Ex}(\sigma)$. This gives an inclusion of $\mathcal{O}_{\tilde{Z}}$-modules which we denote by $\sigma^\# : \sigma^* A \longrightarrow \tilde{A}$. Note that if $(\sigma^* A)_E$ is not maximal, then $\sigma^* A$ is not a normal order (c.f. condition N4), so we need to normalise $\sigma^* A$ at E by choosing a maximal order containing $(\sigma^* A)_E$. [1]

A *birational morphism of orders* is a 4-tuple $(A, A', f, f^\#)$ where A and A' are orders over their respective centres, $f : Z(A') \longrightarrow Z(A)$ is a birational morphism of k-schemes and $f^\#$ is a morphism of $\mathcal{O}_{Z(A)}$-algebras which is an isomorphism away from $\mathrm{Ex}(f)$. The blowup $(\sigma, \tilde{A})$ of A defined above is an example of a birational morphism $(A, \tilde{A}, \sigma, \sigma^\#)$ where $\sigma^\# : \sigma^* A \longrightarrow \tilde{A}$ is the inclusion defined above. A *resolution of singularities* of A is a birational morphism of orders $(A, A', f, f^\#)$ such that A' is terminal (see below for definition) and f is a proper morphism. To construct a resolution of singularities, we first resolve the singularities of $Z(A)$ and apply repeated blowups on $Z(A)$ to obtain a ramification divisor with simple normal crossings. In fact, we can continue blowing up to ensure that its ramification data satisfies condition T.

2.2.3 Discrepancies and singularities

With the above construction of resolution of singularities, we can define the notion of discrepancy for orders. Let A be an order on Z and $(\sigma, \tilde{A})$ a resolution of singularities of A, then we can express $K_{\tilde{A}}$ as

$$K_{\tilde{A}} \equiv \sigma^* K_A + \sum_i a(E_i, A) E_i \qquad (2.2.2)$$

The rational number $a(E_i, A)$ is called the *discrepancy* of A at E_i and the *discrepancy* $\mathrm{discrep}(A)$ of A is defined as the number $\inf\{a(E_i, A) e_{E_i}\}$ where the infimum ranges over all exceptional curves of all resolutions of singularities of A. A resolution $(\sigma, \tilde{A})$ is *minimal*

[1] In [CI05], the definition of blowing up orders incorporates Morita equivalence. We will not need this subtlety, since we always work with orders as sheaves on Z rather than Brauer classes α and their associated log surfaces (Z, Δ_α) (c.f. [CI05], Section 3.3).

if $K_{\tilde{A}}$ is σ-nef (c.f. [CI05], Theorem 3.10). If $(\sigma, \tilde{A})$ is a minimal resolution, then the rational numbers $a(E_i, A)$ appearing in (2.2.2) are either all zero or all strictly negative (c.f. [KM98], Lemma 3.41). Discrepancies provide a definition for terminal orders (c.f. [CI05], Proposition 3.14) which is equivalent to the above characterisation using ramification data. An order is terminal if $\operatorname{discrep}(A) > 0$, *canonical* if $\operatorname{discrep}(A) \geqslant 0$ and *log terminal* if $\operatorname{discrep}(A) > -1$.

Chapter 3

Log terminal orders are numerically rational

The aim of this chapter is to present a generalisation of rational singularities to the noncommutative geometry of orders on surfaces. We call our notion of rational singularities for orders numerical rationality. The main result of this chapter is that the log terminal orders are numerically rational (c.f. Theorem 3.2.1). This analogue of the corresponding fact for varieties provides many examples of numerically rational orders, and for this reason, we think that the notion of numerical rationality is an interesting one. Along the way, we also prove an unpublished formula of Artin and de Jong which computes the Euler characteristic of a terminal order when restricted to divisors.

In our definition of numerical rationality (c.f. Definition 3.1.5), we mimic Artin's numerical condition for rational singularities (c.f. [Art66], Proposition 1) which states that Z is a rational singularity if and only if for some resolution $\tilde{Z} \longrightarrow Z$, every exceptional effective divisor on $\tilde{Z}$ has positive topological Euler characteristic. In the noncommutative generalisation, we are led to consider the Euler characteristic of $\tilde{A}$ restricted to divisors, and arrive at a similar condition. We learned about a computation for this Euler characteristic from an unpublished manuscript [AdJ], of M. Artin and A. J. de Jong. We discover that $\chi(\tilde{A} \otimes \mathcal{O}_E)$ has a nice expression which resembles the adjunction formula for $\chi(\mathcal{O}_E)$, and we call this an adjunction formula for orders.

In this chapter, we will assume that all orders under discussion are normal orders (c.f. Section 2.2).

3.1 Numerical Rationality

The aim of this section is to define and study a generalisation of rational singularities for orders. We will outline some of the difficulties involved in extending such a notion noncommutatively, and hopefully convince the reader that our proposed definition is interesting. Many naturally occurring singularities in birational geometry are rational. The main examples are the log terminal surface singularities, which are quotients of $\mathbb{A}^2$ by a finite subgroup of GL_2, hence are rational singularities. Our point of view is that a version of rational singularities for orders should include the log terminal orders arising from the noncommutative minimal model program of [CI05].

3.1.1 Numerical rationality

The definition of rational singularities for varieties makes essential use of the existence of a resolution of singularities $\sigma : \tilde{Z} \longrightarrow Z$. A resolution $\sigma : \tilde{Z} \longrightarrow Z$ is rational if $\sigma_*\mathcal{O}_{\tilde{Z}} = \mathcal{O}_Z$ and $R^i\sigma_*\mathcal{O}_{\tilde{Z}} = 0$ for $i > 0$; and Z has rational singularities if there exists a rational resolution $\sigma : \tilde{Z} \longrightarrow Z$. This condition does not depend on the resolution, since if Z has a rational resolution, then all resolutions of Z are rational. Orders on surfaces have resolutions of singularities (c.f. [CI05], Corollary 3.6); a resolution of an order A consists of a pair $(\sigma : \tilde{Z} \longrightarrow Z, \tilde{A})$, where $\sigma : \tilde{Z} \longrightarrow Z$ is a resolution of varieties and $\tilde{A}$ is a terminal order on $\tilde{Z}$ with $\sigma^*A \subset \tilde{A}$ such that for any non-exceptional curve D on $\tilde{Z}$, we have $(\sigma^*A)_D = \tilde{A}_D$.

Note that even when σ is given, the terminal order $\tilde{A}$ is not necessarily unique, since in general, we can choose different maximal orders $\tilde{A}_E$ containing $(\sigma^*A)_E$ for each E, and every such choice $\{\tilde{A}_E\}_E$ produces a bona fide resolution of A on $\tilde{Z}$. However, by the Artin-Mumford sequence, the ramification data of A determines the ramification data of its resolutions (c.f. [CI05], Lemma 3.4). In particular, the ramification data of a resolution is independent of the choices of maximal orders at exceptional curves.

To get a good notion for rational resolutions for orders, we generalise Artin's numerical criterion for rational singularities on varieties, which states that Z has rational singularities if and only if for some resolution $\sigma : \tilde{Z} \longrightarrow Z$, we have $\chi(\mathcal{O}_E) > 0$ for all exceptional divisors $E > 0$ on $\tilde{Z}$.

Definition 3.1.1. *Let A be a normal order on a surface Z. A resolution $(\sigma : \tilde{Z} \longrightarrow Z, \tilde{A})$ of A is numerically rational if $\chi(\tilde{A} \otimes \mathcal{O}_E) > 0$ holds for all exceptional divisors $E > 0$ on*

$\tilde{Z}$.

We first show that the above definition has the nice property that if numerical rationality holds for some resolution, then it holds for all resolutions. This generalises the corresponding fact for rational resolutions for varieties. We will need the following adjunction formula for orders.

Theorem 3.1.2. *Let A be a terminal order on Z of rank r^2 and D be an effective divisor whose support is projective. Then*

$$\chi(A \otimes_Z \mathcal{O}_D) = -\frac{r^2}{2}(K_A + D)D$$

where $K_A = K_Z + \Delta_A$ is the canonical divisor of A.

The proof of Theorem 3.1.2 will be given in Section 3.3. We see immediately that $\chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E)$ depends only on the ramification data and the rank of $\tilde{A}$. In particular, if σ is a rational resolution, then numerical rationality is a Morita invariant property (c.f. [Cha], Proposition 4.1).

A resolution $(\sigma : \tilde{Z} \longrightarrow Z, \tilde{A})$ of an order A is minimal if the canonical divisor $K_{\tilde{A}}$ is σ-nef (c.f [CI05], Theorem 3.10).

Proposition 3.1.3. *If an order A admits a numerically rational resolution, then any resolution of A is numerically rational.*

Proof. Suppose $(\sigma, \tilde{A})$ is a resolution of A. If $(\sigma, \tilde{A})$ is not minimal, then there exists a $K_{\tilde{A}}$-negative curve E with $E^2 < 0$. By [CI05], Theorem 3.10, we can factor $\sigma = \tau'\beta'$ through a blowup β' at a point which contracts E, and there exists a terminal order A'_1 such that (τ', A'_1) is a resolution of A. The terminal order A'_1 is obtained by taking the reflexive hull of $\beta'_*\tilde{A}$. Repeating this until we reach a minimal resolution allows us to factor $\sigma = \tau\beta$ where β is a sequence of blowups centred at closed points, and obtain a terminal order A_1 such that (τ, A_1) is a minimal resolution of A.

By Lemma 3.1.4, $(\sigma, \tilde{A})$ is numerically rational if and only if (τ, A_1) is numerically rational. According to Theorem 2.15 of [CHI09], minimal resolutions of A have the same centres and ramification data. Since numerical rationality depends only on the ramification data, the result follows. □

Lemma 3.1.4. *Let $(\sigma : \tilde{Z} \longrightarrow Z, \tilde{A})$, $(\tau : \tilde{Y} \longrightarrow Z, \tilde{B})$ be resolutions of A and suppose $\tau = \sigma\beta$ where β is a blowup at a point $p \in \tilde{Z}$. Then $(\sigma, \tilde{A})$ is numerically rational if and only if $(\tau, \tilde{B})$ is numerically rational.*

Proof. Since both $\tilde{A}$ and $\tilde{B}$ are terminal orders, we can use Theorem 3.1.2 to compute their Euler characteristics when restricted to divisors. Let R denote the ramification divisor of $\tilde{A}$ on $\tilde{Z}$. We have two cases to consider, depending on whether p belongs to the singular locus of $\operatorname{supp} R$.

We denote by $E_0 = \operatorname{Ex}(\beta)$ the exceptional curve on $\tilde{Y}$ contracted by β. If p is not in the singular locus of $\operatorname{supp} R$, we see that E_0 is unramified. If, in addition, $p \in \operatorname{supp} R$ then $\beta^*\Delta_{\tilde{A}} - (1 - 1/e_1)\, E_0 = \Delta_{\tilde{B}}$ where e_1 is the ramification index of the irreducible component of $\operatorname{supp} R$ containing p. If $p \notin \operatorname{supp} R$, then $\beta^*\Delta_{\tilde{A}} = \Delta_{\tilde{B}}$. For convenience, we will write this as $\beta^*\Delta_{\tilde{A}} - (1 - 1/e_1)E_0 = \Delta_{\tilde{B}}$ for $e_1 = 1$.

Now suppose $p \in (\operatorname{supp} R)_{\text{sing}}$. Note that since $\tilde{A}$ is terminal, R only has nodal singularities. Let R_1, R_2 be (not necessarily distinct) irreducible components of R intersecting transversely at p, and denote by e_i the ramification index of R_i. Then $e_1 = se_2$ for some integer s. The Artin-Mumford sequence can be used to show that $\tilde{A}$ is totally ramified at E_0 with $e_0 = e_2$ (c.f. Lemma 3.4, [CI05]). Hence $\beta^*\Delta_{\tilde{A}} - (1 - 1/e_1)\, E_0 = \Delta_{\tilde{B}}$.

We can write a general effective divisor E on $\tilde{Y}$ as $\beta^*\tilde{E} + mE_0$ where $\tilde{E}$ is some effective divisor on $\tilde{Z}$ and $m \in \mathbb{Z}$. Since β is the blowup of a smooth point, we know that $\beta^*K_{\tilde{Z}} + E_0 = K_{\tilde{Y}}$. In each case, we get the following,

$$\chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) \quad = \quad \chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_{\tilde{E}}) + \frac{n^2}{2} m \left(m + \frac{1}{e_1}\right)$$

for some $e_1 > 0$. If $\chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) > 0$ for all $E > 0$, then putting $m = 0$ gives $\chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_{\tilde{E}}) > 0$ for all $\tilde{E} > 0$. Conversely, if $\chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_{\tilde{E}})$ is positive for all $\tilde{E} > 0$, then we can conclude that $\chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) \geqslant 0$ for all $\tilde{E} > 0$ and $m \in \mathbb{Z}$. To see that $\chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) > 0$, we find that the only nontrivial solution of $\chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) = 0$ occurs when $e_1 = 1$, $\tilde{E} = 0$ and $m = -1$. This does not correspond to an effective divisor on $\tilde{Y}$. Hence $\chi(\tilde{B} \otimes_{\tilde{Y}} \mathcal{O}_E) > 0$ for all $E > 0$. □

By Proposition 3.1.3, we can make the following definition.

Definition 3.1.5. *An order A is numerically rational if it admits a numerically rational resolution, or, equivalently, every resolution of A is numerically rational.*

3.1.2 Naïve rational singularities for orders

The most naïve procedure to generalise rational resolutions for orders is to require that a resolution $(\sigma, \tilde{A})$ satisfy $\sigma_*\tilde{A} = A$ and $R^i\sigma_*\tilde{A} = 0$ for $i > 0$. We see easily that this runs into problems. Firstly, as the following example shows, this condition depends on the choice of maximal orders in blowing up, hence is not Morita invariant (c.f. [Cha], Proposition 4.1). From the point of view of noncommutative geometry, this is rather discouraging. However, the existence of a resolution $(\sigma, \tilde{A})$ such that $R^1\sigma_*\tilde{A} = 0$ can still be useful. For instance, it can be used to deduce that the centre of Z of A has rational singularities. We have not been able to prove the analogous fact for numerically rational orders.

Example 3.1.6. Let $Z = \operatorname{Spec} k[[u, v]]$ and

$$A = \begin{pmatrix} \mathcal{O}_Z & \mathcal{O}_Z \\ \mathcal{O}_Z(u^3 - v^2) & \mathcal{O}_Z \end{pmatrix}$$

be a canonical order of type BL_1. The order A is ramified on the curve D defined by the equation $u^3 - v^2$ and can be resolved by a single blowup $\sigma : \tilde{Z} \longrightarrow Z$ at the cusp p of D (c.f. [CHI09], Figure 1). Let $\tilde{D}$ denote the strict transform of D and E be the exceptional curve of σ. There are three non-isomorphic terminal orders on $\tilde{Z}$

$$\tilde{A}_m = \begin{pmatrix} \mathcal{O}_{\tilde{Z}} & \mathcal{O}_{\tilde{Z}}(mE) \\ \mathcal{O}_{\tilde{Z}}(-\tilde{D} - mE) & \mathcal{O}_{\tilde{Z}} \end{pmatrix}$$

for $m = 0, 1, 2$ which are maximal at E and contain σ^*A, so $R^1\sigma_*\tilde{A}_m$ vanishes if and only if $m \neq 2$.

The same example shows that there exists a resolution of a canonical order that is not rational in the naïve sense above. This transgresses our requirement that the canonical orders (which are log terminal) of the noncommutative Mori program should be rational. An alternative is to use the dual formulation of the definition, that is, say that a resolution $(\sigma, \tilde{A})$ is rational if ω_A is a Cohen-Macaulay sheaf and $\sigma_*\omega_{\tilde{A}} = \omega_A$. Unfortunately, this too is susceptible to the same objections as above. We note here that the above two formulations for rational resolutions for orders are not equivalent. This can be demonstrated using Example 3.1.8, recall that $R^1\sigma_*\tilde{A}_m = 0$, but $\sigma_*\omega_{\tilde{A}_1}$ is not Cohen-Macaulay:

$$\sigma_*\omega_{\tilde{A}_1} \simeq \omega_Z \oplus \omega_Z \oplus \omega_Z \otimes \mathcal{I}_p \oplus \omega_Z(D) \otimes \mathcal{I}_p$$

where $\mathcal{I}_p$ is the ideal sheaf of the origin $p \in Z$.

Our definition of numerical rationality is weaker than the naïve generalisation of rational resolutions to orders: if $R^1\sigma_*\tilde{A} = 0$, then we can see by taking the long exact sequence in cohomology associated to the short exact sequence

$$0 \longrightarrow \tilde{A}(-E) \longrightarrow \tilde{A} \longrightarrow \tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E \longrightarrow 0$$

that $h^1(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) = 0$. Since $\mathcal{O}_E \subset \tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E$, we have $h^0(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) > 0$ hence $\chi(\tilde{A} \otimes_{\tilde{Z}} \mathcal{O}_E) > 0$. Example 3.1.8 and Corollary 3.1.7 shows that numerical rationality is strictly weaker than the naïve generalisation of rationality.

3.1.3 Examples

Recall that canonical orders have crepant minimal resolutions (c.f. [CHI09], Proposition 6.1), that is, if $(\sigma, \tilde{A})$ is a minimal resolution of the canonical order A, then $K_{\tilde{A}} = \sigma^* K_A$. It is easy to show that canonical orders are numerically rational.

Corollary 3.1.7. *Let $(\sigma : \tilde{Z} \longrightarrow Z, \tilde{A})$ be a crepant resolution of the $\mathcal{O}_Z$-order A. Then $(\sigma, \tilde{A})$ is a numerically rational resolution. In particular, canonical orders are numerically rational.*

Proof. If $(\sigma, \tilde{A})$ is crepant, then $\chi(\tilde{A} \otimes \mathcal{O}_E) = -n^2 E^2/2$, which is positive for any exceptional divisor E, so $(\sigma, \tilde{A})$ is numerically rational. Let A be a canonical order. The minimal resolution of a canonical order A is crepant, hence A is numerically rational. □

We conclude this section with an example of an order with smooth centre but is not numerically rational. We do not know whether there exists a numerically rational order with non-rational centre.

Example 3.1.8. Let $Z = \operatorname{Spec} k[[u, v]]$ and consider the order

$$A = \begin{pmatrix} \mathcal{O}_Z & \mathcal{O}_Z \\ \mathcal{O}_Z(u^4 - v^4) & \mathcal{O}_Z \end{pmatrix}.$$

A single blowup $\sigma : \tilde{Z} \longrightarrow Z$ at the origin $p \in Z$ resolves this order. The strict transform $\tilde{D}$ of $D = (u^4 - v^4)$ consists of four disjoint lines, each intersecting the exceptional curve E once. There are five nonisomorphic terminal $\mathcal{O}_{\tilde{Z}}$-orders which are blowups of A, namely

$$\tilde{A}_m = \begin{pmatrix} \mathcal{O}_{\tilde{Z}} & \mathcal{O}_{\tilde{Z}}(mE) \\ \mathcal{O}_{\tilde{Z}}(-\tilde{D} - mE) & \mathcal{O}_{\tilde{Z}} \end{pmatrix}$$

for $m = 0, \ldots, 4$. A direct computation shows that

$$\chi(\tilde{A}_m \otimes \mathcal{O}_E) = 0$$

for any m, so A is not numerically rational.

3.2 Log terminal implies numerically rational

In this section, we prove a noncommutative version of the following result: log terminal singularities are rational singularities. There is a notion of log terminal orders developed in the context of the noncommutative Mori theory of [CI05], and the analogue for rational singularities is provided by our notion of numerical rationality (c.f. Definition 3.1.5). The noncommutative version of the above result has the following pleasant statement.

Theorem 3.2.1. *If A is a log terminal order on Z, then A is numerically rational.*

Note that if A is log terminal, the associated log pair (Z, Δ_A) of A is klt ([CI05], Proposition 3.15). It follows from [KM98], Corollary 2.35 that Z has log terminal singularities. So we assume below that Z is the spectrum of a local ring with log terminal singularities. To prove that log terminal orders are numerically rational, by Proposition 3.1.3 we need only study their minimal resolutions. Since all minimal resolutions have the same ramification data ([CHI09], Theorem 2.5) we will use the following characterisation of log terminal orders, which is equivalent to the definition in Section 2.2.3. Let A be an order on Z and $(\sigma : Z' \longrightarrow Z, A')$ be any minimal resolution. We can write

$$K_{A'} \quad = \quad \sigma^* K_A + \sum_i a_i E_i$$

where the E_i's range over the exceptional curves on Z'. Then A is log terminal if and only if $\min\{a_i e_i\} > -1$, where e_i is the ramification index of A' at E_i.

Let A be a log terminal order on Z and $(\sigma : \tilde{Z} \longrightarrow Z, \tilde{A})$ be any resolution. Denote by $\boldsymbol{E}$ the subgroup $\operatorname{Pic} \tilde{Z}$ generated by the exceptional curves $\tilde{E}_i$. For $D \in \boldsymbol{E} \otimes_{\mathbb{Z}} \mathbb{Q}$, we define the function $f_{\sigma,D} : \boldsymbol{E} \longrightarrow \mathbb{Q}$ by $E \longmapsto -(D+E)E$. Let $\boldsymbol{E}^+ = \{\sum a_i \tilde{E}_i \in \boldsymbol{E} \mid a_i \geqslant 0\}$ denote the effective cone of $\boldsymbol{E}$. By Theorem 3.1.2, $(\sigma, \tilde{A})$ is a numerically rational resolution if and only if $f_{\sigma,K_{\tilde{A}}}(E) > 0$ for all $E \in \boldsymbol{E}^+ \backslash \{0\}$.

The function $f_{\sigma,K_{\tilde{A}}}$ is the sum of a positive definite quadratic form $q(E) = -E^2$ and a linear function $\ell(E) = -DE$ on $\boldsymbol{E}$. The log terminal condition on $K_{\tilde{A}}$ puts constraints on the coefficients of ℓ. To get some information out of these constraints, it is profitable to choose a different $\mathbb{Z}$-basis $\{N_i\}$ for $\boldsymbol{E}$. We define

$$N_i \quad = \quad \tau_i^* \tau_{i*} \tilde{E}_i$$

where τ_i is the unique factorisation $\tilde{Z} \xrightarrow{\tau_i} Z' \xrightarrow{\tau_i'} Z$ of σ satisfying the following properties

1. Z' is smooth,

2. $\tilde{E}_i$ is not contracted by τ_i, so $\tau_{i*}\tilde{E}_i$ is a curve on Z', and

3. there are no (-1)-curves on Z' except for possibly $\tau_{i*}\tilde{E}_i$.

It is easy to see that one obtains the factorisation above by sequentially contracting (-1)-curves except for the pushforwards of $\tilde{E}_i$.

We will adopt the following numbering convention the exceptional curves $\tilde{E}_i$ on $\tilde{Z}$: the resolution σ factors through a minimal resolution $\pi_0 : Z_0 \longrightarrow Z$ of Z so that $\sigma = \pi_0\pi$, we label $\tilde{E}_1, \dots, \tilde{E}_r$ the exceptional curves not contracted by π and the rest by $\tilde{E}_{r+1}, \dots, \tilde{E}_{r+\ell}$.

Proposition 3.2.2. *Let $(\sigma : \tilde{Z} \longrightarrow Z, \tilde{A})$ be a minimal resolution of an order A on Z and let $\sigma = \pi_0\pi$ be as above. Then*

$$f_{\sigma,K_{\tilde{A}}}(E) \;=\; f_{\pi_0,\pi_*K_{\tilde{A}}}(\pi_*E) + \left(\sum_{j=1}^{\ell} (N_{r+j}E)\, N_{r+j}\right)(E + K_{\tilde{A}}). \tag{3.2.1}$$

Proof. Since $f_{\pi_0,\pi_*K_{\tilde{A}}}(\pi_*E) = -\pi^*\pi_*E(E + K_{\tilde{A}})$, it suffices to show that

$$E \;=\; \pi^*\pi_*E - \left(\sum_{j=1}^{\ell} (N_{r+j}E)\, N_{r+j}\right). \tag{3.2.2}$$

A simple computation gives

$$N_{r+j}\tilde{E}_i \;=\; \begin{cases} -1 & \text{if } i = r+j \\ 1 & \text{if } \tau_i = \alpha\tau_{r+j} \text{ where } \alpha \text{ is a blowup centered at a point on } \tau_{i*}\tilde{E}_i \\ 0 & \text{otherwise} \end{cases}.$$

Let $\tilde{E}_{r+j_1}, \dots, \tilde{E}_{r+j_s}$ be components of N_i which intersect $\tilde{E}_i$, then $\tilde{E}_i = N_i - N_{r+j_1} - \dots - N_{r+j_s}$. The curve $\tilde{E}_{r+j_i}$ is contracted by τ_i, hence we can factor $\tau_i = \alpha\tau_{r+j}$ for some birational map α. Since $\tilde{E}_{r+j_i}\tilde{E}_i \neq 0$, α must be a single blowup centered at a point on $\tau_{i*}\tilde{E}_i$. Hence (3.2.2) follows from the above computation for $N_{r+j}\tilde{E}_i$. □

We can deduce from (3.2.1) that the inequalities $N_{r+j}K_{\tilde{A}} < 1$ for $j = 1, \dots, \ell$ are necessary conditions for $(\sigma, \tilde{A})$ to be numerically rational, since if $N_{r+j}K_{\tilde{A}} \geqslant 1$, we have

$$f_{\sigma,K_{\tilde{A}}}(N_{r+j}) \;=\; -(N_{r+j}K_{\tilde{A}} - 1) \leqslant 0.$$

Proposition 3.2.3. *Suppose A is log terminal. Then $0 \leqslant N_{r+j}K_{\tilde{A}} < 1$ for $j = 1, \dots, \ell$.*

Proof. Since $(\sigma, \tilde{A})$ is a minimal resolution, $K_{\tilde{A}}$ is σ-nef (c.f. [CI05], Theorem 3.10). Since N_{r+j} is effective, we get the first inequality. Note that $N_{r+j}\tilde{E}_{r+j} = (\tau_{r+j*}\tilde{E}_{r+j})^2 = -1$

and $N_{r+j}\tilde{E}_k = 0$ for any other exceptional curve $\tilde{E}_k \subseteq \operatorname{supp} N_{r+j}$. Clearly the intersection numbers of N_{r+j} with exceptional curves away from its support are non-negative, in fact, if $\tilde{E}_i$ is not an irreducible component of N_{r+j}, then $\tilde{E}_i N_{r+j} = 0$ or 1. This gives

$$N_{r+j} K_{\tilde{A}} \quad = \quad -a_{r+j} + \sum_i a_i$$

where the summation ranges over all i where $\tilde{E}_i$ intersects N_{r+j}. Now since $K_{\tilde{A}}$ is σ-nef, we have by [KM98], Lemma 3.41 that $a_i \leqslant 0$ for all i. Moreover, since A is log terminal, we have $a_{r+j} > -1$, hence $N_{r+j} K_{\tilde{A}} < 1$. $\square$

The above propositions shows that log terminal orders with smooth centres are numerically rational, and that in general, a minimal resolution $(\sigma, \tilde{A})$ of a log terminal order is numerically rational if and only if $f_{\pi_0, \pi_* K_{\tilde{A}}}(\pi_* E) > 0$ for all $E \in \boldsymbol{E}^+ \backslash \{0\}$ such that $\pi_* E > 0$. This allows us to work directly with the minimal resolution $\pi_0 : Z_0 \longrightarrow Z$. Every exceptional curve on Z_0 occur as the pushforward of an exceptional curve on $\tilde{Z}$. Let $E_1 = \pi_* \tilde{E}_1, \ldots, E_r = \pi_* \tilde{E}_r$ be the exceptional curves on Z_0 and denote by $\boldsymbol{E}_0 \subseteq \operatorname{Pic} Z_0$ the subgroup generated by $E_1, \ldots, E_r$.

Recall that the numerical cycle Z_{num} with respect to the birational morphism $\pi_0 : Z_0 \longrightarrow Z$ is defined to be the minimal effective exceptional divisor E on Z_0 such that $-E$ is π_0-nef. Given a connected effective exceptional divisor D, we ccan contract $\operatorname{supp} D$ to get a birational morphism $\pi_D : Z_0 \longrightarrow Z_D$. We define D_{num} to be the numerical cycle with respect to π_D, and call it the *numerical cycle of the support of* D. The values of $f_{\pi_0, \pi_* K_{\tilde{A}}}$ on effective exceptional divisors D satisfying $D = D_{\text{num}}$ determine whether $f_{\pi_0, \pi_* K_{\tilde{A}}}$ is positive for all effective exceptional divisors. Theorem 3.2.1 follows from the next two results.

Proposition 3.2.4. *Let A be a log terminal order on Z. Then $f_{\pi_0, \pi_* K_{\tilde{A}}}(E) > 0$ for all $E \in \boldsymbol{E}_0$ satisfying $E = E_{\text{num}}$.*

Proof. Let $E \in \boldsymbol{E}_0$ such that $E = E_{\text{num}}$. Suppose E_j is not contained in $\operatorname{supp} E$, then $E_{\text{num}} E_j \geqslant 0$. Since $a_i \leqslant 0$ (c.f. proof of Proposition 3.2.3), we have

$$f_{\pi_0, \pi_* K_{\tilde{A}}}(E_{\text{num}}) \quad \geqslant \quad -E_{\text{num}} \left(E_{\text{num}} + \sum_{E_i \subseteq \operatorname{supp} E} a_i E_i \right).$$

Now A is log terminal, so $a_i > -1$ for all i. Moreover, by the definition of E_{num}, we have $-E_{\text{num}} E_i \geqslant 0$ for any $E_i \subseteq \operatorname{supp} E$, so

$$f_{\pi_0, \pi_* K_{\tilde{A}}}(E_{\text{num}}) \quad > \quad -E_{\text{num}}(E_{\text{num}} - E_{\text{red}}),$$

where E_{red} denotes the reduced exceptional divisor with the same support as E_{num}. Since E_{num} is the numerical cycle of its support, we see that $E_{\mathrm{num}} - E_{\mathrm{red}}$ is an effective divisor with support contained in $\operatorname{supp} E$. This gives $f_{\pi_0,\pi_* K_{\tilde{A}}}(E_{\mathrm{num}}) > 0$. $\square$

Theorem 3.2.5. *Let $\pi_0 : Z_0 \longrightarrow Z$ be the minimal resolution of a log terminal singularity and $g : \boldsymbol{E} \longrightarrow \mathbb{Q}$ be a function $g(E) = -E^2 + \ell(E)$ where ℓ is any linear function satisfying $\ell(E_i) \leqslant 0$ for $i = 1, \ldots, r$. Then $g(E) > 0$ for all $E \in \boldsymbol{E}_0^+ \backslash \{0\}$ if and only if $g(E) > 0$ for all $E \in \boldsymbol{E}_0$ satisfying $E = E_{\mathrm{num}}$.*

Note 3.2.6. Note that $f_{\pi_0,\pi_* K_{\tilde{A}}}$ satisfies the above hypotheses for g if $(\sigma, \tilde{A})$ is a minimal resolution. Since in this case $K_{\tilde{A}}$ is σ-nef, we have $\pi_* K_{\tilde{A}} \pi_* E_i = K_{\tilde{A}} N_i \geqslant 0$ for $i = 1, \ldots, r$, hence $\pi_* K_{\tilde{A}}$ is π_0-nef.

The rest of this section is devoted to the proof of the above theorem. We fix notation for the rest of the section: let $\pi_0 : Z_0 \longrightarrow Z$ be the minimal resolution of a log terminal singularity. We denote by $E_1, \ldots, E_r$ the exceptional curves and Z_{num} the numerical cycle on Z_0. Also we will denote by $g(E) = -E^2 + \ell(E)$ a function $\boldsymbol{E}_0 \longrightarrow \mathbb{Q}$ satisfying the hypothesis of Theorem 3.2.5.

3.2.1 Modified numerical cycle

The usual notion of numerical cycle can be modified with respect to a given effective exceptional divisor D.

Proposition 3.2.7. *Let D be an exceptional divisor with $D \geqslant 0$. There exists a unique effective divisor D' satisfying $D \leqslant D'$ and $-D'$ is π_0-nef.*

Proof. To see that D' exists and is unique for a given D, pick a positive integer $n$such that $D \leqslant nZ_{\mathrm{num}}$. Then the divisor $D'' = \gcd\{C \mid D \leqslant C \leqslant nZ_{\mathrm{num}}, -C \text{ is } \pi_0\text{-nef}\}$ is well defined since the gcd is taken over finitely many exceptional divisors, and clearly $D' = D''$. $\square$

We call the divisor D' the *numerical cycle associated to* D. When $D = 0$ then D' is just the usual numerical cycle Z_{num}.

We can construct D' inductively by the following procedure, which is modelled on the construction of Z_{num} (c.f. [Rei97], Section 4.5). We start with $D_0 = D$ and define D_{i+1} recursively as follows. If $-D_i$ is π_0-nef, then we are done; otherwise there exists some irreducible exceptional curve E such that $D_i \cdot E > 0$. Define $D_{i+1} = D_i + E$ and repeat.

We now show that for each i, we have $D_i \leqslant D'$, hence the above procedure terminates at D'.

Suppose $D_i \leqslant D'$ and let E be any exceptional curve. If the effective divisor $D' - D_i$ is supported away from E, then $D_i E \leqslant D' E \leqslant 0$. Hence $D_{i+1} = D_i + \tilde{E}$ where $\tilde{E}$ is an exceptional curve whose multiplicity in D_i is strictly less than its multiplicity in D'. This shows that $D_{i+1} \leqslant D'$.

The following inequality will be useful for bounding $-D^2$ below.

Lemma 3.2.8. *Let D and D' be as above. Then $h^0(\mathcal{O}_D) \geqslant h^0(\mathcal{O}_{D'})$.*

Proof. It suffices to show that $h^0(\mathcal{O}_{D_i}) \geqslant h^0(\mathcal{O}_{D_{i+1}})$. By construction $D_{i+1} = D_i + E$ for some exceptional curve E with $E \cdot D_i > 0$. Applying χ to the exact sequence $0 \longrightarrow \mathcal{O}_E(-D_i) \longrightarrow \mathcal{O}_{D_{i+1}} \longrightarrow \mathcal{O}_{D_i} \longrightarrow 0$ and using the fact that π_0 is a rational resolution, we obtain

$$h^0(\mathcal{O}_{D_{i+1}}) \quad = \quad \chi(\mathcal{O}_E(-D_i)) + h^0(\mathcal{O}_{D_i}).$$

Since $E \simeq \mathbb{P}^1$ and $E \cdot D_i > 0$, we have

$$h^0(\mathcal{O}_{D_{i+1}}) \quad = \quad 1 - E \cdot D_i + h^0(\mathcal{O}_{D_i}) \leqslant h^0(\mathcal{O}_{D_i}).$$

□

3.2.2 Bounding $-D^2$

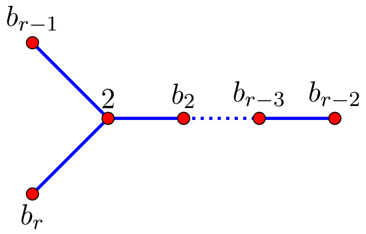
We gather here a few facts about numerical invariants of singularities. Let $\pi_0 : Z_0 \longrightarrow Z$ be a resolution of a rational surface singularity. We denote by $b_i = -E_i^2$ for exceptional curves $E_1, \ldots, E_r$. Recall that its multiplicity can be expressed in terms of the numerical cycle by the formula $m = -Z_{\text{num}}^2$ (c.f. [Rei97], section 4.17). If, in addition, Z has log terminal singularities, then $m = -Z_{\text{num}}^2$ simplifies to

$$m \quad = \quad 2 + \sum_{i=1}^{r} (b_i - 2). \tag{3.2.3}$$

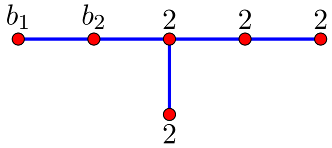
This was observed in [Bri68], proof of Satz 2.11, and can be deduced from the following proposition, which we will also need for the proof of Proposition 3.2.11.

Proposition 3.2.9. *Let $\pi_0 : Z_0 \longrightarrow Z$ be the minimal resolution of a log terminal singularity, with exceptional curves $E_1, \ldots, E_r$ and numerical cycle Z_{num}. If $b_i > 2$, then the multiplicity of E_i in Z_{num} is 1.*

Proof. We refer the reader to Section 2.1.2 for the intersection graphs of the exceptional curves on minimal resolutions of log terminal singularities. Let Γ be such a graph and let $v(i)$ denote the number of edges incident on a vertex i. The first observation is if $v(i) \leqslant b_i$ for all vertices i, then the numerical cycle Z_{num} is reduced. In particular, the proposition holds for such graphs Γ. The graphs Γ for which there exist vertices i, j such that $v(i) < b_i$ and $b_j > 2$ have the following forms

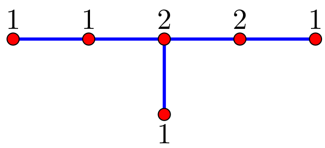
1. 

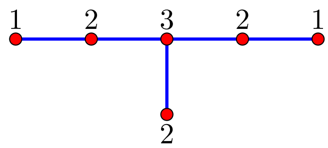
where $b_i \geq 2$ for $i = 1, \ldots, r$.

2. 

where $(b_1, b_2) = (2,3), (3,2)$.

In case 1, let j be the minimal integer such that $b_i = 2$ for all $i < j$. Then the numerical cycle is $2(E_1 + \cdots + E_{j-1}) + E_j + \cdots + E_r$. For case 2, the numerical cycles are, respectively,



and


where the numbers above a vertex indicate its multiplicity in Z_{num}. We have thus shown that for any graph Γ, $b_i > 2$ implies that the multiplicity of E_i in Z_{num} is 1. $\square$

We denote by $\alpha = (\alpha_1, \ldots, \alpha_r)$ the vector whose entries are the discrepancies α_i of the exceptional curves of $\pi_0 : Z_0 \longrightarrow Z$. The vector α can be expressed in terms of the intersection matrix $I = (E_i E_j)$ and the vector $v_I = (E_1^2 + 2, \ldots, E_r^2 + 2)$ as

$$\alpha \quad = \quad -I^{-1} v_I. \tag{3.2.4}$$

Note that these α_i's are different from the discrepancies of the order a_i introduced earlier.

Proposition 3.2.10. *Let $D = \sum_{i=1}^r n_i E_i$ be an effective exceptional divisor on Z_0 and s be the minimal integer such that $D \leqslant sZ_{\text{num}}$. Then*

$$-D^2 \quad \geqslant \quad 2s + \sum_{i=1}^r (b_i - 2) n_i.$$

Proof. The adjunction formula for a divisor on a surface gives

$$-D^2 \;=\; 2h^0(\mathcal{O}_D) + K_{\tilde{Z}}D,$$

and from equation (3.2.4), we compute $K_{\tilde{Z}}D = \sum_{i=1}^{r}(b_i - 2)n_i$. We show by induction that if s is the minimal integer such that $D \leqslant sZ_{\text{num}}$ then $h^0(\mathcal{O}_D) \geqslant s$, which completes the proof. The implication is trivial for $s = 0$, and we suppose that it holds for $s - 1$. By Lemma 3.2.8, we have $h^0(\mathcal{O}_D) \geqslant h^0(\mathcal{O}_{D'})$ where D' is the numerical cycle associated to D. Now $-D'$ is σ-nef, so $D' - Z_{\text{num}}$ is effective. Taking Euler characteristics of the exact sequence, $0 \longrightarrow \mathcal{O}_{D'-Z_{\text{num}}}(-Z_{\text{num}}) \longrightarrow \mathcal{O}_{D'} \longrightarrow \mathcal{O}_{Z_{\text{num}}} \longrightarrow 0$, we obtain $h^0(\mathcal{O}_{D'}) = \chi(\mathcal{O}_{D'-Z_{\text{num}}}(-Z_{\text{num}})) + h^0(\mathcal{O}_{Z_{\text{num}}})$. Since $-Z_{\text{num}}$ is σ-nef and $h^0(\mathcal{O}_{Z_{\text{num}}}) \geqslant 1$ we have

$$h^0(\mathcal{O}_{D'}) \;\geqslant\; h^0(\mathcal{O}_{D'-Z_{\text{num}}}) + 1.$$

Now s is the minimal integer such that $D' \leqslant sZ_{\text{num}}$, so $s - 1$ is the minimal integer such that $D' - Z_{\text{num}} \leqslant (s-1)Z_{\text{num}}$. By the induction hypothesis, we conclude that $h^0(\mathcal{O}_D) \geqslant h^0(\mathcal{O}_{D'}) \geqslant s - 1 + 1 = s$. ☐

3.2.3 Proof of theorem 3.2.5

Our strategy is to decompose D as the sum of two effective divisors $D = D_1 + D_2$ with $D_1D_2 \leqslant 0$. Then we have $g(D) \geqslant g(D_1) + g(D_2) - 2D_1D_2 \geqslant g(D_1) + g(D_2)$ and the problem is reduced to showing $g(D_i) \geqslant 0$ for $i = 1, 2$.

Let $D = \sum_{j\in I} n_jE_j$ with $n_j > 0$ where $I \subseteq [1, r]$, and we assume $\operatorname{supp} D$ is connected. Then $\operatorname{supp} D$ contracts to a log terminal singularity, and we define the multiplicity $m(D)$ of D to be the multiplicity of the contracted singularity. The number $m(D)$ can be computed by modifying equation (3.2.3) appropriately,

$$m(D) \;=\; 2 + \sum_{j\in I}(b_j - 2), \tag{3.2.5}$$

and by definition of D_{num} we have $m(D) = -D_{\text{num}}^2$. Since π_0 is a minimal resolution, we have $m(D) \geqslant 2$. If $m(D) = 2$, then let $D_1 = D$ and $D_2 = 0$. If $m(D) > 2$, then let n be the positive integer $n = \min\{n_i \mid -E_i^2 > 2\}$ and define

$$D_1 \;=\; \gcd(nD_{\text{num}}, D) \quad \text{and} \quad D_2 \;=\; D - D_1. \tag{3.2.6}$$

Proposition 3.2.11. *Let D be a connected effective exceptional divisor on Z_0 and $D = D_1 + D_2$ be the decomposition above. Let $g : \boldsymbol{E}_0 \longrightarrow \mathbb{Q}$ be a function satisfying the hypotheses of Theorem 3.2.5. Then*

1. *$D_1 D_2 \leqslant 0$*

2. *If $g(D_{\text{num}}) > 0$, then $g(D_1) > 0$*

3. *for each connected component C of D_2, we have $m(C) < m(D)$.*

Proof. First note that the effective divisors $nD_{\text{num}} - D_1$ and D_2 have no common components, hence $nD_{\text{num}}D_2 \geqslant D_1D_2$. Since D_2 is supported on $\bigcup_{i\in I} E_i$ and $-D_{\text{num}}E_i \geqslant 0$ for any $i \in I$, we have $D_{\text{num}}D_2 \leqslant 0$. This proves part 1 of the proposition.

Note that $D_1 \leqslant nD_{\text{num}}$, and we now show that n is the minimal integer with this property. By definition of n, there exists an irreducible component E_s of D of multiplicity n and $-E_s^2 > 2$. By Proposition 3.2.9, the multiplicity of E_s in D_{num} is 1. Hence the multiplicities of E_s in nD_{num} and D_1 are equal, so we can conclude that n is the minimal integer such that $D_1 \leqslant nD_{\text{num}}$. Moreover, the multiplicity of E_j in D_1 is equal to n whenever $-E_j^2 > 2$. Now we can apply Proposition 3.2.10 to obtain the inequality

$$-D_1^2 \ \geqslant \ 2n + \sum_{j\in I}(b_j - 2)n$$

and by (3.2.3) the last expression is equal to $nm(D_1)$. Since D is connected, the same is true for D_1, so $m(D_1) = -D_{\text{num}}^2$. Therefore

$$g(D_1) \ \geqslant \ nm(D_1) + \ell(nD_{\text{num}}) \geqslant n(-D_{\text{num}}^2 + \ell(D_{\text{num}})) = ng(D_{\text{num}}) > 0. \quad (3.2.7)$$

This proves part 2 of the proposition.

Since the multiplicities of E_s in D_1 and D are equal, the effective divisor D_2 is supported away from E_s. In particular, any connected component C of D_2 is supported away from E_s. Since $-E_s^2 > 2$, we see from (3.2.5) that $m(C)$ must be strictly less than $m(D)$. This proves part 3 of the proposition. □

The above proposition is enough to show that log terminal orders whose centres have canonical singularities are numerically rational.

Corollary 3.2.12. *Let $\pi_0 : Z_0 \longrightarrow Z$ be a minimal resolution of a canonical surface singularity and $g : \boldsymbol{E}_0 \longrightarrow \mathbb{Q}$ be a function satisfying the hypotheses of Theorem 3.2.5. Then $g(E) > 0$ for all $E \in \boldsymbol{E}_0^+ \backslash \{0\}$ if and only if $g(Z_{\text{num}}) > 0$.*

Proof. Let $D \in \boldsymbol{E}_0^+ \backslash \{0\}$, then the decomposition $D = D_1 + D_2$ defined in (3.2.6) is trivial in that $D = D_1$. By the proof of part 2 of the above proposition (in particular the inequality (3.2.7)) we have $g(D_1) \geqslant ng(D_{\text{num}})$ where n is a positive integer. Now $D_{\text{num}} \leqslant Z_{\text{num}}$ so $g(D_{\text{num}}) \geqslant g(Z_{\text{num}}) > 0$ which proves the proposition. □

Proof of Theorem 3.2.5. We assume as in the hypothesis of Theorem 3.2.5 that g is positive on effective exceptional divisors D satisfying $D = D_{\text{num}}$. We wish to show that for all $D \in \boldsymbol{E}_0^+ \backslash \{0\}$, $g(D) > 0$. Clearly we can assume D is connected. Suppose that $m(D) = 2$, then since $g(D_{\text{num}}) > 0$ by hypothesis, we can conclude from part 2 of Proposition 3.2.11 that $g(D) > 0$. Otherwise, let $D = D_1 + D_2$ be the decomposition from Proposition 3.2.11, we can conclude from parts 1 and 2 of the same proposition that $g(D) \geqslant g(D_1) + g(D_2)$, and $g(D_1) > 0$. It remains to show that $g(D_2) > 0$. To this end, we repeat the above argument on each connected component of D_2. This procedure terminates since by part 3 of Proposition 3.2.11, the connected components of D_2 have multiplicities strictly less than that of D. The theorem then follows from corollary 3.2.12. □

3.3 Adjunction formula for orders

The adjunction formula for a divisor D on a smooth surface Z expresses the Euler characteristic of $\mathcal{O}_D$ in terms of intersection numbers involving D and K_Z, (c.f. 4.11, [Rei97])

$$\chi(\mathcal{O}_D) \quad = \quad -\frac{1}{2}(K_Z + D)D. \tag{3.3.1}$$

Note that for the intersection product above to be well-defined, we require $\operatorname{supp} D$ to be a projective variety, and we keep this assumption below. The aim of this section is to derive a similar adjunction formula for a terminal order A on Z, which expresses the Euler characteristic of A restricted to some divisor D in terms of intersection numbers involving D and K_A. As mentioned in the introduction, the Theorem 3.1.2 appears in the unpublished work of M. Artin and A. J. de Jong. We reproduce the statement of the theorem for the reader's convenience. Let A be a terminal order on Z of rank r^2 and D be an effective divisor whose support is projective. Then

$$\chi(A \otimes_Z \mathcal{O}_D) \quad = \quad -\frac{r^2}{2}(K_A + D)D \tag{3.3.2}$$

where $K_A = K_Z + \Delta_A$ is the canonical divisor of A.

Note that (3.3.1) appears as a special case of (3.3.2) (where $A = \mathcal{O}_Z$), so we feel justified in calling (3.3.2) an adjunction formula. As we have already seen, our motivation for understanding $\chi(A \otimes \mathcal{O}_D)$ is to study the notion of numerical rationality. In that context, the divisor D is exceptional with respect to some birational morphism, hence its support is projective. The rest of this section will be devoted to the proof of Theorem 3.1.2.

3.3.1 Setup

Let A be a terminal order of rank r^2 on a surface Z and C be an irreducible curve in Z. Recall that $Z(A_C/J(A_C))$ is a product of field extensions of $k(C)$ which defines a union of cyclic covers of curves $\pi_C : \tilde{C} \longrightarrow C$. The degree of π_C is the ramification index e_C of A at C. Terminal orders can be characterised using ramification data; an order A is terminal if the ramification divisor $D = \bigcup D_i$ is a normal crossing divisor on a smooth surface Z and the cyclic covers π_{D_i}, π_{D_j} ramify only at nodes $p \in D_i \cap D_j$ with $e_i | e_j$ and π_{D_i} totally ramified at p.

We first compute $\chi(A \otimes \mathcal{O}_C)$ by filtering the sheaf A as follows. Let J_C be the Jacobson radical of $A \otimes k(C)$ and we define J by the following exact sequence

$$0 \longrightarrow J \longrightarrow A \longrightarrow A \otimes k(C)/J_C \longrightarrow 0. \tag{3.3.3}$$

For a positive integer i, we define J^i to be the image of the multiplication map

$$\underbrace{J \otimes_A J \otimes_A \cdots \otimes_A J}_{i \text{ times}} \quad \longrightarrow \quad A.$$

Then $J^e = A \otimes_Z \mathcal{O}_Z(-C) = A(-C)$ where $e = e_C$. This allows us to define $J^{-i} = J^{me-i} \otimes_Z \mathcal{O}_Z(mC)$ where m is an integer such that $me - i \geqslant 0$. We have a filtration $J^e = A(-C) \subset J^{e-1} \subset \cdots \subset J \subset A$, from which we obtain the exact sequences

$$0 \longrightarrow J^{i-1}/J^i \longrightarrow A/J^i \longrightarrow A/J^{i-1} \longrightarrow 0$$

for $i = 1, \ldots, e$. Hence

$$\chi(A \otimes \mathcal{O}_C) \quad = \quad \chi(J^{e-1}/J^e) + \cdots + \chi(J/J^2) + \chi(A/J). \tag{3.3.4}$$

We compute $\chi(A/J)$ below, and to do this, we need to determine the local structure of A/J.

3.3.2 Local structure of A/J

We can use the étale local structure of A to determine the étale local structure of A/J. Let r^2 denote the rank of A as an $\mathcal{O}_Z$-module and we assume that A is ramified at C with index e. Since A is terminal, any other ramification curve D intersects C transversely at a finite number of points and the ramification indices satisfy $e_D|e$ or $e|e_D$.

1. First suppose $p \in C$ is a nonsingular point of the ramification divisor. Let $u \in \mathfrak{m}_C$ be a uniformising parameter for $\mathcal{O}_{Z,C}$ and we denote $\mathcal{O} = \mathcal{O}_p^{sh}$. Then

$$A_p^{sh} = M^{r/e \times r/e} \text{ where } M = \begin{pmatrix} \mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ u\mathcal{O} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{O} \\ u\mathcal{O} & \dots & u\mathcal{O} & \mathcal{O} \end{pmatrix} \subset \mathcal{O}^{e \times e}$$

$$J_p^{sh} = N^{r/e \times r/e} \text{ where } N = \begin{pmatrix} u\mathcal{O} & \mathcal{O} & \dots & \mathcal{O} \\ u\mathcal{O} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathcal{O} \\ u\mathcal{O} & \dots & u\mathcal{O} & u\mathcal{O} \end{pmatrix} \subset \mathcal{O}^{e \times e},$$

so

$$(A/J)_p^{sh} = \left((\mathcal{O}/u\mathcal{O})^{r/e \times r/e}\right)^e.$$

Moreover J_p^{sh} is generated, as a left (or right) A_p^{sh}-module by the regular normal element

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 & 1 \\ u & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

2. Now suppose $p \in C \cap D$ where D is a ramification curve with ramification index e_D. Let $v \in \mathfrak{m}_D$ be a uniformising parameter for $\mathcal{O}_{Z,D}$. Denote by $S = \mathcal{O}_{Z,p}^{sh} \langle x, y \rangle / (x^e -$

$u, y^e - v, xy - \zeta_e yx)$ where ζ_e is a primitive e-th root of unity. If $e|e_D$, then

$$A_p^{sh} = M^{r/e_D \times r/e_D} \text{ where } M = \begin{pmatrix} S & S & \dots & S \\ yS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & S \\ yS & \dots & yS & S \end{pmatrix} \subset S^{e_D/e \times e_D/e}$$

$$J_p^{sh} = N^{r/e_D \times r/e_D} \text{ where } N = \begin{pmatrix} xS & xS & \dots & xS \\ xyS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & xS \\ xyS & \dots & xyS & xS \end{pmatrix} \subset S^{e_D/e \times e_D/e}$$

Let $\overline{S} = S/xS \simeq k\{v\}[y]/(y^e - v)$, where $k\{v\}$ denotes the strict henselisation of $k[v]$ at the origin. Then

$$(A/J)_{\mathfrak{p}}^{sh} = P^{r/e_D \times r/e_D} \text{ where } P = \begin{pmatrix} \bar{S} & \bar{S} & \dots & \bar{S} \\ y\bar{S} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \bar{S} \\ y\bar{S} & \dots & y\bar{S} & \bar{S} \end{pmatrix} \subset \overline{S}^{e_D/e \times e_D/e}.$$

The generator for J_p^{sh} in this case is just $x1_{A_p^{sh}}$.

3. In the case where $e_D|e$, we denote by

$$S = \frac{\mathcal{O}_{Z,p}^{sh}\langle x, y\rangle}{(x^{e_D} - u, y^{e_D} - v, xy - \zeta_{e_D} yx)}.$$

Then

$$A_p^{sh} = M^{r/e \times r/e} \text{ where } M = \begin{pmatrix} S & S & \dots & S \\ xS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & S \\ xS & \dots & xS & S \end{pmatrix} \subset S^{e/e_D \times e/e_D}$$

$$J_p^{sh} = N^{r/e \times r/e} \text{ where } N = \begin{pmatrix} xS & S & \dots & S \\ xS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & S \\ xS & \dots & xS & xS \end{pmatrix} \subset S^{e/e_D \times e/e_D}$$

so

$$(A/J)_p^{sh} = ((S/xS)^{e/e_D})^{r/e \times r/e}.$$

Again J_p^{sh} is generated by a regular normal element

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 0 & 1 \\ x & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Note that in each case above, the ideal J is generated locally by a regular normal element.

Lemma 3.3.1. *Let $\pi : \tilde{C} \longrightarrow C$ be the cover of C determined by the ramification data. Then*

$$\chi(\mathcal{O}_{\tilde{C}}) = e\chi(\mathcal{O}_C) - \frac{e}{2} \sum_{D \in Z^1 \setminus \{C\}} \left(1 - \frac{1}{\min\{e, e_D\}}\right).$$

Proof. The cover π has degree $e = e_C$, so by the Riemann-Hurwitz formula, we have

$$\chi(\mathcal{O}_{\tilde{C}}) = e\chi(\mathcal{O}_C) - \frac{1}{2} \sum_{p \in \tilde{C}} (e_p - 1).$$

If $\pi(p)$ is a nonsingular point of the ramification divisor, then $e_p = 1$. Now suppose $\pi(p) \in C \cap D$ where A is ramified on D with ramification index e_D. If $e_D \geqslant e$, then π is totally ramified at $\pi(p)$, hence $e_p = e$. If $1 < e_D < e$, then in the fibre $\pi^{-1}(\pi(p))$ there are e/e_D points each with ramification index $e_D - 1$. A simple calculation then yields the above formula. □

Lemma 3.3.2. *The sheaf A/J considered as a sheaf on C is a $\pi_*\mathcal{O}_{\tilde{C}}$-module.*

Proof. Firstly, $A/J \otimes k(C)$ is isomorphic to $M_n(k(\tilde{C}))$. So it suffices to show that $(A/J)_{\mathfrak{p}}\mathcal{O}_{\tilde{C},\mathfrak{p}} \subseteq (A/J)_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \in \operatorname{Spec} C$ where we identify everything with their natural images in $A/J \otimes k(C)$. From the étale local structures for A/J above, we can see that $(A/J)_{\mathfrak{p}}^{sh}\mathcal{O}_{\tilde{C},\mathfrak{p}}^{sh} \subseteq (A/J)_{\mathfrak{p}}^{sh}$ for all $\mathfrak{p} \in \operatorname{Spec} C$. Intersecting with $A/J \otimes k(C)$ gives the desired result. □

Proposition 3.3.3. *Let A be a terminal $\mathcal{O}_Z$-order of rank r^2, C be an irreducible curve on Z, e be the ramification index of A at C and J be as defined in (3.3.3). Then*

$$\chi(A/J) = \frac{r^2}{2e}\left(2\chi(\mathcal{O}_C) - C \cdot \Delta_A + \left(1 - \frac{1}{e}\right) C^2\right).$$

Proof. The previous lemma shows that we can consider A/J as a sheaf on $\tilde{C}$. In fact, we can see from the local structure of A/J that it is an order on $\tilde{C}$ in the semi-simple algebra $M_{r/e}(k(\tilde{C}))$ (semi-simple since $k(\tilde{C})$ is a product of fields). Hence we can embed A/J in a maximal order Ω. This gives an exact sequence of $\mathcal{O}_C$-modules

$$0 \longrightarrow A/J \longrightarrow \pi_*\Omega \longrightarrow Q \longrightarrow 0$$

where Q is a torsion sheaf supported on points where A/J is not a maximal order on the corresponding fibre. Since the Brauer group of a curve is trivial, Ω is a maximal order in a matrix algebra, hence is trivial Azumaya. This gives $\chi(\Omega) = (r/e)^2\chi(\mathcal{O}_{\tilde{C}})$ which is equal to $\chi(\pi_*\Omega)$ since π is a finite morphism. The sheaf Q is supported on points, so $\chi(Q)$ is the sum of the lengths of Q_p over $p \in C$. Referring again to the local structure of A/J, we see that A/J is nonmaximal at p if and only if p is a point of intersection of C and a ramification curve D where $e|e_D$. A simple computation shows that

$$Q_p \quad = \quad P^{r/e_D \times r/e_D} \text{ where } P = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ k & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ k & \cdots & k & 0 \end{pmatrix} \subset k^{e_D/e \times e_D/e}$$

hence

$$\dim(Q_p) \quad = \quad \frac{r^2}{e_D^2}\sum_{j=1}^{e_D/e-1} j = \frac{r^2}{2e}\left(\frac{1}{e} - \frac{1}{e_D}\right).$$

So

$$\chi(Q) \quad = \quad \sum_{D\in Z^1\setminus\{C\}} \frac{r^2}{2e}\left(\frac{1}{e} - \frac{1}{\max\{e, e_D\}}\right) C \cdot D.$$

Combining the expressions for $\chi(\Omega)$ and $\chi(Q)$, we obtain

$$\begin{aligned}
\chi(A/J) \quad &= \quad \frac{r^2}{e}\left(\chi(\mathcal{O}_C) - \frac{1}{2}\sum_{D\in Z^1\setminus\{C\}}\left(1 - \frac{1}{\min\{e, e_D\}} + \frac{1}{e} - \frac{1}{\max\{e, e_D\}}\right) C \cdot D\right) \\
&= \quad \frac{r^2}{e}\left(\chi(\mathcal{O}_C) - \frac{1}{2}\sum_{D\in Z^1\setminus\{C\}}\left(1 - \frac{1}{e_D}\right) C \cdot D\right)
\end{aligned}$$

which proves the proposition. □

To finish the computation of $\chi(A \otimes_Z \mathcal{O}_C)$ we need the values of $\chi(J^i/J^{i+1})$. Let B be an $\mathcal{O}_C$-algebra. We say that L is an invertible $(B - B)$-bimodule if there exists a $(B - B)$-bimodule L' such that $L \otimes_B L' \simeq B$ as $(B - B)$-bimodules.

Lemma 3.3.4. *Let C be a projective curve, B be an $\mathcal{O}_C$-algebra which is torsion-free as an $\mathcal{O}_C$-module, and L, L' be invertible $(B-B)$-bimodules. Then*

$$\chi(L \otimes_B L') = \chi(L) + \chi(L') - \chi(B).$$

Proof. First we assume that L is generated as an $\mathcal{O}_C$-module by its sections. Suppose $p \in C$ is a closed point and $s_p \in L_p$ is a regular element, that is $B_p \longrightarrow L_p$ given by $b \longmapsto s_p b$ is injective. Then since L is generated by sections, we can lift this to a section $s \in H^0(C, L)$. Now B is torsion-free as an $\mathcal{O}_C$-module, so the map $B \longrightarrow L$ given by s is also injective. This gives an exact sequence $0 \longrightarrow B \longrightarrow L \longrightarrow Q \longrightarrow 0$ of right B-modules. Since $- \otimes_B L'$ induces an equivalence of categories, it is exact, so we have the exact sequence

$$0 \longrightarrow L' \longrightarrow L \otimes_B L' \longrightarrow Q \otimes_B L' \longrightarrow 0.$$

Note that since L' is invertible, its rank as an $\mathcal{O}_C$-module is the same as the $\mathcal{O}_C$-rank of B. Thus the sheaf Q is supported on points, and $\chi(Q \otimes_B L') = \chi(Q)$. This gives

$$\chi(L \otimes_B L') = \chi(L') + \chi(Q) = \chi(L') + \chi(L) - \chi(B).$$

In general, let $\mathcal{O}_C(1)$ be a very ample line bundle on C so that $\mathcal{O}_C(n) \otimes_C L$ is generated by sections for some $n \gg 0$. We denote by r the rank of L as an $\mathcal{O}_C$-module, and note that $L \otimes_B L'$ has the same rank. Then

$$\begin{aligned} \chi(\mathcal{O}_C(n) \otimes_C L \otimes_B L') &= \chi(\mathcal{O}_C(n) \otimes_C L) + \chi(L') - \chi(B) \\ &= rn + \chi(L) + \chi(L') - \chi(B). \end{aligned}$$

But $\chi(\mathcal{O}_C(n) \otimes_C L \otimes_B L') = rn + \chi(L \otimes_B L')$ so we are done. □

The following lemma follows from the local structure of A/J.

Lemma 3.3.5. *Let $\mathcal{L}_i = J^i/J^{i+1}$, then $\mathcal{L}_i \otimes_{A/J} \mathcal{L}_j \simeq \mathcal{L}_{i+j}$ as $(A/J - A/J)$-bimodules. In particular, $\mathcal{L}_i$ is an invertible $(A/J - A/J)$-bimodule and is locally generated by a regular normal element.*

Proof. Since J is locally generated by a regular normal element, so is J^i for any integer i. This shows that $\mathrm{Tor}_1^A(A/J^k, J^i) = 0$, hence $J^i \otimes_A J^k = J^{i+k}$ for any integers i, k. We can write $\mathcal{L}_i = A/J \otimes_A J^i$, so we have the following isomorphisms

$$\begin{aligned}
\mathcal{L}_i \otimes_{A/J} \mathcal{L}_k &= A/J \otimes_A J^i \otimes_{A/J} A/J \otimes_A J^k \\
&\simeq A/J \otimes_A J^i \otimes_A J^k \\
&\simeq A/J \otimes_A J^{i+k} \\
&= \mathcal{L}_{i+k}.
\end{aligned}$$

of $(A/J - A/J)$-bimodules. It follows then $\mathcal{L}_i \otimes_{A/J} \mathcal{L}_{-i} = A/J$, hence $\mathcal{L}_i$ is invertible. Finally, since J^i is locally generated by a regular normal element, so is $\mathcal{L}_i$. □

Corollary 3.3.6. *Let A be a terminal $\mathcal{O}_Z$-order of rank r^2 and C be an irreducible curve on C. Then*

$$\chi(A \otimes_Z \mathcal{O}_C) = \frac{r^2}{2}\left(2\chi(\mathcal{O}_C) - C \cdot \Delta_A\right).$$

Proof. By Lemma 3.3.5, we have $(J/J^2)^{\otimes e} \simeq J^e/J^{e+1} \simeq A/J \otimes_Z \mathcal{O}_Z(-C)$. By Lemma 3.3.4, we have $\chi(A/J \otimes_Z \mathcal{O}_Z(-C)) = \chi((J/J^2)^{\otimes e}) = e\chi(J/J^2) - (e-1)\chi(A/J)$. Note that the rank of A/J as an $\mathcal{O}_C$-module is r^2/e, so $\chi(A/J \otimes_Z \mathcal{O}_Z(-C)) = \chi(A/J) - r^2C^2/e$. Putting these together, we get

$$\chi(J/J^2) = -\frac{r^2C^2}{e^2} + \chi(A/J).$$

Recall (3.3.4) from the beginning of this section,

$$\begin{aligned}
\chi(A \otimes_Z \mathcal{O}_C) &= \chi(J^{e-1}/J^e) + \chi(J^{e-2}/J^{e-1}) + \cdots + \chi(J/J^2) + \chi(A/J) \\
&= \chi(A/J) + \sum_{i=1}^{e-1}\left(i\chi(J/J^2) - (i-1)\chi(A/J)\right) \\
&= \chi(A/J) + \frac{e(e-1)}{2}\left(-\frac{r^2C^2}{e^2} + \chi(A/J)\right) - \frac{(e-1)(e-2)}{2}\chi(A/J) \\
&= -\frac{e-1}{2e}r^2C^2 + \left(1 + \frac{e(e-1)}{2} - \frac{(e-1)(e-2)}{2}\right)\chi(A/J) \\
&= -\frac{e-1}{2e}r^2C^2 + e\chi(A/J) \\
&= \frac{r^2}{2}\left(-(1-\frac{1}{e})C^2 + \left(2\chi(\mathcal{O}_C) - C \cdot \Delta_A + \left(1 - \frac{1}{e}\right)C^2\right)\right) \\
&= \frac{r^2}{2}\left(2\chi(\mathcal{O}_C) - C \cdot \Delta_A\right)
\end{aligned}$$

□

To prove Theorem 3.1.2 for a general effective divisor E, we first need the following lemma.

Lemma 3.3.7. *Let $E = n_1E_1 + \ldots + n_sE_s$ be an effective divisor on Z and V be a rank r vector bundle on Z. Then*

$$\chi(V \otimes_Z \mathcal{O}_E) = -\frac{rE^2}{2} + \sum_{i=1}^{s} n_i \left(\frac{rE_i^2}{2} + \chi(V \otimes_Z \mathcal{O}_{E_i}) \right). \tag{3.3.5}$$

Proof. We prove (3.3.5) by induction. For irreducible E, there is nothing to prove. Suppose (3.3.5) holds for E, we show that it holds too for $E+E_j$. Using the following exact sequence

$$0 \longrightarrow V \otimes_Z \mathcal{O}_{E_j}(-E) \longrightarrow V \otimes_Z \mathcal{O}_{E+E_j} \longrightarrow V \otimes_Z \mathcal{O}_E \longrightarrow 0$$

we obtain

$$\chi(V \otimes_Z \mathcal{O}_{E+E_j}) = \chi(V \otimes_Z \mathcal{O}_{E_j}(-E)) - \frac{rE^2}{2} + \sum_{i=1}^{s} n_i \left(\frac{rE_i^2}{2} + \chi(V \otimes_Z \mathcal{O}_{E_i}) \right).$$

Since $\chi(V \otimes_Z \mathcal{O}_{E_j}(-E)) = \chi(V \otimes_Z \mathcal{O}_{E_j}) - rE_j \cdot E$, we get

$$\chi(V \otimes_Z \mathcal{O}_{E+E_j}) = \chi(V \otimes_Z \mathcal{O}_{E_j}) + \frac{rE_j^2}{2} - \frac{r(E+E_j)^2}{2} + \sum_{i=1}^{s} n_i \left(\frac{rE_i^2}{2} + \chi(V \otimes_Z \mathcal{O}_{E_i}) \right)$$

which shows that (3.3.5) holds for $E + E_j$. □

of Theorem 3.1.2. The theorem follows from Corollary 3.3.6, Lemma 3.3.7, and the following computation

$$\begin{aligned}
\chi(A \otimes_Z \mathcal{O}_E) &= -\frac{r^2E^2}{2} + \sum_{i=1}^{s} n_i \left(\frac{r^2E_i^2}{2} + \frac{r^2}{2} \left(2\chi(\mathcal{O}_{E_i}) - E_i \cdot \Delta_A\right) \right) \\
&= \frac{r^2}{2} \left(-E^2 + \sum_{i=1}^{s} n_i \left(E_i^2 - (K_A + E_i)E_i\right) \right) \\
&= -\frac{r^2}{2}(K_A + E)E.
\end{aligned}$$

□

3.4 Concluding Remarks

We have shown that the notion of numerical rationality includes many interesting examples of orders which arise naturally in the context of noncommutative birational geometry. Our definition is natural in that it does not depend on the choice of resolution, nor does it depend on the choice of representative in a Morita equivalence class (if the centre has rational singularities). Moreover, the adjunction formula for orders makes it easy to check whether an order is numerically rational.

Chapter 4

Orders over discrete valuation rings

In this chapter, we investigate the question of finding maximal orders containing a given order in a central simple algebra. The motivation is the construction of blowing up orders. Recall that to blowup an $\mathcal{O}_Z$-order A along a birational morphism $\sigma : \tilde{Z} \longrightarrow Z$, we find maximal orders Ω_i containing $(\sigma^* A)_{E_i}$ for each i where $\mathrm{Ex}(\sigma) = \bigcup_i E_i$ is the irreducible decomposition of $\mathrm{Ex}(\sigma)$. We develop a method to simplify the calculations involved in finding all such maximal orders Ω_i.

Let R be a discrete valuation ring with K its field of fractions and Λ be an R-order in the central simple K-algebra Λ_K. By [CR81], Theorem 26.21 there is a bijective correspondence between the maximal R-orders in Λ_K and maximal $\hat{R}$-orders in $\hat{\Lambda}_K$, so we will assume that R is complete local. Orders over R will generally be denoted by capital Greek letters $\Lambda, \Gamma, \ldots$ and we will only consider left modules unless otherwise stated. Our approach to finding maximal orders containing Λ involves determining all left Λ-modules which are R-free of a certain rank satisfying some closure condition. The representation theory of Λ is captured by data involving a pair of artin rings. We then express the closure condition in terms of these artin rings to obtain a computationally friendly method to find maximal orders containing Λ.

4.1 Review of orders over complete discrete valuation rings

We review the classical theory of orders over complete discrete valuation rings. An R *-order* Λ is an R-subalgebra of a central simple K-algebra B such that Λ is a finite R-module and $\Lambda_K := \Lambda \otimes_R K$. An order is *maximal* if it is maximal with respect to inclusions in Λ_K, and it is *hereditary* if every left ideal (or, equivalently, right ideal) is Λ-projective. An order Λ is *Gorenstein* if its dual $\mathrm{Hom}_R(\Lambda, R)$ is a projective right Λ-module, and it is *Bass* if every order (including Λ itself) containing Λ is Gorenstein. Note that maximal orders are hereditary, hereditary orders are Bass, and of course Bass orders are Gorenstein. Finally, we say that Ω is a *minimal order* containing Λ if $\Omega \neq \Lambda$ and for any chain of orders $\Omega \supseteq \Gamma \supseteq \Lambda$ in Λ_K we have $\Gamma = \Omega$.

An important invariant associated to an R-order Λ is its *discriminant ideal*, denoted $d(\Lambda)$. It is defined as follows: let $\{e_i\}$ be a free R-basis for Λ and consider the determinant d of the matrix $(\mathrm{tr}(e_i e_j))_{ij}$ where tr is the reduced trace map $\mathrm{tr} : \Lambda \longrightarrow R$ (c.f. [Rei03], equation (9.6a)). The element d of R is always a square, and the discriminant ideal $d(\Lambda)$ is defined as the ideal of R generated by $\sqrt{d}$. The discriminant ideal is a rough measure of how far the order Λ differs from a maximal order. The following basic result will be useful in Chapters 5 and 6: given orders $\Lambda \subseteq \Lambda'$ we have $d(\Lambda) \subseteq d(\Lambda')$ with $\Lambda = \Lambda'$ if and only if $d(\Lambda) = d(\Lambda')$ (c.f. [CR81], Proposition 26.3, iii). More generally, if $\Lambda \subseteq \Lambda'$ and Λ'/Λ is a Λ-module of length m, then $d(\Lambda) = u^m d(\Lambda')$ where u is any uniformising parameter of R.

A Λ-module is called a Λ *-lattice* or a *Cohen-Macaulay* Λ-module, if it is torsion-free as an R-module. We will always work with left modules and left lattices unless stated otherwise. The Λ-lattices form an additive category, which we denote by $CM(\Lambda)$. Since R is complete, the category $CM(\Lambda)$ is Krull-Schmidt. Denote by $\mathrm{Ind}(\Lambda)$ the set of isomorphism classes of indecomposable objects in $\mathrm{CM}(\Lambda)$. We say that Λ has *finite representation type* if $\mathrm{Ind}(\Lambda)$ is a finite set. Let L be a Λ-lattice. Define the *left order $O_l(L)$ associated to L* to be the order in Λ_K consisting of elements $\{x \in \Lambda_K \mid xL \subseteq L\}$. If L is a right Λ-module, then the *right order* $O_r(L)$ is defined similarly. It is clear that Λ is contained in $O_l(L)$ and $O_l(L)$ depends only on the isomorphism class of L. If $L \simeq L' \oplus L''$ as Λ-lattices, then by definition $O_l(L) = O_l(L') \cap O_l(L'')$. We can naturally identify $O_l(L)$ with an endomorphism ring as follows.

Lemma 4.1.1. *Let L be a Λ-lattice. Then $O_l(L) \simeq \mathrm{End}_{\mathrm{End}_\Lambda(L)}(L)$.*

Proof. Let $B = \Lambda_K$, then the map $\mu : B \longrightarrow \mathrm{End}_K(L \otimes_R K)$ which sends λ to left multiplication by λ embeds B as a subalgebra of the matrix algebra $M = \mathrm{End}_K(L \otimes_R K)$. The endomorphism ring $E = \mathrm{End}_B(L \otimes_R K)$ is naturally a subalgebra of M, and is equal to the centraliser $C_M(B)$ of B. Similarly, $\mathrm{End}_E(L \otimes K)$ when considered as a subalgebra of M is equal to $C_M(C_M(B))$. Since B is simple, by the double centraliser theorem we have $C_M(C_M(B)) = B$. Hence the map μ induces an isomorphism $\Lambda_K \simeq C_M(C_M(B)) = \mathrm{End}_E(L \otimes K)$. Therefore $\mathrm{End}_{\mathrm{End}_\Lambda(L)}(L)$ can be considered as a subalgebra of Λ_K consisting of elements λ such that $\lambda L \subseteq L$. $\square$

We say that an order Λ is *local* (some authors say primary) if $\Lambda/J(\Lambda)$ is a division ring, and *quasi-local* if $\Lambda/J(\Lambda)$ is a simple artin ring. An order is maximal if and only if it is hereditary and quasi-local (c.f. [AG60], Proposition 2.3). Maximal orders in the same central simple algebra are conjugates, that is, if Λ and Λ' are maximal orders in Λ_K, then there is an invertible element $a \in \Lambda_K$ such that $\Lambda = a\Lambda' a^{-1}$ (c.f. [AG60], Proposition 3.5). If Λ is hereditary and $\Lambda/J(\Lambda)$ is a local ring, then Λ is the unique maximal order in Λ_K, and Λ_K is a division algebra. Conversely, if D is a division algebra, then there is a unique maximal order in D. These results characterise maximal orders completely: if $\Lambda_K \simeq M_n(D)$ for some division K-algebra D and Δ is the unique maximal order in D, then any maximal order in Λ_K is conjugate to $M_n(\Delta)$.

There is a similar structure theorem for hereditary orders (c.f. [Rei03], Theorem 39.14). An hereditary order Λ in $M_n(D)$ is conjugate to

$$\Lambda \quad = \quad \begin{pmatrix} \Delta & \Delta & \cdots & \Delta \\ J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Delta \\ J & \cdots & J & \Delta \end{pmatrix}^{[m_1,\ldots,m_p]} \tag{4.1.1}$$

where $J = J(\Delta)$, $m_1 + \cdots + m_p = n$, and the superscript $[m_1, \ldots, m_p]$ indicates that we replace the (i,j)-th entry Λ_{ij} by $\Lambda_{ij}^{m_i \times m_j}$. For example,

$$\begin{pmatrix} \Delta & \Delta & \Delta \\ J & \Delta & \Delta \\ J & J & \Delta \end{pmatrix}^{[1,2,3]} = \left(\begin{array}{c|cc|ccc} \Delta & \Delta & \Delta & \Delta & \Delta & \Delta \\ \hline J & \Delta & \Delta & \Delta & \Delta & \Delta \\ J & \Delta & \Delta & \Delta & \Delta & \Delta \\ \hline J & J & J & \Delta & \Delta & \Delta \\ J & J & J & \Delta & \Delta & \Delta \\ J & J & J & \Delta & \Delta & \Delta \end{array}\right).$$

An hereditary order Λ is called *basic* if $m_1 = \cdots = m_n = 1$. In this case, Λ has n maximal ideals, then the semisimple algebra $\Lambda/J(\Lambda)$ decomposes into n simple components. By [Har63], Theorem 3.3, there are exactly s maximal orders and s minimal orders containing Λ. The following proposition is well-known, and is an easy consequence of the structure theorem for hereditary orders.

Proposition 4.1.2. *Let Λ be an hereditary order and suppose $\Lambda/J(\Lambda)$ has s simple components. Then Λ has exactly s isomorphism classes of projective indecomposables, $P_1, \ldots, P_s$, and the maximal orders containing Λ are of the form $O_l(P_1), \ldots, O_l(P_s)$.*

Proof. Since Λ is a finite module over a complete local ring, there is a one-to-one correspondence between the simple components of $\Lambda/J(\Lambda)$ and indecomposable projective Λ-modules. Since we know that there are s maximal orders containing Λ, we just have to show that $O_l(P_i)$ is maximal for each i. This follows from the structure theorem for hereditary orders, for example if

$$\Lambda \;=\; \begin{pmatrix} \Delta & \Delta & \Delta \\ J & \Delta & \Delta \\ J & J & \Delta \end{pmatrix}$$

where $J = J(\Delta)$, and

$$P_0 = \begin{pmatrix} \Delta \\ \Delta \\ \Delta \end{pmatrix}, \quad P_1 = \begin{pmatrix} \Delta \\ \Delta \\ J \end{pmatrix}, \quad P_2 = \begin{pmatrix} \Delta \\ J \\ J \end{pmatrix}$$

then

$$O_l(P_0) = \Delta^{3\times 3}, \quad O_l(P_1) = \begin{pmatrix} \Delta & J^{-1} \\ J & \Delta \end{pmatrix}^{[2,1]}, \quad O_l(P_2) = \begin{pmatrix} \Delta & J^{-1} \\ J & \Delta \end{pmatrix}^{[1,2]}.$$

Note that J^{-1} is well defined since maximal orders are principal ideal rings (c.f. [AG60], Corollary to Proposition 3.3). The general case is similar. □

We conclude this overview with some results about Gorenstein orders. If Λ is a local Gorenstein order, then the left order $O_l(J(\Lambda))$ associated to $J(\Lambda)$ is the unique minimal order containing Λ, and that $O_l(J(\Lambda)) = O_r(J(\Lambda))$ (c.f. [Nis92], Lemma 1.2). For non-local Gorenstein orders, we have the following (c.f. [DK72], Lemma 3.4).

Proposition 4.1.3. *Let Λ be a Gorenstein order and $P_1, \ldots, P_n$ be the full set of nonisomorphic indecomposable projective Λ-lattices. Then every minimal order containing Λ is of the form $\Omega_j = \bigcap_{i\neq j} O_l(P_i)$ for $j = 1, \ldots, n$.*

4.2 Finding maximal orders

The aim of this section is to develop a method to find, for any given R-order Λ, all the maximal orders in Λ_K containing Λ. If Λ is hereditary, this is known (c.f. Proposition 4.1.2 and [Har63], Theorem 3.3); and in special cases such as when Λ is a local Bass order, where one can obtain all the maximal orders containing Λ using local Bass chains (c.f. [Iya01], 4.1) and Proposition 4.1.2. In general, we can associate to any Λ-lattice L the order $O_l(L)$, and through this correspondence one can relate properties of L to properties of $O_l(L)$. Iyama has some deep results which relate the lattice of orders containing Λ to certain subgraphs of its AR-quiver (c.f. [Iya98]). The relevant question for us is, can we identify a set $\{L_i\}_i$ of Λ-lattices such that $\{O_l(L_i)\}_i$ contains all the maximal orders?

We first show that one only has to consider indecomposable Λ-lattices of a certain R-rank. Denote by $\mathrm{Ind}_s(\Lambda)$ the isomorphism classes of irreducible Cohen-Macaulay Λ-modules of rank s.

Proposition 4.2.1. *Let Λ be an R-order and suppose $\Lambda_K \simeq M_n(D)$ where D is a central division K-algebra. Then every maximal order containing Λ is the left order of some $L \in \mathrm{Ind}_r(\Lambda)$ where $r = n \dim_K D$.*

Proof. Let Ω be a maximal order containing Λ and $e_1, \dots, e_n$ be a full set of primitive idempotent elements of Ω. Then there is a decomposition $\Omega \simeq \oplus_{i=1}^n \Omega e_i$ of Ω into indecomposable left Ω-ideals. Since $\Omega = O_l(\Omega) = \bigcap_{i=1}^n O_l(\Omega e_i)$, by maximality we have $\Omega = O_l(\Omega e_1) = \dots = O_l(\Omega e_n)$. By the structure theory for maximal orders, the R-rank of Ωe_i is equal to r. Moreover Ωe_i is an indecomposable left Ω-lattice, so it is also indecomposable as a Λ-lattice. $\square$

4.2.1 Reduction to artin rings

To find the Λ-lattices of rank r, we use the method of [RR79], which essentially reduces the representation theory of Λ to the representation theory of a pair of artin rings. Let Λ' be any hereditary order containing Λ, then any Λ-lattice M is naturally a submodule of $\Lambda' M$. So to study Λ-lattices, it suffices to study Λ-sublattices of (necessarily projective) Λ'-lattices, and Theorem 4.2.3 (c.f. [RR79], (1.2) Theorem I) states that we can work modulo some common ideal I of Λ and Λ'.

The above ideas can be formalised as follows. Given an hereditary order Λ' containing Λ and an ideal I of Λ' contained in $J(\Lambda)$, we denote $B = \Lambda/I$ and $B' = \Lambda'/I$. We construct the category $\mathcal{C} = \mathcal{C}(B \longrightarrow B')$ whose objects are given by B-module monomorphisms $\sigma : T \hookrightarrow T'$ where T is a B-module and T' is a projective B'-module such that $B'\sigma(T) = T'$. A morphism from $T \hookrightarrow T'$ to $S \hookrightarrow S'$ is a commutative diagram

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S' \end{array}$$

where $T \longrightarrow S$ is an B-module homomorphism and $T' \longrightarrow S'$ is a B'-module homomorphism. The category $\mathcal{C}$ is additive, since the product of $T \hookrightarrow T'$ and $S \hookrightarrow S'$ exists and is given by $T \times S \hookrightarrow T' \times S'$. We define a functor $F : CM(\Lambda) \longrightarrow \mathcal{C}$ by

$$M \longmapsto (M/IM \hookrightarrow \Lambda' M/IM). \tag{4.2.1}$$

Note that F is an additive functor.

Note 4.2.2. The construction of $\mathcal{C}$ involves choices of an hereditary order Λ' and a Λ'-ideal I. An hereditary order Λ' containing Λ can be found by iterating the embedding $\Lambda_0 := \Lambda \subseteq O_l(J(\Lambda)) := \Lambda_1$. Since $O_l(J(\Lambda_i)) = \Lambda_i$ if and only if Λ_i is hereditary (c.f. [Jac71], Proposition 1), this process will terminate at an hereditary order $\Lambda' = \Lambda_n$. For the ideal I of Λ', we take the minimal power of $J(\Lambda')$ which is contained in $J(\Lambda)$.

The functor F defined above is almost an equivalence of categories. To be more precise, we need the following definition. Let C and D be categories, a functor $G : C \longrightarrow D$ is a *representation equivalence* if it is full, essentially surjective, and for $T_1, T_2 \in \mathrm{ob}(C)$ we have $G(T_1) \simeq G(T_2)$ if and only if $T_1 \simeq T_2$. Note that the map $\mathrm{Hom}_C(T_1, T_2) \longrightarrow \mathrm{Hom}_D(GT_1, GT_2)$ on hom-sets is not required to be injective, so G is not in general an equivalence of categories. However, since G reflects and preserves isomorphisms, it induces a bijection between the isomorphism classes of objects of C and D.

Theorem 4.2.3. *([RR79], (1.2) Theorem I) Let R be a complete discrete valuation ring and Λ be an R-order. Then the functor $F : CM(\Lambda) \longrightarrow \mathcal{C}$ defind in (4.2.1) is a representation equivalence. In particular, F induces a bijective correspondence between the sets of isomorphism classes of objects in $CM(\Lambda)$ and $\mathcal{C} = \mathcal{C}(\Lambda/I \hookrightarrow \Lambda'/I)$.*

Since $\mathcal{C}$ is representation equivalent to $CM(\Lambda)$, $\mathcal{C}$ is Krull-Schmidt, and we denote by $\mathrm{Ind}(\mathcal{C})$ the set of isomorphism classes of indecomposable objects in $\mathcal{C}$. The functor F

induces a map of sets $F : \mathrm{Ind}(\Lambda) \longrightarrow \mathrm{Ind}(\mathcal{C})$. The inverse map $F^{-1} : \mathrm{Ind}(\mathcal{C}) \longrightarrow \mathrm{Ind}(\Lambda)$ is defined as follows; let $\sigma : T \hookrightarrow T'$ be an object in $\mathcal{C}$ and L' be the Λ'-projective cover of T', then $F^{-1}(\sigma) = L$ is defined (up to unique isomorphism) by the following cartesian square

$$\begin{array}{ccc} L & \longrightarrow & L' \\ \downarrow & \square & \downarrow \\ T & \hookrightarrow & T' \end{array} \tag{4.2.2}$$

in the category of left Λ-modules.

Let r be the integer defined in Proposition 4.2.1 and we denote by $\mathrm{Ind}_r(\mathcal{C})$ the image of $\mathrm{Ind}_r(\Lambda)$ under F.

Proposition 4.2.4. *Let $\sigma : T \hookrightarrow T'$ be an object in $\mathrm{Ind}(\mathcal{C})$. Then $\sigma \in \mathrm{Ind}_r(\mathcal{C})$ if and only if T' is an indecomposable projective Λ'/I-module.*

Proof. Let $(\sigma : T \hookrightarrow T') \in \mathrm{Ind}(\mathcal{C})$, $L = F^{-1}(\sigma) \in \mathrm{Ind}(\Lambda)$ and $L' = \Lambda' L \in \mathrm{Ind}(\Lambda')$. By [CR81], Proposition 6.17 iv, $T' \simeq L'/IL$ is an indecomposable Λ'/I-module if and only if L' is indecomposable as a Λ'-lattice. Since Λ' is an hereditary order, the Λ'-lattice L' is indecomposable if and only if the rank of L' is equal to r. The Λ'-lattice L' has the same rank as L since $L'/L \simeq \frac{\Lambda' L/IL}{L/IL}$ is R-torsion. Hence T' is indecomposable if and only if $\sigma \in \mathrm{Ind}_r(\mathcal{C})$. □

Note 4.2.5. The above proof also shows that r is the minimal R-rank of Λ-lattices.

4.2.2 Maximality condition

We have shown, in Proposition 4.2.1, that maximal orders containing Λ are of the form $O_l(L)$ where L is an indecomposable Λ-lattice of minimal rank. Next we give a condition in terms of the finite dimensional module L/IL to determine whether $O_l(L)$ is a maximal order.

Let Λ be an R-order and Λ' be an hereditary order containing Λ. Let L be an indecomposable left Λ-lattice of rank r, where r is the integer defined in Proposition 4.2.1, and suppose $L' = \Lambda' L$ is isomorphic to $\Lambda' e$ for some primitive idempotent e. This idempotent is not uniquely determined, so we fix some choice of e in the following discussion. The left Λ'-module $\Lambda' e$ is naturally a right $e\Lambda' e$-module, and identifying L as a submodule of $\Lambda' e$, we can define a morphism

$$m : L \otimes_R e\Lambda' e \longrightarrow \Lambda' e \tag{4.2.3}$$

by restricting the multiplication of $e\Lambda'e$ on $\Lambda'e$ to L. Recall that every maximal order containing Λ is of the form $O_l(L)$ for some $L \in \mathrm{Ind}_r(\Lambda)$ (c.f. Proposition 4.2.1). The following proposition provides a converse by identifying the $L \in \mathrm{Ind}_r(\Lambda)$ for which $O_l(L)$ is maximal.

Proposition 4.2.6. *Let $L \in \mathrm{Ind}_r(\Lambda)$, then $O_l(L)$ is a maximal order if and only if the image of m is equal to L.*

Proof. We first identify $\Omega = O_l(L)$ with $\mathrm{End}_{\mathrm{End}_\Omega(L)}(L)$ under the isomorphism from Lemma 4.1.1. Then [AG60], Theorem 3.6, states that Ω is a maximal order if and only if $\mathrm{End}_\Omega(L)$ is maximal. The endomorphism ring $\mathrm{End}_{\Omega_K}(L_K)$ a division algebra and is equal to $\mathrm{End}_{\Omega_K}((\Lambda'e)_K) \simeq (e\Lambda'e)_K$, where Λ' and e are as defined above. This provides an embedding

$$i : \mathrm{End}_\Omega(L) \quad \longrightarrow \quad (e\Lambda'e)_K$$

which identifies the action of $\mathrm{End}_\Omega(L)$ on $L \subseteq \Lambda'e$ with right multiplication by elements of $(e\Lambda'e)_K$. Suppose that Ω is maximal, then $\mathrm{End}_\Omega(L)$ is maximal. Since $(e\Lambda'e)_K$ is a division algebra, its unique maximal order is $e\Lambda'e$. Hence i sends $\mathrm{End}_\Omega(L)$ to $e\Lambda'e$, so the image of m is L. Conversely, if $\mathrm{im}(m) = L$, then we have a map $e\Lambda'e \longrightarrow \mathrm{End}_\Omega(L)$. Since Λ' is an hereditary order, $e\Lambda'e$ is maximal. Hence $\mathrm{End}_\Omega(L)$ is maximal, so Ω is also maximal. □

Now let I be an ideal of Λ' contained in $J(\Lambda)$ and consider the pair category $\mathcal{C} = \mathcal{C}(\Lambda/I \longrightarrow \Lambda'/I)$. Let $\sigma : T \hookrightarrow T'$ be an element of $\mathrm{Ind}_r(\mathcal{C})$, denote $A' = \Lambda'/I$ and suppose that $T' = A'\varepsilon$ for some primitive idempotent in A'. The natural multiplication of $\varepsilon A'\varepsilon$ makes $A'\varepsilon$ a right $\varepsilon A'\varepsilon$-module, and restricting the multiplication to T, we obtain a morphism

$$\mu : T \otimes_{R/I\cap R} \varepsilon A'\varepsilon \quad \longrightarrow \quad A'\varepsilon. \tag{4.2.4}$$

We can associate the order $O_l(F^{-1}(\sigma))$ to σ. By Proposition 4.2.1 and Theorem 4.2.3, if $O_l(F^{-1}(\sigma))$ is maximal then $\sigma \in \mathrm{Ind}_r(\mathcal{C})$.

Corollary 4.2.7. *Let $(\sigma : T \hookrightarrow T') \in \mathrm{Ind}_r(\mathcal{C})$, then $O_l(F^{-1}(\sigma))$ is a maximal order if and only if the image of μ is equal to T.*

Proof. Let e be a lift of the idempotent ε to Λ', then $L = F^{-1}(\sigma)$ can be identified with a submodule of $\Lambda'e$, and let m be the morphism defined in (4.2.3). By Proposition 4.2.6, it suffices to show that if $\mathrm{im}(\mu) = T$ if and only if $\mathrm{im}(m) = L$. To this end, we have the following commutative diagram

$$\begin{array}{ccc} L \otimes_R e\Lambda' e & \xrightarrow{m} & \Lambda' e \\ \downarrow & & \downarrow \\ T \otimes_{R/I\cap R} \varepsilon A'\varepsilon & \xrightarrow{\mu} & A'\varepsilon \end{array}$$

where the second vertical map is $\Lambda' e \longrightarrow \Lambda' e/Ie \simeq A\varepsilon$, and the first vertical map is defined as follows. Applying $- \otimes_R e\Lambda' e$ to the surjection $L \longrightarrow L/IL \simeq T$ we obtain $L \otimes_R e\Lambda' e \longrightarrow T \otimes_R e\Lambda' e$, which is again surjective since $e\Lambda' e$ is R-free. Since $T \otimes_R (I \cap e\Lambda' e) \subseteq Ie$ and $e\Lambda' e/(I \cap e\Lambda' e) \simeq \varepsilon A'\varepsilon$, we have an isomorphism $T \otimes_R e\Lambda' e \xrightarrow{\sim} T \otimes_{R/I\cap R} \varepsilon A'\varepsilon$. Composing the last two maps gives a surjective morphism

$$L \otimes_R e\Lambda' e \longrightarrow T \otimes_R e\Lambda' e \xrightarrow{\sim} T \otimes_{R/I\cap R} \varepsilon A'\varepsilon.$$

Note that the image of L under $\Lambda' e \longrightarrow A'\varepsilon$ is T, so by the commutative diagram above we have $\mathrm{im}(\mu) = T$ if and only if $\mathrm{im}(m) = L$. □

Let $\mathrm{maxord}(\Lambda)$ denote the set of maximal orders containing Λ. We say that $(\sigma : T \hookrightarrow T') \in \mathrm{Ind}_r(\mathcal{C})$ is *saturated* if the morphism μ defined in (4.2.4) with respect to some choice of idempotents has image equal to T. This condition is independent of the choice of idempotents. We denote by $\mathcal{S}$ the subset of $\mathrm{Ind}_r(\mathcal{C})$ consisting of saturated elements.

Corollary 4.2.8. *Let Λ be an R-order, Λ' be an hereditary order containing Λ and I be an ideal of Λ' contained in Λ. The map* $\mathrm{maxord}(\Lambda) \longrightarrow \mathcal{S}$ *given by* $\sigma \longmapsto O_l(F^{-1}(\sigma))$ *is a bijective set morphism, with inverse given by* $\Omega \longmapsto (\Omega e \longrightarrow \Lambda'\Omega e/I\Omega e)$ *where e is any primitive idempotent of* Ω.

Chapter 5

Blowing up terminal orders

The category theoretic point of view of noncommutative geometry comes from the observation that the category of quasi-coherent sheaves on a scheme captures all of its intrinsic geometry. In [VdB01], the noncommutative analogue of a scheme is called a *quasi-scheme* and is defined simply as a Grothendieck category. This definition is motivated by the following fact, shown in [TT90], Appendix B, that for a quasi-compact and quasi-separated scheme X, the category of quasi-coherent $\mathcal{O}_X$-modules is a Grothendieck category. A similar argument shows that for a coherent $\mathcal{O}_X$-algebra A on a quasi-compact and quasi-separated scheme X, the category of quasi-coherent left A-modules is a Grothendieck category. Hence an order A over a surface Z can be considered as a quasi-scheme via the category $\operatorname{Mod} A$ of quasi-coherent left A-modules.

The noncommutative blowup of [VdB01] is defined for a class of quasi-schemes which are the noncommutative analogues of smooth surfaces. These are characterised by the existence of an embedded smooth commutative curve with "divisorial" behaviour. This chapter will investigate to what extent Van den Bergh's blowup can inform us about the blowup of orders. We find our most satisfying answer in the case of terminal orders. Using the techniques developed in Chapter 4, we determine the blowups of terminal orders arising from different choices of maximal orders at the exceptional curve. We then describe for terminal orders a one-to-one correspondence between their blowups as orders and their blowups as quasi-schemes, thus drawing a connection between the two theories of blowing up.

5.1 Blowups of quasi-schemes

We describe below the noncommutative blowup of [VdB01] applied to $\operatorname{Mod} A$ where A is an order over a surface Z. For this purpose, we do not require the full generality of the category theoretic machinery for blowing up quasi-schemes. In this spirit, we express Van den Bergh's construction in the language of sheaves of algebras on a commutative scheme rather than the language of category theory. A similar approach can be found in Artin's example (c.f [Art97], section 9) of noncommutative blowing up.

The noncommutative blowup of $\operatorname{Mod} A$ mimics the commutative blowup in that it involves a Rees algebra construction. First recall the ingredients for blowing up an ideal $\mathfrak{m}$ on a variety Z: we construct the Rees algebra

$$\mathcal{R} = \operatorname{Rees}(\mathcal{O}_Z, \mathfrak{m}) \quad = \quad \mathcal{O}_Z \oplus \mathfrak{m} \oplus \mathfrak{m}^2 \oplus \cdots$$

and the blowup of Z at $\mathfrak{m}$ is defined as $\operatorname{Proj}(\mathcal{R})$. Note that the Rees algebra and Proj still makes sense for $\mathfrak{m}L \subset L$ in place of $\mathfrak{m}$ where L is any invertible $\mathcal{O}_Z$-bimodule. This is uninteresting if $\mathcal{O}_Z$ is commutative and the left and right actions on L are equal, whence $\operatorname{Proj}(\mathcal{R}) \simeq \operatorname{Proj}(\operatorname{Rees}(\mathcal{O}_Z, \mathfrak{m} \otimes L))$, but is important for noncommutative blowups.

Now we replace $\mathcal{O}_Z$ with an $\mathcal{O}_Z$-algebra A. The blowup of $\operatorname{Mod} A$ is defined for points on a commutative divisor in $\operatorname{Mod} A$. We explain these terms below. Let I be an ideal of A and suppose the category $\operatorname{Mod}(A/I)$ is equivalent to the category of quasi-coherent $\mathcal{O}_C$-modules, where C is a (smooth) commutative curve. Then we say that C is a *curve embedded in* $\operatorname{Mod} A$. If, in addition, the ideal I is an invertible ideal of A, we say that the embedded curve C is a *divisor* in $\operatorname{Mod} A$. A point p on C corresponds to an ideal M_p of A containing I such that A/M_p is isomorphic to a matrix algebra over k. We will denote the object A/M_p in $\operatorname{Mod} A$ by $\mathcal{O}_p$. For a sub-A-bimodule $\mathcal{M}$ of an invertible A-bimodule $\mathcal{L}$, the *Rees algebra* $\operatorname{Rees}(A, \mathcal{M})$ is defined by

$$\operatorname{Rees}(A, \mathcal{M}) \quad = \quad A \oplus \mathcal{M} \oplus \mathcal{M}^2 \oplus \cdots$$

where $\mathcal{M}^n$ is the image of $\mathcal{M}^{\otimes n} \longrightarrow \mathcal{L}^{\otimes n}$. The invertible ideal I plays the role of a "twisting parameter," in that $I \otimes_A \mathcal{O}_p \not\simeq \mathcal{O}_p$ and $\mathcal{O}_p \otimes_A I \not\simeq \mathcal{O}_p$ (c.f. [VdB01], equation (5.5)). In particular, $\operatorname{Rees}(A, M_p)$ is different from $\operatorname{Rees}(A, M_p I)$.

Let $\mathcal{B}$ be a sheaf of graded $\mathcal{O}_Z$-algebras. We define the category $\operatorname{Proj} \mathcal{B}$ to be the quotient category $\operatorname{Gr}(\mathcal{B})/\operatorname{Tors}(\mathcal{B})$ where $\operatorname{Gr} \mathcal{B}$ denotes the category of graded $\mathcal{B}$-modules and $\operatorname{Tors} \mathcal{B}$ denotes the full subcategory of $\operatorname{Gr} \mathcal{B}$ whose objects are unions of right bounded modules.

This is a Grothendieck category, so $\operatorname{Proj} \mathcal{B}$ is a quasi-scheme. The blowup of $\operatorname{Mod} A$ at $\mathcal{O}_p \in \operatorname{Mod} C$ is then defined as the quasi-scheme $\operatorname{Proj}(\mathcal{R})$ where $\mathcal{R} = \operatorname{Rees}(A, M_p I)$.

The definition of Proj (c.f. [AZ94], Section 2) is inspired by Serre's theorem, which states the following. Let A be a commutative, connected, graded algebra over a field k which is finitely generated in degree 1. Then we have an equivalence of categories $\operatorname{Mod}(\operatorname{Proj} A) \simeq \operatorname{Gr}(A)/\operatorname{Tors}(A)$. The definition of $\operatorname{Proj} A$ includes some extra structure, corresponding to the autoequivalence induced by the natural shift in $\operatorname{Gr}(A)$ and a distinguished object obtained from A. We will not need this extra structure, since in our application, we will always use Veronese subalgebras to derive equivalences between Proj categories. Let $B = \bigoplus_i B_i$ be a graded $\mathcal{O}_Z$-algebra. The m-th Veronese subalgebra $B^{(m)}$ of B is defined as $B_i^{(m)} = B_{mi}$ for all i. The following is well known (c.f. [VdB01], Lemma 3.12.1).

Proposition 5.1.1. *Let Z be a noetherian scheme and A be a noetherian graded $\mathcal{O}_Z$-algebra which is generated in degree* 1*. Then the functors*

$$\begin{array}{rcl} \operatorname{Gr}(A) & \longrightarrow & \operatorname{Gr}(A^{(n)}) \\ M & \longmapsto & M^{(n)} \end{array}$$

and

$$\begin{array}{rcl} \operatorname{Gr}(A^{(n)}) & \longrightarrow & \operatorname{Gr}(A) \\ N & \longmapsto & N \otimes_{A^{(n)}} A \end{array}$$

factor over $\operatorname{Tors}(A)$ *and* $\operatorname{Tors}(A^{(n)})$*, and induce an equivalence of categories* $\operatorname{Proj}(A) \simeq \operatorname{Proj}(A^{(n)})$.

The category equivalence above will not respect the autoequivalence induced by the shift in Gr on the respective Proj categories. However this is expected since we are changing polarisations: this phenomenon can be observed in the degree n embedding of a rational curve $\varphi : \mathbb{P}^1 \longrightarrow \mathbb{P}^{n+1}$. The natural shift on $\mathfrak{Qcoh}(\varphi(\mathbb{P}^1))$ is of course $- \otimes \mathcal{O}_{\mathbb{P}^1}(n)$ and this does not agree with $- \otimes \mathcal{O}_{\mathbb{P}^1}(1)$. In any case, we consider $\operatorname{Proj} A$ and $\operatorname{Proj} A^{(m)}$ to be equivalent.

5.2 Blowing up orders using the blowup of quasi-schemes

In this section, we construct the blowups of terminal orders using Van den Bergh's blowup of quasi-schemes. We fix notation for the rest of the chapter. Let $Z = \operatorname{Spec} k[[u, v]]$ and

$\sigma : \tilde{Z} \longrightarrow Z$ be the blowup of Z at the origin with $E = \mathrm{Ex}(\sigma)$. We denote by K the fraction field of $k[[u, v]]$. Since we are mainly interested in the category $\mathrm{Mod}\, A$ where A is a terminal $k[[u, v]]$-order, we can work up to Morita equivalence, that is, we can assume that

$$A = A(r, e) \quad = \quad \begin{pmatrix} S & \dots & \dots & S \\ xS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ xS & \dots & xS & S \end{pmatrix} \subset S^{r \times r}, \tag{5.2.1}$$

where

$$S \quad = \quad \frac{k[[u, v]] \langle x, y \rangle}{(x^e - u, y^e - v, xy - \zeta yx)} \tag{5.2.2}$$

and ζ is a primitive e-th root of unity.

Note 5.2.1. We distinguish between the two different notions of blowing up by the notation with which we refer to the order A. When we write "a blowup of $\mathrm{Mod}\, A$", we are referring to a blowup of A as a quasi-scheme, which is a Proj category; whereas "a blowup of A" refers to the blowup of A as an order, which is a pair $(\sigma, \tilde{A})$ where $\sigma : \tilde{Z} \longrightarrow Z$ is a birational morphism and $\tilde{A}$ is a normal order on $\tilde{Z}$ (c.f. Section 2.2.2).

Following the discussion in the previous section, to construct blowups of $\mathrm{Mod}\, A$ amounts to producing some Rees algebras from the data $(I, \mathcal{M})$ where I is an invertible ideal of A and $\mathcal{M}$ is an ideal of A containing I.

We show that the cyclic cover of a ramification curve defined by the ramification data is a divisor in $\mathrm{Mod}\, A$. Consider the ramification curves D and D' of A in Z with equations $u = 0$ and $v = 0$. To these ramification curves we can associate the ideals

$$I := J(A_D) \cap A \quad = \quad \begin{pmatrix} xS & S & \dots & S \\ xS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & S \\ xS & \dots & xS & xS \end{pmatrix}$$

$$I' := J(A_{D'}) \cap A \quad = \quad \begin{pmatrix} yS & \dots & \dots & yS \\ xyS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ xyS & \dots & xyS & yS \end{pmatrix}$$

of A. We first show that the ideal I' does not give rise to a commutative divisor in $\mathrm{Mod}\, A$.

Proposition 5.2.2. *The category* $\operatorname{Mod} A/I'$ *is not equivalent to the category* $\mathfrak{Qcoh}(C')$ *of quasi-coherent sheaves on a smooth commutative affine curve* C'.

Proof. Suppose $C' = \operatorname{Spec} R'$ where R' is some commutative k-algebra. Then A/I' is Morita equivalent to R' and since R' is commutative, standard Morita theory (c.f. [MR87], Theorem 5.9, iii) gives $Z(A/I') \simeq R'$. Hence $R' \simeq k[[x,u]]/(x^e - u) \simeq k[[x]]$. Now $\operatorname{Mod} k[[x]]$ has a unique indecomposable projective object, up to isomorphism. Since the category $\operatorname{Mod} A/I'$ has r non-isomorphic indecomposable projective objects, $\operatorname{Mod} A/I'$ is not equivalent to $\operatorname{Mod} k[[x]]$. $\square$

Proposition 5.2.3. *The embedding* $\operatorname{Mod} A/I \longrightarrow \operatorname{Mod} A$ *defines a divisor* C *in* $\operatorname{Mod} A$.

Proof. The quotient A/I is isomorphic to the commutative algebra $\prod_{i=1}^r S/xS$, and since $S/xS \simeq k[[v,y]]/(y^e - v)$, the scheme $\operatorname{Spec} A/I$ has dimension one. Moreover, $I^r = xA$, so the ideal I is invertible. $\square$

Note 5.2.4. The morphism $\operatorname{Spec} A/I \longrightarrow D$ induced by the inclusion of k-algebras $k[[v]] \longrightarrow S/xS \simeq k[[v,y]]/(y^e - v)$ gives a degree re cover of D.

There are r maximal ideals $M_1, \ldots, M_r$ containing I such that $\mathcal{O}_{p_1} := A/M_1, \ldots, \mathcal{O}_{p_r} := A/M_r$ are supported at the origin. We describe M_i as follows. Let $(x,y) \subset S$ be the Jacobson radical of S and let

$$M = \begin{pmatrix} (x,y) & S & \dots & S \\ xS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & S \\ xS & xS & xS & (x,y) \end{pmatrix}.$$

The quotient A/M is a product of fields $k \times \cdots \times k$. Let π_i be the i-th projection $A/M \longrightarrow k$, then the maximal ideals M_i is the kernel of the composition $A \longrightarrow A/M \xrightarrow{\pi_i} k$. The blowup of $\operatorname{Mod} A$ at $\mathcal{O}_{p_i} \in \operatorname{Mod} C$ is defined as $\operatorname{Proj}(\operatorname{Rees}(A, M_i I^{-1}))$.

We can now state our comparison result between the blowups of $\operatorname{Mod} A$ and A. As mentioned above, we assume that $A = A(r,e)$, where $A(r,e)$ is defined in (5.2.1). Note that $A_K \simeq S_K^{r\times r}$ where S is the terminal order in (5.2.2), and S_K remains a division algebra after completion at E. Let Δ be the unique maximal order in $S \otimes_{k[[u,v]]} \hat{K}_E$ and J its Jacobson radical. For a quasi-coherent $\mathcal{O}_{\tilde{Z}}$-module F, we denote by $\Gamma_\bullet(F)$ the graded algebra $\bigoplus_{s=0}^{\infty} H^0(\tilde{Z}, F(-sE))$.

Theorem 5.2.5. *Let $A = A(r,e)$ be a terminal order. Then there are r blowups $(\sigma, \tilde{A}_1)$, $\ldots, (\sigma, \tilde{A}_r)$ of A where*

$$(\tilde{A}_i)_{\widehat{E}} \quad = \quad \mathrm{End}_\Delta(\Delta^{\oplus i} \oplus J^{\oplus r-i})$$

and the equality indicates equality as subalgebras of $A \otimes_{k[[u,v]]} \hat{K}_E$.

Theorem 5.2.6. *Let $A = A(r,e)$ be a terminal order and $(\sigma, \tilde{A}_1), \ldots, (\sigma, \tilde{A}_r)$ be the blowups of A appearing in Theorem 5.2.5. Then, after relabelling the $\tilde{A}_i$'s if necessary, there are equivalences of categories*

$$\mathrm{Proj}(\Gamma_\bullet(\tilde{A}_i)) \quad \simeq \quad \mathrm{Proj}(\mathrm{Rees}(A, M_i I^{-1}))$$

for $i = 1, \ldots, r$.

We explain this correspondence as follows. The order $(\sigma^* A)_E$ is contained in a unique hereditary order Λ' (c.f. Corollary 5.4.3), so the maximal orders containing $(\sigma^* A)_E$ are parameterised by the projective indecomposables of Λ', or equivalently, the simple objects in $\mathrm{Mod}\,\Lambda'$. There are r simple objects $\mathcal{O}_{p_1}, \ldots, \mathcal{O}_{p_r}$ in $\mathrm{Mod}\,A$ which are supported at the origin. The functor $F : \mathrm{Mod}\,A \longrightarrow \mathrm{Mod}\,\Lambda'$ given by base change, induces a bijective correspondence between $\{O_{p_1}, \ldots, \mathcal{O}_{p_r}\}$ and the simple objects in Λ'. This allows us to interpret the different blowups of A arising from the choices of maximal orders containing $(\sigma^* A)_E$ as blowing up different points on the quasi-scheme $\mathrm{Mod}\,A$. The rest of this chapter is devoted to the proof of the above two results.

5.3 Blowups of local terminal orders

In this section, we prove Theorems 5.2.5 and 5.2.6 for local terminal orders. Let $R = \widehat{\mathcal{O}}_{\tilde{Z},E}$ and let u be its uniformising parameter. We also denote by κ the residue field $k(v/u)$ of R. A terminal order is local if and only if it is Morita equivalent to $S = A(1,e)$ for some e, and we construct the blowups of S below. Let $\mathcal{B} = \{x^i y^j \mid i,j = 0, \ldots, e-1\}$ denote the free $k[[u,v]]$-basis of S given by monomials in the generators x and y.

Proposition 5.3.1. *The discriminant ideal $d(S)$ of S is given by*

$$d(S) \quad = \quad (uv)^{e(e-1)/2} k[[u,v]].$$

Proof. First note that the only monomial in $\mathcal{B}$ with nonzero trace is $x^0 y^0 = 1$, so for $b = x^i y^j \in \mathcal{B} \backslash \{x^0 y^0\}$ there is a unique element $b' = x^{e-i} y^{e-j} \in \mathcal{B} \backslash \{x^0 y^0\}$ such that

$\mathrm{tr}(bb') \neq 0$. Hence the $e^2 \times e^2$ matrix $T = (\mathrm{tr}(x_i x_j))_{i,j}$ is equal to a permutation matrix multiplied by the diagonal matrix $T' = \mathrm{diag}(bb' \mid b \in \mathcal{B})$. Hence

$$\begin{aligned} \det(T')R &= \left(\prod_{i=1}^{e-1} x^i \cdot x^{e-i}\right)\left(\prod_{i=1}^{e-1} y^i \cdot y^{e-i}\right)\left(\prod_{i,j=1}^{e-1} x^i y^j \cdot x^{e-i} y^{e-j}\right) R \\ &= u^{e-1} v^{e-1} (uv)^{(e-1)^2} R \\ &= (uv)^{e(e-1)} R \end{aligned}$$

so $d(S)$ is generated by $\sqrt{\det(T')} = (uv)^{e(e-1)/2}$. □

Let Γ be the R-order $(\sigma^* S)_E$ and for a subset P of Λ_K we denote by $\Lambda\langle P\rangle$ the smallest subalgebra of Λ_K containing Λ and P.

Proposition 5.3.2. *The unique maximal order containing* Γ *is*

$$\Delta := \Gamma\left\langle (x,y)^e u^{-1}\right\rangle = \bigoplus_{0 \leqslant i+j < e} R x^i y^j \oplus \bigoplus_{i+j \geqslant e} u^{-1} R x^i y^j. \tag{5.3.1}$$

The Jacobson radical $J := J(\Delta)$ *of* Δ *is*

$$J = x\Delta \tag{5.3.2}$$

and the discriminant ideal $d(\Delta)$ *of* Δ *is*

$$d(\Delta) = u^{e(e-1)/2} R.$$

Proof. It is clear that the finite R-module appearing on the right hand side of (5.3.1) is closed under multiplication, so Δ is an order. We show that $J(\Delta) = x\Delta$. Note that x is a normal element of Δ, so $(x\Delta)^e = x^e\Delta = u\Delta$ so $x\Delta \subseteq J(\Delta)$. We have an isomorphism

$$\kappa[X]/(X^e - (v/u)) \xrightarrow{\sim} \kappa \times \kappa x^{e-1} y u^{-1} \times \cdots \times \kappa x y^{e-1} u^{-1} \simeq \Delta/x\Delta$$

given by sending $X \longmapsto x^{e-1} y u^{-1}$, so $x\Delta = J(\Delta)$. This shows that Δ is local and hereditary, so is the unique maximal order in $\Delta \otimes \hat{K}_E$. The discriminant ideal of Δ is easily calculated from the definition and Proposition 5.3.1. □

Corollary 5.3.3. *Let* $\sigma : \tilde{Z} \longrightarrow Z$ *be the blowup at the origin. The unique blowup* $\tilde{S}$ *of* S *along* σ *is a direct sum of line bundles*

$$\tilde{S} = \bigoplus_{i,j=0}^{e-1} \tilde{S}_{ij} x^i y^j \tag{5.3.3}$$

where $\tilde{S}_{ij} = \mathcal{O}_{\tilde{Z}}$ *if* $i + j < e$ *and* $\tilde{S}_{ij} = \mathcal{O}_{\tilde{Z}}(E)$ *if* $i + j \geqslant e$.

We now prove Theorem 5.2.6 for local terminal orders. Let $D_1 = (u)$ and $I = xS$. Note that x is a normal element of S and $S/xS \simeq k[[v, y]]/(y^e - v)$ is a commutative curve. Let $\mathfrak{m} = (u, v) \in \operatorname{Spec} k[[u, v]]$, $M_p = (x, y)$ and $\mathcal{O}_p = S/M_p$. Theorem 5.2.6 for local terminal orders follows from the following proposition.

Proposition 5.3.4. *The e-th Veronese subalgebra $\mathcal{D}$ of* $\operatorname{Rees}(S, M_p I^{-1})$ *is isomorphic to* $\Gamma_\bullet(\tilde{S})$ *as graded algebras.*

Proof. First note that $\mathcal{D}_0$ is isomorphic to $\Gamma(\tilde{S})$ as k-algebras. We show by induction that

$$\mathcal{D}_t \simeq u^{-t}\left(\bigoplus_{0\leqslant i+j<e} \mathfrak{m}^t x^i y^j \oplus \bigoplus_{e\leqslant i+j} \mathfrak{m}^{t-1} x^i y^j\right) \qquad (5.3.4)$$

for all positive integers t, which proves that $\mathcal{D}_t = u^{-t}\Gamma_t(\tilde{S})$ under the identification $\mathcal{D}_0 \simeq \Gamma_0(\tilde{S})$.

The multiplication in $\mathcal{D}$ can be described as follows: note that $u^t\mathcal{D}_t$ is an ideal of $\mathcal{D}_0$, given $\xi_t \in \mathcal{D}_t$ and $\xi_s \in \mathcal{D}_s$, then we have $\xi_t\xi_s = u^{-(t+s)}(u^t\xi_t) \cdot (u^s\xi_s)$ where the $\cdot$ denotes multiplication in $\mathcal{D}_0$. The multiplication in $\Gamma_\bullet(\tilde{S})$ can be described similarly. Since $\mathcal{D}_0$ and $\Gamma_0(\tilde{S})$ are isomorphic algebras, the equality $\mathcal{D}_t = u^{-t}\Gamma_t(\tilde{S})$ shows that $\mathcal{D}$ and $\Gamma_\bullet(\tilde{S})$ are isomorphic as graded algebras.

It remains to prove (5.3.4). First note that $xM_p = M_p x$, so the ideals M_p and I^{-1} commute. Since

$$\begin{aligned}(x, y)^e &= (u, x^{e-1}y, \ldots, xy^{e-1}, v)\\ &= \left(\bigoplus_{0\leqslant i+j<e} \mathfrak{m} x^i y^j \oplus \bigoplus_{e\leqslant i+j} k[[u, v]] x^i y^j\right)\end{aligned}$$

the equation (5.3.4) holds for $t = 1$. Now suppose (5.3.4) holds for some t. Since

$$\begin{aligned}(x, y)^{2e} &= (u^2, ux^{e-1}y, \ldots, uxy^{e-1}, uv, vx^{e-1}y, \ldots, vxy^{e-1}, v^2)\\ &= \mathfrak{m}(x, y)^e,\end{aligned}$$

and

$$\mathcal{D}_{t+1} = u^{-(t+1)}(x, y)^{(t+1)e} = u^{-(t+1)}\mathfrak{m}^{t-1}(x, y)^{2e},$$

the equation (5.3.4) holds for $t + 1$. $\square$

5.4 Blowups of terminal orders in general

In this section, we prove Theorems 5.2.5 and 5.2.6 for a general terminal order $A = A(r, e)$. We let $\Lambda = (\sigma^* A)_E$ and keep the notation of the previous section. We first find all the maximal orders containing Λ and for each such maximal order, determine the $\mathcal{O}_{\tilde{Z}}$-module structure of the corresponding blowup. We first embed Λ into a suitable hereditary order Λ', and identify an ideal I' of Λ' such that $I' \subseteq J(\Lambda)$. With this data, we can then make use of the correspondence in Corollary 4.2.8 to find all the maximal orders.

Since

$$\Lambda = \begin{pmatrix} \Gamma & \cdots & \cdots & \Gamma \\ x\Gamma & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ x\Gamma & \cdots & x\Gamma & \Gamma \end{pmatrix} \subset \Gamma^{r\times r}$$

and $x\Gamma \subseteq x\Delta = J$ (c.f. Proposition 5.3.2, (5.3.2)), so the order Λ is contained in the following hereditary order

$$\Lambda' = \begin{pmatrix} \Delta & \cdots & \cdots & \Delta \\ J & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ J & \cdots & J & \Delta \end{pmatrix} \subset \Delta^{r\times r}. \tag{5.4.1}$$

Proposition 5.4.1. *Let $I' = J(\Lambda')^{r(e-1)}$, then the Λ'-ideal I' is contained in $J(\Lambda)$.*

Proof. A simple calculation shows that

$$I' = \begin{pmatrix} J^{e-1} & J^{e-1} & \cdots & J^{e-1} \\ J^e & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & J^{e-1} \\ J^e & \cdots & J^e & J^{e-1} \end{pmatrix} \subset \Lambda'.$$

By Proposition 5.3.2, we have $\Delta = \Gamma \left\langle (x, y)^e u^{-1} \right\rangle$ and $J = x\Delta = \Delta x$. Since $x^{e-1}(x, y)^e u^{-1} \subseteq \Gamma$, we have $J^{e-1} = x^{e-1}\Delta \subseteq \Gamma$. It follows that $J^e \subseteq x\Gamma$, so $I' \subseteq \Lambda$. Since I' is a power of $J(\Lambda')$, some power of I' is in $u^2\Delta \subset J(\Lambda)$, hence I' is in $J(\Lambda)$. □

Next we find the maximal orders containing Λ. Let $A = \Lambda/I'$, $A' = \Lambda'/I'$, and $e_1, \ldots, e_r$ be the standard matrix idempotents in Λ' (c.f. (5.4.1)). We denote by $\bar{e}_i$ the image of e_i in A' and $A'_i = \bar{e}_i A' \bar{e}_i$.

Proposition 5.4.2. *Let N be a left A-submodule of $A'\overline{e}_i$. If $A'N = A'\overline{e}_i$ and $NA'_i = N$, then $N = A'\overline{e}_i$.*

Proof. Since N is an A-module and $\overline{e}_1, \dots, \overline{e}_r \in A$, we can decompose $N = N_1 \oplus \cdots \oplus N_r$ such that $N_j \subseteq \overline{e}_j A' \overline{e}_i$. Each N_j is a right A'_i-module, so is equal to $x^{k_j}\overline{e}_j A'\overline{e}_i$ for some non-negative integer k_j. Since N generates $A'\overline{e}_i$ as a left A'-module, a simple computation shows that $k_j = 0$ for all j. Hence $N = A'\overline{e}_i$. □

Corollary 5.4.3. *There are r blowups of A given by $(\sigma, \tilde{A}_i)$ for $i = 1, \dots, r$ such that*

$$(\tilde{A}_i)_E \quad = \quad O_l(\Lambda' e_i) \quad = \quad \begin{pmatrix} \Delta & J^{-1} \\ J & \Delta \end{pmatrix}^{[i,r-i]} .$$

Proof. The above proposition and Corollary 4.2.8 shows that there are r maximal orders containing $\Lambda = (\sigma^* A)_E$. Since every maximal order containing Λ is a maximal order containing Λ', the local structure of $\tilde{A}_i$ at E follows from Proposition 4.1.2. □

Note 5.4.4. The projective indecomposable left Λ'-module $\Lambda' e_i$ is (c.f. Proposition 4.1.2)

$$\Lambda' e_i \quad = \quad \Delta^{\oplus i} \oplus J^{\oplus r-i}.$$

5.4.1 The $\mathcal{O}_{\tilde{Z}}$-module structure of $\tilde{A}_i$

We now describe $\tilde{A}_i$ as a subsheaf of a certain sheaf of endomorphism rings. Let $\tilde{S}$ be the blowup of S at the origin of $Z = \operatorname{Spec} k[[u,v]]$ (c.f. Corollary 5.3.3) and $\tilde{J}$ be the subsheaf of $\tilde{S}$ defined by $J(\tilde{S}_E) \cap \tilde{S}$. Let $V_i = \tilde{S}^{\oplus i} \oplus \tilde{J}^{\oplus r-i}$ for $i = 1, \dots, r$ and consider the sheaves of endomorphism rings $\mathcal{E}_i = \mathcal{E}nd_{\tilde{S}}(V_i)$. Using the notation from Chapter 4, equation (4.1.1), we can express $\mathcal{E}_i$ as follows

$$\mathcal{E}_i \quad = \quad \begin{pmatrix} \tilde{S} & \tilde{J}^\vee \\ \tilde{J} & \tilde{S} \end{pmatrix}^{[i,r-i]} ,$$

where $\tilde{J}^\vee$ denotes the dual $\mathcal{H}om_{\tilde{S}}(\tilde{J}, \tilde{S})$ of $\tilde{J}$ as an $\tilde{S}$-module.

To state our result, we first define $\mathcal{T} := (\sigma^* A)(\tilde{J}^\vee)$ where $\tilde{J}^\vee$ embeds diagonally into $\tilde{A}_i \otimes k(\tilde{Z})$ and the multiplication is taken in $\tilde{A}_i \otimes k(\tilde{Z})$. Since $\tilde{S}\tilde{J}^\vee = \tilde{J}^\vee$, we can write $\mathcal{T}$ as follows

$$\mathcal{T} \quad = \quad \begin{pmatrix} \tilde{J}^\vee & \cdots & \cdots & \tilde{J}^\vee \\ x\tilde{J}^\vee & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ x\tilde{J}^\vee & \cdots & x\tilde{J}^\vee & \tilde{J}^\vee \end{pmatrix} \subset \tilde{A}_i \otimes k(\tilde{Z}).$$

Proposition 5.4.5. *For $i = 1, \ldots, r$, we have*

$$\tilde{A}_i \quad = \quad \mathcal{E}_i \cap \mathcal{T} \tag{5.4.2}$$

where the intersection is taken in $\tilde{A}_i \otimes k(\tilde{Z})$.

Proof. Since both $\tilde{A}_i$ and $\mathcal{E}_i \cap \mathcal{T}$ are locally free subsheaves of $\tilde{A}_i \otimes k(\tilde{Z})$, to show equality, it suffices to show that $(\tilde{A}_i)_C = (\mathcal{E}_i \cap \mathcal{T})_C$ in $\tilde{A}_i \otimes k(\tilde{Z})$ for all $C \in \tilde{Z}^1$. By Proposition 5.3.2, we have $J = x\Delta = \Delta x$, so $(\tilde{J}^\vee)_E = \mathrm{Hom}_\Delta(J, \Delta) = J^{-1}$ and $(x\tilde{J}^\vee)_E = \Delta$. This shows that at E, we have

$$(\mathcal{E}_i \cap \mathcal{T})_E \quad = \quad \begin{pmatrix} \Delta & J^{-1} \\ J & \Delta \end{pmatrix}^{[i,r-i]} \cap \begin{pmatrix} J^{-1} & \cdots & \cdots & J^{-1} \\ \Delta & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \Delta & \cdots & \Delta & J^{-1} \end{pmatrix} = \begin{pmatrix} \Delta & J^{-1} \\ J & \Delta \end{pmatrix}^{[i,r-i]}$$

which is equal to $(\tilde{A}_i)_E$ by Corollary 5.4.3. For $C \in \tilde{Z}^1 \backslash \{E\}$, we have $\tilde{J}_C = \tilde{S}_C$, so

$$(\mathcal{E}_i \cap \mathcal{T})_C \quad = \quad \tilde{S}_C^{r \times r} \cap (\sigma^* A)_C.$$

Hence $(\mathcal{E}_i \cap \mathcal{T})_C = (\sigma^* A)_C = (\tilde{A}_i)_C$. □

To describe the $\mathcal{O}_{\tilde{Z}}$-module of $\tilde{A}_i$ more explicitly, we record below the $\mathcal{O}_{\tilde{Z}}$-module structures of the $\tilde{S}$-module summands of $\tilde{A}_i$. Note that $\tilde{A}_i$ is a direct sum of the following four $\tilde{S}$-modules, $\tilde{S}$, $\tilde{J}^\vee$, $x\tilde{J}^\vee \cap \tilde{S}$, $x\tilde{J}^\vee \cap \tilde{J}$, and these are all direct sums of invertible $\mathcal{O}_{\tilde{Z}}$-modules. The following proposition is obtained by routine computations.

Proposition 5.4.6. *Let*

$$\tilde{J} \quad = \quad \bigoplus_{j,k=0}^{e-1} \tilde{J}_{jk} x^j y^k$$

with similar notation for the decomposition of $\tilde{J}^\vee_{jk}$ and $x\tilde{J}^\vee$ into direct sums of line bundles. Then

$$\tilde{J}_{jk} \quad = \quad \begin{cases} \mathcal{O}_{\tilde{Z}}(-E) & j = k = 0 \\ \mathcal{O}_{\tilde{Z}} & \textit{if} \quad 0 < j + k < e + 1 \\ \mathcal{O}_{\tilde{Z}}(E) & j + k \geqslant e + 1 \end{cases} \tag{5.4.3}$$

$$\tilde{J}^\vee_{jk} \quad = \quad \begin{cases} \mathcal{O}_{\tilde{Z}} & \textit{if} \quad 0 \leqslant j + k < e \\ \mathcal{O}_{\tilde{Z}}(E) & j + k \geqslant e \end{cases} \tag{5.4.4}$$

$$(x\tilde{J}^\vee)_{jk} \quad = \quad \begin{cases} \mathcal{O}_{\tilde{Z}}(-\tilde{D}) & j = 0 \\ \mathcal{O}_{\tilde{Z}} & \textit{if} \quad j > 0 \quad \textit{and} \quad 0 < j + k < e + 1 \\ \mathcal{O}_{\tilde{Z}}(E) & j > 0 \quad \textit{and} \quad j + k \geqslant e + 1 \end{cases}$$

where $-\sigma^*(D) = -\tilde{D} - E$ *is the divisor on* $\tilde{Z}$ *on which* x^e *vanishes.*

We can conclude from the above proposition that $x\tilde{J}^\vee \cap \tilde{S} = x\tilde{J}^\vee$, and

$$(x\tilde{J}^\vee \cap \tilde{J})_{jk} \quad = \quad \begin{cases} \mathcal{O}_{\tilde{Z}}(-\tilde{D} - E) & \quad j = 0 \quad \text{and} \quad k = 0 \\ \mathcal{O}_{\tilde{Z}}(-\tilde{D}) & \text{if} \quad j = 0 \quad \text{and} \quad k > 0 \\ \mathcal{O}_{\tilde{Z}} & \quad j > 0 \quad \text{and} \quad 0 < j + k < e + 1 \\ \mathcal{O}_{\tilde{Z}}(E) & \quad j > 0 \quad \text{and} \quad j + k \geqslant e + 1 \end{cases} . \qquad (5.4.5)$$

Example 5.4.7. We write out the conclusions of Propositions 5.4.5 and 5.4.6 explicitly in the case $r = 3$, $e = 3$. The three maximal orders containing $(\sigma^* A)_E$ are (c.f. Proposition 5.4.3)

$$O_l(\Lambda' e_1) = \begin{pmatrix} \Delta & J^{-1} \\ J & \Delta \end{pmatrix}^{[1,2]}, \quad O_l(\Lambda' e_2) = \begin{pmatrix} \Delta & J^{-1} \\ J & \Delta \end{pmatrix}^{[2,1]}, \quad O_l(\Lambda' e_3) = \Delta^{3\times 3}.$$

These correspond to blowups $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{A}_3$ where

$$\begin{aligned} \tilde{A}_1 &= \begin{pmatrix} \tilde{S} & \tilde{J}^\vee & \tilde{J}^\vee \\ x\tilde{J}^\vee \cap \tilde{J} & \tilde{S} & \tilde{S} \\ x\tilde{J}^\vee \cap \tilde{J} & x\tilde{J}^\vee & \tilde{S} \end{pmatrix} \\ \tilde{A}_2 &= \begin{pmatrix} \tilde{S} & \tilde{S} & \tilde{J}^\vee \\ x\tilde{J}^\vee & \tilde{S} & \tilde{J}^\vee \\ x\tilde{J}^\vee \cap \tilde{J} & x\tilde{J}^\vee \cap \tilde{J} & \tilde{S} \end{pmatrix} \qquad (5.4.6) \\ \tilde{A}_3 &= \begin{pmatrix} \tilde{S} & \tilde{S} & \tilde{S} \\ x\tilde{J}^\vee & \tilde{S} & \tilde{S} \\ x\tilde{J}^\vee & x\tilde{J}^\vee & \tilde{S} \end{pmatrix}. \end{aligned}$$

We will use the notation for invertible $\mathcal{O}_{\tilde{Z}}$-modules L_{ij},

$$\bigoplus \begin{bmatrix} L_{00} & L_{10} & L_{20} \\ L_{01} & L_{11} & L_{21} \\ L_{02} & L_{12} & L_{22} \end{bmatrix} = \bigoplus_{j,k=0}^{2} L_{jk} x^j y^k.$$

The $\mathcal{O}_{\tilde{Z}}$-module structures of the $\tilde{S}$-modules appearing in the summands of (5.4.6) are as follows:

$$\tilde{S} = \bigoplus \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} & \mathcal{O}(E) \\ \mathcal{O} & \mathcal{O}(E) & \mathcal{O}(E) \end{bmatrix} \quad (5.4.7)$$

$$\tilde{J}^{\vee} = \bigoplus \begin{bmatrix} \mathcal{O} & \mathcal{O} & \mathcal{O}(E) \\ \mathcal{O} & \mathcal{O}(E) & \mathcal{O}(E) \\ \mathcal{O}(E) & \mathcal{O}(E) & \mathcal{O}(E) \end{bmatrix}$$

$$x\tilde{J}^{\vee} \cap \tilde{S} = \bigoplus \begin{bmatrix} \mathcal{O}(-\tilde{D}) & \mathcal{O} & \mathcal{O} \\ \mathcal{O}(-\tilde{D}) & \mathcal{O} & \mathcal{O}(E) \\ \mathcal{O}(-\tilde{D}) & \mathcal{O}(E) & \mathcal{O}(E) \end{bmatrix}$$

$$x\tilde{J}^{\vee} \cap \tilde{J} = \bigoplus \begin{bmatrix} \mathcal{O}(-E-\tilde{D}) & \mathcal{O} & \mathcal{O} \\ \mathcal{O}(-\tilde{D}) & \mathcal{O} & \mathcal{O} \\ \mathcal{O}(-\tilde{D}) & \mathcal{O} & \mathcal{O}(E) \end{bmatrix}.$$

5.4.2 Proof of Theorem 5.2.6

Let M_i and I be as in Section 5.2 and denote $\mathfrak{m} = (u,v) \in \operatorname{Spec} k[[u,v]]$.

Lemma 5.4.8. *For any positive integer m, we have*

$$(M_i I^{-1})^{mre} = u^{m-1}\mathfrak{m}^{m-1}(M_i I^{-1})^{re}. \quad (5.4.8)$$

Proof. By direct computation, we can obtain the following formula for computing the graded pieces of $\operatorname{Rees}(A, M_i I^{-1})$: for any integer s, we have

$$(M_i I^{-1})^s = M_i M_{i+1} \dots M_{i+s} I^{-s},$$

where $M_0 = M_r$ and the indices are taken modulo r (c.f. [VdB01], Proposition 6.1.3). Using this, one can show that

$$(M_i I^{-1})^r = \begin{pmatrix} \mathcal{I}(i) & S^{i\times(r-i)} \\ (x(x,y))^{(r-i)\times i} & \mathcal{I}(r-i) \end{pmatrix} I^{-r}$$

where

$$\mathcal{I}(k) = \begin{pmatrix} (x,y) & \dots & \dots & (x,y) \\ xS & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ xS & \dots & xS & (x,y) \end{pmatrix} \subset S^{k\times k}.$$

A routine computation shows that

$$(M_iI^{-1})^{2r} \quad = \quad (x,y)(M_iI^{-1})^rI^{-r}.$$

This gives

$$\begin{aligned}(M_iI^{-1})^{2re} &= (x,y)^{2e-1}(M_iI^{-1})^rI^{-(2e-1)r}\\ &= \mathfrak{m}(x,y)^{e-1}(M_iI^{-1})^rI^{-(2e-1)r}\\ &= \mathfrak{m}(M_iI^{-1})^{re}I^{-er}\end{aligned}$$

from which (5.4.8) follows, since $I^{-er} = u^{-1}A$. □

By Proposition 5.1.1, Theorem 5.2.6 is equivalent to the following.

Corollary 5.4.9. *Let $\mathcal{D}(i)$ be the re-th Veronese subalgebra of* $\text{Rees}(A, M_iI^{-1})$. *Then*

$$\begin{aligned}\mathcal{D}(i) &= A \oplus (M_iI^{-1})^{re} \oplus u^{-1}\mathfrak{m}(M_iI^{-1})^{re} \oplus u^{-2}\mathfrak{m}^2(M_iI^{-1})^{re} \oplus \cdots\\ &\simeq \Gamma_\bullet(\tilde{A}_i).\end{aligned}$$

Proof. The first equality follows from Lemma 5.4.8.

To describe $\Gamma_\bullet(\tilde{A}_i)$, note that by Proposition 5.4.6 and (5.4.5), the $\mathcal{O}_{\tilde{Z}}$-module $\tilde{A}_i$ is a direct sum of the line bundles $\mathcal{O}_{\tilde{Z}}$, $\mathcal{O}_{\tilde{Z}}(E)$, $\mathcal{O}_{\tilde{Z}}(-\tilde{D})$ and $\mathcal{O}_{\tilde{Z}}(-\tilde{D}-E)$. If L is any of these four line bundles on $\tilde{Z}$, then by the projection formula, we have $H^0(\tilde{Z}, L(-sE)) = \mathfrak{m}^{s-1}H^0(\tilde{Z}, L(-E))$ for any integer $s \geqslant 1$. This shows that

$$\Gamma_\bullet(\tilde{A}_i) \quad = \quad A \oplus H^0(\tilde{Z}, \tilde{A}_i(-E)) \oplus \mathfrak{m}H^0(\tilde{Z}, \tilde{A}_i(-E)) \oplus \mathfrak{m}^2H^0(\tilde{Z}, \tilde{A}_i(-E)) \oplus \cdots.$$

The isomorphism $\mathcal{D}(i) \simeq \Gamma_\bullet(\tilde{A}_i)$ follows from the following lemma. □

Lemma 5.4.10. *There is a canonical isomorphism of A-modules*

$$u^{-1}H^0(\tilde{Z}, \tilde{A}_i(-E)) \quad \simeq \quad (M_iI^{-1})^{re},$$

hence $D(i) \simeq \Gamma_\bullet(\tilde{A}_i)$ as graded algebras.

Proof. Comparing $(M_iI^{-1})^{re} = (x,y)^{e-1}(M_iI^{-1})^rI^{-r}$ and $H^0(\tilde{Z}, \tilde{A}_i(-E))$, it suffices to show the following,

$$\begin{aligned}(x,y)^e &= H^0(\tilde{Z}, \tilde{S}(-E))\\ x(x,y)^e &= H^0(\tilde{Z}, (x\tilde{J}^\vee \cap \tilde{J})(-E))\\ (x,y)^{e-1} &= H^0(\tilde{Z}, \tilde{J}^\vee(-E))\\ x(x,y)^{e-1} &= H^0(\tilde{Z}, (x\tilde{J}^\vee \cap \tilde{S})(-E)).\end{aligned}$$

These all follow from direct computation. Finally, the isomorphism $\mathcal{D}(i) \simeq \Gamma_\bullet(\tilde{A}_i)$ follows from the same argument which shows that $\mathcal{D}$ and $\Gamma_\bullet(\tilde{S})$ are isomorphic as graded algebras in the proof of Proposition 5.3.4. $\square$

Example 5.4.11. For the case $r = e = 3$, we have

$$\begin{aligned}
(M_1 I^{-1})^9 &= \begin{pmatrix} (x,y)^3 & (x,y)^3 & (x,y)^3 \\ x(x,y)^2 & (x,y)^3 & (x,y)^3 \\ x(x,y)^2 & x(x,y)^2 & (x,y)^3 \end{pmatrix} u^{-1} \qquad (5.4.9) \\
(M_2 I^{-1})^9 &= \begin{pmatrix} (x,y)^3 & (x,y)^3 & (x,y)^2 \\ x(x,y)^2 & (x,y)^3 & (x,y)^2 \\ x(x,y)^3 & x(x,y)^3 & (x,y)^3 \end{pmatrix} u^{-1} \\
(M_3 I^{-1})^9 &= \begin{pmatrix} (x,y)^3 & (x,y)^2 & (x,y)^2 \\ x(x,y)^3 & (x,y)^3 & (x,y)^3 \\ x(x,y)^3 & x(x,y)^2 & (x,y)^3 \end{pmatrix} u^{-1}.
\end{aligned}$$

(compare with (5.4.6)), and

$$\begin{aligned}
(x,y)^3 &= \bigoplus \begin{bmatrix} \mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ \mathfrak{m} & \mathfrak{m} & k[[u,v]] \\ \mathfrak{m} & k[[u,v]] & k[[u,v]] \end{bmatrix} \\
(x,y)^2 &= \bigoplus \begin{bmatrix} \mathfrak{m} & \mathfrak{m} & k[[u,v]] \\ \mathfrak{m} & k[[u,v]] & k[[u,v]] \\ k[[u,v]] & k[[u,v]] & k[[u,v]] \end{bmatrix} \\
x(x,y)^2 &= \bigoplus \begin{bmatrix} (u) & \mathfrak{m} & \mathfrak{m} \\ (u) & \mathfrak{m} & k[[u,v]] \\ (u) & k[[u,v]] & k[[u,v]] \end{bmatrix} \\
x(x,y)^3 &= \bigoplus \begin{bmatrix} u\mathfrak{m} & \mathfrak{m} & \mathfrak{m} \\ (u) & \mathfrak{m} & \mathfrak{m} \\ (u) & \mathfrak{m} & k[[u,v]] \end{bmatrix}.
\end{aligned}$$

Comparing with (5.4.7), we see that the above are equal to $\Gamma_1(\tilde{S})$, $\Gamma_1(\tilde{J}^\vee)$, $\Gamma_1(x\tilde{J}^\vee \cap \tilde{S})$ and $\Gamma_1(x\tilde{J}^\vee \cap \tilde{J})$ respectively.

Chapter 6

Blowing up canonical orders

Recall that the blowup of orders involves a choice of maximal orders containing a given order over a discrete valuation ring. In this chapter, we apply the techniques of Chapter 4 to construct blowups of canonical orders. The original motivation for constructing blowups of canonical orders was to define rational singularities for orders and, to this end, the examples in this chapter served as our first test cases. Although our definition of numerical rationality (c.f. Definition 3.1.5) does not depend on the choice of maximal orders involved in blowing up, the construction of blowing up orders is interesting in its own right. Knowledge about sheaf theoretic structure of the blown up order $\tilde{A}$ allow us to ask interesting questions about $\tilde{A}$, for instance, questions regarding the moduli space of locally projective $\tilde{A}$-modules, or the structure of the derived category of $\tilde{A}$. We saw in Chapter 5 that the blowups of a terminal order A can be obtained by blowing up different points on the quasi-scheme $\operatorname{Mod} A$. While we do not yet have such an interpretation for non-terminal orders, we hope that knowledge of the possible blowups of canonical orders will help in understanding the blowups of orders more generally.

6.1 Preliminaries

The setup for this chapter will be as follows: let A be a canonical order on $Z = \operatorname{Spec} k[[u, v]]$ and $\sigma : \tilde{Z} \longrightarrow Z$ be the blowup of Z at the origin with $E = \operatorname{Ex}(\sigma)$. Let R be the completion $\widehat{\mathcal{O}}_{\tilde{Z},E}$ of $\mathcal{O}_{\tilde{Z},E}$ with uniformising parameter u and residue field $\kappa = k(v/u)$. We denote by K the field of fractions of R. Finally, we denote by Λ the R-order $(\sigma^* A)_{\widehat{E}} = A \otimes_{k[[u,v]]} R$. As in Chapter 4, we will reserve the capital Greek letters $\Lambda, \Gamma, \ldots$ to denote R-orders. All ideals are two-sided ideals unless indicated otherwise. The aim of this chapter is to find all the blowups of the canonical orders A in the cases where $\Lambda = A \otimes_{k[[u,v]]} R$ is a local ring.

The canonical orders are classified by their ramification data in [CHI09], and we will refer to a canonical order as having a certain ramification type. For convenience, we will refer to the R-order $\Lambda = A \otimes_{k[[u,v]]} R$ as having the same ramification type of A. Blowups of local canonical orders of each ramification type with smooth centre will be constructed below, and in doing so, we will need generators and relations for low rank canonical orders given in Appendix A.

Lemma 6.1.1. *Let A be a canonical order with Gorenstein centre and $\Lambda = A \otimes_{k[[u,v]]} R$, then Λ is a Gorenstein order.*

Proof. By [CHI09], Section 7, canonical orders are Gorenstein, that is, $\mathcal{H}om_{\mathcal{O}_Z}(A, \omega_Z)$ is locally isomorphic to A as a left A-module and as a right A-module. If Z is Gorenstein, then $\omega_Z \simeq \mathcal{O}_Z$. By hom-tensor adjunction, we have

$$\begin{aligned} \mathrm{Hom}_R(A \otimes_{\mathcal{O}_Z} R, R) &\simeq \mathrm{Hom}_{\mathcal{O}_Z}(A, \mathrm{Hom}_R(R, R)) \\ &\simeq \mathrm{Hom}_{\mathcal{O}_Z}(A, \mathcal{O}_Z) \otimes_{\mathcal{O}_Z} R \\ &\simeq A \otimes_{\mathcal{O}_Z} R. \end{aligned}$$

Since all the isomorphisms above are isomorphisms of right Λ-modules, it follows that Λ is a Gorenstein order. □

Lemma 6.1.2. *Let Ω be an R-order containing Λ. If Ω/Λ is a length 1 Λ-module, then Ω is a minimal order containing Λ. If, furthermore, Λ is a non-hereditary local Gorenstein order, then $\Omega = O_l(J(\Lambda))$.*

Proof. Since Ω/Λ has length 1, we have $d(\Omega) = u^{-1}d(\Lambda)$. If Γ is an order such that $\Lambda \subseteq \Gamma \subseteq \Omega$, then $d(\Lambda) \subseteq d(\Gamma) \subseteq d(\Omega)$, so $d(\Gamma)$ is equal to $d(\Lambda)$ or $d(\Omega)$. By [CR81], Proposition 26.3 iii, $\Gamma = \Omega$, hence Ω is minimal. If Λ is also a non-hereditary local Gorenstein order, then by [Nis92], Lemma 1.2 the minimal order containing Λ is unique and is equal to $O_l(J(\Lambda))$. Hence $\Omega = O_l(J(\Lambda))$. □

6.2 Canonical quaternion orders

We first classify quaternion orders over a complete discrete valuation ring which are generically split and whose discriminant ideal is the square of the maximal ideal. Then we apply the results of Chapter 4 to determine the numbers of maximal orders containing such orders.

Proposition 6.2.1. *Let R be a complete discrete valuation ring with field of fractions K, residue field κ and uniformising parameter u. Let Λ be an R-order in $K^{2\times 2}$ with $d(\Lambda) = u^2R$. Then*

$$\Lambda \simeq \begin{pmatrix} R & u^2R \\ R & R \end{pmatrix} \text{ or } \Lambda_1 \tag{6.2.1}$$

where Λ_1 is the subalgebra of $R^{2\times 2}$ generated by

$$x = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.2.2}$$

Proof. We can assume that, after applying an appropriate inner automorphism, that Λ is contained in $\Omega = R^{2\times 2}$. If $d(\Lambda) = u^2R$, then $\Lambda/J(\Lambda) \simeq \kappa \times \kappa$ or κ. We obseve that if $\Lambda/J(\Lambda)$ has idempotents, then we can lift these to Λ, and by the Pierce decomposition, we have

$$\Lambda \simeq \begin{pmatrix} R & u^2R \\ R & R \end{pmatrix}.$$

Now suppose that $\Lambda/J(\Lambda) \simeq \kappa$. Assuming that $u\Omega$ is an ideal contained in Λ, the quotient $\Lambda/u\Omega$ has length 2. Since Ω has no idempotents, we have $\Lambda/u\Omega \simeq \kappa[\varepsilon]/(\varepsilon^2)$. After applying an inner automorphism, we can assume that $\Lambda/u\Omega$ embeds as the subalgebra generated by e_{21} in $\Omega/u\Omega \simeq \kappa^{2\times 2}$. A simple calculation then shows that $\Lambda = \Lambda_1$.

It remains to show that $u\Omega$ is an ideal contained in Λ. First we have the following chain of inclusions $J(\Lambda) \subseteq \Lambda \subseteq \Lambda' = O_l(J(\Lambda))$. Note that since Λ is Gorenstein so $O_l(J(\Lambda)) = O_r(J(\Lambda))$, that is, $J(\Lambda)$ is an ideal of Λ'. Since Λ'/Λ has length 1, we see that $\Lambda'/J(\Lambda)$ has length 2. Now $d(\Lambda') = uR$, so Λ' is hereditary. Hence $\Lambda'/J(\Lambda)$ must have idempotents, so $\Lambda'/J(\Lambda) \simeq \kappa \times \kappa$. This shows that $J(\Lambda) = J(\Lambda')$. Now we apply an inner automorphism and assume that

$$\Lambda' = \begin{pmatrix} R & uR \\ R & R \end{pmatrix} \subseteq \Omega. \tag{6.2.3}$$

The the chain of inclusions $J(\Lambda') \subset \Lambda \subset \Lambda' \subset \Omega$ and $u\Omega \subset J(\Lambda')$ shows that $u\Omega$ is an ideal contained in Λ. ☐

Proposition 6.2.2. *With the notation and hypotheses of the above proposition, if $\Lambda \simeq \Lambda_1$, then there are exactly two maximal orders,*

$$\begin{pmatrix} R & uR \\ u^{-1}R & R \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} R & R \\ R & R \end{pmatrix},$$

in $K^{2\times 2}$ containing Λ. Otherwise, there are exactly three such maximal orders,

$$\begin{pmatrix} R & u^2R \\ u^{-2}R & R \end{pmatrix}, \begin{pmatrix} R & uR \\ u^{-1}R & R \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} R & R \\ R & R \end{pmatrix}.$$

Proof. First suppose that $\Lambda \simeq \Lambda_1$. Let Λ' be the hereditary order defined in (6.2.3), so that we have the inclusions $J \subseteq \Lambda \subseteq \Lambda'$ where $J = J(\Lambda') = J(\Lambda)$. Note that $\Lambda/J \simeq \kappa$, $\Lambda'/J \simeq \kappa \times \kappa$ and the morphism $\delta : \kappa \longrightarrow \kappa \times \kappa$ induced by the inclusion $\Lambda \subseteq \Lambda'$ is just the diagonal map. Let e_{11}, e_{22} be the standard matrix idempotents in Λ_1 and denote by ε_1, ε_2 their images in $\Lambda'/J \simeq \kappa \times \kappa$. We also denote by p_1, p_2 the projective Λ'/J modules $(\Lambda'/J)\varepsilon_1$, $(\Lambda'/J)\varepsilon_2$. It is clear that $\text{Ind}_2(\mathcal{C}(\delta)) = \{p_i \hookrightarrow p_i\}_{i=1}^2$ and both elements are saturated. By Corollary 4.2.8, there are exactly two maximal orders containing Λ. The following table shows the correspondence between maxord(Λ), $\text{Ind}_2(\Lambda)$ and $\text{Ind}_2(\mathcal{C}(\delta))$.

$O_l(L) \in \text{maxord}(\Lambda)$	$L \in \text{Ind}_2(\Lambda)$	$F(L) \in \text{Ind}_2(\mathcal{C}(\delta))$
$\begin{pmatrix} R & R \\ R & R \end{pmatrix}$	$\begin{pmatrix} R & 0 \\ R & 0 \end{pmatrix}$	$p_1 \hookrightarrow p_1$
$\begin{pmatrix} R & uR \\ u^{-1}R & R \end{pmatrix}$	$\begin{pmatrix} 0 & uR \\ 0 & R \end{pmatrix}$	$p_2 \hookrightarrow p_2$

, (6.2.4)

where $F : \text{Ind}_2(\Lambda) \longrightarrow \text{Ind}_2(\mathcal{C}(\delta))$ is the map induced by the functor defined in (4.2.1).

If Λ is not isomorphic to Λ_1, then we can embed Λ in the following hereditary order

$$\Lambda \simeq \begin{pmatrix} R & u^2R \\ R & R \end{pmatrix} \subseteq \begin{pmatrix} R & uR \\ R & R \end{pmatrix} = \Lambda'$$

In this case $J(\Lambda')^2 \subseteq J(\Lambda)$, and the morphism $\Lambda/J(\Lambda')^2 \longrightarrow \Lambda'/J(\Lambda')^2$ can be identified with embedding of lower triangular matrices

$$\iota : \begin{pmatrix} \kappa & 0 \\ \kappa & \kappa \end{pmatrix} \longrightarrow \kappa^{2\times 2}$$

in the full 2×2 matrix κ-algebra. The following table shows the correspondence between maxord(Λ), $\text{Ind}_2(\Lambda)$ and $\text{Ind}_2(\mathcal{C}(\iota))$.

$O_l(L) \in \mathrm{maxord}(\Lambda)$	$L \in \mathrm{Ind}_2(\Lambda)$	$F(L) \in \mathrm{Ind}_2(\mathcal{C}(\iota))$
$\begin{pmatrix} R & R \\ R & R \end{pmatrix}$	$\begin{pmatrix} R & 0 \\ R & 0 \end{pmatrix}$	$\begin{pmatrix} \kappa & 0 \\ \kappa & 0 \end{pmatrix} \hookrightarrow \begin{pmatrix} \kappa & 0 \\ \kappa & 0 \end{pmatrix}$
$\begin{pmatrix} R & uR \\ u^{-1}R & R \end{pmatrix}$	$\begin{pmatrix} 0 & uR \\ 0 & R \end{pmatrix}$	$\begin{pmatrix} 0 & \kappa \\ 0 & \kappa \end{pmatrix} \hookrightarrow \begin{pmatrix} 0 & \kappa \\ 0 & \kappa \end{pmatrix}$
$\begin{pmatrix} R & u^2R \\ u^{-2}R & R \end{pmatrix}$	$\begin{pmatrix} 0 & u^2R \\ 0 & R \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & \kappa \end{pmatrix} \hookrightarrow \begin{pmatrix} 0 & \kappa \\ 0 & \kappa \end{pmatrix}$

(6.2.5)

One can check that each element of $\mathrm{Ind}_2(C(\iota))$ is saturated. By Corollary 4.2.8, the maximal orders containing Λ are exactly the ones appearing in the above table. □

Now we apply the above results to find the number of blowups for canonical quaternion orders with smooth centre. We first recall the table of ramification types of canonical orders and their discriminant divisors D from [CHI09], Theorem 6.5.

ramification type of A	range of n	equation of D	$\mathrm{mult}_p D$
BL_n	$n \geqslant 1$	$u^{2n+1} - v^2$	2
B_n	$n \geqslant 1$	$(v - u^n)(v + u^n)$	2
L_n	$n \geqslant 1$	$(v - u^{n+1})(v + u^{n+1})$	2
DL_1		$u(v^2 - u)$	2
DL_n	$n \geqslant 2$	$u(v^2 - u^{2n-1})$	3
BD_n	$n \geqslant 2$	$u(v - u^{n-1})(v + u^{n-1})$	3

(6.2.6)

Theorem 6.2.3. *Let A be a canonical quaternion order on $Z = \mathrm{Spec}\, k[[u, v]]$ with discriminant divisor D. Denote by $\sigma : \tilde{Z} \longrightarrow Z$ be the blowup of Z at the origin p and $E = \mathrm{Ex}(\sigma)$.*

1. *Suppose that D has a double point at p. If A is local, then there are two blowups of A on $\tilde{Z}$; otherwise A has three blowups on $\tilde{Z}$.*

2. *Suppose that D has a triple point p, then there is a unique blowup of A on $\tilde{Z}$.*

Proof. First note that a canonical quaternion order A can be written in the form

$$A \simeq \frac{k[[u, v]]\,\langle x, y\rangle}{(x^2 - a, y^2 - b, xy + yx - 2c)} \tag{6.2.7}$$

for some $a, b, c \in k[[u, v]]$ (c.f. Appendix A). The reduced discriminant ideal of A is given by $d(A) = (ab - c^2)$, so $d(\sigma^* A)_{\widehat{E}} = u^m \widehat{\mathcal{O}}_{Z,E}$ where $m = \mathrm{mult}_p D$.

Suppose that $\mathrm{mult}_p D = 2$, then from (6.2.6) we see that A has ramification type BL_n, B_n, L_n or DL_1. We show that in all these cases $(\sigma^* A)_{\widehat{E}}$ is generically split. For ramification types BL_n and B_n, note that the secondary ramification indices of the irreducible components of D at p are all equal to 1, so we can conclude from [CI05], Lemma 3.4 that a blowup of A on $\tilde{Z}$ is unramified at E. Hence $(\sigma^* A)_E$ is generically split. For ramification types L_n and DL_1, we show that $(\sigma^* A)_{\widehat{E}}$ has a nontrivial idempotent. By the generators and relations given in (A.1.4) and (A.1.6), we see that a canonical quaternion order of type L_n or DL_1 can be written in the form (6.2.7) with $ab \in u^3 R$ and $c = \lambda u$ for some unit $u \in \widehat{\mathcal{O}}_{\tilde{Z},E}$. A simple calculation shows that the image of $xy/(2c)$ in $(\sigma^* A)_{\widehat{E}}/u(\sigma^* A)_{\widehat{E}}$ is an idempotent, which lifts to an idempotent in $(\sigma^* A)_{\widehat{E}}$ since $(\sigma^* A)_{\widehat{E}}$ is finite over a complete local ring. This shows that $(\sigma^* A)_{\widehat{E}}$ is generically split. By the construction of blowing up for orders, the number of blowups of A on $\tilde{Z}$ is equal to the number of maximal orders containing $(\sigma^* A)_{\widehat{E}}$, hence the first assertion follows from Proposition 6.2.2.

Next suppose that $\mathrm{mult}_p D = 3$, we show that there is a unique maximal order containing $(\sigma^* A)_{\widehat{E}}$. We assume that A has the generators and relations given in (A.1.6) or (A.1.8). We claim that

$$\Omega = (\sigma^* A)_{\widehat{E}} \left\langle u^{-1} y \right\rangle \tag{6.2.8}$$

is the unique maximal order in Ω_K. As an R-module, Ω is generated by $1, x, u^{-1}y, u^{-1}xy$, and a simple computation shows that

$$\frac{\Omega}{u\Omega} \simeq \frac{\kappa \langle x, y \rangle}{(x^2, y^2 - \nu, xy + yx)}$$

where ν is a squarefree element of κ. This shows that $\Omega/J(\Omega)$ is a quadratic field extension of κ, hence Ω is local. Furthermore, we have $d(\Omega) = u^{-2} d((\sigma^* A)_{\widehat{E}}) = uR$, so Ω is hereditary. By [AG60], Theorem 3.11, Ω_K is a division ring and Ω is the unique maximal order in Ω_K.

□

Corollary 6.2.4. *With the hypotheses and notation of the theorem above, the following table shows the number of blowups associated to each canonical quaternion order of a given* *ramification type.*

ramification type of A	*range of n*	*equation of D*	$\text{mult}_p D$	*number of blowups*
BL_n	$n \geqslant 1$	$u^{2n+1} - v^2$	2	2 *if local,* 3 *otherwise*
B_n	$n \geqslant 1$	$(v - u^n)(v + u^n)$	2	2 *if local,* 3 *otherwise*
L_n	$n \geqslant 1$	$(v - u^{n+1})(v + u^{n+1})$	2	2
DL_1		$u(v^2 - u)$	2	2
DL_n	$n \geqslant 2$	$u(v^2 - u^{2n-1})$	3	1
BD_n	$n \geqslant 2$	$u(v - u^{n-1})(v + u^{n-1})$	3	1

Proof. The equations of D are given in [CHI09], Theorem 6.5, or can be computed from the generators and relations given in Appendix A. We note here that if A is a canonical quaternion order of type L_n, DL_n or BD_n for $n \geqslant 1$, then it is always local (c.f. (A.1.4), (A.1.6) and (A.1.8)). The only non-local quaternion orders appear with ramification types B_n (in (A.1.3) with $i = 0$ or $2n$) or BL_n (in (A.1.2), with $i = 0$). The result then follows from Theorem 6.2.3. □

We conclude this section by writing down explicitly the maximal orders containing $(\sigma^* A)_{\widehat{E}}$ when $\text{mult}_p D = 2$. Note that when $\text{mult}_p D = 3$, the unique maximal order containing $(\sigma^* A)_{\widehat{E}}$ is given in (6.2.8).

Corollary 6.2.5. *Let A be a canonical quaternion order on $Z = \operatorname{Spec} k[[u,v]]$ whose discriminant divisor D has a double point at the origin. Denote by $\sigma : \tilde{Z} \longrightarrow Z$ be the blowup of Z at the origin p and $E = \operatorname{Ex}(\sigma)$. We assume that A has the generators and relations as given in Appendix A and we denote $\Lambda = (\sigma^* A)_{\widehat{E}}$.*

1. *If A is local, then the maximal orders containing Λ are*

$$\Lambda \left\langle u^{-1}x \right\rangle \quad \textit{and} \quad \Lambda \left\langle u^{-1}y \right\rangle .$$

2. *If A is not local, then*

$$A \quad \simeq \quad \frac{k[[u,v]] \langle x, y \rangle}{(x^2 - 1, y^2 - u^m, xy + yx - 2v)}$$

for some positive integer m. The maximal orders containing Λ are

$$\Lambda \left\langle u^{-2}(1+x)(y - vx) \right\rangle, \Lambda \left\langle u^{-1}y \right\rangle \quad \textit{and} \quad \Lambda \left\langle u^{-2}(1-x)(y - vx) \right\rangle .$$

Proof. We assume that A has the form (6.2.7), and denote the images of $a, b, c \in k[[u, v]]$ in $\mathcal{O}_{\tilde{Z},E}$ or $\widehat{\mathcal{O}}_{\tilde{Z},E}$ by the same notation. Suppose that A is local with $a = p^2$ for some $p \in \mathcal{O}_{\tilde{Z},E}$. The map $\varphi : \Lambda_K \longrightarrow k(\mathcal{O}_{\tilde{Z},E})^{2\times 2}$ defined by

$$x \longmapsto \begin{pmatrix} p & 2c \\ 0 & -p \end{pmatrix} \quad \text{and} \quad y \longmapsto \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

sends Λ to Λ_1, where Λ_1 is the order defined in Proposition 6.2.1. We can obtain the maximal orders containing Λ by taking φ^{-1} of the two maximal orders containing Λ_1 (c.f. (6.2.4)), and one checks readily that

$$\begin{aligned} \varphi^{-1}\begin{pmatrix} \mathcal{O}_{\tilde{Z},E} & \mathcal{O}_{\tilde{Z},E} \\ \mathcal{O}_{\tilde{Z},E} & \mathcal{O}_{\tilde{Z},E} \end{pmatrix} &= \Lambda\left\langle u^{-1}x\right\rangle \\ \varphi^{-1}\begin{pmatrix} \mathcal{O}_{\tilde{Z},E} & u\mathcal{O}_{\tilde{Z},E} \\ u^{-1}\mathcal{O}_{\tilde{Z},E} & \mathcal{O}_{\tilde{Z},E} \end{pmatrix} &= \Lambda\left\langle u^{-1}y\right\rangle. \end{aligned}$$

If A is local but neither a nor b are squares, then we need to work complete locally to get an embedding $\Lambda \longrightarrow k(\widehat{\mathcal{O}}_{\tilde{Z},E})^{2\times 2}$. We will not write down such an embedding explicitly, and simply note that $\Lambda\left\langle u^{-1}x\right\rangle$ and $\Lambda\left\langle u^{-1}y\right\rangle$ are generated by the elements $1, u^{-1}x, y, u^{-1}xy$ and $1, x, u^{-1}y, u^{-1}xy$ respectively. This shows that $d(\Lambda\left\langle u^{-1}x\right\rangle) = d(\Lambda\left\langle u^{-1}y\right\rangle) = u^{-2}d(\Lambda) = \mathcal{O}_{\tilde{Z},E}$, hence both are maximal orders.

Now suppose that A is not local, then we define a map $\psi : \Lambda_K \longrightarrow k(\mathcal{O}_{\tilde{Z},E})^{2\times 2}$ by

$$x \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad y \longmapsto \begin{pmatrix} v & b - v^2 \\ 1 & -v \end{pmatrix}.$$

The map ψ sends Λ to $\begin{pmatrix} \mathcal{O}_{\tilde{Z},E} & u^2\mathcal{O}_{\tilde{Z},E} \\ \mathcal{O}_{\tilde{Z},E} & \mathcal{O}_{\tilde{Z},E} \end{pmatrix}$, and taking the inverse images of the three maximal orders in (6.2.5), we see that

$$\begin{aligned} \varphi^{-1}\begin{pmatrix} \mathcal{O}_{\tilde{Z},E} & \mathcal{O}_{\tilde{Z},E} \\ \mathcal{O}_{\tilde{Z},E} & \mathcal{O}_{\tilde{Z},E} \end{pmatrix} &= \Lambda\left\langle u^{-2}(1+x)(y-vx)\right\rangle, \\ \varphi^{-1}\begin{pmatrix} \mathcal{O}_{\tilde{Z},E} & u\mathcal{O}_{\tilde{Z},E} \\ u^{-1}\mathcal{O}_{\tilde{Z},E} & \mathcal{O}_{\tilde{Z},E} \end{pmatrix} &= \Lambda\left\langle u^{-1}y\right\rangle, \\ \varphi^{-1}\begin{pmatrix} \mathcal{O}_{\tilde{Z},E} & u^2\mathcal{O}_{\tilde{Z},E} \\ u^{-2}\mathcal{O}_{\tilde{Z},E} & \mathcal{O}_{\tilde{Z},E} \end{pmatrix} &= \Lambda\left\langle u^{-2}(1-x)(y-vx)\right\rangle. \end{aligned}$$

□

Corollary 6.2.6. *With the hypotheses and notation of Corollary 6.2.5,*

1. *if A is local, then the blowups of A are*

$$\mathcal{O}_{\tilde{Z}} \oplus \mathcal{O}_{\tilde{Z}}(E)x \oplus \mathcal{O}_{\tilde{Z}}y \oplus \mathcal{O}_{\tilde{Z}}(E)xy \quad \text{and} \quad \mathcal{O}_{\tilde{Z}} \oplus \mathcal{O}_{\tilde{Z}}x \oplus \mathcal{O}_{\tilde{Z}}(E)y \oplus \mathcal{O}_{\tilde{Z}}(E)xy$$

2. *If A is not local, then the blowups of A are*

$$\begin{pmatrix} \mathcal{O}_{\tilde{Z}} & \mathcal{O}_{\tilde{Z}}(-\tilde{D} - iE) \\ \mathcal{O}_{\tilde{Z}}(iE) & \mathcal{O}_{\tilde{Z}} \end{pmatrix}$$

   for $i = 0, 1, 2$.

6.3 Local canonical orders of rank 16

The only ramification types for which there exist local orders of rank 16 are BD_n and DL_n (c.f. [CHI09], Section 8). For such a canonical order A, we show that $(\sigma^* A)_{\widehat{E}}$ is an order in the central simple algebra $D^{2\times 2}$ where D is a quaternion division algebra. We begin with some basic facts about orders in $D^{2\times 2}$. Let R denote a complete discrete valuation ring with uniformising parameter u, residue field κ, and field of fractions K. Let Δ denote the unique maximal order in D. Firstly, an order Ω in $D^{2\times 2}$ is maximal if and only if $d(\Omega) = u^4 R$ since $d(\Delta^{2\times 2}) = u^4 R$. We say that an order Λ is *tiled* if it contains a complete set of orthogonal idempotents of Λ_K. As in the quaternion case, for tiled orders in $D^{2\times 2}$, we can detect the hereditary property using the discriminant ideal.

Lemma 6.3.1. *Suppose κ has trivial Brauer group, then $\Delta/J(\Delta)$ is a quadratic extension of κ.*

Proof. Since $\mathrm{Br}(\kappa)$ is trivial there are three possibilities for $\Delta/J(\Delta)$: κ, a quadratic extension of κ, or $\kappa^{2\times 2}$. By [Rei03], Theorem 13.2, $\Delta/J(\Delta)$ is a quadratic extension of κ. □

Proposition 6.3.2. *Suppose that κ has trivial Brauer group. Let D be a quaternion division algebra and Λ be a tiled order in $D^{2\times 2}$. If $d(\Lambda)$ is generated by u^6, then Λ is hereditary.*

Proof. Let Δ be the unique maximal order in D. We can choose orthogonal idempotents e_1 and e_2 in Λ such that the Pierce decomposition $\Lambda = (\Lambda_{ij})$ embeds naturally in $\Omega =$

$(\Omega_{ij}) = \Delta^{2\times 2}$, and we will identify Λ as a subalgebra of Ω. Note that $d(\Omega)$ is generated by u^4, so Λ is properly contained in Ω.

First suppose that $\Lambda_{11} = \Omega_{11}$ and $\Lambda_{22} = \Omega_{22}$. Then Λ_{12} is a sub $(\Delta - \Delta)$-bimodule of $\Omega_{12} \simeq \Delta$, so $\Lambda_{12} \simeq J(\Delta)^{k_{12}}$, and similarly $\Lambda_{21} \simeq J(\Delta)^{k_{21}}$, for non-negative integers k_{12}, k_{21}. Since $d(\Lambda) = u^6 R$, we see that the only possibilities are $\Lambda_{12} \simeq \Delta$, $\Lambda_{21} \simeq J(\Delta)$ or $\Lambda_{12} \simeq J(\Delta)$, $\Lambda_{21} \simeq \Delta$. This shows that Λ is hereditary.

Next we show that Λ_{11} cannot be a proper subalgebra of Ω_{11}, and the same proof shows the analogous statement for Λ_{22}. Suppose that Λ_{11} is a proper subalgebra of Ω_{11}. Since $d(\Lambda) = u^6 R$, at most one of the inclusions $\Lambda_{st} \subseteq \Omega_{st}$ for $(s,t) \neq (1,1)$ is proper. If $\Lambda_{st} = \Omega_{st}$ for all $(s,t) \neq (1,1)$, then $\Lambda_{11} \supseteq \Lambda_{12}\Lambda_{21} = \Omega_{12}\Omega_{21} = \Omega_{11}$, which is a contradiction. The same argument shows that $\Lambda_{22} \subseteq \Omega_{22}$ cannot be a proper inclusion. Hence the only possibilities are

$$\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Delta & \Delta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \Lambda_{11} & \Delta \\ \Lambda_{21} & \Delta \end{pmatrix}.$$

In the first case, since Λ is a subalgebra of Ω, we have $\Lambda_{11}\Lambda_{12} + \Lambda_{12}\Delta \subset \Lambda_{12}$. So Λ_{12} is a proper right Δ-submodule of $\Omega_{12} \simeq \Delta$, hence $\Lambda_{12} \simeq J(\Delta)^{k_{12}}$ for some positive integer k_{12}. A discriminant calculation shows that $u^6 \notin d(\Lambda)$, which is a contradiction. A similar argument shows that the second case also does not occur. □

For the remainder of this chapter, let $R = \widehat{\mathcal{O}}_{\tilde{Z},E}$. We assume below that A is a canonical order of rank 16 such that $\Lambda = A \otimes_{k[[u,v]]} R$ is a local ring. As mentioned above, the only ramification types which have orders satisfying these assumptions are BD_n and DL_n. We show that there exists a unique hereditary order Λ' containing Λ and there are exactly two maximal orders containing Λ'.

Let A be a local canonical order of rank 16. We refer the reader to Section A.1.2, (A.1.9) and (A.1.10) for the generators and relations of A. As in the quaternion case, we first need to compute the discriminant ideal $d(\Lambda)$ of Λ. To do this, we first find the discriminant ideal $d(A)$ of A, and since taking discriminant ideals commutes with localisations and completions, this determines $d(\Lambda)$. The discriminant ideal of A is defined in the same way as that of Λ: take a free $\mathcal{O}_Z$-basis $\{e_i\}$ and we define $d(A) = (\sqrt{\det((\mathrm{tr}(e_i e_j))_{i,j})})$. Then

$$d(A) \quad = \quad \bigcap_{C \in Z^1} d(\hat{A}_C). \tag{6.3.1}$$

The discriminant ideal of A_C can be determined directly from the ramification data. Recall that the cyclic covers $\pi_C : \tilde{C} \longrightarrow C$ associated with the ramification data have degree equal to the ramification index e_C of A at C. The cyclic covers π_C ramify only at the singular points of the discriminant divisor, and the ramification indices $e_{C,p}$ of π_C at p are called the secondary ramification indices at p. Since we are in the complete local setting, we can assume the discriminant divisor has a singularity only at the origin of $Z(A) = \operatorname{Spec} k[[s,t]]$. For canonical orders the integers e_{C_i} and $e_{C_i,p}$ are given in [CHI09], Theorem 6.5. According to [CHI09], Theorem 7.1, the secondary ramification e_p is equal to the order of $A \otimes \hat{K}_C$ in the Brauer group. Since A is normal, the order $\hat{A}_C$ is Morita equivalent to a basic hereditary order, and we show below that the ramification index e_C determines the isomorphism class of $\hat{A}_C$ in $A \otimes \hat{K}_C$.

Proposition 6.3.3. *Let A be a local canonical order with rank 16.*

1. *If A has ramification type DL_n, then the discriminant ideal $d(A)$ of A is generated by $(u(v^2 - u^{2n-2}))^4$.*

2. *If A has ramification type BD_n, then the discriminant ideal $d(A)$ of A is generated by $(u(v^2 - u^{2(n-1)}))^4$.*

Proof. The discriminant divisor of A has three smooth components C_1, C_2, C_3 intersecting at the origin p of Z. Let $f_i = 0$ denote a local equation for C_i. The ramification indices $(e_{C_i}, e_{C_i,p})$ for $i = 1,2,3$ are $(2,2)$, $(2,2)$, and $(2,1)$. In the first two cases, the central simple algebra $A \otimes \hat{K}_{C_i}$ has index 2, so is isomorphic to 2×2 matrices over a quaternion division $\hat{K}_{C_i}$-algebra. Since $e_{C_i} = 2$, we conclude by Lemma 6.3.1 that $\hat{A}_{C_i}$ is a maximal order, hence its discriminant ideal is generated by f_i^4 in $\widehat{\mathcal{O}}_{Z,C_i}$. In the last case $(e_{C_3}, e_{C_3,p}) = (2,1)$, the central simple algebra $A \otimes \hat{K}_{C_3}$ is split. Now A is normal, so $\hat{A}_{C_3}$ is a standard hereditary order in a matrix algebra. By the structure theorem for hereditary orders and $e_{C_3} = 2$, we conclude that

$$\hat{A}_{C_3} \simeq \begin{pmatrix} \widehat{\mathcal{O}}_{Z,C_3} & \widehat{\mathcal{O}}_{Z,C_3} \\ \mathfrak{m}_{C_3} & \widehat{\mathcal{O}}_{Z,C_3} \end{pmatrix}$$

where $\mathfrak{m}_{C_3}$ denotes the unique maximal ideal of $\widehat{\mathcal{O}}_{Z,C_3}$. Hence the discriminant ideal in this case is also f_3^4. We can thus conclude by (6.3.1) that $d(A)$ is generated by $(f_1 f_2 f_3)^4$. □

The only local canonical orders with rank 16 have ramification types DL_n and BD_n, so the corollary below follows directly from Proposition 6.3.3.

Corollary 6.3.4. *Let A be a local canonical order of rank 16 on $Z = \operatorname{Spec} k[[u,v]]$. Denote by $\sigma : \tilde{Z} \longrightarrow Z$ the blowup of Z at the origin with $E = \operatorname{Ex}(\sigma)$, and u the uniformising* *parameter of the complete local ring* $\widehat{\mathcal{O}}_{\tilde{Z},E}$. *Then*

$$d(\sigma^* A)_{\widehat{E}}) \quad = \quad u^{12}\widehat{\mathcal{O}}_{\tilde{Z},E}.$$

Proposition 6.3.5. *Let* A *be a local canonical order of type* DL_n *or* BD_n *with rank* 16 *on* $Z = \operatorname{Spec} k[[u,v]]$*, and we assume that* A *has the presentation given in (A.1.9), (A.1.10). Denote by* Λ *the* $\widehat{\mathcal{O}}_{\tilde{Z},E}$*-order* $(\sigma^* A)_{\widehat{E}}$ *where* $\sigma : \tilde{Z} \longrightarrow Z$ *is the blowup at the origin with* $E = \operatorname{Ex}(\sigma)$*; and* $\Omega = \Lambda \langle x t_1 u^{-1}, t_1 t_2 t_1 u^{-1} \rangle$ *be the* Λ*-algebra generated by* $x t_1 u^{-1}$ *and* $t_1 t_2 t_1 u^{-1}$.

1. *The* Λ*-algebra* Ω *is generated by the sixteen elements*

$$\begin{array}{ccccccc} 1 & , & t_2 & , & t_2 t_1 & , & t_2 t_1 t_2 & , \\ t_1 t_2 t_1 t_2 u^{-1} & , & t_1 & , & t_1 t_2 & , & t_1 t_2 t_1 u^{-1} & , \\ x & , & x t_2 & , & x t_2 t_1 u^{-1} & , & x t_2 t_1 t_2 u^{-1} & , \\ x t_1 t_2 t_1 t_2 u^{-1} & , & x t_1 u^{-1} & , & x t_1 t_2 u^{-1} & , & x t_1 t_2 t_1 u^{-1} & \end{array}$$

   as an R*-module. In particular,* Ω *is an* R*-order.*

2. *We have an isomorphsm* $\Omega / u\Omega \xrightarrow{\sim} Q^{2\times 2}$ *where*

$$Q \quad = \quad \frac{\kappa \langle \varepsilon, \zeta \rangle}{(\zeta^2 - \nu, \varepsilon^2, \varepsilon\zeta + \zeta\varepsilon)}$$

   and ν *is a squarefree element of* κ*. In particular,* Ω *is a maximal* R*-order.*

3. *We have the inclusion* $u\Omega \subseteq \Lambda$ *and the morphism* $i : \Lambda / u\Omega \longrightarrow \Omega / u\Omega$ *induced by the inclusion* $\Lambda \subseteq \Omega$ *embeds* $\Lambda / u\Omega$ *as a subalgebra of* $Q^{2\times 2}$ *generated by the following three matrices*

$$\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, \zeta\varepsilon e_{12} = \begin{pmatrix} 0 & \zeta\varepsilon \\ 0 & 0 \end{pmatrix}, e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{6.3.2}$$

Proof. The generators for Ω as an R-module are obtained by direct computation. We define a map $\pi : \Omega \longrightarrow Q^{2\times 2}$ by

$$x \longmapsto \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon \end{pmatrix}, t_1 \longmapsto \begin{pmatrix} 0 & \zeta\varepsilon \\ 0 & 0 \end{pmatrix}, t_2 \longmapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

It is easy to verify that π is an algebra morphism and $u\Omega \subseteq \ker(\pi)$. Since $\Omega / u\Omega$ and $Q^{2\times 2}$ both have dimension 16 over κ, the morphism π is an isomorphism. Now

$$Q^{2\times 2} / J(Q^{2\times 2}) \quad \simeq \quad (\kappa[\zeta]/(\zeta^2 - \nu))^{2\times 2}$$

so by lemma ? $\Omega_K \simeq D^{2\times 2}$ where D is a division quaternion algebra. An order in Ω_K maximal if and only if its discriminat ideal is equal to $u^4\widehat{\mathcal{O}}_{\tilde{Z},E}$. From the generators of Ω as an R-module and Proposition 6.3.3, we see that $d(\Omega) = u^{-8}d(\Lambda) = u^4\widehat{\mathcal{O}}_{\tilde{Z},E}$, hence Ω is maximal. Finally, the matrices in (6.3.2) are the images of x, t_1 and t_2 under π, hence they generate $\pi(\Lambda)$ as a κ-algebra. □

We denote by $\mathcal{C}$ the pair category $\mathcal{C}(\iota)$.

Proposition 6.3.6. *With the hypotheses and notation of the proposition above, there are exactly two maximal orders containing* Λ.

Proof. Let e_{11}, e_{22} be the standard matrix idempotents of $Q^{2\times 2}$. We denote $P_1 = Q^{2\times 2}e_{11}$ and consider P_1 as a subset of $Q^{2\times 2}$. By Corollary 4.2.8, there is a bijective correspondence between maximal orders containing Λ and saturated elements of $\mathrm{Ind}_8(\mathcal{C})$. Let $\sigma \in \mathrm{Ind}_8(\mathcal{C})$ be a saturated element, and we first show that $e_{22}P_1 \subseteq \sigma(T)$. Note that σ satisfies $Q^{2\times 2}\sigma(T) = P_1$, so there exists an element $x \in \sigma(T)$ not in εP_1. This means that we can write $x = e_{11}x + e_{22}x$ such that

$$e_{ii}xe_{11}P_1 \quad = \quad e_{ii}P_1 \tag{6.3.3}$$

holds for $i = 1$ or 2. Suppose that (6.3.3) holds for $i = 1$. Recall that σ is said to be saturated if the submodule $\sigma(T)$ of P_1 is preserved by right multiplication by $Q \simeq e_{11}P_1$. Since $e_{21}x \in \sigma(T)$, we have $e_{21}xe_{11}P_1 = e_{21}P_1 = e_{22}P_1 \subseteq \sigma(T)$. Now suppose that (6.3.3) holds for $i = 2$ but not for $i = 1$. Note that $(\zeta\varepsilon e_{12})x \in \sigma(T)$, so $(\zeta\varepsilon e_{12})xe_{11}P_1 = \zeta\varepsilon e_{12}(e_{22}x)e_{11}P_1 = \zeta\varepsilon e_{22}P_1 = \varepsilon e_{22}P_1 \subseteq \sigma(T)$. Since we assumed that (6.3.3) does not hold for $i = 1$, we have $e_{11}x \in \varepsilon e_{22}P_1$, hence $e_{22}x \in \sigma(T)$ and we can conclude that $e_{22}P_1 \subseteq \sigma(T)$.

In particular, $e_{21} \in \sigma(T)$, so $(\zeta\varepsilon e_{12})e_{21} = \zeta\varepsilon e_{11} \in \sigma(T)$ and $\varepsilon e_{11}P_1 \subseteq \sigma(T)$. Hence we see that the saturated elements of $\mathrm{Ind}_8(\mathcal{C})$ are

$$\begin{pmatrix} \varepsilon Q & 0 \\ Q & 0 \end{pmatrix} \hookrightarrow \begin{pmatrix} Q & 0 \\ Q & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Q & 0 \\ Q & 0 \end{pmatrix} \hookrightarrow \begin{pmatrix} Q & 0 \\ Q & 0 \end{pmatrix}.$$

Therefore there are exactly two maximal orders containing Λ. □

Corollary 6.3.7. *With the hypotheses and notation of Proposition 6.3.5, the two maximal orders containing* Λ *are*

$$\Lambda\left\langle xt_1u^{-1}, t_1t_2t_1u^{-1}\right\rangle \quad \textit{and} \quad \Lambda\left\langle xt_2u^{-1}, t_2t_1t_2u^{-1}\right\rangle.$$

Proof. We showed that the order $\Lambda\left\langle xt_1u^{-1}, t_1t_2t_1u^{-1}\right\rangle$ is maximal in Proposition 6.3.5 and the same proof works to show that $\Lambda\left\langle xt_2u^{-1}, t_2t_1t_2u^{-1}\right\rangle$ is also maximal. $\square$

Corollary 6.3.8. *With the hypotheses and notation of Proposition 6.3.5, the blowups of A are*

$$\begin{array}{ccccccccc}
\mathcal{O}_{\tilde{Z}} & \oplus & \mathcal{O}_{\tilde{Z}}t_2 & \oplus & \mathcal{O}_{\tilde{Z}}t_2t_1 & \oplus & \mathcal{O}_{\tilde{Z}}t_2t_1t_2 & \oplus \\
\mathcal{O}_{\tilde{Z}}(E)t_1t_2t_1t_2 & \oplus & \mathcal{O}_{\tilde{Z}}t_1 & \oplus & \mathcal{O}_{\tilde{Z}}t_1t_2 & \oplus & \mathcal{O}_{\tilde{Z}}(E)t_1t_2t_1 & \oplus \\
\mathcal{O}_{\tilde{Z}}x & \oplus & \mathcal{O}_{\tilde{Z}}xt_2 & \oplus & \mathcal{O}_{\tilde{Z}}(E)xt_2t_1 & \oplus & \mathcal{O}_{\tilde{Z}}(E)xt_2t_1t_2 & \oplus \\
\mathcal{O}_{\tilde{Z}}(E)xt_1t_2t_1t_2 & \oplus & \mathcal{O}_{\tilde{Z}}(E)xt_1 & \oplus & \mathcal{O}_{\tilde{Z}}(E)xt_1t_2 & \oplus & \mathcal{O}_{\tilde{Z}}(E)xt_1t_2t_1 &
\end{array}$$

and

$$\begin{array}{ccccccccc}
\mathcal{O}_{\tilde{Z}} & \oplus & \mathcal{O}_{\tilde{Z}}t_2 & \oplus & \mathcal{O}_{\tilde{Z}}t_2t_1 & \oplus & \mathcal{O}_{\tilde{Z}}(E)t_2t_1t_2 & \oplus \\
\mathcal{O}_{\tilde{Z}}(E)t_1t_2t_1t_2 & \oplus & \mathcal{O}_{\tilde{Z}}t_1 & \oplus & \mathcal{O}_{\tilde{Z}}t_1t_2 & \oplus & \mathcal{O}_{\tilde{Z}}t_1t_2t_1 & \oplus \\
\mathcal{O}_{\tilde{Z}}x & \oplus & \mathcal{O}_{\tilde{Z}}(E)xt_2 & \oplus & \mathcal{O}_{\tilde{Z}}(E)xt_2t_1 & \oplus & \mathcal{O}_{\tilde{Z}}(E)xt_2t_1t_2 & \oplus \\
\mathcal{O}_{\tilde{Z}}(E)xt_1t_2t_1t_2 & \oplus & \mathcal{O}_{\tilde{Z}}xt_1 & \oplus & \mathcal{O}_{\tilde{Z}}(E)xt_1t_2 & \oplus & \mathcal{O}_{\tilde{Z}}(E)xt_1t_2t_1 &
\end{array}$$

Appendix A

Presentations for canonical orders

A.1 Local canonical orders

We obtain generators and relations for canonical orders of ranks 4 and 16 using their description as invariant rings. The setup involves a finite subgroup G of $GL_2(k)$ acting on $k[[s,t]]$ extending to an action on the matrix ring $M = k[[s,t]] \otimes M_n(k)$ for some $n \in \mathbb{N}$. By [CHI09], Lemma 4.1, we can assume that the action of G on M is the tensor product of the linear representation on $k[[s,t]]$ and some projective representation $\beta : G \longrightarrow \mathrm{Aut}_{k\text{-Alg}}(M_n(k)) \simeq PGL_n(k)$. Then any canonical order is isomorphic to $A = M^G$ where the possible projective representation β are given in [CHI09], Section 8.

The construction of canonical orders by invariant rings is engineered, using the Artin cover, from the classification of their ramification data (c.f. [CHI09], Theroems 3.1 and 6.5). We choose groups G for which the morphism $\operatorname{Spec} k[[s,t]] \longrightarrow \operatorname{Spec} k[[s,t]]^G$ has ramification data coinciding with a canonical order; and choose projective representations β of G such that M^G is a normal order. This recipe produces, for suitable G and β, a canonical order M^G with centre $k[[s,t]]^G$. Since our main interest is canonical orders with smooth centre, we require that $G \subset GL_2(k)$ be generated by pseudo-reflections.

To do computations with the projective representation β, we need to lift it to a representation b by introducing some roots of unity. We can consider β as an element of $H^1(G, PGL_n(k))$. Let d denote the connecting homomorphism in the long exact sequence for group cohomology associated to the exact sequence

$$1 \longrightarrow k^* \longrightarrow GL_n(k) \longrightarrow PGL_n(k) \longrightarrow 1.$$

If the element $d\beta \in H^2(G, k^*)$ has order e, then by the Kummer exact sequence $d\beta \in$

$H^2(G, \mu_e)$. This defines a central extension

$$1 \longrightarrow \mu_e \longrightarrow G' \longrightarrow G \longrightarrow 1$$

of G by μ_e and the projective representation β lifts to some representation b of G'. Note that the order of $d\beta$ in $H^2(G, k^*)$ is equal to the order of the Brauer class of A_K (c.f. [Art86], Proposition 4.9).

Proposition A.1.1. *Suppose that M^G has smooth centre, then M^G is local if and only if b is an irreducible representation of G'.*

Proof. Note that the Jacobson radical $J(M^G)$ of M^G is equal to $J(M) \cap M^G$ provided $|G|$ is invertible in M (c.f. [Mon76]). Since $J(M)$ is generated by the maximal ideal (s, t) of $k[[s, t]]$, any idempotent of M^G is lifted from an idempotent of $M_n(k)^G$. The invariant ring M^G is local if and only if $M_n(k)^G$ has no nontrivial idempotents, which occurs if and only if b is irreducible. $\square$

By considering the projective representation $\beta : G \longrightarrow PGL_n(k)$ appearing in Section 8 of [CHI09] which lift to irreducible representations, we see that the local canonical orders have ranks 4 or 16. Furthermore, the above proof shows that M^G is local if and only if $M^G/J(M^G) \simeq k$.

In the following sections, we will give G' as a subgroup of $GL_2(k)$ (in the rank 4 case) or $GL_4(k)$ (in the rank 16 case) using the following notation: suppose b is a representation of G' which lifts β. Then we denote by b_σ the image of σ under b. It turns out that in each case, if S generates G then $\{b_s | s \in S\}$ generates G'. Hence it suffices to write down a matrix b_σ for each generator $\sigma \in G$ to specify G' completely. Finally, the canonical orders below all have smooth centres and the isomorphism $k[[u, v]] \simeq k[[s, t]]^G$ will be given explicitly in each case.

A.1.1 Rank 4

The generators and relations for the canonical orders of rank 4 were computed by Paul Hacking in an unpublished manuscript in order to understand the conic bundles constructed from their Brauer-Severi varieties. We fix notation for this subsection. We denote $R = k[[s, t]]$, $M = R \otimes k^{2\times 2}$, and G be a finite subgroup of $GL_2(k)$. Let $\beta : G \longrightarrow PGL_2(k)$ be a projective representation of G and $b : G' \longrightarrow GL_2(k)$ be a lift of β to a representation of central extension G' of G. We compute M^G where the action of G on M is $\rho \otimes \beta$ where $\rho : G \hookrightarrow GL_2(k)$.

Type BL_n

See Section 8.2 of [CHI09]. Let G be the dihedral group D_r considered as a subgroup of $GL_2(k)$ generated by the matrices

$$\sigma = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.1.1}$$

where ζ is a primitive r-th root of unity and $r = 2n + 1$. For this ramification type A_K is trivial in the Brauer group, so β lifts to an actual representation of G. Hence we can take $G' = G$ and the possible representations b_i are given by

$$b_{i,\sigma} = \begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^{-i} \end{pmatrix} \quad \text{and} \quad b_{i,\tau} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

for $0 \leqslant i \leqslant n$. The invariant ring M^G corresponding to b_i is generated by the matrices

$$\begin{aligned} u &= \begin{pmatrix} st & 0 \\ 0 & st \end{pmatrix}, & v &= \begin{pmatrix} s^r + t^r & 0 \\ 0 & s^r + t^r \end{pmatrix} \\ x_i &= \begin{pmatrix} 0 & t^{2i} \\ s^{2i} & 0 \end{pmatrix}, & y_i &= \begin{pmatrix} 0 & s^{r-2i} \\ t^{r-2i} & 0 \end{pmatrix} \end{aligned}$$

so M^G is isomorphic to the $k[[u,v]]$-algebra

$$A(i) = \frac{k[[u,v]] \langle x_i, y_i \rangle}{(x_i^2 - u^{2i}, y_i^2 - u^{2n+1-2i}, x_i y_i + y_i x_i - 2v)}. \tag{A.1.2}$$

Note that $A(i)$ is local if and only if $i > 0$.

Type B_n

See Section 8.3 of [CHI09]. In this case $G = D_{2n}$. Since A_K is again trivial in the Brauer group, we can take $G' = G$ and the calculations are identical. So a rank 4 canonical order with ramification type B_n is isomorphic to

$$A(i) = \frac{k[[u,v]] \langle x_i, y_i \rangle}{(x_i^2 - u^{2i}, y_i^2 - u^{2n-2i}, x_i y_i + y_i x_i - 2v)} \tag{A.1.3}$$

for $0 \leqslant i \leqslant n$. The $k[[u,v]]$-algebra $A(i)$ is local if and only if $0 < i < n$.

Type L_n

See Section 8.4 of [CHI09]. Here $G = D_r$ where $r = 2n+2$ and in this case A_K has order 2 in the Brauer group. We take G' to be the subgroup of $GL_2(k)$ generated by the matrices

$$b_{i,a,\sigma} = \begin{pmatrix} \zeta^i & 0 \\ 0 & \zeta^{a-i} \end{pmatrix} \quad \text{and} \quad b_{i,a,\tau} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where ζ is an r-th root of unity, $i = 0, \dots, 2n+2$ and a is an odd integer. It suffices to consider $b_{i,a}$ where $i \leqslant a \leqslant 2i$ since $b_{i,2i+b} \simeq b_{i+b,2i+b}$ and $b_{i,i-b} \simeq b_{2n+2-b,2n+2+i-b}$ for any integer b. The invariant ring M^G corresponding to $b_{i,a}$ is generated by the matrices

$$u = \begin{pmatrix} st & 0 \\ 0 & st \end{pmatrix}, \quad v = \begin{pmatrix} s^r + t^r & 0 \\ 0 & s^r + t^r \end{pmatrix}$$

$$x_i = \begin{pmatrix} 0 & t^{2i-a} \\ s^{2i-a} & 0 \end{pmatrix}, \quad y_i = \begin{pmatrix} 0 & s^{r-(2i-a)} \\ t^{r-(2i-a)} & 0 \end{pmatrix}$$

so the invariant ring is isomorphic to the algebra

$$\frac{k[[u,v]]\langle x_i, y_i\rangle}{(x_i^2 - u^{2i-a}, y_i^2 - u^{2n+2-(2i-a)}, x_iy_i + y_ix_i - 2v)}.$$

The integer $2i - a$ ranges over all odd integers between 0 and $2n+2$ as i and a varies. So a rank 4 canonical order with ramification type L_n is local, and is isomorphic to

$$A(i) = \frac{k[[u,v]]\langle x_i, y_i\rangle}{(x_i^2 - u^{2i+1}, y_i^2 - u^{2n+1-2i}, x_iy_i + y_ix_i - 2v)} \tag{A.1.4}$$

for $0 \leqslant i \leqslant n$.

Type DL_n

See Section 8.5 of [CHI09]. Let G be the subgroup of $GL_2(k)$ generated by the matrices

$$\sigma = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \pi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1.5}$$

where $r = 2n-1$ for $n > 0$ and ζ be a primitive $2r$-th root of unity. Let G' be the subgroup of $GL_2(k)$ generated by the matrices

$$b_\sigma = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad b_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad b_\pi = \begin{pmatrix} 0 & -\mathrm{i}\nu \\ \nu & 0 \end{pmatrix}$$

where $\nu \in k$ and $\mathrm{i} = \sqrt{-1}$. The invariant ring M^G is generated by the matrices

$$u = \begin{pmatrix} s^2t^2 & 0 \\ 0 & s^2t^2 \end{pmatrix}$$

$$v = \frac{1}{2}\begin{pmatrix} s^{2r} + t^{2r} & 0 \\ 0 & s^{2r} + t^{2r} \end{pmatrix}$$

$$x = \frac{1}{\sqrt{2\mathrm{i}}}\begin{pmatrix} 0 & s^r + \mathrm{i}t^r \\ \mathrm{i}s^r + t^r & 0 \end{pmatrix}$$

$$y = \frac{1}{\sqrt{2\mathrm{i}}}\begin{pmatrix} 0 & t^{r+1}s + \mathrm{i}s^{r+1}t \\ \mathrm{i}t^{r+1}s + s^{r+1}t & 0 \end{pmatrix}$$

so a rank 4 canonical order with ramification type DL_n is local, and is isomorphic to

$$A = \frac{k[[u,v]]\langle x,y\rangle}{(x^2 - v, y^2 - uv, xy + yx - 2u^n)}. \tag{A.1.6}$$

Type BD_n

See Section 8.6 of [CHI09]. Let G be the subgroup of $GL_2(k)$ generated by the matrices

$$\sigma = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \pi = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.1.7}$$

where ζ be a primitive $4r$-th root of unity. Let $r = n-1$ where $n > 1$ and G' be the subgroup of $GL_2(k)$ generated by

$$b_\sigma = \begin{pmatrix} \zeta^i & 0 \\ 0 & -\zeta^i \end{pmatrix}, \quad b_\tau = \begin{pmatrix} \pm\zeta^{ir} & 0 \\ 0 & \pm\mu\zeta^{-ir} \end{pmatrix}, \quad b_\pi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\mu = \pm 1$. The invariant ring M^G is generated by the matrices

$$\begin{aligned}
u &= s^2t^2 \begin{pmatrix} s^2t^2 & 0 \\ 0 & s^2t^2 \end{pmatrix} \\
v &= \frac{1}{2} \begin{pmatrix} s^{4r} + t^{4r} & 0 \\ 0 & s^{4r} + t^{4r} \end{pmatrix} \\
x &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & t^{2r} + (-1)^i \mu s^{2r} \\ t^{2r} + (-1)^i \mu s^{2r} & 0 \end{pmatrix} \\
y &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & st^{1+2r} + (-1)^{i+1} \mu s^{1+2r} t \\ (-1)^i \mu s^{1+2r} t - st^{1+2r} & 0 \end{pmatrix}
\end{aligned}$$

so M^G is isomorphic to

$$\frac{k[[u,v]]\langle x,y\rangle}{(x^2 - (v + (-1)^i \mu u^r), y^2 + u(v - (-1)^i \mu u^r), xy + yx)}.$$

Note that any combination of $i = 1, 2$ and $\mu = -1, 1$ gives isomorphic algebras. Hence a canonical order of rank 4 with ramification type BD_n is local, and is isomorphic to

$$A = \frac{k[[u,v]]\langle x,y\rangle}{(x^2 - (v + u^{n-1}), y^2 + u(v - u^{n-1}), xy + yx)}. \tag{A.1.8}$$

A.1.2 Rank 16

In this subsection, we denote $R = k[[s,t]]$, and $N = k[[s,t]] \otimes k^{4\times 4}$. We compute N^G where the action of G on N is as described in the first paragraph of this chapter.

Type DL_n

Let G be the subgroup of $GL_2(k)$ generated by the matrices in (A.1.5). Recall that ζ is a primitive $2r$-th root of unity, and $r = 2n - 1$. Let G' be the subgroup of $GL_4(k)$ generated by the following matrices

$$b_\sigma = \begin{pmatrix} \zeta^i & 0 & 0 & 0 \\ 0 & -\zeta^i & 0 & 0 \\ 0 & 0 & -\zeta^{a-i} & 0 \\ 0 & 0 & 0 & \zeta^{a-i} \end{pmatrix}$$

$$b_\tau = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$b_\pi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm\mathrm{i} \\ 0 & 0 & \mp\mathrm{i} & 0 \end{pmatrix}$$

where $\mathrm{i} = \sqrt{-1}$. Let a be an odd integer and $a' = 2i - a$. Assume that $0 < a' < r$ and $a' \not\equiv r \bmod 2r$. The invariant ring N^G is generated by the matrices

$$
\begin{aligned}
u &= s^2t^2 \cdot 1_M \\
v &= \frac{s^{2r}+t^{2r}}{2} \cdot 1_M \\
x &= \begin{pmatrix} st & 0 & 0 & 0 \\ 0 & -st & 0 & 0 \\ 0 & 0 & -st & 0 \\ 0 & 0 & 0 & st \end{pmatrix} \\
t_1 &= \begin{pmatrix} 0 & 0 & 0 & t^{a'} \\ 0 & 0 & \mathrm{i}t^{a'} & 0 \\ 0 & \mathrm{i}s^{a'} & 0 & 0 \\ s^{a'} & 0 & 0 & 0 \end{pmatrix} \\
t_2 &= \begin{pmatrix} 0 & 0 & s^{r-a'} & 0 \\ 0 & 0 & 0 & \mathrm{i}s^{r-a'} \\ \mathrm{i}t^{r-a'} & 0 & 0 & 0 \\ 0 & t^{r-a'} & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

One can show that the generators x, t_1 and t_2 generate a rank 16 algebra over $k[[u,v]]$ with the following relations

$$
\begin{aligned}
x^2 &= u \\
t_1^2 &= xu^{\frac{a'-1}{2}} \\
t_2^2 &= \mathrm{i}u^{\frac{r-a'}{2}} \\
t_1t_2t_1t_2 + t_2t_1t_2t_1 &= -2v \\
xt_1 - t_1x &= 0 \\
xt_2 + t_2x &= 0.
\end{aligned} \tag{A.1.9}
$$

Type BD_n

Let G be the subgroup of $GL_2(k)$ generated by the matrices in (A.1.7). Recall that ζ is a primitive $4r$-th root of unity, where $r = n-1$ with $n > 1$. Let G' be the subgroup of

$GL_4(k)$ generated by the following matrices

$$b_\sigma = \begin{pmatrix} \zeta^i & 0 & 0 & 0 \\ 0 & -\zeta^i & 0 & 0 \\ 0 & 0 & -\zeta^{a-i} & 0 \\ 0 & 0 & 0 & \zeta^{a-i} \end{pmatrix}$$

$$b_\tau = \begin{pmatrix} 0 & 0 & 0 & (-1)^i \\ 0 & 0 & (-1)^i & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$b_\pi = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu(-1)^i \\ 0 & 0 & \mu(-1)^i & 0 \end{pmatrix}$$

where $\mu^2 = \pm 1$, a is an even integer. Assume the following congruences modulo $4r$ do not hold

$$\begin{aligned} 2i &\equiv a \\ a - 2i &\equiv 2r. \end{aligned}$$

Let $a' = 2i - a$. The invariant ring N^G is generated by the matrices

$$
\begin{aligned}
u &= s^2t^2 \cdot 1_M \\
v &= \frac{s^{4r}+t^{4r}}{2} \cdot 1_M \\
x &= \begin{pmatrix} st & 0 & 0 & 0 \\ 0 & -st & 0 & 0 \\ 0 & 0 & st & 0 \\ 0 & 0 & 0 & -st \end{pmatrix} \\
t_1 &= \begin{pmatrix} 0 & 0 & 0 & t^{a'} \\ 0 & 0 & \mu(-1)^i t^{a'} & 0 \\ 0 & \mu s^{a'} & 0 & 0 \\ (-1)^i s^{a'} & 0 & 0 & 0 \end{pmatrix} \\
t_2 &= \begin{pmatrix} 0 & 0 & s^{2r-a'} & 0 \\ 0 & 0 & 0 & \mu(-1)^i s^{2r-a'} \\ \mu t^{2r-a'} & 0 & 0 & 0 \\ 0 & (-1)^i t^{2r-a'} & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

One can show that the generators x, t_1 and t_2 generate a rank 16 algebra over $k[[u,v]]$ with the following relations

$$
\begin{aligned}
x^2 &= u \\
t_1^2 &= (-1)^i u^{a'/2} \\
t_2^2 &= \mu u^{r-a'/2} \\
t_1t_2t_1t_2 + t_2t_1t_2t_1 &= 2v \\
xt_2 - t_2x &= 0 \\
xt_1 + t_1x &= 0.
\end{aligned} \tag{A.1.10}
$$

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