THE RESOLUTION OF TORIC SINGULARITIES

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November 2006

Submitted in partial fulfillment of the requirements of the degree of
Bachelor of Science with Honours

Supervisor: Dr Daniel Chan

Acknowledgements

The greatest thanks must go to my supervisor Daniel Chan. Daniel was on leave for 6 months this year, yet he nonetheless committed a lot of time to helping me. His enthusiasm and intelligence is very inspiring.

Thanks also to my dear parents, who have supported me all the way through university.

Thanks to everyone in the honours room! I appreciated sharing Wirkunggeschicte with Ambros and Maike. Coffee with Doug was great. The complaining of Gina and the steadfastness of Michael will be much missed!

James MacLaurin, 3 November 2006.

Introduction

Algebraic geometry is the study of the zeros of systems of polynomials. We denote these sets *algebraic varieties*. For example, hyperbolas, parabolas and hyperplanes are algebraic varieties.

In algebraic geometry a phenomenon of particular interest is the *singularity*. Informally, singularities occur when the tangent space to a particular point in a variety is not well defined. Consider for example the curve $y^2 = x^3 - x^2$ in $\mathbb{R}^2$. The curve crosses itself at 0, and as a consequence if we intersect any line through the origin with the curve we get a double root.

A question of great interest to the algebraic geometer is this: given any algebraic variety X how may I resolve its singularities? Suppose Y is a nonsingular variety. A resolution of singularities is a map $\phi : X \to Y$ with the following characteristics. The map $\phi = (\phi_1, \ldots, \phi_k)$ must have rational components, and in addition ϕ must be defined almost everywhere on X, such that the set of points where ϕ isn't defined forms an algebraic subvariety.

The Japanese mathematician Heisuke Hironaka provided an answer to our question in 1964. Hironaka's celebrated theorem is that a resolution of singularities of X always exists, as long as the variety X is over a field of characteristic zero. He was awarded a Fields Medal for his efforts in 1970. However, although we know a resolution of singularities of any algebraic variety X must exist, it is often exceedingly difficult to determine. Moreover Hironaka's Theorem employs a lot of sophisticated mathematics which goes well beyond the scope of this thesis.

The beauty of *toric varieties* is that we can reduce the problem of finding a resolution of singularities to relatively straightforward combinatorics. Toric varieties are algebraic varieties which are constructed in a special way, using *convex polyhedral cones*. (A cone is the positive span of a set of vectors (see Figure 3.1).) Some examples of toric varieties are $\mathbb{P}^n$ and the quadric surface: $\{(x, y, z) : y^2 - xz = 0\}$. We will see in chapter 6 that we can tell whether a toric variety is singular or not merely from studying the cone that generates it. We will also see that refining a cone (that is, subdividing the cone into new cones) induces a rational map on the toric variety associated with the cone. We will prove that we can always find a resolution of singularities that is induced by a refinement of the cone. Thus the problem of resolving the singularities of toric varieties has been reduced to the combinatorial problem of finding a refinement of a cone.

In Chapter 1 we overview classical algebraic geometry. We state Hilbert's Theorems and explore the interplay between algebras, maximal ideals and algebraic sets. In Chapter 2 we define rational maps and singularities, before taking a brief look at Hironaka's Theorem. In Chapter 3 we provide a detailed overview of cones. The reader is advised not to spend too long on this chapter, but to refer back to it when necessary. In Chapter 4 we define the toric variety. In Chapter 5 we define the action of the torus $(\mathbb{C}^*)^n$ on a toric variety, and we use this to analyse the structure of toric varieties. Chapter 6 represents the climax of this thesis. We bring together the theory of Chapters 2-5: we use the combinatorics developed in chapter 4 to prove that there exists a nonsingular refinement of a singular fan. We look at the *toric morphism* induced by the refinement and prove that it is a resolution of singularities. We describe the toric morphism in terms of the orbits of the torus action of Chapter 5. Finally in Chapter 7 we explore some examples; focussing on 2-dimensional toric varieties, which are particularly elegant.

The principal text I have drawn from is Fulton's *Introduction to Toric Varieties* [4]. Fulton is extremely terse and rarely provides systematic proofs. As a consequence, while I have drawn on many of Fulton's results, most of the actual proofs are my own. I have also extensively referred to Ewald's *Combinatorial Convexity and Algebraic Geometry* [6], Oda's *Convex Bodies and Algebraic Geometry*[11] and the papers by Voight [17] and Reid [12] listed in the bibliography.

I assume the reader has a grasp of the basics of algebra, such as rings, ideals, algebras, homomorphisms and linear algebra.

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Chapter 1

Introductory Algebraic Geometry

In this chapter we review classical algebraic geometry.

1.1 Algebraic Sets

Algebraic Geometry is the study of the zeros of polynomials. We will study polynomials in $\mathbb{C}[x_1, x_2, \ldots x_n]$, and their corresponding zero-sets within $\mathbb{C}^n$.

It is somewhat idiosyncratic to confine ourselves to polynomials over $\mathbb{C}$. Many take a more general perspective and look at polynomials over a closed field k. In any case, the proof of the general case of most of these theorems is straightforward.

We denote the algebra of polynomials in n dimensions by

$$\mathbb{C}[x_1, x_2, \ldots, x_n]$$

Note that $\mathbb{C}[x_1, \ldots, x_n]$ is also a ring. We denote the ideal generated by the polynomials $\{p_1, \ldots, p_k\}$ as $(p_1, \ldots, p_k)$.

Definition 1.1.1. For any $S \subset \mathbb{C}[x_1, \ldots x_n]$, the **vanishing set** of S is

$$V(S) = \{(\xi_1, \ldots \xi_n) \in \mathbb{C}^n : \forall f \in S \;\; f(\xi_1, \ldots \xi_n) = 0\}$$

Vanishing sets are geometric objects which may be specified algebraically (as the common zeros of a set of polynomials). For this reason they are also known as *algebraic sets*. The vanishing set of a single polynomial is known as a *hypersurface*.

Definition 1.1.2. Conversely, for any $X \subset \mathbb{C}^n$ we may define the ideal of all polynomials which are zero on X:

$$I(X) = \{f \in \mathbb{C}[x_1, \ldots x_n : \forall \alpha \in X \;\; f(\alpha) = 0\}$$

Lemma 1.1.3.

i. If $F \subset F' \subset \mathbb{C}[x_1, \dots x_n]$ then $V(F') \subset V(F)$.

ii. If $X \subseteq X' \subset \mathbb{C}^n$ then $I(X') \subset I(X)$.

Proof. i. Let $\xi \in V(F')$. Then $\forall f \in F'$, $f(\xi) = 0$. But since $F \subset F'$, it immediately follows that $\forall f \in F$, $f(\xi) = 0$ and hence $\xi \in V(F)$. Thus $V(F') \subset V(F)$.

ii. If $f \in I(X')$ then as $X \subset X'$, $f(x) = 0$ for all x in X. So $f \in I(X)$.

□

Corollary 1.1.4. *If $X \subset \mathbb{C}_m$ and $S \subset \mathbb{C}[x_1 \dots x_n]$ then*

i. $V(I(X)) \supseteq X$

ii. $I(V(S)) \supseteq S$

iii. $V(I(V(S))) = V(S)$

iv. $I(V(I(X))) = I(X)$

Proof. (i) follows from applying Lemma 1.1.3 to X twice.

(ii) follows from applying Lemma 1.1.3 to S twice.

To prove (iii), note that $S \subseteq I(V(S)) \Rightarrow V(I(V(S))) \subseteq V(S)$ and $V(I(V(S))) \supseteq V(S)$ follows from (i). Thus $V(I(V(S))) = V(S)$.

(iv) follows similarly. □

The fact that $V(I(V(S))) = V(S)$ means that if $X = V(S)$ is any algebraic set then X may always be expressed as the vanishing set of the ideal $I(V(S))$. Thus from now on we will refer to algebraic sets as the set of common zeros of an ideal.

Lemma 1.1.5. *[5, §1.5] Suppose $\{J_\alpha\}_{\alpha \in P} \triangleleft \mathbb{C}[x_1, \dots x_n]$. Then*

i. An intersection of algebraic sets is an algebraic set: $\cap_\alpha V(J_\alpha) = V(\sum_\alpha J_\alpha)$

ii. Likewise a finite union of algebraic sets is another algebraic set. If $\{J_1, \dots, J_k\}$ are all ideals in $\mathbb{C}[x_1, \dots x_n]$ then $\cup_i V(J_i) = V(J_1.J_2 \dots J_k)$

iii. $V(0) = \mathbb{C}^n$ and $V(\mathbb{C}[x_1, \dots x_n]) = \phi$.

iv. If $J = (\eta_1, \dots, \eta_k)$ then $V(J) = \bigcap_{i=1}^k V(\eta_i)$.

Proof. i. ($\supseteq$). Fix $\beta \in P$. Now $J_\beta \subseteq \sum_{\alpha \in P} J_\alpha \Rightarrow V(J_\beta) \supseteq V(\sum_{\alpha \in P} J_\alpha)$ (by Lemma 1.1.3). It follows that $\cap_{\alpha \in P} V(J_\alpha) \supseteq V(\sum_{\alpha \in P} J_\alpha)$.

($\subseteq$). Suppose $\xi \in \bigcap_{\alpha \in P} V(J_\alpha)$. Then $\forall \alpha \in P$, $\forall f \in J_\alpha$, $f(\xi) = 0$. But every member of $\sum_{\alpha \in P} J_\alpha$ must be expressible in the form $g = \sum_{\alpha \in P} r_\alpha f_\alpha$

$(f_\alpha \in J_\alpha)$. Since $\forall \alpha,\ f_\alpha(\xi) = 0$, it immediately follows that $g(\xi) = 0$ and $\xi \in V(\sum_\alpha J_\alpha)$. Thus $V(\sum_\alpha J_\alpha) = \bigcap_\alpha V(J_\alpha)$.

ii. Let $x \in \mathbb{C}^n$. Then $x \in \cup_i V(J_i)$ means there exists j such that $x \in V(J_j)$. Hence $\forall f \in J_j,\ f(x) = 0 \Rightarrow x \in V(J_1 J_2 \dots J_k)$. Conversely suppose $\forall i \leq k,\ x \notin V(J_i)$. Then $\forall i \leq k,\ \exists f_i \in J_i$ such that $f_i(x) \neq 0$. But then $f = f_1 f_2 \dots f_k \in J_1 J_2 \dots J_k$ and $f(x) \neq 0$. Thus $x \notin V(J_1 \dots J_k)$.

iii. Any point is a root of the zero polynomial, so $V(0) = \mathbb{C}^n$. Since the polynomial $f \equiv 1$, which clearly has no roots, belongs to $\mathbb{C}[x_1, \dots, x_n]$ we find $V(\mathbb{C}[x_1, \dots, x_n]) = \emptyset$.

iv. $(\supseteq)$ Suppose $x \in \bigcap_{i=1}^k V(\eta_i)$ and let $f = \alpha_1\eta_1 + \dots \alpha_k\eta_k \in J$ be arbitrary. Now $\forall i,\ \eta_i(x) = 0$. Hence $f(x) = 0$ and $x \in V(J)$. Thus $\bigcap_{i=1}^k V(\eta_i) \subseteq V(J)$. Conversely $\bigcup_{i=1}^k \eta_i \subseteq J \Rightarrow V(J) \subseteq \bigcap_{i=1}^k V(\eta_i)$. Thus $V(J) = \bigcap_{i=1}^k V(\eta_i)$.

□

Example 1.1.6. We construct some examples in $\mathbb{C}^2$. Let J be the ideal generated by $(z_1^2 + z_2^2 - 1)$. $V(J)$ is a (complex) circle. Let K be the ideal generated by $(z_1 - z_2)$. $V(K)$ is a line. It can be seen that $V((J, K))$ is the intersection of the circle $V(J)$ with the line $V(K)$. □

Lemma 1.1.5 shows us that both intersections and unions of algebraic sets leave us with algebraic sets. In fact we will define a topology on algebraic sets in Section 1.3 called the Zariski Topology. In the case of the algebraic set $\mathbb{C}^n$, the closed sets of the Zariski Topology on $\mathbb{C}^n$ are the set of all algebraic sets. The open sets are the complements of algebraic sets.

Definition 1.1.7. [6, §VI.1], [8, I.1] An affine algebraic set $X \subset \mathbb{C}^n$ is called *irreducible* or an *affine algebraic variety* if it is not the union of two proper algebraic subsets. If $X_1 \subset X_2 \subset \mathbb{C}^n$ we say that X_1 is a *subvariety* of X_2.

Example 1.1.8. Let L be the ideal in $\mathbb{C}[z_1, z_2, z_3]$ generated by $(z_1 - 1)(z_2 - 1)(z_3 - 1)$. $V(L)$ is the union of the three planes $z_1 = 1$, $z_2 = 1$ and $z_3 = 1$. Thus $V(L)$ is not irreducible. Referring back to Example 1.1.6, both $V(J)$ and $V(K)$ are irreducible. We can't prove this yet, but in Section 1.2 we will see that this is because each of J and K are prime. □

Lemma 1.1.9. *An algebraic set X is irreducible iff $I(X)$ is prime.*

Proof. Suppose $I(X)$ is not prime. Let $J = I(X)$ be the ideal generated by X. Then $\exists f, g \notin J$ such that $fg \in J$. Then $fg \subseteq J \Rightarrow V(fg) = V(f) \cup V(g) \supseteq V(J) = X$. On intersecting both sides of this equation with $V(J)$ we find

$$(V(f) \cap V(J)) \cup (V(g) \cap V(J)) = V(J)$$

That is, $V(f, J) \cup V(g, J) = V(J) = X$. This means X is not irreducible.

Conversely suppose that X is not irreducible - i.e. $X = S_1 \cup S_2$, where $S_i \subsetneq X$. There must exist a polynomial f_1 that vanishes on S_1 but not on X, because if X and S_1 are the vanishing sets of the same polynomials then they would be equal. We can similarly find a polynomial f_2 that vanishes on S_2 but not on X. Then $V(f_i) \supseteq S_i$. This means $V(f_1 f_2) = V(f_1) \cup V(f_2) \supseteq X$. Thus $f_1 f_2 \in I(X)$. In other words $I(X)$ is not prime. $\square$

We make some brief comments on projective space. To construct projective space, define an equivalence relation on $\mathbb{C}^{n+1}$: $x \sim y$ if $\exists \lambda \in (\mathbb{C} - \{0\})$ such that $y = \lambda x$. We set $\mathbb{P}^n$ to be the set of all nonzero equivalence classes under this relation.

An element of $\mathbb{P}^n$ is a *point*. If P is a point then any point $a = (a_0, \ldots a_n) \in \mathbb{C}^{n+1}$ in its equivalence class is called a *set of homogeneous coordinates for P*. We write $p = (a_0 : \ldots : a_n)$. Let $\mathbb{C}_i = \{(a_0 :, \ldots : a_n) : a_i = 1\}$. It is easy to check that $\mathbb{C}_i$ is homeomorphic to $\mathbb{C}^n$, and for this reason $\mathbb{C}_i$ is known as the i^{th} affine piece. Note also that the set $\{\mathbb{C}_i\}_{i=1}^n$ forms a cover of $\mathbb{P}^n$. We will see that $\mathbb{P}^n$ can be constructed as a toric variety.

1.2 Hilbert's Theorems and their Consequences

Hilbert's *Bassisatz* (Basis Theorem) and *Nullstellensatz* (Zeros-Theorem) allow us to establish several very important correspondences between ideals and algebraic sets. Herein lies the beauty of algebraic geometry, for it allows us to study algebraic sets through the polynomials that define them.

The first of Hilbert's Theorems we will prove is that every algebraic set may be expressed as the union of a finite number of irreducible algebraic sets.

1.2.1 Bassisatz

We make some introductory definitions.

Definition 1.2.1. A ring is said to be *Noetherian* if every ideal in the ring is finitely generated. In particular, fields are Noetherian rings.

Theorem 1.2.2. Hilbert's Basis Theorem. *If R is a Noetherian Ring, then $R[x_1, \ldots x_n]$ is a Noetherian ring.*

Proof. Since $R[X_1, \ldots X_n]$ is isomorphic to $R[X_1, \ldots X_{n-1}][X_n]$, the theorem will follow by induction if it can be proved that $R[X]$ is Noetherian whenever R is Noetherian.

For the proof of this refer to [5, §1.4]. $\square$

Since $\mathbb{C}$ is a field, and hence is Noetherian, the following Corollary is clear.

Corollary 1.2.3. *$\mathbb{C}[x_1, \ldots x_n]$ is Noetherian.*

Theorem 1.2.2 has many implications. We list several.

Corollary 1.2.4. *Every algebraic set is the intersection of a finite number of hypersurfaces.*

Proof. Let $X = V(J)$ be an algebraic set. Then $J = (\eta_1 \ldots \eta_k)$ is finitely generated and so $V(J) = \bigcap_{i=1}^{k} V(\eta_i)$ by Lemma 1.1.5. $\square$

Corollary 1.2.5. *If $X_1 \supset X_2 \supset \ldots$ is a sequence of nested algebraic sets, then there is an integer r such that $X_r = X_{r+1} = X_{r+2} \ldots$.*

This is known as the *descending chain condition.*

Proof. [6, §VI, 1.9] Let $I(X_1) \subset I(X_2) \subset \ldots$ be the corresponding chain of ideals in $\mathbb{C}[\xi_1, \ldots \xi_n]$ and let $J = \bigcup_{i=1}^{\infty} I(X_i)$. By Theorem 1.2.2 $J \lhd \mathbb{C}$ is Noetherian and therefore $J = (p_1, \ldots p_k)$ for some polynomials $\{p_i\}$. Furthermore $p_i \in I(X_{j_i})$ for some $j_i \in \mathbb{N}$. Let $m = max\{j_i | j = 1, \ldots k\}$. Then $p_i \in I(X_{j_i}) \subseteq I(X_m)$. Hence $(p_1, \ldots, p_k) \subseteq I(X_m)$. Since $J = (p_1, \ldots, p_k)$, this means $J = I(X_m) = I(X_{m+1}) = \ldots$ and therefore $X_m = X_{m+1} = \ldots$. $\square$

Lemma 1.2.6. *Every algebraic set X is a finite union of irreducible algebraic sets.*

Proof. [6, §VI, 1.10] Suppose X is not a finite union of algebraic sets. Then $X = Y \bigcup Y'$, where $Y, Y' \subsetneqq X$ are algebraic sets. Now if both of Y and Y' could be expressed as a finite union of algebraic sets, then X could be as well, contradicting our definition. So at least one of Y and Y - say Y' - can not be expressed as a finite union of algebraic sets. By repeating this reasoning and using induction we can find an infinite sequence of algebraic sets $X \supsetneqq Y \supsetneqq Y_1 \supsetneqq Y_2 \ldots$. But this contradicts

Corollary 1.2.5. Hence every algebraic set may be expressed as a finite union of algebraic sets. □

Corollary 1.2.7. *The decomposition of any algebraic set X into a finite union of irreducible algebraic sets is essentially unique.*

Proof. [6, §VI.1][5, §1] Suppose $X = X_1 \cup \ldots X_j$ and $X = Y_1 \cup \ldots Y_k$ are both finite unions of irreducible algebraic sets. Suppose further that there are no redundant sets: that is $X_h \nsubseteq X_i$ for all $h \neq i$ (and similarly for the Y's). For each p,

$$X_p = X \cap X_p = (\bigcup_i^k Y_i) \cap X_p = \bigcup_i^k (Y_i \cap X_p)$$

But since X_p is irreducible it follows that for some q, $X_p = Y_q \cap X_p$ and therefore $X_p \subseteq Y_q$. We can analogously find a p' such that $Y_q \subseteq X_{p'}$ and so $X_p \subseteq Y_q \subseteq X_{p'}$. Since we assumed there are no redundant sets, it follows that $X_p = X_{p'} = Y_q$. Thus each X_p is equal to a Y_q. The converse follows by exactly the same reasoning: for each Y_r we can find an X_s such that $Y_r = X_s$. Therefore we can conclude that the decompositions are the same. □

1.2.2 Nullstellensatz

We move onto an equally useful theorem of Hilbert's: the Nullstellensatz.

Definition 1.2.8. If $J \triangleleft R$, then the *radical* of I is $rad(J) = \sqrt{J} = \{f \in R | \exists p \in \mathbb{N}, f^p \in J\}$

Note that $\sqrt{J}$ is an ideal because $f, g \in R \Rightarrow f^p, g^q \in J$ for some $p, q \in \mathbb{N}$. Thus if $r \geq p + q - 1$ then $(f+g)^r = \sum_{i=1}^r f^i g^{r-i} \in J$.

Lemma 1.2.9. *If J is prime then $\sqrt{J} = J$.*

Proof. Suppose for a contradiction that J is prime but $\sqrt{J} \neq J$. Then $\exists f \in \sqrt{J}$, $f \notin J$ and $f^p \in J$ for some minimal p. However $f.f^{p-1} \in J \Rightarrow f \in J$ or $f^{p-1} \in J$ (as J is prime). This contradicts the minimality of p. □

Hilbert's Nullstellensatz comes in a variety of manifestations: we list three. The last is perhaps the most common; it is often known at the 'Strong Nullstellensatz'.

Theorem 1.2.10. Hilbert's Nullstellensatz:

i. Let $J \triangleleft \mathbb{C}[x_1, \ldots x_n]$ be maximal. Then J is of the form $m_a = (x_1 - a_1, \ldots x_n - a_n)$ for some point $a = (a_1, \ldots a_n) \in \mathbb{C}^n$. In other words $J = V((a))$.

ii. *Suppose* $J \triangleleft \mathbb{C}[x_1, \dots x_n]$ *is any proper ideal. Then* $V(J) \neq \emptyset$.

iii. *Let* $J \triangleleft \mathbb{C}[x_1, \dots x_n]$. *Then* $I(V(J)) = \sqrt{J}$.

For a proof refer to [12] or [2, §10.7-10.8].

Corollary 1.2.11. *There is a one-to-one correspondence between radical ideals and algebraic subsets.*

Proof. The correspondence is:

$$\begin{array}{rcl} \text{radical ideals} & \leftrightarrow & \text{algebraic sets} \\ J & \rightarrow & V(J) \\ I(X) & \leftarrow & X \end{array}$$

We check that these maps are inverses. If J is a radical ideal then it follows from Theorem 1.2.10 that $J = I(V(J))$. Conversely if X is an algebraic set then by Corollary 1.1.4 $X = V(I(X))$. Since $I(X) = I(V(I(X)))$, it follows by Theorem 1.2.10 that $I(X)$ is a radical ideal. □

Corollary 1.2.12. *If J is a prime ideal, then $V(J)$ is irreducible. Thus there is a one-to-one correspondence between prime ideals and algebraic varieties (irreducible algebraic sets).*

Proof. Suppose J is a prime ideal. Then $I(V(J)) = \sqrt{(}J) = J$, thus $I(V(J))$ is prime and $V(J)$ is irreducible by Lemma 1.1.9. Conversely given any irreducible set $V(J)$ it follows by the same lemma that $I(V(J))$ is prime. □

Finally it is evident from Theorem 1.2.10 (i) that

Corollary 1.2.13. *There is a one-to-one correspondence between points in $\mathbb{C}^n$ and maximal ideals of $\mathbb{C}[x_1, \dots, x_n]$.*

$$x \longleftrightarrow m_x$$

Remark 1.2.14. From algebra we know there is a one-one correspondence between maximal ideals of $\mathbb{C}[x_1, \dots x_n]$ and $\mathbb{C}$-algebra homomorphisms $\mathbb{C}[x_1, \dots x_n] \to \mathbb{C}$. Each maximal ideal is the kernel of an algebra homomorphism. Conversely each algebra homomorphism has a maximal ideal as its kernel. But by Corollary 1.2.13

there is a 1-1 correspondence between maximal ideals and $\mathbb{C}^n$, so there must be a 1-1 correspondence between $Hom_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1, \dots x_n] \to \mathbb{C})$ and $\mathbb{C}^n$ as well.

Let $\phi : \mathbb{C}[x_1, \dots x_n] \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism. If p is an arbitrary polynomial, then it follows from the definition of an algebra homomorphism that $\phi(p) = p(\phi(x_1), \dots, \phi(x_n))$. That is, ϕ is the evaluation of p at the point $x = (\phi(x_1), \dots \phi(x_n))$. It can be seen that the kernel of ϕ is m_x. Thus our correspondence is

$$\begin{aligned} Hom_{\mathbb{C}\text{-alg}}(\mathbb{C}[x_1, \dots, x_m] \to \mathbb{C}) \quad &\leftrightarrow \quad \mathbb{C}^n \\ \phi \quad &\leftrightarrow \quad \phi(x) \end{aligned}$$

□

1.3 Maximal Ideals and the Zariski Topology

We explore further correspondences between maximal ideals and algebraic sets. We will define a topology, called the Zariski Topology, which will be heavily used in subsequent chapters. The Zariski Topology is named after the late Oscar Zariski, former Professor of Harvard and one of the greatest algebraic geometers of the twentieth century.

Lemma 1.3.1. *If $J \lhd \mathbb{C}[x_1, \dots, x_n]$ then $V(J) = \{x \in \mathbb{C}^n | J \subseteq m_x\}$*

Proof. ($\supseteq$). Suppose $y \in \mathbb{C}^n$ is such that $J \subseteq m_y$. Then $\forall f \in J\ f(y) = 0$ and so $y \in V(A)$.

($\subseteq$). Conversely suppose $x \in V(J)$. Then $V(m_x) = \{x\} \subseteq V(J)$ and so $I(V(J)) \subseteq I(V(m_x))$. By Theorem 1.2.10 (the Nullstellensatz) m_x is maximal. This means $I(V(m_x)) = m_x$. Hence $J \subseteq I(V(J)) \subseteq m_x$. □

Definition 1.3.2. If X is an algebraic set then $\mathbb{C}[X] := \mathbb{C}[x_1, \dots, x_n]/I(X)$ is the *co-ordinate ring* or the ring of regular functions of the affine algebraic set X.

Suppose that $f \in \mathbb{C}[x_1, \dots x_n]$. If we take any two arbitrary polynomials $g, g' \in f + I(X)$ then $g|_X = g'|_X$. Hence we can equivalently think of $\mathbb{C}[X]$ as $\mathbb{C}[x_1, \dots x_n]|_X$.

This leads us to a generalisation of Corollary 1.2.13. But firstly we state a lemma concerning the algebra of rings.

Lemma 1.3.3. *Let $J \lhd R$. The maximal ideals of the ring R/J are precisely those of the form m/J where m is maximal in R and $J \subset m$.*

Proof. Suppose $m \lhd R$ is maximal, $J \subset m$ but m/J is not maximal in R/J. Then $\exists m'$ such that $m/J \lhd m'/J$, which would mean that $m \lhd (m', J)$. This contradicts the maximality of m. Conversely if m/J is maximal in R/J then it is straightforward to check m is maximal in R. $\square$

Lemma 1.3.4. *If X is an affine algebraic set with coordinate ring $\mathbb{C}[X]$ then there is a one-to-one correspondence:*

$$\begin{array}{rcl} X & \leftrightarrow & \{\textit{maximal ideals in } \mathbb{C}[X]\} \\ x & \leftrightarrow & m_x/I(X) \end{array}$$

Proof. By Lemma 1.3.3 the maximal ideals of $\mathbb{C}[X]$ are of the form $m/I(X)$ where m is maximal in $\mathbb{C}[x_1, \ldots, x_n]$ and $I(X) \subset m$. But by Lemma 1.3.1 the set of maximal ideals in $\mathbb{C}[x_1, \ldots, x_n]$ containing X is $\{m_x : x \in X\}$. This means that the set of maximal ideals of $\mathbb{C}[X]$ is $\{m_x/I(X) : x \in X\}$. $\square$

We reach a generalisation of Remark 1.2.14.

Corollary 1.3.5. *The following is a 1-1 correspondence:*

$$\begin{array}{rcl} Hom_{\mathbb{C}\text{-alg}}(\mathbb{C}[X], \mathbb{C}) & \leftrightarrow & X \\ \phi & \rightarrow & (\phi(x_1), \ldots, \phi(x_n)) \end{array} \tag{1.1}$$

Proof. Let $f \in \mathbb{C}[X]$. As in Remark 1.2.14 we find $\phi(f) = f(\phi(x_1), \ldots, \phi(x_n))$. Thus if $f \in I(X)$ then $\phi(f) = 0 \Leftrightarrow (\phi(x_1), \ldots, \phi(x_n)) \in X$. The fact that (1.1) is a bijection follows from the fact that the polynomials $\{x_i\}$ generate $\mathbb{C}[X]$, which means ϕ is uniquely determined by $\{\phi(x_i)\}$. $\square$

Let X be an *affine algebraic set*. If Y is an algebraic set such that $Y \subseteq X$ we say Y is an *algebraic subset* of X. This leads us to the *Zariski Topology* on X:

Proposition 1.3.6. *Algebraic subsets of X form the closed sets of the Zariski Topology. The open sets of the Zariski Topology on X are the complements in X of algebraic subsets. A* principal open subset *is the complement in X of $V(\eta)$, where $\eta \in \mathbb{C}[x_1, \ldots, x_n]$. We set $D(\eta) := X - V(\eta)$.*

Proof. The fact that this is a topology follows from the results of Lemma 1.1.5. Consider a family of closed sets
$\{V(J_\alpha)\}_{\alpha\in P}$. Their intersection is $\cap_{\alpha\in P}V(J_\alpha) = V(\cup_{\alpha\in P}J_\alpha)$. This is also a closed set. Consider a finite set of closed sets $\{V(J_1), \dots V(J_p)\}$. Their union is $\cup_{i=1}^{p}V(J_i) = V(J_1J_2\dots J_p)$. This is also a closed set. □

Zariski Open sets are also known as *quasi-affine* varieties. Unless stated otherwise, when we make topological references such as "open", "closed" or "dense" we are referring to the Zariski Topology. Note that X is a quasi-affine variety, as $X = X - V(\mathbb{C}[x_1, \dots, x_n])$.

Lemma 1.3.7.

i. *The Zariski Topology is non-Hausdorff.*

ii. *Every open subset of an algebraic variety X is dense.*

Proof.
i. Suppose U_1 and U_2 are two open sets of an algebraic variety X. If $U_1 \cap U_2 = \emptyset$ then $X = (X - U_1) \cup (X - U_2)$ would be reducible, which is a contradiction. Thus $U_1 \cap U_2 \neq \emptyset$. This means that if x_1 and x_2 are distinct points of X, there are never disjoint neighbourhoods containing them.

ii. Firstly we prove this result with respect to the Zariski Topology. Suppose U is open in X; so $U = X - X_2$ for some subvariety X_2. If $U \subseteq Y \subsetneq X$ for some subvariety Y then $X = X_2 \cup Y$ would not be reducible. So the smallest closed set containing U must be X.

□

We denote the topology induced by the standard norm on $\mathbb{C}^n$ the "ordinary" topology. Although the Zariski Topology is our default, the ordinary topology will be used at various places in this thesis.

Lemma 1.3.8. *If X contains a set U which is "ordinary" open then $X = \mathbb{C}^n$. Thus if W is a Zariski open set of X, then the smallest "ordinary" closed set containing W is X.*

Proof. Since $\forall f \in I(X)$, $f(U) = 0$, we find by analytic continuation that $f \equiv 0$. This means $I(X)$ contains only the zero polynomial and therefore $X = \mathbb{C}^n$. Thus if W is a Zariski open subset of X then $X - W$ contains no "ordinary" open sets, which means that every point in $X - W$ is an "ordinary" topological boundary point and the "ordinary" closure of $X - W$ is X. □

We have seen that Zariski open sets are dense with respect to both topologies. To anticipate some results, we will see in Chapter 4 that the n-dimensional torus $(\mathbb{C} - \{0\})^n$ is a Zariski open subset of a *toric variety*. Lemmas 1.3.8 and 1.3.7 thus implies that the torus is a dense open subset of toric varieties with respect to both topologies.

1.4 The Spectrum

Let X be an affine algebraic set. From Lemma 1.3.4 we know that there is a one-to-one correspondence between maximal ideals $m \supseteq I(X)$ and points in X. But points in X are closed sets (the zeros of maximal ideals) in the Zariski Topology. This suggests that we can define a topology homeomorphic to the Zariski Topology on the set of all maximal ideals of $I(X)$. This is indeed the case. Since $I(X)$ is a finitely generated algebra, we define our topology on finitely generated algebras.

Let A be a finitely generated $\mathbb{C}$-algebra.

Definition 1.4.1. [6, §VI.1.2.4] The *Spectrum* of A is the set of all maximal ideals of A. Write *Spec*(A).

Later on we will introduce other, equivalent, definitions of the spectrum. If a distinction between these is needed, we refer to this definition as *Specm*(A).

We define a topology on *Spec*(A):

Definition 1.4.2. Let $J \triangleleft A$. Let sets of the form $\overline{V}(J) = \{m \in spec(A) : J \subseteq m\}$ be the closed sets of our topology. Complements of closed sets are open sets.

Lemma 1.4.3. *[8, §II.2.1] The topology is well-defined:*

i. If $K, L \triangleleft A$ then $\overline{V}(K) \cup \overline{V}(L) = \overline{V}(KL)$.

ii. If $\{J_\alpha\}_{\alpha \in P}$ is any set of ideals of A then $\cap_{\alpha \in P} \overline{V}(J_\alpha) = \overline{V}(J)$ where J is the smallest ideal containing every ideal in $\{J_\alpha\}_{\alpha \in P}$.

Proof. We follow [8, §II.2.1].

i. Let $m \in Spec(A)$. If $m \in \overline{V}(K) \cup \overline{V}(L)$ then $m \supseteq K$ or $m \supseteq L$, which means $m \supseteq KL$ and $m \in \overline{V}(KL)$. For the converse, suppose $m \in \overline{V}(KL)$. If $m \supseteq K$ then $m \in \overline{V}(K)$. Otherwise suppose $m \not\supseteq K$. Then there is a $k \in K$ such that $k \notin m$. Now for any $l \in L$, as $KL \subseteq m$ we must have $kl \in m$. Since maximal ideals are prime this implies $l \in m$. Thus $L \subseteq m$ and $m \in \overline{L}$.

ii. m is in $\cap \overline{V}(J_\alpha) \Leftrightarrow m$ contains each $J_\alpha \Leftrightarrow m$ contains J (because J is by definition the smallest ideal containing every J_α).

$\square$

We come to an obvious Lemma.

Lemma 1.4.4. *If $\phi : A \to B$ is an isomorphism then $Spec(A)$ is homeomorphic to $Spec(B)$.*

Proof. ϕ induces a bijection between the maximal ideals of A and B. This is a homeomorphism because ϕ must preserve ideal inclusions (that is, if $J \subseteq K$ then $\phi(J) \subseteq \phi(K)$). $\square$

Thus far we have only studied the spectrum as an algebraic entity. We now see that it can also be studied as a geometric entity: in fact the spectrum is homeomorphic to an algebraic set.

Let A be a finitely generated $\mathbb{C}$-algebra with generators $\{a_1, \dots a_k\}$. We assume A is an integral domain (that is, $xy = 0 \Rightarrow x = 0$ or $y = 0$). Define a homomorphism $\phi : \mathbb{C}[x_1, \dots x_k] \to A$ by setting $\phi(x_i) = a_i$ and naturally extending ϕ to sums and multiples of the x_i. By the Ring Isomorphism Theorem we find

$$\mathbb{C}[x_1, \dots, x_k]/ker(\phi) \simeq A$$

Since A is an integral domain, $ker(\phi)$ is prime.

Thus we can understand the spectra of finitely generated $\mathbb{C}$-algebra*s* through studying the spectra of algebras of the form $\mathbb{C}[x_1, \dots, x_k]/J$, where J is prime. From now on we assume $A = \mathbb{C}[x_1, \dots x_n]/J$, where J is prime. This leads us to an extension of Lemma 1.3.4, which will facilitate a geometric interpretation of the spectrum.

Proposition 1.4.5. *The bijection $\gamma : V(A) \to Spec(A)$ that takes $x \to m_x$ is a homeomorphism.*

Proof. It suffices to show that γ identifies closed sets with closed sets. Closed sets in $V(A)$ are of the form $V(J)$ $(J \triangleleft A)$ and closed sets in $Spec(A)$ are of the form $\overline{V}(K)$ $(K \triangleleft A)$. By Lemma 1.3.1 $V(J) = \{x : J \subseteq m_x\}$. Thus $\gamma(V(J)) = \{m_x : J \subseteq m_x\} = \overline{V}(J)$. $\square$

Thus the spectrum of a finitely generated algebra A (where A is an integral domain) is homeomorphic to an affine algebraic set. This suggests an alternative realisation of the spectrum:

Definition 1.4.6. *Geometric Realisation of the Spectrum.* Let A be a finitely generated algebra. The *coordinate realisation of the spectrum*, or just *spectrum*, of A is $V(J)$ where $A \simeq \mathbb{C}[x_1, \ldots x_k]/J$.

This calls for some remarks.

Remark 1.4.7. A quick word on our notation. Suppose A is a $\mathbb{C}$-algebra with generators $\{w_1, \ldots, w_k\}$. Often we stipulate $(x_1, \ldots, x_k)$ as *coordinates* for A. This is a shorthand way of referring to the vanishing set $V(J) \subseteq \mathbb{C}^k$, where J is such that $\mathbb{C}[x_1, \ldots, x_k]/J \simeq A$. That is, when we stipulate $(x_1, \ldots, x_k)$ as *coordinates* for A we are pointing to an algebraic set $V(J)$ in $\mathbb{C}^k$ that is homeomorphic to *Spec*(A). □

Remark 1.4.8. The coordinate realisation of the spectrum is thus far only defined up to homeomorphism. Suppose $A \simeq \mathbb{C}[y_1, \ldots, y_l]/J'$ and $A \simeq \mathbb{C}[x_1, \ldots x_k]/J''$. Thus $V(J')$ and $V(J'')$ are both homeomorphic to *Spec*(A). In Chapter 2 we will see that $V(J')$ and $V(J'')$ more than just homeomorphic; they are *isomorphic* in the sense that there is a bijective "morphism" between them. □

By Corollary 1.3.5 there is a bijective relationship between $Hom_{\mathbb{C}\text{-alg}}(A, \mathbb{C})$ and *Spec*(A). This leads us to a third understanding of the Spectrum.

Definition 1.4.9. $\mathbb{C}$-*algebra Homomorphism Definition of the Spectrum.* The $\mathbb{C}$-algebra homomorphism spectrum of A is $Hom_{\mathbb{C}\text{-alg}}(A, \mathbb{C})$.

If $A = \mathbb{C}[x_1, \ldots, x_n]/J$ then by Remark 1.3.5 the correspondence is: $(y_1, \ldots, y_n) \in V(J) \to \xi \in Hom_{\mathbb{C}\text{-alg}}(A, \mathbb{C})$, where $\xi(x_i) = y_i$.

Remark 1.4.10. All these definitions of the spectrum are useful. The coordinate definition is sometimes more intuitive as one can "visualise" it. However the coordinate definition also requires the introduction of co-ordinates: which can mean a lot of cumbersome notation (as is perhaps the case in Ewald ([6])). On the other hand, the algebraic definition of the spectrum often makes for more elegant proofs. Furthermore it is more consistent with the definition used in more advanced algebraic geometry, where the spectrum is defined as the set of all *prime ideals* (see [8, p 70]). The $\mathbb{C}$-algebra homomorphism definition is useful when A can not be easily expressed as the quotient of a polynomial ring by a radical ideal. It also dovetails elegantly with a fourth realisation of the Spectrum to be introduced in Chapter 4: the Semigroup-Homomorphism realisation of the Spectrum. □

Chapter 2

Morphisms and Singularities

In the previous chapter we defined the coordinate ring $\mathbb{C}[X]$ of an affine variety X. We saw that the coordinate ring is essentially the restriction of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ to X. In this chapter we define additional functions on affine varieties, and also look at functions on quasi-affine varieties. We conclude with a statement of Hironaka's "Resolution of Singularities" theorem.

2.1 Regular Functions on Quasi-Affine Varieties

In this section we will be looking at functions defined on quasi-affine varieties $X \subset \mathbb{C}^n$. More specifically, we want to take fractions of polynomials of affine and quasi-affine sets. In order for our definition to be consistent we need to take a slight detour and look at the algebra of rings. Let R be a ring.

Definition 2.1.1. A *multiplicatively closed subset* $T \subset R$ is a subset which satisfies

i. $s, t \in T \Rightarrow st \in T$
ii. $1 \in T$

Lemma 2.1.2. *[3, §4.8],[8, Introduction] Let $T \subset R$ be a multiplicatively closed subset. Define $R[T^{-1}]$ to be the set of equivalence classes of the set $\left\{\frac{r}{t} : r \in R, t \in T\right\}$, with equivalence relation $\frac{r}{t} \sim \frac{r'}{t'} \Leftrightarrow t"(rt' - r't) = 0$ for some $t" \in T$. $R[T^{-1}]$ is a ring with the operations:*

$$\frac{r}{t} + \frac{r'}{t'} = \frac{rt' + r't}{tt'} \text{ and } \frac{r}{t} . \frac{r'}{t'} = \frac{rr'}{tt'}$$

Further we have a natural ring homomorphism $R \to R[T^{-1}]$: $r \to \frac{r}{1}$.

The proof of this requires routine checking; refer to [3, §4] for the details.

Consider $T = \{1, t, t^2, \ldots\}$ for some $t \in R$. It is clear that T is multiplicatively closed. We write $R[t^{-1}] := R[T^{-1}]$. This leads us to a lemma we will use in Chapter 4.

Lemma 2.1.3. *Let $x \in R$. Then $R[x^{-1}] \simeq R[y]/(xy-1)$*

Proof. Let $\phi : R[y] \to R[x^{-1}]$ be the homomorphism that maps $y \to x^{-1}$ and fixes all other elements of R. Note that ϕ is surjective. We prove $ker(\phi)$ is the ideal generated by $(xy-1)$. Note that $(xy-1) \in ker(\phi)$. Let $p = a_o + a_1 y + \ldots + a_k y^k \in ker(\phi)$. We prove p belongs to the ideal generated by $(xy-1)$. Now $\phi(p) = a_0 + a_1 x^{-1} + \ldots + a_k x^{-k} = 0$. On multiplying through by x^k we find $a_k = -xg$ where $g = a_0 x^{k-1} + a_1 x^{k-2} + \ldots + a_{k-1}$. It follows that $p = a_0 + a_1 y + \ldots + (-xg)y^k$. Hence $q(y) := p + (xy-1).g(x)y^{k-1} = a_0 + a_1 y + \ldots + a_{k-1}y^{k-1} - g(x)y^{k-1}$. Observe that $q(y) \in ker(\phi)$ and is of degree $k-1$. Also $p = q(y) - (xy-1)g(x)y^{k-1}$. Thus if $q(y)$ belongs to the ideal generated by $(xy-1)$ then p does too. We repeat this reasoning with $q(y)$ to obtain a polynomial $q_2(y)$ of degree $k-2$, and similarly find that $q(y)$ (and therefore $p(y)$) belongs to the ideal generated by $(xy-1)$ if $q_2(y)$ does. Continuing inductively we eventually find that $p(y)$ belongs to the ideal generated by $(xy-1)$ if a polynomial of degree 0, i.e. a constant c, does. But $\phi(c) = 0 \Rightarrow c = 0$. Thus p can be written as a sum of powers of $(xy-1)$. This means $ker(\phi)$ is the ideal generated by $(xy-1)$. By the Ring Isomorphism Theorem $R[x^{-1}] \simeq R[y]/(xy-1)$. □

Let X be an affine variety, and let $T := \{f \in \mathbb{C}[X] : f \neq 0\}$. Now T must be a multiplicatively closed set. To see this, suppose for a contradiction that $0 \neq f, g \in \mathbb{C}[X]$ and $fg \notin T$ (i.e. $fg = 0$). Then $X = (V(f) \cap X) \cup (V(g) \cap X)$. That is, if T is not multiplicatively closed then X is not irreducible, which contradicts our stipulation that X is irreducible. Thus T is multiplicatively closed, and we can make the following definition:

Definition 2.1.4. Let $\mathbb{C}(X) := \mathbb{C}[X][T^{-1}]$. This is called the *function field* or *ring of rational functions.*

i. A function f is *rational* if $f \in \mathbb{C}(X)$.
ii. A rational function f is *regular* at $p \in Y$ if there exist $g, h \in \mathbb{C}[X]$ with $h(p) \neq 0$ and $f(p) = \frac{g(p)}{h(p)}$. Note that f is well-defined on the neighbourhood $\{x : h(x) \neq 0\}$.
iii. If f is regular everywhere on X then f is denoted a *regular function* on X.

Remark 2.1.5. Note that if $f \in \mathbb{C}(X)$ then f may not be defined at every point in X. In fact f is defined precisely where f is regular. □

We can also define regular functions on quasi-affine varieties. Let $U \subset X \subseteq \mathbb{C}^n$ be a quasi-affine variety.

Definition 2.1.6. Suppose f is rational on X. We say f is *regular* on U if f is regular at every point in U.

Let $D(f) \subseteq X$ be a principal open set. Recall that $D(f) = X - V(f)$.

Lemma 2.1.7. *[3, §5.7] The set of regular functions on $D(f)$ is $\mathbb{C}[X][f^{-1}] \subseteq \mathbb{C}(X)$.*

Proof. Refer to [3, §5.7] for a proof. □

We obtain the following corollary by setting $f = 1$:

Corollary 2.1.8. *The set of regular functions on X is $\mathbb{C}[X]$.*

That is, if f_2 is regular on X and at every point $x \in X$, we can write $f_2 = \frac{g_x}{h_x}$ for $f_x, g_x \in \mathbb{C}[X]$ on some open subset containing x, then $f_2 \in \mathbb{C}[X]$.

Example 2.1.9. Let $f := z_1 \in \mathbb{C}[x_1, \ldots, x_n]$ and let $X = V(f)$. Observe that X is the plane $z_1 = 0$, and $D(f)$ is the complement of this plane. Consider the rational function in $\mathbb{C}(X)$: $h = \frac{1}{z_1 z_2}$. Observe that h is not regular when $z_2 = 0$, but is regular everywhere else. Now let U be the quasi-affine variety $X - V(z_2)$. h is regular everywhere on U. More generally, we find by Lemma 2.1.7 that the regular functions on U are $\{\frac{g}{z_2^p} : p \in \mathbb{N},\ g \in \mathbb{C}]X]\}$. □

2.2 Regular Maps

In this section we naturally define regular maps between quasi-affine varieties as composites of regular functions. Remember that affine varieties are also quasi-affine varieties: so our definitions apply to affine varieties as well.

Let $U \subseteq X \subseteq \mathbb{C}^n$ be a quasi-affine variety.

Definition 2.2.1. Let $W \subset \mathbb{C}^m$ be another quasi-affine variety. A *regular map* or *morphism of quasi-affine varieties* is a map $\phi : U \to W$ of the form

$$\phi(u) = (\phi_1(u), \ldots, \phi_m(u))$$

where the $\{\phi_i\}$ are regular functions on U.

This leads us to a notion of isomorphism.

Definition 2.2.2. A morphism of varieties $\phi : U \to W$ is an *isomorphism* if there is an inverse morphism $\psi : W \to U$ such that $\phi \circ \psi = id_W$ and $\psi \circ \phi = id_U$. In such a case we write $U \simeq W$.

If $U \simeq W \subset W'$ where W is a Zariski open or closed subset of W' we write $U \hookrightarrow W'$. If W is an open set we say ϕ is an *open embedding* of U into W', and if W is a closed set we say ϕ is a *closed embedding* of U into W'.

Definition 2.2.3. Let Y be an affine variety. A *rational map* $\phi : X \rightsquigarrow Y$ is a regular function defined on a dense open subset U of X. ϕ is *birational* if it has a rational inverse $\psi : Y \rightsquigarrow X$ such that $\phi\psi$ and $\psi\phi$ are the identity map on dense open sets.

Suppose $\phi : X \rightsquigarrow Y$ and $\xi : X \rightsquigarrow Y$ are birational maps. If ϕ is defined on X_ϕ and ξ is defined on X_ξ, we say that ϕ and ξ are birationally equivalent if they are equal on the open set $X_\xi \cap X_\phi$.

2.2.1 Morphisms of Affine Varieties

In later chapters we will predominantly be studying affine varieties, so we turn to concentrating on these. Let X and Y be affine varieties. The major result of this section is Proposition 2.2.7: X and Y are isomorphic if and only if $\mathbb{C}[X]$ and $\mathbb{C}[Y]$ are isomorphic.

Straightforward checking reveals that the pull-back is a well-defined $\mathbb{C}$-algebra homomorphism.

Proposition 2.2.4. *Give any morphism of affine varieties $\phi : X \to Y$, then we define the pullback $\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$, $\phi^*(f) := f \circ \phi$. ϕ^* is a $\mathbb{C}$-algebra homomorphism.*

Proof. Let $f, g \in \mathbb{C}[Y]$ and $\lambda \in \mathbb{C}$. We check linearity: $(f+g)^*(x) = (f+g)(\phi(x)) = (f \circ \phi)(x) + (g \circ \phi)(x) = (f^* + g^*)(x)$. Also $(\lambda f)^* = (\lambda f \circ \phi) = \lambda \phi^*(f)$. Finally, we check the pullback preserves multiplication: $(f.g)^* = (f.g) \circ \phi = (f \circ \phi).(g \circ \phi) = f^*.g^*$. □

As the components of a morphism are rational functions, it is clear that a morphism is continuous with respect to the "ordinary" topology. Indeed the morphism is also continuous with respect to the Zariski Topology.

Lemma 2.2.5. *A morphism of varieties $\phi : X \to Y$ is continuous with respect to the Zariski Topology. If ϕ is an isomorphism then ϕ is also a homeomorphism.*

Proof. We prove that the preimage of any closed set is closed. Let $S \subset Y$ be a closed; we need to show $\phi^{-1}(S)$ is closed. Let $J = \phi^*(I(S))$. It follows that

$\phi^{-1}(S) = V(J)$, because if $x \in X$ then $x \in V(J) \Leftrightarrow \forall g \in J,\ g(x) = 0 \Leftrightarrow \forall f \in I(S),\ (f \circ \phi)(x) = 0 \Leftrightarrow \phi(x) \in S$.

The second assertion follows from the fact that ϕ^{-1} is a morphism and therefore is continuous. $\square$

There is an inverse $^\wedge$ to the pullback.

Proposition 2.2.6. *[3, §6] Let $\psi : \mathbb{C}[Y] \to \mathbb{C}[X]$ be an algebra homomorphism. Define the wedge of ψ, $\psi^\wedge : X \to Y$ as follows*

$$(\psi^\wedge)(x) = ((\psi \circ y_1)(x), \ldots, (\psi \circ y_m)(x))$$

We claim that $\psi^\wedge$ is a morphism. Further, $(\psi^\wedge)^* = \psi$. Conversely if ϕ is a morphism from X to Y then $(\phi^*)^\wedge = \phi$.

Proof. We need to check $im(\psi^\wedge) \subseteq Y$. It is sufficient to show that for any $x \in X$ and any $f \in I(Y)$, $f(\psi^\wedge(x)) = 0$. However since $\psi \circ f = 0$ (as $f \in I(Y)$), it suffices to show that

$$(\psi \circ f) = (f \circ \psi^\wedge) \tag{2.1}$$

That is, we need to show $\forall x \in X$, $(\psi(f))(x) = f(\psi(y_1)(x), \ldots, \psi(y_m)(x))$. We prove this for the special case that f is a monomial, because if it is true for all monomials then it must be true for all polynomials. Let $f = \prod_{i=1}^{k} y_i^{a_i}$, then

$$\begin{aligned} (\psi(f))(x) &= \psi(\prod_{i=1}^{k} y_i^{a_i}(x)) = \prod_{i=1}^{k} (\psi(y_i)(x))^{a_i} \\ &= f(\psi(y_1)(x), \ldots \psi(y_m)(x)) \end{aligned}$$

as required.

Secondly we prove that the wedge and the pullback are inverses of each other. Let $\phi : X \to Y$ be a homomorphism of varieties and let $\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ be its pullback. Then

$$\begin{aligned} (\phi^*)^\wedge(x_1, \ldots x_n) &= (\phi^*(y_1)(x), \ldots, \phi^*(y_m)(x)) \\ &= ((y_1 \circ \phi)(x) \ldots (y_m \circ \phi)(x)) \\ &= \phi(x) \end{aligned}$$

Hence $\phi_2 = \phi$ as required. To prove the converse consider $(\psi^\wedge)^*(f) = f \circ \psi^\wedge$. By (2.1) this equals $\psi \circ f$. Thus $(\psi^\wedge)^* = \psi$. $\square$

This brings us to the most important result of the section: the bijective correspondence between algebra isomorphisms and isomorphisms of varieties.

Proposition 2.2.7. *[8, §I.3], [6, §VI.1] Let $\phi : X \to Y$ be a morphism of affine varieties, and let $\phi^* : \mathbb{C}[Y] \to \mathbb{C}[X]$ be the corresponding homomorphism of rings. Then ϕ^* is an isomorphism if and only if ϕ is an isomorphism.*

Proof. We use the bijection between morphisms and algebra homomorphisms of Proposition 2.2.6.

$$\begin{aligned} \phi^* \text{is an isomorphism} \quad &\Leftrightarrow \quad \text{there exists an inverse homomorphism } \psi : \mathbb{C}[X] \to \mathbb{C}[Y] \\ &\Leftrightarrow \quad \text{there exists a morphism } \psi^\wedge : Y \to X \\ &\Leftrightarrow \quad \phi \text{ is an isomorphism} \end{aligned}$$

□

Remark 2.2.8. This result will be particularly useful in chapter 4. It means that if two finitely generated algebras are isomorphic then their geometric spectra are isomorphic as well. If we had only required X and Y to be homeomorphic in order that $X \simeq Y$ then we would have been unable to define the pullback, and therefore unable to prove Proposition 2.2.7. This is why we needed to define the morphism in terms of regular functions. □

2.2.2 Morphisms of Spectra

Let X and Y be affine varieties, and $\phi : X \to Y$ a morphism. In Proposition 1.4.5 we saw that X and Y are homeomorphic to $Specm(\mathbb{C}[X])$ and $Specm(\mathbb{C}[Y])$ respectively. Thus under this homeomorphism, ϕ can also be viewed as a morphism of spectra, $\phi : Specm(X) \to Specm(Y),\ m_x \to m_{\phi(x)}$. In fact we will see that this morphism can be described without reference to the geometric realisations of the spectra.

Lemma 2.2.9. *If $x \in X$ then $m_{\phi(x)} = (\phi^*)^{-1}(m_x)$*

Proof. Let $f \in \mathbb{C}[Y]$. Then $f \in m_{\phi(x)} \Leftrightarrow f(\phi(x)) = 0 \Leftrightarrow (\phi^* f)(x) = 0 \Leftrightarrow \phi^*(f) \in m_x$. □

This Lemma allows us to understand a morphism purely algebraically. Suppose A and B are each a finitely generated $\mathbb{C}$-algebra and an integral domain, and suppose

$$A \simeq \mathbb{C}[x_1, \dots, x_n]/I(X) \text{ and } B \simeq \mathbb{C}[y_1, \dots, y_m]/I(Y) \tag{2.2}$$

where X and Y are affine algebraic sets. Let $\psi : A \to B$ be a $\mathbb{C}$-algebra homomorphism. Under the isomorphisms of (2.2) we can write this as $\psi_1 : \mathbb{C}[x_1, \dots, x_n]/I(X)$ $\to \mathbb{C}[y_1, \dots, y_m]/I(Y)$. This induces the morphism of spectra

$$\psi_1^\wedge : Spec(\mathbb{C}[y_1, \dots, y_m]/I(Y)) \to Spec(\mathbb{C}[x_1, \dots, x_n]/I(X))$$

defined such that $\psi_1^\wedge(m_y) = \psi_1^{-1}(m_y)$. Translating this result back to A and B it can be seen that $\psi_1^\wedge$ is the continuous map $\psi^\wedge : Spec(B) \to Spec(A),\ \psi^\wedge(m) = \psi^{-1}(m)$. Thus we can make this definition / proposition:

Proposition 2.2.10. *Let $\psi : A \to B$ be a homomorphism, where A and B are each a finitely generated $\mathbb{C}$-algebra and an integral domain. Then ψ induces a continuous map $\psi^\wedge : Spec(B) \to Spec(A),\ \psi^\wedge(m) = \psi^{-1}(m)$ which we denote a morphism of spectra.*

We can also realise the morphism in terms of $\mathbb{C}$-algebra homomorphisms. Suppose $\psi : A \to B$ is the homomorphism of Proposition 2.2.10. If $\xi \in Hom_{\mathbb{C}\text{-alg}}(B, \mathbb{C})$ has kernel $m \in Spec(B)$ then $\xi \circ \psi \in Hom_{\mathbb{C}\text{-alg}}(A, \mathbb{C})$ has kernel $\psi^{-1}(m)$. Thus the morphism $Spec(B) \to Spec(A)$ realised in terms of $\mathbb{C}$-algebra homomorphisms is $\xi \to \xi \circ \psi$.

Thus we can understand the induced morphism in terms of coordinates, in terms of maximal ideals and in terms of $\mathbb{C}$-algebra homomorphisms. The latter two are useful because they allow us to work with the induced morphism without setting up coordinates.

2.3 The Resolution of Singularities

In this section we state one of the most important results of algebraic geometry: Hironaka's proof that there is a resolution of singularities of any algebraic variety over a field of nonzero characteristic. However before we do this we need to make some comments on the *dimension of a variety*, *singularities* and *projective space*. It is possible to go into great depth in these topics. However our discussion will not be detailed, as detail is not required to characterise the singularities of toric varieties.

Let X be a quasi-affine algebraic variety.

Definition 2.3.1. [8, I.1],[6, §VI.1] The dimension of X is the supremum of all integers n such that there exists a chain

$$\emptyset \neq X_0 \subset X_1 \subset \ldots \subset X_n = X$$

of distinct irreducible closed sets.

Note that the dimension must be finite because by Corollary 1.2.5 an infinite sequence of nested distinct algebraic sets $X_1 \subsetneq X_2 \subsetneq X_3 \subsetneq \ldots$ does not exist.

We list without proof some consequences of this definition. Consult [8, §I.1] or [6, §VI.1] for more details.

Proposition 2.3.2. *If $U \subset X$ is a quasi-affine variety then $dim(U) = dim(X)$. Also $dim(\mathbb{C}^n) = n$. Since the torus $(\mathbb{C}^*)^n$ is an open subset of $\mathbb{C}^n$, the dimension of $(\mathbb{C}^*)^n$ is n.*

Now that we have a notion of dimension we can move onto *singularities.* If $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $P \in \mathbb{C}^k$ is such that $f(P) = 0$, define $d_P f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(P).(x_i - p_i)$. This is known as the *differential* of f at P. The *tangent space* $T_P V(f)$ to $V(f)$ at P is the vanishing set of $d_P f$. If X is a variety and $I(X)$ is generated by $\{f_1, \ldots, f_k\}$ then we define the tangent space to $P \in X$ to be $T_P X = \bigcap_{i=1}^{k} T_P V(d_P f_i)$. We say that P is a *nonsingular point* if the dimension of $T_P X$ (as a vector space) equals the dimension of X; otherwise we say P is a *singularity.*

This is the classical definition of singularity. However it is not the definition most suited to our purposes. It requires an understanding of the geometry of $X \hookrightarrow \mathbb{C}^n$; however this is not easy to determine in toric varieties. Instead we will work with a more algebraic definition of the singularity.

Definition 2.3.3. A point $P \in X$ is *nonsingular* if the (vector space) dimension of $\frac{m_P}{m_P^2}$ equals the dimension of X, otherwise P is *singular.*

It is straightforward to prove that this definition is equivalent to the tangent space definition. The proof involves recognising that the kernel of $d_P : m_P \to T_P X$ is m_P^2, which follows from the fact that the differential of any monomial $(x_1 - p_1)^{a_1} \ldots (x_n - p_n)^{a_n}$ equals zero if $\sum_i a_i > 1$. Thus by the Ring Isomorphism Theorem $\frac{m_P}{m_P^2} \simeq (T_P X)^*$.

We now have a sufficient understanding of singularities to characterise the singularities of toric varieties in Chapter 6. We turn to Hironaka's celebrated theorem. Since it encompasses mathematics which goes well beyond the scope of this thesis, we only state a limited version of it.

Definition 2.3.4. Let X be a singular quasi-affine variety. If Y is a nonsingular quasi-affine variety and $f : Y \to X$ is a surjective birational map then we say f is a *resolution of singularities.*

Theorem 2.3.5. Hironaka. *If X is a singular quasi-affine variety then there always exists a resolution of its singularities.*

The beauty of toric varieties is that we do not need the sophisticated mathematics of Hironaka's proof to understand the resolution of their singularities. We can ascertain whether a toric variety is singular or not merely through studying the *fan* that generates it. Moreover we will see in Chapter 6 that the problem of resolving the singularities of a toric variety can be reduced to the problem of finding a refinement of its fan. In Chapter 7 we will specifically compute the resolution of singularities of 2-dimensional cones.

Chapter 3

Cones and Fans

We now turn to studying convex cones and fans, which might on first appearances seem completely unrelated to algebraic varieties. However we will show one can define special types of ideals from convex fans and from there we will look at the algebraic varieties generated from these. In fact toric varieties can largely be understood through studying the fans that generate them. Many of the lemmas and theorems might seem tangential to the reader who confined themself to this chapter alone. However they will all be needed in later chapters. We will employ the combinatorics of cones and fans in the resolution of toric singularities in Chapters 6 and 7.

We haven't the space to build our study of cones from scratch. So in certain places we refer to more general theorems from the study of convex sets. A detailed overview of convex set theory may be obtained from Arne [1], Grunbaum [7], Rockafellar [14] or Ewald [6]. We have incorporated many results from Fulton [4] into this chapter. However Fulton is extremely terse, so the majority of the proofs are my own.

The reader is advised not to spend too much time on this chapter and to refer back to it later as necessary.

3.1 Convex Sets and Cones

Let V be an n-dimensional vector space over $\mathbb{R}$, with dual space V^*.

Definition 3.1.1. If $u \in V$, $v \in V^*$ then $\langle u, v\rangle = u(v)$

Remark 3.1.2. There is a reason we write this as an inner product. If $u = \sum_{i=1}^n u_i e_i$ and $v = \sum_{j=1}^n v_i e_i^*$ then $u(v) = \sum_{i=1}^n u_i v_i$. That is, $u(v) = (u_1, \ldots, u_n).(v_1, \ldots, v_n)$. □

Definition 3.1.3. [4, p 12] A *convex polyhedral cone*, or *cone* for short, is a set

$$\sigma = \{r_1 v_1 + \ldots r_s v_s \in V : r_i \geq 0\}$$

Equivalently this may be written $\sigma = \mathbb{R}_{\geq 0} v_1 + \ldots + \mathbb{R}_{\geq 0} v_s$. The vectors $\{v_i\}$ are called *generators* of the cone σ. We write $\mathbb{R}.\sigma$ for the smallest vector space containing σ. The *dimension* of σ is defined to be the dimension of $\mathbb{R}.\sigma$.

As a convention, we assume throughout this chapter that σ is a convex polyhedral cone in $V = \mathbb{R}^n$ generated by $S = \{v_1, \ldots v_s\}$. We also assume none of the generators are redundant. That is, if we remove any generator from S then we are left with a different cone. Observe that σ is saturated: if $x \in \sigma$ then $\lambda x \in \sigma$ for all $\lambda \in \mathbb{R}_{\geq 0}$.

Definition 3.1.4. Let B be the matrix with the generators $\{v_i\}$ as columns. We denote B the *generating matrix* of the cone σ.

Observe that $\sigma = \{Bx | x \in \mathbb{R}^k_{\geq 0}\}$.

Definition 3.1.5. [1, pp 8-9] We introduce some concepts from the theory of convex sets.

i. If $u \in V^*$ and $\alpha \in \mathbb{R}$, then the $n-1$-dimensional subspace of V, $H = \{x : \langle x, u \rangle = \alpha\} := V \cap u^{\perp}$, is denoted a *hyperplane*.
ii. A set of the form $\{x : \langle x, u \rangle \geq \alpha\}$ or $\{x : \langle x, u \rangle \leq \alpha\}$ is a *halfspace* bounded by H.
iii. Suppose K is a convex set. Then H is a *supporting hyperplane* of K if $K \cap H \neq \emptyset$ and either $\langle x, u \rangle \leq \alpha$ for all $x \in K$ or $\langle x, u \rangle \geq \alpha$ for all $x \in K$.

In other words, a supporting hyperplane of K is a hyperplane intersecting K such that K lies entirely in one of its halfspaces.

Lemma 3.1.6. *If H is a supporting hyperplane of a cone σ then $0 \in H$. Hence every supporting hyperplane of a cone is of the form $H = \{v : \langle u, v \rangle = 0\}$.*

Proof. Let $H = \{v : \langle v, u \rangle = \alpha\}$ for some $u \in V^*$ and $\alpha \in \mathbb{R}$. Let $x \in \sigma \cap H$. Suppose there exists $\lambda \neq 1$ such that $\lambda x \in H$. Then $\langle u, \lambda x \rangle = \langle u, x \rangle = \alpha$, and hence $\langle u, (\lambda - 1)x \rangle = (\lambda - 1)\langle u, x \rangle = 0$. Thus $\alpha = 0$ and therefore $0 \in H$.

Otherwise for all $\lambda \neq 1$, $\lambda x \notin H$. In particular, $2x$ and $\frac{1}{2}x$ must lie in different halfspaces, as $\langle 2x, u \rangle > \alpha$ and $\langle \frac{1}{2}x, u \rangle < \alpha$. Since both these points lie in σ (as σ is saturated), this contradicts our assumption that H is a supporting hyperplane. $\square$

Figure 3.1: On the left is the cone generated by $(0,1)$ and $(2,-1)$, and on the right is its dual.

For reasons which will become apparent later we will mostly be working with the *dual* of a cone:

Definition 3.1.7. The *dual* $\sigma^\vee$ of σ is

$$\sigma^\vee = \{u \in V^* : \langle x, u\rangle \geq 0 \ \forall x \in \sigma\}$$

In Section 3.5 we will see that $\sigma^\vee$ is also a convex polyhedral cone.

We now state two fundamental results from the theory of convex sets. These are sufficient for us to build up an understanding of the combinatorics of cones. The first proposition reveals that for any point $v \notin \sigma$ we can always find a hyperplane separating v from σ. For a proof refer to [1, §4.4.4].

Proposition 3.1.8. *[4, p 9], [1, §4.4.4] If σ is a convex polyhedral cone and $v \notin \sigma$, then there is some $u \in \sigma^\vee$ with $\langle v, u\rangle < 0$.*

Although we haven't yet proved $\sigma^\vee$ is a cone, we can naturally define the dual of $\sigma^\vee$ to be $(\sigma^\vee)^\vee := \{v \in \sigma : \langle v, u\rangle \geq 0, \ \forall u \in \sigma^\vee\}$.

Theorem 3.1.9. *[11, §A.4],[4, p 9] The dual of $\sigma^\vee$ is σ: $(\sigma^\vee)^\vee = \sigma$.*

Proof. If $v \in \sigma$ then by definition $\forall u \in \sigma^\vee$ $\langle v, u\rangle \geq 0$. So $v \in (\sigma^\vee)^\vee$ and hence $\sigma \subseteq (\sigma^\vee)^\vee$. Conversely, if $v \notin \sigma$ then Proposition 3.1.8 implies there exists $u \in \sigma^\vee$ such that $\langle v, u\rangle < 0$. This implies $v \notin (\sigma^\vee)^\vee$. Thus $(\sigma^\vee)^\vee = \sigma$. □

3.2 Faces

The faces of a cone shed considerable insight into its structure. For this reason we will spend some time exploring the relationship between a cone and its faces. Our results will be of considerable value in the construction of toric varieties in Chapter 4, in the analysis of the torus action in Chapter 5 and in the resolution of toric singularities in Chapter 6.

Definition 3.2.1. A *face* τ of σ is the intersection of σ with a supporting hyperplane. We write $\tau \prec \sigma$.

Because every supporting hyperplane goes through the origin (Lemma 3.1.6), τ is of the form:

$$\tau = \sigma \cap u^{\perp} = \{v \in \sigma : \langle u, v\rangle = 0\}$$

where $u \in \sigma^{\vee}$. The dimension of τ is the dimension of the smallest subspace containing τ. A cone is regarded as a face of itself, while other faces are called *proper* faces. A *facet* is a face of codimension 1.

Definition 3.2.2. A cone σ is *strongly convex* if $\sigma \cap (-\sigma) = \{0\}$. Note that this is equivalent to saying that σ contains no nonzero linear subspace.

Lemma 3.2.3. *Let W be the subspace generated by σ.*

i. If $dim(W) \geq 2$ and $\sigma \neq W$ then σ possesses at least one proper nonzero face.

ii. If σ has the above properties but in addition is strongly convex then σ possesses at least two proper nonzero faces.

Proof. i. We consider σ as a cone in W. Then $\sigma \neq W$ and so in the topology of W, σ possess a boundary point $x \neq 0$. From the theory of convex geometry x must lie in a supporting hyperplane $H \subset W$ (see [1, §1.4.3]). Hence $\sigma \cap H$ is a face τ. As $dim(\sigma) = dim(W)$ is greater than $dim(H)$, τ is proper.

ii. If σ only has one proper face τ then τ must be its topological boundary, which can only mean that σ is a hyperplane. This contradicts our stipulation that σ is not strongly convex.

□

Proposition 3.2.4. *A face $\tau = \sigma \cap u^{\perp}$ is also a convex polyhedral cone. τ is generated by $\tau \cap S$.*

Proof. Let $S_2 = \tau \cap S$. We prove τ is the cone generated by S_2. Let $y = \sum_{i=1}^{s} \mu_i v_i \in \tau$, (where $\mu_i \geq 0$). Now

$$\langle y, u\rangle \quad = \quad \langle \sum_{i=1}^{s} \mu_i v_i, u\rangle = \sum_{i=1}^{s} \mu_i \langle v_i, u\rangle = 0 \text{ as } y \in \tau$$

But for all i, $\langle v_i, u\rangle \geq 0$ because $u \in \sigma^{\vee}$. Since the entire sum is zero, it must be the case that $\mu_i\langle v_i, u\rangle = 0$. If $v_i \notin \tau$ then $\langle v_i, u\rangle \neq 0$ and hence $\mu_i = 0$. But this means y belongs to the cone generated by S_2. Conversely, suppose $y = \sum_{i=1}^{s} \mu_i v_i \in \tau$ ($v_i \in S_2$). Because $\langle v_i, u\rangle = 0$ it follows that $\langle y, u\rangle = 0$ and $y \in \tau$. □

Lemma 3.2.5.

i. Any intersection of faces is also a face.

ii. Any face of a face is a face.

Proof. i. Let $\tau_1 = \sigma \cap u_1^{\perp}$ and $\tau_2 = \sigma \cap u_2^{\perp}$ ($u_i \in \sigma^{\vee}$) be two faces. We prove $\tau_1 \cap \tau_2 = \sigma \cap (u_1 + u_2)^{\perp}$. Suppose $x \in \tau_1 \cap \tau_2$. Then $\langle u_1 + u_2, x\rangle = \langle u_1, x\rangle + \langle u_2, x\rangle = 0$. So $x \in \sigma \cap (u_1 + u_2)^{\perp}$. Conversely suppose $x \in \sigma \cap (u_1 + u_2)^{\perp}$. Then $\langle u_1 + u_2, x\rangle = \langle u_1, x\rangle + \langle u_2, x\rangle = 0$. Since $\langle u_i, x\rangle \geq 0$, it must be the case that $\langle u_1, x\rangle = \langle u_2, x\rangle = 0$. So $x \in \tau_1 \cap \tau_2$.

ii. We follow [4, §1.2.4]. Suppose $\tau = \sigma \cap u^{\perp}$ and $\gamma = \tau \cap (u')^{\perp}$ ($u \in \sigma^{\vee}$ and $u' \in \tau^{\vee}$). We find a $p \in \mathbb{R}$ such that $u' + pu \in \sigma^{\vee}$. Consider $\langle u' + pu, v_i\rangle = \langle u', v_i\rangle + \langle pu, v_i\rangle$, where v_i is a generating vector. We need this expression to be greater than or equal to zero. Note that if $v_i \in \tau$ then the inner product is zero. We choose p such that:

$$p \geq min\left\{\frac{-\langle u', v_i\rangle}{\langle u, v_i\rangle} : v_i \notin \tau\right\}$$

It immediately follows that $\langle u' + pu, v_i\rangle \geq 0$ for all generating vectors v_i, and therefore this result holds for all $v \in \sigma$. In fact $\gamma = \sigma \cap (u' + pu)^{\perp}$. To see this, let $x \in \sigma \cap (u' + pu)^{\perp}$. Then $\langle u' + pu, x\rangle = 0$ and $\langle u', x\rangle + p\langle u, x\rangle = 0$. Since both these summands must be greater than or equal to zero, they must both be zero. So $x \in \gamma$.

□

The following partial ordering is nonstandard.

Proposition 3.2.6. *If σ is strongly convex, then we define the following partial ordering on σ. If $u, w \in \sigma$ write*

$$u \leq w \textit{ if there exists } c \in \sigma \textit{ such that } w = u + c$$

Proof. We check this is well-defined. The reflexivity and transitivity of $(\leq)$ are clear: so we only need to check its anti-symmetry. Suppose $w \leq u$ and $u \leq w$. Then $\exists b, c \in \sigma$ such that $w = u + b$ and $u = w + c$. Hence $w = w + c + b \Rightarrow b + c = 0$. Since σ is strongly convex this means $b = c = 0$ and $u = w$. □

Lemma 3.2.7. *Suppose x and y are in σ.*

i. If $\tau \prec \sigma$ then $x + y \in \tau \Rightarrow x, y \in \tau$.

ii. If σ is strongly convex and $x \leq y$ for some $y \in \tau$ then $x \in \tau$.

Proof. i. Suppose $\tau = \sigma \cap u^{\perp}$ where $u \in \sigma^{\vee}$. If $x + y \in \tau$ then $\langle u, x + y\rangle = \langle u, x\rangle + \langle u, y\rangle = 0$. But $u \in \sigma^{\vee} \Rightarrow \langle u, x\rangle \geq 0$ and $\langle u, y\rangle \geq 0$. Hence $\langle u, x\rangle = \langle u, y\rangle = 0$ and $x, y \in \tau$.

ii. Follows immediately.

□

3.3 The Quotient Cone

It turns out that the quotient of a cone by one of its faces is also a cone. We will use quotient cones to characterise the orbits of the torus action in Chapter 5.

Suppose $\tau \prec \sigma$. Let $W = \mathbb{R}.\tau$ be the smallest subspace of V containing τ. Note that $W^{\perp}$ is the subspace of all vectors in V that are perpendicular to W (we momentarily break with our convention of treating $W^{\perp}$ as a subspace of V^*). We adopt this terminology to be consistent with Fulton [4] and Oda [11]. Let $(W^{\perp})^*$ be the dual of $W^{\perp}$.

Definition 3.3.1. If $x \in V$ define $\overline{x} = proj_{W^{\perp}} x$. Similarly, if $y \in V^*$ define $\overline{y} = proj_{(W^{\perp})^*} y$. Define the convex polyhedral cone in $W^{\perp}$

$$\overline{\sigma} = \{\sum_{i=1}^{s} \lambda_i \ \overline{v_i} : \lambda_i \geq 0\}$$

Note that $\overline{\sigma}$ is the image of σ under the projection $V \to W^{\perp}$.

We call $\overline{\sigma}$ the *quotient* of σ by W, or sometimes just the *quotient cone.*

Remark 3.3.2. There is a reason for this terminology. Since $W^{\perp} \simeq V/W$, we can also think of $\overline{\sigma}$ as a cone in V/W. This is how Fulton [4] treats the quotient cone. □

Lemma 3.3.3. *The dimension of $\overline{\sigma}$ equals $dim(\sigma) - dim(\tau)$.*

Proof. We choose $dim(\sigma)$ linearly independent vectors in σ, which we denote G. G clearly spans $\mathbb{R}.\sigma$. The projection of $\mathbb{R}.\sigma$ onto $W^{\perp}$ yields a subspace of dimension $(dim(\sigma) - dim(W))$ spanned by $proj_{W^{\perp}} G$. This subspace is $\mathbb{R}.\overline{\sigma}$. Since $dim(\tau) = dim(W)$, we find that $dim(\overline{\sigma}) = dim(\sigma) - dim(\tau)$. □

Lemma 3.3.4. *$\overline{\sigma}$ is strongly convex.*

Proof. . Suppose $\exists z \in \overline{\sigma}$ such that $-z \in \overline{\sigma}$. It follows that $\exists y, y_2 \in \sigma$ such that $z = proj_{W^\perp}(y) = y - proj_W(y)$ and $-z = proj_{W^\perp}(y_2) = y_2 - proj_W(y_2)$ and $-z = proj_{W^\perp}(y_2) = y_2 - proj_W(y_2)$. Together these equations imply $y + y_2 = proj_W(y + y_2)$. This means $y + y_2 \in W$. But since y and y_2 are in σ, this means $y + y_2$ is in τ. By Lemma 3.1.7 y and y_2 are in τ. This means $y = proj_W(y)$ and $z = 0$. So $\overline{\sigma}$ is strongly convex. □

We find that the faces of $\overline{\sigma}$ are precisely the quotients of the faces of σ containing τ.

Proposition 3.3.5. *Let $\tau \prec \sigma$ be a face that is not a facet. Then:*

i. $\overline{\sigma}^\vee = \sigma^\vee \cap (W^\perp)^*$.
ii. Every face of $\overline{\sigma}$ is of the form $\overline{\alpha}$, where $\tau \prec \alpha \prec \sigma$.
iii. If $\tau \prec \kappa \prec \sigma$, then $\alpha \prec \kappa$ if and only if $\overline{\alpha} \prec \overline{\kappa}$.
iv. If $\overline{\alpha} \neq 0$ then $\tau \neq \alpha$.

Proof. i. Suppose $v \in (W^\perp)^*$. Then $v \in \overline{\sigma}^\vee \Leftrightarrow \forall x \in \sigma,\ \langle \overline{x}, v \rangle \geq 0 \Leftrightarrow \forall x \in \sigma,\ \langle x - proj_W x, v \rangle \geq 0 \Leftrightarrow \forall x \in \sigma,\ \langle x, v \rangle \geq 0 \Leftrightarrow v \in \sigma^\vee$. Thus $\overline{\sigma}^\vee = \sigma^\vee \cap (W^\perp)^*$.

ii. Let $\gamma = \overline{\sigma} \cap u_o^\perp$ be a face of $\overline{\sigma}$, where $u_o \in \overline{\sigma}^\vee$. Now by (i) above, $u_0 \in \sigma^\vee$. Thus $\alpha = \sigma \cap u_0^\perp$ is a face of σ; indeed it is the face we are looking for. As $u_0 \in (W^\perp)^*$ it is clear that $\tau \prec \alpha$. Also $\gamma = \overline{\alpha}$ because $x \in \alpha \Leftrightarrow \langle x, u_0 \rangle = 0 \Leftrightarrow \langle \overline{x}, u_0 \rangle + \langle proj_W(x), u_0 \rangle = 0 \Leftrightarrow \langle \overline{x}, u_0 \rangle = 0 \Leftrightarrow \overline{x} \in \gamma$.

iii. It is clear that $\alpha \subseteq \kappa \Leftrightarrow \overline{\alpha} \subseteq \overline{\kappa}$. If $\alpha \subseteq \kappa$ then $\alpha = \sigma \cap u_\alpha^\perp \Rightarrow \alpha = \kappa \cap u_\alpha^\perp$, and so $\alpha \prec \kappa$. Thus $\alpha \subseteq \kappa$ if and only if $\alpha \prec \kappa$. Similarly $\overline{\alpha} \subseteq \overline{\kappa}$ if and only if $\overline{\alpha} \prec \overline{\kappa}$.

iv. If $\overline{\alpha} \neq 0$ then $\exists \overline{x} \neq 0 \in \overline{\alpha}$. But then $x \notin \tau$. So $\alpha \neq \tau$.

□

We can also use the quotient cone to study σ.

Lemma 3.3.6. *Suppose $\tau \prec \sigma$ is not a facet.*

i. There exists a proper face $\gamma \neq \tau$ such that $\tau \prec \gamma \prec \sigma$.
ii. Every proper face belongs to some facet.
iii. Any proper face is the intersection of all facets containing it.

Proof. i. By Lemma 3.3.4 $\overline{\sigma}$ is strongly convex and therefore $\sigma^\vee \neq 0$. Furthermore from Lemma 3.3.3 the dimension of $\overline{\sigma}$ equals the dimension of σ minus the dimension of τ. Since τ is not a facet this means $dim(\overline{\sigma}) \geq 2$. It follows by Lemma 3.2.3 that $\overline{\sigma}$ has a proper nonzero face $\overline{\alpha} = \overline{\sigma} \cap u_0^\perp$, where $u_0 \in \overline{\sigma}^\perp$. Proposition 3.3.5 implies that there is a corresponding proper face $\alpha \prec \sigma$ such that $\alpha \neq \tau$.

ii. Suppose $\tau \prec \sigma$. By part (i) above we can find faces $\tau \prec \tau_1 \prec \tau_2 \prec \ldots \prec \tau_r \prec \sigma$, where the τ_i are of increasing dimension and $dim(\tau_r) = n - 1$

iii. Let τ be of dimension $r < n-1$, and let its generating set be $R \subset S$ (remember that S is the generating set of σ). Let $\Gamma = \{\gamma_i\}$ be the set of all faces of dimension $r+1$ containing τ. We prove that τ is the intersection of all the faces in Γ. Now from our proof of (i), $|\Gamma| \geq 2$. Hence $\overline{\sigma}$ is strongly convex and is of dimension at least 2 (as τ is not a facet), so it has at least two distinct one-dimensional faces. Denote them $\mathbb{R}_{\geq 0}\overline{v_i}$ and $\mathbb{R}_{\geq 0}\overline{v_j}$. These correspond to faces γ_i and γ_j in σ. By Proposition 3.3.5

$$\begin{aligned} x \in \gamma_i \cap \gamma_j \quad &\Leftrightarrow \quad \overline{x} \in (\mathbb{R}_{\geq 0}\overline{v}_i) \cap (\mathbb{R}_{\geq 0}\overline{v}_j) \\ &\Leftrightarrow \quad \overline{x} = 0 \ \Leftrightarrow \ x \in \tau \end{aligned}$$

Since $\tau \subset \bigcap_{\gamma\in\Gamma} \gamma$ is clear, it must be the case that $\tau = \bigcap_{\gamma\in\Gamma} \gamma$. We repeat this reasoning with each $\gamma_i \in \Gamma$. Each γ_i can be written as the intersection of all proper faces of dimension one higher. Continuing this way we can eventually write τ as the intersection of all facets containing it.

□

3.4 The Topology of a Cone

We now introduce topological considerations, which will allow us to employ general theorems concerning the topology of convex sets. Thus many of the results of this section, such as Lemma 3.4.2, hold for general convex sets as well.

Definition 3.4.1. We say that x is in the *relative interior* of σ if x is in the topological interior of the subspace $\mathbb{R}.\sigma$.

Lemma 3.4.2. *[4] The topological boundary of a cone with dimension $dim(V)$ is the union of its facets.*

Proof. If σ has no topological boundary then $\sigma = V$ and has no facets. Otherwise it is a fundamental result in the theory of convex geometry that every point x on the topological boundary of σ lies in a supporting hyperplane H (see [1, §1.4.3]). But $\sigma \cap H$ is a face, and by Lemma 3.2.6 $\sigma \cap H$ lies in a facet. So x lies in a facet. □

Corollary 3.4.3. *If $x \in \sigma$ and $x \neq 0$ then there exists a unique face $\tau \prec \sigma$ such that x is in the relative interior of τ.*

Proof. If x is in the relative interior of σ then Lemma 3.4.2 implies that x isn't in any facets. Since every face belongs to a facet, x doesn't lie in any face at all. So σ is the unique face such that x lies in its relative interior. Otherwise suppose x lies in some facet γ. We then repeat the above reasoning: x either lies in the relative interior of γ, or x belongs to a facet of γ. Eventually either we find a face τ such that x lies in the relative interior of τ, or we find that $x = 0$. □

This leads us to an alternative way of defining cones: as the intersection of half-spaces bounded by hyperplanes.

Proposition 3.4.4. *σ is the intersection of the half-spaces $H_\tau = \{v \in V : \langle v, u_\tau \rangle \geq 0\}$ as τ ranges over the facets of σ.*

Proof. We follow [4, p 11]. It is clear that $\forall v \in \sigma$ and $\forall \tau \prec \sigma$, $\langle v, u_\tau \rangle \geq 0$. Conversely, suppose v is in the intersection of the half-spaces but not in σ. Let v' be in the interior of σ. Consider the intersection I of the interval from $[v, v']$ with σ. I must be a closed and convex subset of the interval $[v, v']$, and therefore must be a closed interval with v' one endpoint (as $v' \in \sigma$). Let w be the other endpoint of I. w is on the boundary of σ, and so by Lemma 3.1.2 w lies in some facet τ. Now $\langle v', u_\tau \rangle > 0$ (because $u_\tau \in \sigma^\vee$) and $\langle w, u_\tau \rangle = 0$. But because the hyperplane perpendicular to u_τ is a supporting hyperplane, it follows that $\langle v, u_\tau \rangle < 0$. This contradicts our assumption. □

We obtain some useful results concerning interior points:

Lemma 3.4.5. *[11, §A.4],[4, Exercise p 14] If $v \in \sigma$ then the following are equivalent:*

i. *v is in the relative interior of σ.*
ii. $\langle v, u \rangle > 0 \ \forall u \in \sigma^\vee - \sigma^\perp$.
iii. $\sigma^\vee \cap v^\perp = \sigma^\perp$.
iv. $\forall x \in \sigma$, $\exists p \in \mathbb{R}_{>0}$ *such that* $x \leq p.v$.

Proof. (i) $\Leftrightarrow$ (ii): v is in the relative interior if and only v isn't in a face. In turn v is not in a face if and only if $\langle v, u \rangle > 0$, $\forall u \in \sigma^\vee - \sigma^\perp$.

(ii) $\Rightarrow$ (iii): If $u \in \sigma^\vee \cap v^\perp$ then $\langle v, u \rangle = 0$ and hence by (ii) $u \notin \sigma^\vee - \sigma^\perp$, so $u \in \sigma^\perp$.

(iii) $\Rightarrow$ (i): If (i) does not hold then v is contained in a proper face $\tau \prec \sigma$, where $\tau = \sigma \cap u_\tau^\perp$. u_τ cannot be in $\sigma^\perp$ because then $\sigma = \tau$. Hence $u_\tau \in \sigma^\vee \cap v^\perp$. However $u_\tau \notin \sigma^\perp$; hence $\sigma^\vee \cap v^\perp \neq \sigma^\perp$.

(i) $\Rightarrow$ (iv): Suppose (iv) does not hold, but that v lies in the relative interior of σ. Then there exists $x \in \sigma$ such that $\forall p \in \mathbb{N}$, $p.v - x \notin \sigma$. By Proposition 3.4.4 this means that for all $p \in \mathbb{N}$, there exists a face $\tau = \sigma \cap u_\tau^\perp$ such that

$$\langle p.v - x, u_\tau \rangle < 0 \tag{3.1}$$

Because v doesn't lie in a face the line $(\mathbb{R}.v - x)$ intersects each face at most once (as a line intersects a hyperplane it doesn't lie in exactly once). Choose $p' > 0$ large enough so that for every face τ, the set $\{p.v - x : p \geq p'\}$ lies entirely within one of the halfspaces bounded by $u_\tau^\perp$. Let γ be a face such that $\langle p'.v - x, u_\gamma \rangle < 0$ (γ must exist by (3.1)). It follows that $\forall p \geq p'$, $\langle p.v - x, u_\gamma \rangle < 0$. It follows that for all $p > p'$, $p\langle v, u_\gamma \rangle < \langle x, u_\gamma \rangle$. This is a contradiction as $\langle v, u_\tau \rangle$ and $\langle x, u_\tau \rangle$ are both greater than zero.

(iv) $\Rightarrow$ (i): Suppose (i) does not hold. Then v belongs to a face. Choose a generating vector v_i that isn't in the same face as v. Then if (iv) is to hold there must exist $y \in \sigma$ and $p \in \mathbb{N}$ such that $v_i + y = p.v$. Lemma 3.2.7 implies v_i is in the same face as v - a contradiction. $\square$

We will use the following Lemma in Chapter 5.

Lemma 3.4.6. *Let $\alpha \subset \sigma$ be a convex set such that $x + y \in \alpha \Rightarrow x, y \in \alpha$. Then α is a face.*

Proof. Suppose α contains a point v in the relative interior of σ. Then part (iv) of Lemma 3.4.5 implies that $\forall x \in \sigma$, $\exists p \in \mathbb{N}$ such that $x \leq p.v$. So $\exists y \in \sigma$ such that $x + y = p.v$. But $p.v = v + v + v + \ldots v \in \alpha$. So $x, y \in \alpha$. Since x is arbitrary this means $\alpha = \sigma$ and so α is an (improper) face.

Otherwise α lies in the topological boundary of σ. Suppose α does not lie wholly within a facet. Then there exist points x and y in α that lie in different facets. The convexity of α implies that the interval joining x and y must lie in α. At least two points in the interval must lie in the same face κ, and therefore the entire interval lies in κ - contradicting our assumption. Thus α lies wholly within a facet γ. We then repeat the above reasoning with respect to γ: either $\alpha = \gamma$ or α is a subset of a facet γ^1 of γ. We continue until we find $\alpha = \gamma^k \prec \gamma^{k-1} \dots \gamma \prec \sigma$. By Lemma 3.2.5 α is a face of σ. □

3.5 The Dual

For reasons which will become clear later, we construct toric varieties using the dual of a cone, rather than the cone itself.

Theorem 3.5.1. *[11, §A.1], [4, p 12] $\sigma^\vee$ is a cone in V^*.*

Proof. We follow [4, p 11]. Let $W = \mathbb{R}.\sigma$ and let $\tau = \sigma \cap w_\tau^\perp$ be a facet. Define $u_\tau = proj_{W^*}(w_\tau)$ and note that $\tau = \sigma \cap u_\tau^\perp$. Let X' be a basis for the vector space $(W^\perp)^*$ and let $X = X' \cup (-X')$. We claim $\sigma^\vee$ is generated by $\{u_\tau : \tau \text{ is a facet}\} \cup X$. Let α be the cone $\sum_{\tau \text{ a facet}} \mathbb{R}_{\geq 0} u_\tau + \sum_{x \in X} \mathbb{R}_{\geq 0} x$. Now if $x \in X$ then $\forall v \in \sigma, \langle v, x\rangle = 0$. So $X \subset \sigma^\vee$. Since α is generated by $\{u_\tau : \tau \text{ a facet}\} \cup X$, it is apparent that $\alpha \subset \sigma^\vee$. It remains to prove $\sigma^\vee \subset \alpha$. Suppose $u \in \sigma^\vee$. Write $u = u_1 + u_2$, where $u_1 = proj_{W^*}(u)$ and $u_2 = proj_{(W^\perp)^*}(u)$. Since $u_2 \in span(X)$, it is sufficient to prove $u_1 \in \alpha'$, where $\alpha' = \sum_{\tau \text{ a facet}} \mathbb{R}_{\geq 0} u_\tau$. If $u_1 \notin \alpha'$ then Proposition 3.1.8 applied to α' implies there exists $w \in \alpha'^\vee$ such that

$$\langle w, u_1 \rangle < 0 \tag{3.2}$$

But since $w \in \alpha'^\vee$, $\langle w, u_\tau\rangle \geq 0$. By Proposition 3.4.4 $w \in \sigma$. But $u_1 \in \sigma^\vee$ because $\forall y \in \sigma, 0 \leq \langle u, y\rangle = \langle u_1, y\rangle + \langle u_2, y\rangle = \langle u_1, y\rangle$. Hence $\langle w, u_1\rangle \geq 0$, which contradicts (3.2). So $u_1 \in \alpha'$ and $u \in \alpha$. This means $\sigma^\vee \subset \alpha$, as required. □

There is an elegant correspondence between faces of σ and faces of $\sigma^\vee$. We will use these results in Chapter 5 to describe the orbits of the action of a torus $(\mathbb{C}^*)^p$ on a toric variety.

Proposition 3.5.2. *[4, p 12],[11, §A.6] There is a bijective correspondence between faces of σ and faces of $\sigma^\vee$. Let $\tau \prec \sigma$.*

i. For any v in the relative interior of τ, $\sigma^\vee \cap \tau^\perp = \sigma^\vee \cap v^\perp$

ii. The faces of $\sigma^\vee$ are precisely the sets of the form $\sigma^\vee \cap \tau^\perp$.

iii. If $\tau \prec \sigma$ define $\tau^* = \sigma^\vee \cap \tau^\perp$. Then $*$ is a bijective order-reversing map between faces of σ and faces of $\sigma^\vee$.

iv. The smallest face of σ is $\sigma \cap (-\sigma)$.

v. $dim(\tau) + dim(\sigma^\vee \cap \tau^\perp) = n = dim(V)$

Proof. i. By Lemma 3.4.5 $\tau^\vee \cap v^\perp = \tau^\perp$. On intersection with $\sigma^\vee$ we find $\sigma^\vee \cap \tau^\vee \cap v^\perp = \sigma^\vee \cap v^\perp = \sigma^\vee \cap \tau^\perp$ (as $\sigma^\vee \subseteq \tau^\vee$).

ii. Let $\sigma^\vee \cap w^\perp$ be an arbitrary face. By Corollary 3.4.3 there exists a unique face γ such that w is in its relative interior. By (i) $\sigma^\vee \cap w^\perp = \sigma^\vee \cap \gamma$.

iii. We define $\tau^* = \sigma^\vee \cap \tau^\perp$. This map is order reversing because if $\gamma \prec \tau$ then $\gamma^* = \sigma \cap \gamma^\perp \supset \sigma \cap \tau^\perp = \tau^*$.

Further if $(\tau^*) = \sigma^\vee \cap \tau^\perp$, and $W = \mathbb{R}.\tau$ then

$$\begin{aligned} (\tau^*)^* &= \sigma \cap (\sigma^\vee \cap \tau^\perp)^\perp \\ &= \sigma \cap ((\sigma^\vee)^\perp \cup W) \text{ as } (\tau^\perp)^\perp = W \\ &= (\sigma \cap (\sigma^\vee)^\perp) \cup (\sigma \cap W) \\ &= (\sigma \cap (\sigma^\vee)^\perp) \cup \tau \end{aligned}$$

So to prove $(\tau^*)^* = \tau$ it is sufficient to prove $(\sigma \cap (\sigma^\vee)^\perp) \subset \tau$. Let $x \in (\sigma \cap (\sigma^\vee)^\perp)$ and let $\tau = \sigma \cap u^\perp$, $(u \in \sigma^\vee)$. Then because $x \in \sigma$, $\langle u, x\rangle = 0$. Hence $x \in \tau$. Note that if $\tau \prec \sigma^\vee$, then $(\tau^*)^* = \tau$ because $\sigma^\vee$ is also a cone. Thus $*$ is a one-one order reversing correspondence between the faces of σ and the faces of $\sigma^\vee$.

iv. Let $\alpha \prec \sigma$ be arbitrary. Then $\sigma^\vee \cap \alpha^\perp \prec \sigma^\vee$. We take the duals of $\sigma^\vee \cap \alpha^\perp$ and $\sigma^\vee$, and note that the inclusions are reversed. Hence $\sigma \cap (\sigma^\vee)^\perp \prec \alpha$. That is, $\sigma \cap (-\sigma) \prec \alpha$.

v. This equality clearly holds for $\tau = \sigma$ and for $\tau = \sigma \cap (-\sigma)$. Define $\sigma' := \sigma + \mathbb{R}.\tau$. Clearly the smallest face of σ' is $\sigma' \cap (-\sigma') = \tau \cap (-\tau)$. This has dimension $dim(\tau)$. The dual to σ' is $\sigma^\vee \cap (-\tau)^\vee = \sigma^\vee \cap \tau^\perp = \tau^*$. Since $dim(\sigma' \cap (-\sigma')) + dim((\sigma' \cap (-\sigma'))^*) = n$, we find $dim(\tau) + dim(\tau^*) = n$.

□

If σ is strongly convex then we can speak of *the* generators of σ, as the following lemma demonstrates.

Lemma 3.5.3. *If σ is strongly convex then its generators are uniquely determined (up to scalar multiplication).*

Proof. Suppose $\{w_1, \dots, w_k\}$ is a second set of (nonredundant) generators of σ. Then the one-dimensional faces of σ can be written in two ways: as $\{\mathbb{R}_{\geq 0} w_i\}$ or $\{\mathbb{R}_{\geq 0} v_i\}$. Since these sets are equal, we find $w_j = c v_i$ for some constant $c > 0$. □

Lemma 3.5.4. *If u is in the relative interior of τ^* then $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}.(-u)$*

Proof. We partly follow [4, p 13]. Since both $\tau^\vee$ and $\sigma^\vee + \mathbb{R}_{\geq 0}.(-u)$ are cones, it suffices by the Duality Theorem (Theorem 3.1.9) to prove that their duals are equal. The dual of $\tau^\vee$ is τ by the Duality Theorem.

We prove the dual of $\sigma^\vee + \mathbb{R}_{\geq 0}.(-u)$ is also τ. Let $x \in (\sigma^\vee + \mathbb{R}_{\geq 0}.(-u))^\vee$. Then as $\mathbb{R}.u \subset (\sigma^\vee + \mathbb{R}_{\geq 0}.(-u))$, we find $\langle x, u \rangle = 0$. Since $x \in (\sigma^\vee)^\vee$, we find $x \in (\sigma^\vee)^\vee \cap u^\perp$, and the latter set by Proposition 3.5.2 equals τ, so $x \in \tau$. Conversely suppose $x \in \tau$. Let $y + ku \in \sigma^\vee + \mathbb{R}_{\geq 0}.(-u)$. Then $\langle x, y + ku \rangle = \langle x, y \rangle + k\langle x, u \rangle = \langle x, y \rangle \geq 0$. So $x \in (\sigma^\vee + \mathbb{R}_{\geq 0}.(-u))^\vee$. Thus $\tau = (\sigma^\vee + \mathbb{R}_{\geq 0}.(-u))^\vee$. □

Lemma 3.5.5. *[4, p 13] Suppose σ and σ' are convex polyhedral cones whose intersection τ is a face of each. We can find a u in the relative interior of $\sigma \cap \tau^\perp$ such that $(-u)$ is in the relative interior of $\sigma' \cap \tau^\perp$, and $\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp$.*

Proof. We follow [4, p 13]. Let γ be the cone $\sigma + (-\sigma')$. Choose u in the relative interior of $\gamma^\vee$. Then by Proposition 3.5.2

$$\gamma \cap u^\perp = \gamma \cap (\gamma^\vee)^\perp = \gamma \cap (-\gamma) = (\sigma - \sigma') \cap (\sigma' - \sigma)$$

We claim $\tau = \sigma \cap u^\perp$. Since $\sigma \subset \gamma$, $u \in \sigma^\vee$. Since $\tau \subseteq \gamma \cap (-\gamma)$, $\tau \subseteq \sigma \cap u^\perp$.

Conversely if $v \in \sigma \cap u^\perp$ then $v \in \sigma' - \sigma$ so $v = w' - w$ ($w' \in \sigma'$, $w \in \sigma$). Then $v + w \in \sigma$ and $v + w = w' \in \sigma'$, so $v + w \in \tau$. Lemma 3.2.7 implies $v \in \tau$. Thus $\sigma \cap u^\perp = \tau$. An analagous argument proves that $\sigma' \cap u^\perp = \tau$ as well. □

3.6 The Lattice within a Cone

We are not so much interested in cones as in the *lattice points* within cones. We introduce some notation (the same that is used in [4]).

Definition 3.6.1. We denote the embedded lattices in V and V^*:

i. $N = \mathbb{Z}^n \subset V$
ii. $N_\sigma = (\mathbb{R}.\sigma) \cap N$
iii. $M = N^*$
iv. $S_\sigma = \sigma^\vee \cap M$

Note that because σ is saturated, N_σ is saturated, and therefore we can decompose N as the direct product $N_\sigma \oplus N''$ (and similarly for M). Note also that $\sigma \cap N$ and S_σ are semigroups.

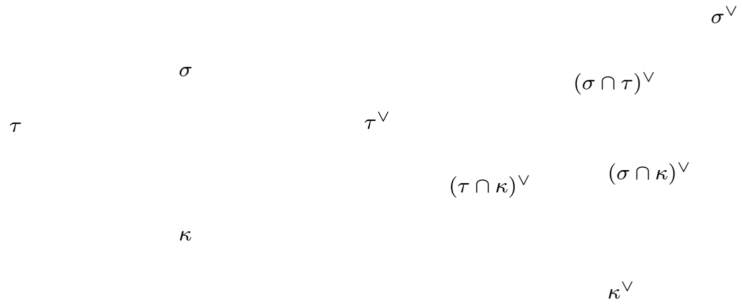

Figure 3.2: The fan Σ and its dual.

Definition 3.6.2.

i. A cone σ is *rational* if its generators can be taken from N.

ii. A cone σ is *simplicial* if its generators are linearly independent.

We assume every cone we are dealing with is rational. We also assume that if $v_i = (v^1, \dots, v^n)$ is a generator of σ then $gcd(v^1, \dots, v^n) = 1$. We come to the important notion of a *fan*.

Definition 3.6.3. A *fan* Σ is a set of rational cones in V such that

i. If $\sigma \in \Sigma$ then every face of σ is in Σ.

ii. The intersection of any two cones in Σ is again a cone in Σ.

If $\sigma \in \Sigma$ is not the face of another cone in Σ then we say σ is *maximal*.

A fan is strongly convex if each cone it contains is strongly convex.

Example 3.6.4. We construct the following fan of cones Σ (refer to Figure 3.2). The maximal cones are σ (generated by $\{e_1, e_2\}$), κ (generated by $\{e_1, -e_1 - e_2\}$) and τ (generated by $\{e_2, -e_1 - e_2\}$). The 1-dimensional cones in Σ are $\sigma \cap \kappa = \mathbb{R}_{\geq 0} e_1$, $\sigma \cap \tau = \mathbb{R}_{\geq 0} e_2$ and $\tau \cap \kappa = \mathbb{R}_{\geq 0}(-e_1 - e_2)$. The last cone in Σ, which is just a point, is $\{0\}$. We will see in Chapter 4 that the toric variety corresponding to this fan is the projective plane. $\square$

In the next chapter we construct the "toric variety" corresponding to Σ via the sets $\{S_\sigma : \sigma \in \Sigma\}$. However before we do this we need some more lemmas concerning fans.

Lemma 3.6.5.

i. If $\tau \prec \sigma$ then τ is rational.

ii. If σ is rational then $\sigma^\vee$ is rational.

iii. We can choose $u \in M \cap \sigma^\vee$ such that $\tau = \sigma \cap u^\perp$.

Proof. i. By Proposition 3.2.4 τ is generated by $S \cap \tau$. Since every vertex in S is rational, so is every vertex in $S \cap \tau$.

ii. Let $W = \mathbb{R}.\sigma$ and let $k = dim(W)$. Let X' be a basis for W^* and $X = X' \cup (-X')$. In the proof of Theorem 3.5.1 we saw that $\sigma^\vee$ is generated by $X \cup \{u_\tau : \tau \prec \sigma \text{ is a facet}\}$ (note that u_τ is in W^*). We need to prove that we can choose a generating set for $\sigma^\vee$ in M.
Let τ be a facet. Let $Z \subseteq \mathbb{Q}^n$ be a basis for W and let the $k-1$ generators of τ with respect to Z be Z_τ. Let $u' \in \mathbb{Q}^k$ be the completion of Z, i.e. the vector such that $Z \cup u'$ is a basis for $\mathbb{Q}^n$. We multiply u' by a sufficiently large integer to obtain a vector in N, and set $u_\tau := (u')^* \in M$. Thus $\{u_\tau\} \subset M$, and we have found a generating set for $\sigma^\vee$ in M.

iii. Write $\tau = \sigma \cap (\tau^*)^\perp$. By Proposition 3.5.2 it is sufficient to choose $u \in M$ in the relative interior of τ^*.

□

This brings us to a result which is essential for the construction of gluing maps in Chapter 4.

Corollary 3.6.6. *[4, p 13] If $\tau \prec \sigma$ then*

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$$

for any $u \in M$ in the interior of $\tau^* = \sigma^\vee \cap \tau^\perp$.

Proof. By Lemma 3.5.4 $\tau^\vee = \sigma^\vee + \mathbb{R}_{\geq 0}.(-u)$, where u is any vertex in the interior of $\sigma^\vee \cap \tau^\perp$. We choose $u \in M$. Then on intersecting this equation with M we get:

$$\tau^\vee \cap M = (\sigma^\vee + \mathbb{R}_{\geq 0}.(-u)) \cap M$$

The left hand side is S_τ. We need to prove the right hand side equals $S_\sigma + \mathbb{Z}_{\geq 0}(-u)$. Let $w = x + p(-u)$ be an arbitrary member of $(\sigma^\vee + \mathbb{R}_{\geq 0}.(-u))$. Let $q = p - \lfloor p \rfloor$. Then $w = (x + qu) + (p - q)u \in S_\sigma + \mathbb{Z}_{\geq 0}(-u)$. □

The following Proposition is necessary for us to prove toric varieties are Hausdorff in Chapter 4.

Proposition 3.6.7. *Suppose σ and σ' are rational cones whose intersection τ is a face of each. We can find a $u \in M$ such that $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$ and $S_\tau = S_{\sigma'} + \mathbb{Z}_{\geq 0}(u)$. As a corollary, $S_\tau = S_\sigma + S_{\sigma'}$.*

Proof. We follow [4, p 14]. Since $S_\sigma, S_{\sigma'} \subset S_\tau$, $S_\sigma + S'_\sigma \subset S_\tau$ is clear. For the other inclusion, we can choose $u \in \sigma^\vee \cap (-\sigma')^\vee \cap M$ so that by Lemma 3.5.5 $\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp$. Hence $u \in \tau^* \cap M$ and $-u \in \tau^* \cap M$. It follows by Corollary 3.6.6 that $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u) = S_{\sigma'} + \mathbb{Z}_{\geq 0}u$. The corollary $S_\tau = S_\sigma + S_{\sigma'}$ is clear. □

3.7 The Minimal Generating Set of the Lattice Within a Cone

Our aim is to find a minimal generating set of the semigroup $N \cap \sigma$. Minimal generating sets shed insight into the resolution of singularities in chapters 6 and 7. As before we assume that σ is a rational strongly convex cone generated by $S = \{v_1, \ldots v_s\}$, where none of the generating vectors are redundant. Since $\sigma^\vee$ is also a cone, the results of this chapter can also be used to find a minimal generating set of $\sigma^\vee \cap M$ (as long as $\sigma^\vee$ is strongly convex).

Definition 3.7.1. The *extraneous set* is

$$K = \left\{ \sum_{i=1}^{n} t_i v_i | 0 \leq t_i < 1 \right\} \cap N$$

In Chapter 6 we will prove that $|K|$ equals what we call the *multiplicity* of σ.

Lemma 3.7.2. *Every $u \in \sigma \cap N$ may be uniquely written as $u = v + w$ where $v \in \sum_{i=1}^n \mathbb{N}v_i$ and $w \in K$.*

Proof. Let $w \in \sigma \cap N$ and write $w = \sum_{i=1}^n r_i v_i$, $r_i \in \mathbb{R}_{\geq 0}$. Let $s_i = \lfloor r_i \rfloor$ and $t_i = r_i - s_i$. Then $w = \sum_{i=1}^n s_i v_i + \sum_{i=1}^n t_i v_i$ as required. □

Corollary 3.7.3. *If σ is a rational, convex cone then $\sigma \cap N$ is a finitely generated semigroup. One such finite set of generators is $S \cup K$.*

Proof. (This proof draws on [4, p 12]). Let $K' = \{\sum_{i=1}^n t_i v_i : 0 \leq t_i \leq 1\}$. Then K' is compact and M is discrete, so the intersection $K' \cap M$ is finite. Since $K \subset K' \cap M$, K is also finite. Lemma 3.7.2 implies that $S \cup K$ generates $\sigma \cap N$.
□

The following definition is nonstandard.

Definition 3.7.4. If $x \in \sigma \cap N$ then x is *irreducible in σ*, or just *irreducible*, if $y \leq x \Rightarrow y = 0$ or $y = x$ (where $y \in \sigma \cap N$). If x is not irreducible we say that

x is *reducible*. We say a point x is irreducible in Δ if x is irreducible in some cone $\delta \in \Delta$ containing x.

Irreducibility in $\sigma^\vee \cap M$ is defined analogously. Note that the generating vectors $\{v_i\}$ must all be irreducible. The reason is that $\mathbb{R}_{\geq 0} v_i$ is a face, and so by Lemma 3.2.7 $y \leq v_i \Rightarrow y \in \mathbb{R}_{\geq 0} v_i$. If $y \in N$ then this implies $y = v_i$ or 0.

Lemma 3.7.5. *Suppose $\tau \prec \sigma$ and $x \in \tau$. Then*

i. x is irreducible in τ if and only if x is irreducible in σ.

ii. Suppose Δ is a fan. If x is irreducible in Δ, then x is locally irreducible in every cone in Δ to which it belongs.

Proof. If x is irreducible in σ it is obvious that x is irreducible in τ. Conversely, suppose x is irreducible in τ. If $y \in \sigma$ and $y \leq x$ then by Lemma 3.2.7 $y \in \tau$. Since x is irreducible in τ, $y = x$ or $y = 0$. Hence x is irreducible in σ.

For the second assertion, suppose $x \in \delta \in \Delta$ and x is irreducible in δ. Let κ be another cone in Δ containing x. Then as $\delta \cap \kappa$ is a face of κ, we find by (i) that x is irreducible in $\delta \cap \kappa$. Since $\delta \cap \kappa$ is also a face of κ, we find x is irreducible in κ as well. $\square$

This Proposition will be very useful in Chapter 7.

Proposition 3.7.6. *There is a unique minimal generating set $L \subseteq K \cup V$ of the semigroup $\sigma \cap N$ (that is, a generating set of minimal cardinality). L consists of every irreducible vertex.*

Proof. Observe that any generating set must contain every irreducible vertex. Note also that every irreducible vertex x must belong to $K \cup V$, because otherwise Lemma 3.7.2 implies $\exists v \in V$ such that $x - v \in \sigma \cap N$. Thus because $K \cup V$ generates $\sigma \cap N$, it suffices for us to prove that the set of all irreducible vertices generates $K \cup V$.

Let $x \in K \cup V$ be arbitrary. Call $J_x = \{z \leq x\}$ the set of *smaller vertices*. Note that $|J_x|$ is finite. We prove that x belongs to the (semigroup) span of $L_x = \{y | y \leq x,\ y \text{ is irreducible}\}$. If x is irreducible this is clear. If x is not irreducible, then there exists $\{y_1, y_2\} \neq \{0, x\}$ such that $x = y_1 + y_2$. That is, x belongs to the semigroup span of $J_{y_1} \cup J_{y_2}$. Now $J_{y_1} \cup J_{y_2} \subseteq J_x - \{x\}$, so $|J_{y_1} \cup J_{y_2}| < |J_x|$. We continue this process for both y_1 and y_2 until we can express x as a finite sum of irreducible vertices. The reason this proces must eventually terminate is that every time we reach a reducible vertex the cardinality of the set of smaller vertices decreases by at least one.

In conclusion, $K \cup V$ is generated by the set of all irreducible vertices L. $\square$

Remark 3.7.7. It is important not to confuse the generators of σ with the minimal generating set of $N \cap \sigma$. The former generate σ, whilst the latter generate the semigroup $N \cap \sigma$ within σ. Proposition 3.7.6 implies that the generators of σ belong to the minimal generating set of $N \cap \sigma$. $\square$

Chapter 4

Toric Varieties

Let σ be a *strongly convex* polyhedral cone. We assume $\sigma \neq \{0\}$.

We associate a spectrum $Spec(\sigma)$ with σ. This involves constructing an algebra A_σ which is isomorphic (as a semigroup) to σ. We then define the spectrum of σ to be the spectrum of A_σ. In Sections 4.2 and 4.3 we patch together the spectra of cones in a fan to form a *toric variety*.

4.1 Preliminary Considerations

In Section 1.4 we encountered three different realisations of the spectrum of A_σ: the set of all maximal ideals of A_σ, the coordinate realisation of the spectrum and the set of all $\mathbb{C}$-algebra homomorphisms $A_\sigma \to \mathbb{C}$. In this section we establish preliminary definitions and get a handle on the relationship between these three different realisations. We also establish a fourth realisation of the spectrum of σ: the set of all semigroup homomorphisms $S_\sigma \to \mathbb{C}$. Each of these realisations of the spectrum will be useful in subsequent chapters.

Definition 4.1.1. Define $\mathbb{C}[z, z^{-1}]$ to be the algebra $\mathbb{C}[z_1, \ldots z_n, z_1^{-1}, \ldots z_n^{-1}]$.

Recall that M is a semigroup generated by $\{e_1^*, \ldots e_n^*, -e_1^*, \ldots - e_n^*\}$. This leads us to the definition of a semigroup mapping from M to $\mathbb{C}[z, z^{-1}]$.

Definition 4.1.2. Let χ be the semigroup mapping $\chi : M \to \mathbb{C}[z, z^{-1}]$. If $a = \sum_{i=1}^n a_i e_i^*$ then $\chi^a := \Pi_{i=1}^n z_i^{a_i}$. We denote χ the *algebra construction map.*

Which leads us to the algebra 'generated' by the cone:

Definition 4.1.3. A_σ is the algebra generated by $\{\chi^a : a \in S_\sigma\}$.

Note that A_σ is a subalgebra of $\mathbb{C}[z, z^{-1}]$.

Lemma 4.1.4. *A_σ is the set of all finite sums of monomials of the form $\lambda\chi^a$ where $a \in S_\sigma$ and $\lambda \in \mathbb{C}$. Multiplication in A_σ corresponds to addition in S_σ.*

Proof. It is clear that any finite sum of monomials of the form $\lambda\chi^a$ is in A_σ. To prove the converse, it suffices to prove that the multiplication of two polynomials of this form is again in A_σ. In turn, to prove this it suffices to prove that the multiplication of two monomials in A_σ is again in A_σ.

Let $a, b \in S_\sigma$. Then $\chi^a.\chi^b = \chi^{a+b}$. Since $a+b \in S_\sigma$, $\chi^{a+b} \in A_\sigma$. □

By Proposition 3.7.6, S_σ is a finitely generated semigroup. Let $W = \{w_1, \dots w_j\}$ be a generating set for S_σ.

Remark 4.1.5. In many of our applications we choose W to be unique minimal generating set of S_σ. (We proved in Proposition 3.7.6 that the unique minimal generating set is the set of all irreducible elements in S_σ.) However the construction of $Spec(\sigma$ does not depend on W being the unique minimal generating set. □

Any element of S_σ can be written in the form $\sum_{i=1}^{j} a_i w_i$. By Lemma 4.1.4 any element in A_σ can be written as a finite sum of monomials of the form

$$\chi^{\sum_{i=1}^{j} a_i w_i} = \Pi_{i=1}^{j} (\chi^{w_j})^{a_i}$$

Thus we find

Lemma 4.1.6. $A_\sigma = \mathbb{C}[\chi^{w_1}, \dots, \chi^{w_j}]$.

Definition 4.1.7. The spectrum of σ is the spectrum of the algebra A_σ.

We saw in Section 1.4 that the spectrum can be realised geometrically. We review this construction. Refer back to Section 1.4 for more details. Consider the algebra of polynomials over j variables $\mathbb{C}[x_1, \dots, x_j]$. Define $\phi : \mathbb{C}[x_1, \dots, x_j] \to A_\sigma$ such that

$$\phi(x_i) = \chi^{w_i} \tag{4.1}$$

Let I_σ be the kernel of ϕ. Then by the First Ring Isomorphism Theorem

$$A_\sigma \simeq \mathbb{C}[x_1, \dots, x_j]/I_\sigma$$

The geometric realisation of the spectrum is then $V(I_\sigma) \hookrightarrow \mathbb{C}^j$. Coordinates for the spectrum are $(x_1, \dots, x_j)$.

We also saw that we can realise the spectrum as $Hom_{\mathbb{C}\text{-alg}}(A_\sigma, \mathbb{C})$. A point in the spectrum with coordinates $(y_1, \dots, y_j)$ corresponds to the homomorphism $\xi \in Hom_{\mathbb{C}\text{-alg}}(A_\sigma, \mathbb{C})$ via $\xi(\chi^{w_i}) = y_i$.

Remark 4.1.8. Of the three realisations of $Spec(\sigma)$ encountered thus far, the maximal ideal realisation seems the most fundamental, as it is defined directly in terms of A_σ. It is however the hardest to work with; we will employ the coordinate realisation of the spectrum much more often. The $\mathbb{C}$-algebra homomorphism realisation of the spectrum is useful because it is closely related to the soon to be defined semigroup homomorphism realisation of the spectrum. □

The following proposition provides us with a method for finding a finite set of generators of I_σ.

Proposition 4.1.9. *I_σ is generated by polynomials of the form*

$$x_1^{a_1}x_2^{a_2}\dots x_j^{a_j} - x_1^{b_1}x_2^{b_2}\dots x_j^{b_j}$$

where

i. $a_1, \dots a_j, b_1, \dots b_j$ are nonnegative integers satisfying the equation $a_1w_1 + \dots a_jw_j = b_1w_1 + \dots b_jw_j$.

ii. For all $i \le j$, a_i or b_i equals zero.

iii. The greatest common divisor of the set $(\{a_i\} \cup \{b_i\})$ equals one.

Firstly we prove this weaker result, which is stated as an exercise in [4, p 19]

Lemma 4.1.10. *I_σ is generated by polynomials of the form*

$$x_1^{a_1}x_2^{a_2}\dots x_j^{a_j} - x_1^{b_1}x_2^{b_2}\dots x_j^{b_j} \tag{4.2}$$

where $a_1, \dots a_j, b_1, \dots b_j$ are nonnegative integers satisfying the equation

$$a_1w_1 + \dots a_jw_j = b_1w_1 + \dots b_jw_j$$

Proof. Let ϕ be as in (4.1).

Note that every polynomial of the form (4.2) belongs to the kernel, because $\phi(x_1^{a_1}\dots x_j^{a_j} - x_1^{b_1}\dots x_j^{b_j}) = \chi^{a_1u_1+\dots+a_ju_j} - \chi^{b_1u_1+\dots+b_ju_j} = 0$.

We prove that any polynomial in $ker(\phi)$ can be written as a linear combination of polynomials of the form (4.2). Let $f = \sum_{a\in\mathbb{Z}^j_{\ge 0}} \lambda_a X^a$ be an arbitrary polynomial in $ker(\phi)$, where $X^a = x_1^{a_1} x_2^{a_2} \dots x_j^{a_j}$. For any $u \in S_\sigma$, define $u' = \{a \in \mathbb{Z}^j_{\geq 0} : a_1 w_1 + \ldots + a_j w_j = u\}$. Then

$$\phi(f) = \sum_{a \in \mathbb{Z}^j_{\geq 0}} \lambda_a \chi^{a_1 w_1 + \ldots + a_j w_j} = \sum_{u \in S_\sigma} \chi^u (\sum_{a \in u'} \lambda_a)$$

Thus $\phi(f) = 0 \Rightarrow \forall u \in S_\sigma,\ \sum_{a \in u'} \lambda_a = 0$.

Now we can write $f = \sum_{u \in S_\sigma} f_u$, where $f_u = (\sum_{a \in u'} \lambda_a X^a)$. To prove our proposition it is sufficient to show that each of the polynomials f_u can be written as a sum of polynomials of the form (4.2).

Fix $u \in S_\sigma$. Since f_u is a finite polynomial, the set $\{a : a_1 w_1 + \ldots a_j w_j = u,\ \lambda_a \neq 0\}$ is finite. Write it as $\{a^1, \ldots, a^e\}$. Then as $\sum_i^e \lambda_{a^i} = 0$,

$$\begin{aligned}\sum_{i=1}^{e} \lambda_{a^i} X^{a^i} =& \lambda_{a^1}(X^{a^1} - X^{a^2}) + (\lambda_{a^1} + \lambda_{a^2})(X^{a^2} - X^{a^3}) + \\ & (\lambda_{a^1} + \lambda_{a^2} + \lambda_{a^3})(X^{a^3} - X^{a^4}) + \ldots + (\sum_{i}^{e-1} \lambda_{a^i})(X^{a^{e-1}} - X^{a^e}) + \\ & (\sum_{i}^{e} \lambda_{a^i})(X^{a^e} - X^{a^1})\end{aligned}$$

as required. □

Now we prove Proposition 4.1.9.

Proof. Our general strategy is to prove that the generating polynomials of Lemma 4.1.10 can be divided by polynomials satisfying the conditions of Proposition 4.1.9.

Suppose $a_1, \ldots a_j, b_1, \ldots b_j$ are nonnegative integers satisfying the equation

$$a_1 w_1 + \ldots a_j w_j = b_1 w_1 + \ldots b_j w_j \tag{4.3}$$

and let

$$h = x_1^{a_1} \ldots x_j^{a_j} - x_1^{b_1} \ldots x_j^{b_j} \tag{4.4}$$

be the corresponding polynomial.

Firstly we show that h can be divided by a polynomial satisfying condition (ii) of Proposition 4.1.9.

Let $c_i = max(a_i, b_i)$ and $c = \sum_{i=1}^{j} c_i w_i$. On subtracting c from both sides of (4.3) we obtain

$$(a_1 - c_1)w_1 + \ldots (a_j - c_j)w_j = (b_1 - c_1)w_1 + \ldots + (b_j - c_j)w_j$$

such that for all $i \leq j$, $a_i - c_i$ or $b_i - c_i$ equals zero. Furthermore

$$x_1^{a_1} x_2^{a_2} \ldots x_j^{a_j} - x_1^{b_1} x_2^{b_2} \ldots x_j^{b_j} = x_1^{c_1} x_2^{c_2} \ldots x_j^{c_j} (x_1^{a_1 - c_1} \ldots x_j^{a_j - c_j} - x_1^{b_1 - c_1} \ldots x_j^{b_j - c_j})$$

where $(x_1^{a_1 - c_1} \ldots x_j^{a_j - c_j} - x_1^{b_1 - c_1} \ldots x_j^{b_j - c_j})$ satisfies condition (ii) of Proposition 4.1.9, as required.

Thus we can now assume h satisfies condition (ii). It remains to prove h can be divided by a polynomial satisfying conditions (ii) and (iii).

Suppose that $gcd(\{a_i\} \cup \{b_i\}) = p \neq 1$. If we let $c_i = \frac{a_i}{p}$ and $d_i = \frac{b_i}{p}$ then on dividing (4.3) by p we obtain the equation

$$c_1 w_1 + \ldots + c_j w_j = d_1 w_1 + \ldots + d_j w_j \tag{4.5}$$

with corresponding polynomial $g_1 - g_2$ where $g_1 = x_1^{c_1} \ldots x_j^{c_j}$ and $g_2 = x_1^{d_1} \ldots x_j^{d_j}$. Now $x_1^{a_1} x_2^{a_2} \ldots x_j^{a_j} = g_1^p$ and $x_1^{b_1} \ldots x_j^{b_j} = g_2^p$. This means $(g_1 - g_2) | (g_1^p - g_2^p)$, and $(g_1 - g_2)$ satisfies conditions (ii) and (iii) of Proposition 4.1.9. $\square$

The following Corollary is a straightforward implication of Proposition 4.1.9. It gives us necessary and sufficient conditions for determining whether a point belongs to the spectrum or not.

Corollary 4.1.11. *A point with coordinates $(y_1, \ldots, y_j)$ belongs to the spectrum of σ if and only if for all nonnegative integers $a_1, \ldots a_j, b_1, \ldots b_j$ satisfying the equation $a_1 w_1 + \ldots a_j w_j = b_1 w_1 + \ldots b_j w_j$*

$$y_1^{a_1} y_2^{a_2} \ldots y_j^{a_j} = y_1^{b_1} y_2^{b_2} \ldots y_j^{b_j}$$

Example 4.1.12. Suppose that W is a linearly independent set. Then $a_1 u_1 + \ldots a_j u_j = b_1 u_1 + \ldots b_t u_t$ implies $a_i = b_i$. Thus $I_\sigma = \{0\}$ and $V(I_\sigma) = \mathbb{C}^j$. $\square$

We now reach a fourth definition of the spectrum. The spectrum of σ can be thought of as the set of all homomorphisms from S_σ to $\mathbb{C}$.

Proposition 4.1.13. Fourth Definition of the Spectrum. *We identify a point with coordinates* $(y_1, \dots, y_j)$ *with* $\mu \in Hom_{sg}(S_\sigma, \mathbb{C})$ *via* $\mu(w_i) = y_i$*.* μ *corresponds to the same point in* $Spec(A_\sigma)$ *as* $\xi \in Hom_{\mathbb{C}\text{-alg}}(A_\sigma, \mathbb{C})$*, where* $\xi(\chi^x) = \mu(x)$*. We denote this the semigroup homomorphism realisation of the spectrum, or just SH spectrum.*

Proof. We check μ is well-defined. Let $u \in S_\sigma$. Suppose $u = a_1w_1 + \ldots + a_jw_j = b_1w_1 + \ldots + b_jw_j$. Then we need $\mu(u) = y_1^{a_1} \dots y_j^{a_j} = y_1^{b_1} \dots y_j^{b_j}$. But this follows from Corollary 4.1.11.

Conversely let $\mu \in Hom_{sg}(S_\sigma, \mathbb{C})$. Let y be the point $(\mu(w_1), \dots, \mu(w_j))$. If $a_1w_1+\ldots+a_jw_j = b_1w_1+\ldots+b_jw_j$, then $\mu(a_1w_1+\ldots+a_jw_j) = \mu(b_1w_1+\ldots+b_jw_j)$. This means $\mu(w_1)^{a_1} \dots \mu(w_j)^{a_j} = \mu(w_1)^{b_1} \dots \mu(w_j)^{b_j}$ and by Corollary 4.1.11 $y \in V(I_\sigma)$.

It is straightforward to check that ξ corresponds to the same coordinate vector as μ. □

Remark 4.1.14. This fourth interpretation of the spectrum has the advantage of avoiding algebras and vanishing sets altogether. It is defined directly in terms of the cone. Thus in some respects, the SH realisation of the Spectrum is the most fundamental of all. The *SH* realisation of the spectrum makes many of the proofs of Chapter 5 particularly elegant.

The SH spectrum is the primary realisation employed by Oda [11]. Ewald [6] on the other hand primarily works with the geometric realisation of the spectrum. Fulton [4] employs all four definitions and regularly interchanges between them. □

We have four equivalent definitions of the spectrum of a cone: the set of maximal ideals in A_σ, the algebraic set $V(I_\sigma)$, $Hom_{\mathbb{C}\text{-alg}}(A_\sigma, \mathbb{C})$ and $Hom_{sg}(S_\sigma, \mathbb{C})$. We identify these as U_σ.

Definition 4.1.15. [4, p 19] Define U_σ to be the spectrum of σ. We denote U_σ an *affine toric variety.*

We will routinely switch between these different realisations of the spectrum.

Example 4.1.16. Let σ be the cone in $\mathbb{R}^2$ generated by $2e_1 - e_2$ and e_2 (see Figure 3.1). Observe $\sigma^\vee$ is the cone generated by e_1^* and $e_1^* + 2e_2^*$. Minimal generators for S_σ are $\{e_1^*, e_1^* + e_2^*, e_1^* + 2e_2^*\}$. Thus $A_\sigma = \mathbb{C}[z_1, z_1z_2, z_1z_2^2]$. Let $\phi : \mathbb{C}[x, y, z] \to \mathbb{C}[z_1, z_1z_2, z_1z_2^2]$ be the map of (4.1). We need to find the nontrivial linear combinations of the generators that satisfy the conditions of Proposition

4.1.4. There are 3 types of equation which need to be checked. Let $a, b, c \in \mathbb{N}$. 1) Equations of the form $ae_1^* = b(e_1^* + e_2^*) + c(e_1^* + 2e_2^*)$. On comparing the coefficients of e_2^* we find $b + 2c = 0 \Rightarrow b = c = 0 \Rightarrow a = 0$.
2) Equations of the form $a(e_1^* + e_2^*) = be_1^* + c(e_1^* + 2e_2^*)$. Comparing the coefficients of e_2^* implies $a = 2c$. Comparison of the coefficients of e_1^* then implies $c = b$. Since the $gcd(a, b, c) = 1$ we find $a = 2, b = 1, c = 1$.
3) Equations of the form $a(e_1^* + 2e_2^*) = be_1^* + c(e_1^* + e_2^*)$. We similarly find $2a = c$ and therefore $a = b + 2a \Rightarrow b + a = 0 \Rightarrow a = b = c = 0$.

Hence the only nontrivial equation is $2(e_1^* + e_2^*) = 1(e_1)^* + 1(e_1^* + 2e_2^*)$. Thus $I_\sigma = (y^2 - xz)$, and the coordinate realisation of U_σ is $\{(x, y, z) : xz = y^2\}$. This is a *quadric cone*. We will see later that U_σ is singular at the origin. □

Remark 4.1.17. We haven't directly defined a topology on $Hom_{\mathbb{C}\text{-alg}}(A_\sigma, \mathbb{C})$ or $Hom_{sg}(S_\sigma, \mathbb{C})$: in fact when we need topological considerations we will use coordinates or maximal ideals. The topologies on $Hom_{\mathbb{C}\text{-alg}}(A_\sigma, \mathbb{C})$ and $Hom_{sg}(S_\sigma, \mathbb{C})$ are inherited through the bijections of these sets with $V(I_\sigma)$. □

4.2 Morphisms of Spectra

We saw in Chapter 2 that an algebra homomorphism determines a morphism of spectra. In this section we push the 'causal chain' back further: we see that homomorphisms of cones determine algebra homomorphisms, which determine morphisms of spectra. In the next chapter we construct toric varieties by gluing together the spectra of cones in a fan. These gluing maps are merely the morphisms induced by the inclusion homomorphism of a face into a cone.

Lemma 4.2.1. *[4, p 18,p 22] Let τ and σ be strongly convex cones, sitting inside vector spaces V_1 and V_2 with lattices N_1 and N_2. Let $\gamma : N_1 \to N_2$ be a semigroup homomorphism, such that $\gamma(\tau) \subseteq \sigma$. Then γ determines an homomorphism of algebras $\phi : A_\sigma \to A_\tau$, which in turn determines a morphism of spectra $U_\tau \to U_\sigma$.*

Proof. Let γ^* be the dual of γ. Observe that if $s \in \sigma^\vee$ then $\gamma^*(s) \in \tau^\vee$, because if $t \in \tau$ then $\gamma^*(s)(t) = s(\gamma(t)) \geq 0$. So γ^* is a well-defined map from $S_\sigma \to S_\tau$. Let χ and χ_2 be the algebra construction maps of S_σ and S_τ respectively. Define ϕ be the algebra homomorphism that takes monomials of the form χ^s (where $x \in S_\sigma$) to monomials of the form $\chi_2^{\gamma^*(x)}$. It is straightforward to check ϕ is an algebra homomorphism. As we saw in Proposition 2.2.10, ϕ determines a morphism of spectra $\phi^\wedge : U_\tau \to U_\sigma$. □

In Proposition 2.2.10 we saw that the induced morphism of spectra can be easily described using maximal ideals or $\mathbb{C}$-algebra homomorphisms. In terms of maximal ideals, the induced morphism is $m \in Spec(\tau) \to \phi^{-1}(m)$; in terms of $\mathbb{C}$-algebra homomorphisms, the induced morphism is $\xi \to \xi \circ \phi$. We can also describe the induced morphism using the SH realisation of the specturm. From Proposition 4.1.13 it is clear that the induced morphism maps $x \in Hom_{sg}(S_\tau, \mathbb{C}) \to x \circ \gamma^*$.

Definition 4.2.2. A *lattice automorphism* is a bijective homomorphism $C : N \to N$.

From linear algebra we know $C \in GL_n(\mathbb{Z})$ and $det(C) = \pm 1$. Consult [2, §12.2.2] for more details. We reach a corollary to Lemma 4.2.1. Let $C\sigma = \{C.a : a \in \sigma\}$.

Corollary 4.2.3. *Suppose σ is a cone in N and C is a lattice automorphism on N. Then $\sigma \simeq C\sigma$, $A_\sigma \simeq A_{C\sigma}$ and $U_\sigma \simeq U_{C\sigma}$.*

Proof. Suppose the generating matrix of σ is A (see Definition 3.1.4). That is, $\sigma = \{Av : v \in (\mathbb{R}_{\geq 0})^n\}$. Then $C(\sigma)$ is the cone $\{CAv : v \in (\mathbb{R}_{\geq 0})^n\}$. Because C^{-1} defines an inverse homomorphism $C\sigma \to \sigma$, C is a semigroup isomorphism. The other isomorphisms follows from Lemma 4.2.1. □

This means we can apply lattice automorphisms without altering the spectra of cones (up to isomorphism). Thus in particular we can apply row operations to the generating matrix. This is essential to the resolution of toric singularities because it allows us to transform a cone into a canonical form for which there is a method to resolve its singularities.

If $\tau \prec \sigma$ then by Corollary 3.6.6 $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-u)$ for some $u \in \sigma^\vee \cap \tau^\perp$. The proposition below forms the cornerstone of the "gluing maps" by which we construct toric varieties. As it is so central we describe it using each of the four realisations of the spectrum.

Proposition 4.2.4. *The inclusion homomorphism $\tau \hookrightarrow \sigma$ determines an algebra inclusion $A_\sigma \hookrightarrow A_\tau$ and a morphism $\delta : U_\tau \hookrightarrow U_\sigma$ such that $U_\tau \simeq \delta(U_\tau)$.*

- *i. Algebraically, δ takes maximal ideals $m \in A_\tau$ to $m \cap A_\sigma$.*
- *ii. In coordinates, δ isomorphically embeds U_τ as a principal open subset $D(q)$ of U_σ (where q is such that $\phi(q) = \chi^u$).*
- *iii. In terms of $\mathbb{C}$-algebra homomorphisms, δ is a bijective map between $Hom_{\mathbb{C}\text{-alg}}(A_\tau, \mathbb{C})$ and $\{\epsilon \in Hom_{\mathbb{C}\text{-alg}}(A_\sigma, \mathbb{C}) : \epsilon(\chi^u) \neq 0\}$. If*

$\xi \in Hom_{\mathbb{C}\text{-alg}}(A_\tau, \mathbb{C})$ then $\delta(\xi) = \xi|_{A_\sigma}$. Conversely if $\epsilon \in A_\sigma$ and $\epsilon(\chi^u) \neq 0$ then $\delta^{-1}(\epsilon)$ is the natural extension of ϵ to A_τ obtained by defining $\delta^{-1}(\epsilon)(\chi^{-u}) = \frac{1}{\epsilon(\chi^u)}$.

iv. *In terms of semigroup homormorphisms, δ is a bijection between $Hom_{sg}(S_\tau, \mathbb{C})$ and $\{y \in Hom_{sg}(S_\sigma, \mathbb{C}) : y(u) \neq 0\}$. If $x \in Hom_{sg}(S_\tau, \mathbb{C})$ then $\delta(x) = x|_{S_\sigma}$. Conversely if $y \in Hom_{sg}(S_\sigma, \mathbb{C})$ and $y(u) \neq 0$ then $\delta^{-1}(y)$ is the natural extension of y obtained by defining $y(-u) = \frac{1}{y(u)}$.*

Proof. i. We construct A_σ following the method in Section 4.1. Let generators of S_σ be $\{w_1, \ldots w_j\}$. It follows from Corollary 3.6.6 that $\exists u \in S_\sigma$ such that $\{w_1, \ldots, w_j, -u\}$ are generators of S_τ. Let $A_\sigma = \mathbb{C}[\chi^{w_1}, \ldots, \chi^{w_j}]$. Then $A_\tau = A_\sigma[(\chi^u)^{-1}]$.

The inclusion $\tau \hookrightarrow \sigma$ determines the natural algebra inclusion $\psi : A_\sigma \hookrightarrow A_\tau$. By Proposition 2.2.10 the induced morphism is $m \to m \cap A_\sigma$.

ii. Let $\phi : \mathbb{C}[x_1, \ldots, x_j] \to A_\sigma$ be the homomorphism of (4.1). By the Ring Isomorphism Theorem $A_\sigma \simeq \mathbb{C}[x_1, \ldots x_j]/I_\sigma$, where $I_\sigma = ker(\phi)$. Let $q \in \mathbb{C}[x_1, \ldots, x_j]$ be such that $\phi(q) = \chi^u$.

By Lemma 2.1.3

$$A_\tau \simeq A_\sigma[y]/((q + I_\sigma)y - 1) \tag{4.6}$$

It is straightforward to check $A_\sigma[y] \simeq \mathbb{C}[x_1, \ldots, x_j, x_{j+1}]/I_\sigma$ via the homomorphism that maps $y \to x_{j+1} + I_\sigma$ and $x_i + I_\sigma \to x_i + I_\sigma$. Under this homomorphism $((q+I_\sigma)y-1) \to J$, where J is the ideal in $\mathbb{C}[x_1, \ldots, x_{j+1}]/I_\sigma$ generated by $(qx_{j+1}-1)+I_\sigma$. In other words J is the ideal $(I_\sigma, (qx_{j+1}-1))/I_\sigma$. Thus (4.6) becomes

$$\begin{aligned} A_\tau &\simeq \frac{\mathbb{C}[x_1, \ldots, x_j, x_{j+1}]/I_\sigma}{J} \\ &\simeq \frac{\mathbb{C}[x_1, \ldots, x_j, x_{j+1}]/I_\sigma}{(I_\sigma, (qx_{j+1} - 1))/I_\sigma} \\ &\simeq \frac{\mathbb{C}[x_1, \ldots, x_j, x_{j+1}]}{(I_\sigma, (qx_{j+1} - 1))} \end{aligned} \tag{4.7}$$

where (4.7) follows from the Third Ring Isomorphism Theorem.

Thus $U_\tau = V(I_\sigma) \cap V(qx_{j+1} - 1) \hookrightarrow \mathbb{C}^{j+1}$. Construct the regular map $\gamma : U_\sigma - V(q) \to U_\tau$ that maps $y = (y_1, \ldots, y_j) \to (y_1, \ldots, y_j, \frac{1}{q(y)})$. We prove that γ is an isomorphism. It is clear that $\gamma(y) \in U_\tau$. We need to check γ is surjective. Let $z = (z_1, \ldots, z_{j+1}) \in U_\tau$. Since $I_\sigma \subset I_\tau$, we find $(z_1, \ldots, z_j) \in U_\sigma$. Also

$q(z_1, \dots, z_j)z_{j+1} - 1 = 0 \Rightarrow z_{j+1} = \frac{1}{q(z_1,\dots z_j)}$. Thus $z = \gamma(z_1, \dots, z_j)$, and hence γ is surjective. We can therefore define an inverse rational map that takes $(z_1, \dots, z_{j+1}) \to (z_1, \dots, z_j)$. Thus γ is an isomorphism. This allows us to geometrically identify U_τ as the principal open subset $\{y \in U_\sigma : q(y) \neq 0\}$.

iii. We use the correspondence between kernels of $\mathbb{C}$-algebra homomorphisms and maximal ideals. By (i) δ in terms of maximal ideals must map $ker(\xi)$ to $ker(\xi) \cap A_\sigma$. Since the kernel of $\xi|_{A_\sigma}$ is clearly $ker(\xi) \cap A_\sigma$, it must be the case $\delta(\xi) = \xi|_{A_\sigma}$.

iv. Follows by (iii) and the correspondence between $\mathbb{C}$-algebra homomorphisms and semigroup homomorphisms of Proposition 4.1.13.

□

Remark 4.2.5. Thus the spectrum inclusion corresponds to the face inclusion: $\tau \prec \sigma \Rightarrow U_\tau \hookrightarrow U_\sigma$. This is why we construct the spectrum of a cone σ using the dual of σ, rather than σ itself. □

Example 4.2.6. The zero face $\{0\}$ is a face of every cone. We find its spectrum. Since $\{0\}^\vee = M$, we find $A_{\{0\}} = \mathbb{C}[z_1, z_1^{-1}, \dots, z_n, z_n^{-1}]$. It follows from Lemma 2.1.3 that $A_{\{0\}} \simeq$
$\mathbb{C}[z_1, \dots, z_{2n}]/(z_1 z_{n+1} - 1) \dots (z_n z_{2n} - 1)$. Then $U_{\{0\}}$ has coordinates
$(z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1})$. Define $\mathbb{C}^* = \mathbb{C} - \{0\}$. It can be seen that $U_{\{0\}} \simeq (\mathbb{C}^*)^n$ via the map $(y_1, \dots, y_n, y_1^{-1}, \dots, y_n^{-1}) \to (y_1, \dots, y_n)$. $(\mathbb{C}^*)^n$ is known as the *n-dimensional torus*. We denote it T_N, as it is the torus generated by the dual of the lattice N. We break with the convention of Section 4.1 and use $(z_1, \dots, z_n)$ as coordinates for U_0. The only constraint for these to be valid coordinates is $\forall i,\ z_i \neq 0$. □

Since $\{0\}$ is a face of every cone, under the open embedding of Proposition 4.2.4 we find:

Lemma 4.2.7. *The spectrum of a cone σ contains the n-dimensional torus $T_N = (\mathbb{C}^*)^n$ embedded as a principal open subset. T_N is dense, with respect to both the Zariski topology and the ordinary topology.*

The fact that T_N is dense follows from Lemma 1.3.8.

4.3 The Construction of a Toric Variety from a Fan

Let Σ be a fan. We construct a Toric variety by gluing together the spectra of the cones of Σ in the following way. If σ and τ are neighbouring cones, then $\sigma \cap \tau$ is a face of each of them. Thus by Proposition 4.2.4, $U_{\sigma\cap\tau}$ has two isomorphic coordinate realisations: as a principal open subset U_1 of U_σ and a principal open subset U_2 of U_τ. Let $\Psi_{\sigma,\tau} : U_1 \to U_2$ be the isomorphism between these two equivalent coordinate representations. Since U_1 and U_2 are dense open subsets, it follows that $\Psi_{\sigma,\tau}$ is a birational map $U_\sigma \to U_\tau$. We denote Ψ the *gluing map*. Observe that every gluing map is the composition of two of the open embedding maps of Proposition 4.2.4. That is, $\Psi_{\sigma,\tau} = \Psi_{\sigma\cap\tau,\tau} \circ \Psi_{\sigma,\sigma\cap\tau}$.

Definition 4.3.1. In the disjoint union $\cup_{\sigma\in\Sigma} U_\sigma$ we identify two points $x \in U_\sigma$ and $y \in U_\tau$ if there is a gluing map $\Psi_{\sigma,\tau}$ that maps x to y. The set of points thus obtained is called the *toric variety* determined by Σ and is denoted U_Σ.

We check that the gluing maps are consistent.

Proposition 4.3.2. *Suppose σ and δ are arbitrary cones in Σ. Let $\Psi_1 : U_\sigma \to U_\delta$ and $\Psi_2 : U_\sigma \to U_\delta$ be the composition of gluing maps, where*

$$\Psi_1 := \Psi_{\kappa_p,\delta}\Psi_{\kappa_{p-1},\kappa_p} \dots \Psi_{\kappa_1,\kappa_2}\Psi_{\sigma,\kappa_1}$$
$$\Psi_2 := \Psi_{\gamma_q,\delta}\Psi_{\gamma_{q-1},\gamma_q} \dots \Psi_{\gamma_1,\gamma_2}\Psi_{\sigma,\gamma_1}$$

for cones $\{\kappa_i\}_{i=1}^p$ and $\{\gamma_i\}_{i=1}^q$ in Σ. We claim Ψ_1 and Ψ_2 are birationally equivalent: that is, if Ψ_1 is defined on the open set U_1 and Ψ_2 is defined on the open set U_2 then Ψ_1 and Ψ_2 are equal on the open set $U_1 \cap U_2$.

To prove Proposition 4.3.2 we need the following lemma. Refer to [4, p 21] for a justification.

Lemma 4.3.3. *Suppose $\tau, \alpha, \beta \in \Sigma$. Then $\Psi_{\alpha,\tau} = \Psi_{\beta,\tau} \circ \Psi_{\alpha,\beta}$.*

Now we prove Proposition 4.3.2.

Proof. Recall $\Psi_1 = \Psi_{\kappa_p,\delta}\Psi_{\kappa_{p-1},\kappa_p} \dots \Psi_{\kappa_1,\kappa_2}\Psi_{\sigma,\kappa_1}$. By Lemma 4.3.3 $\Psi_{\kappa_1,\kappa_2}\Psi_{\sigma,\kappa_1} = \Psi_{\sigma,\kappa_2}$. By making p such substitutions we find $\Psi_1 = \Psi_{\sigma,\tau}$. By doing the same for Ψ_2 we find $\Psi_2 = \Psi_{\sigma,\tau}$. Hence Ψ_1 and Ψ_2 are birationally equivalent. □

We now briefly switch back to the "ordinary" complex topology.

Lemma 4.3.4. *[6, §VI.3.6] With respect to the ordinary complex topology U_Σ is Hausdorff.*

Proof. Let u and v be any two points in U_Σ. If u and v belong to the same variety U_σ then they clearly possess disjoint open neighbourhoods. Suppose u and v do not

Figure 4.1: The fan Σ by which we construct $\mathbb{P}^1$, and its dual.

belong to the same variety. Then there exist varieties such that $u \in U_\sigma - U_\kappa$ and $v \in U_\kappa - U_\sigma$. Let $\tau = \sigma \cap \kappa$. We find disjoint open neighbourhoods (with respect to the ordinary topology) of u and v.

By Proposition 3.6.7 we can find an element w such that $w \in S_\sigma$, $-w \in S_\kappa$ and $S_\tau = S_\sigma + \mathbb{Z}_{\geq 0}(-w) = S_\kappa + \mathbb{Z}_{\geq 0}(w)$. We therefore choose generators $\{w_1, \ldots, w_j, w\}$ of S_σ and generators $\{v_1, \ldots, v_k, -w\}$ of S_κ. Let $(x_1, \ldots, x_j, x_{j+1})$ be the corresponding coordinates of U_σ and $(y_1, \ldots, y_{k+1})$ the coordinates of U_κ. From Proposition 4.2.4 we know U_τ is embedded in U_σ as the principal open set $x_{j+1} \neq 0$, and U_τ is embedded in U_κ as the principal open set $y_{j+1} \neq 0$. Further $\Psi_{\sigma,\kappa}(x_1, \ldots, x_j, x_{j+1}) = (\ldots, \frac{1}{x_{j+1}})$. Note that the coordinates of u are $(u_1, \ldots, u_{j+1})$, where $u_{j+1} = 0$ (as $u \notin \tau$). Similarly the coordinates of v are $(v_1, \ldots, v_{k+1})$, where $v_{k+1} = 0$ (as $v \notin \tau$).

Let $U = \{u' : \|u - u'\| < 1\}$ and $V = \{v' : \|v - v'\| < 1\}$. Thus if $(p_1, \ldots, p_{j+1}) \in U$ then $|p_{j+1}| < 1$. This means $\Psi_{\sigma,\kappa}(p) = (\ldots, \frac{1}{p_{j+1}}) \notin V$ as $|\frac{1}{p_{j+1}}| > 1$. Thus U and V are the required disjoint open neighbourhoods. □

By Proposition 3.5.2 the smallest face of σ is $\sigma \cap (-\sigma)$, which is $\{0\}$ as σ is strongly convex. Hence Σ contains the cone $\{0\}$. Since $U_{\{0\}} = T_N$ is a dense open subset of U_κ for every cone $\kappa \in \Sigma$, we find:

Lemma 4.3.5. *U_Σ contains an embedded n-dimensional torus T that is dense, with respect to both the Zariski topology and the "ordinary" topology.*

Remark 4.3.6. This is why U_Σ is referred to as a *toric* variety. □

We construct some examples of toric varieties.

Example 4.3.7. We construct the projective line as a toric variety. Let V be the 1-dimensional vector space $\mathbb{R}$. Let Σ contain 3 cones (pictured in Figure 4.1): $\sigma = \mathbb{R}_{\geq 0}e_1$, $\kappa = \mathbb{R}_{\geq 0} - e_1$, $\tau = \sigma \cap \kappa = \{0\}$. We tabulate our working: information concerning σ is in the first column, τ in the second and κ in the third.

$$\begin{array}{ccc}
\sigma^\vee = \mathbb{R}_{\geq 0}e_1^* & \tau^\vee = \mathbb{R}.e_1^* & \kappa^\vee = \mathbb{R}.(-e_1^*) \\
S_\sigma = \mathbb{Z}_{\geq 0}e_1^* & S_\tau = \mathbb{Z}e_1^* & S_\kappa = \mathbb{Z}_{\geq 0}(-e_1^*) \\
A_\sigma = \mathbb{C}[z_1] & A_\tau = \mathbb{C}[z_1, z_1^{-1}] & A_\kappa = \mathbb{C}[z_1^{-1}] \\
U_\sigma = \mathbb{C} & U_\tau = \mathbb{C}^* & U_\kappa = \mathbb{C}
\end{array}$$

We define gluing maps $\Psi_{\tau,\sigma}(z) = z$, $\Psi_{\tau,\kappa}(z) = z^{-1}$. $\Psi_{\sigma,\kappa}(z) = \Psi_{\tau,\kappa}(\Psi_{\sigma,\tau}(z)) = z^{-1}$.

It can be seen that U_Σ is the projective line $\mathbb{P}^1$ with affine covers U_σ and U_κ.

Note that although we have described the algebras and the spectra of all three cones in terms of z_1, they are technically distinct apart from the gluing maps. We may as well have used (u) as coordinates for σ, (v, w) as coordinates for τ and (x) as coordinates for κ to emphasize that they are different. We use z_1 for all three because it makes the gluing maps more clear. $\square$

Example 4.3.8. Now consider a fan Σ in 2-dimensions, containing the following cones: $\sigma := \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}e_2$, $\kappa := \mathbb{R}_{\geq 0}e_1 + \mathbb{R}_{\geq 0}(-e_1 - e_2)$ and $\tau := \mathbb{R}_{\geq 0}e_2 + \mathbb{R}_{\geq 0}(-e_1 - e_2)$, as well as their faces. Refer to Figure 3.2 for a diagram. We find

$$\begin{array}{ccc} A_\sigma = \mathbb{C}[z_1, z_2] & A_\tau = \mathbb{C}[z_1^{-1}, z_1^{-1}z_2] & A_\kappa = \mathbb{C}[z_2^{-1}, z_1 z_2^{-1}] \\ U_\sigma = \mathbb{C}^2 & U_\tau = \mathbb{C}^2 & U_\kappa = \mathbb{C}^2 \end{array}$$

We also have $A_{\sigma\cap\tau} = \mathbb{C}[z_1, z_1^{-1}, z_2]$ and therefore $U_{\sigma\cap\tau} = \mathbb{C}\times\mathbb{C}^*$. Thus we can define gluing maps. $\Psi_{\sigma,\sigma\cap\tau}(z_1, z_2) = (z_1, z_1^{-1}, z_2)$ and $\Psi_{\sigma\cap\tau,\tau}(z_1, z_1^{-1}, z_2) = (z_1^{-1}, z_1^{-1}z_2)$. Their composition gives us the gluing map $\Psi_{\sigma,\tau}(z_1, z_2) = (z_1^{-1}, z_1^{-1}z_2)$, which is defined for $z_1 \neq 0$. We similarly find $A_{\sigma\cap\kappa} = \mathbb{C}[z_1, z_2, z_2^{-1}]$ and $U_{\sigma\cap\kappa} = \mathbb{C} \times \mathbb{C}^*$. This gives us the gluing maps $\Psi_{\sigma,\sigma\cap\kappa}(z_1, z_2) = (z_1, z_2, z_2^{-1})$ and $\Psi_{\sigma\cap\kappa,\kappa}(z_1, z_2, z_2^{-1}) = (z_2^{-1}, z_1 z_2^{-1})$. Their composition yields the gluing map $\Psi_{\sigma,\kappa}(z_1, z_2) = (z_2^{-1}, z_1 z_2^{-1})$ $(z_2 \neq 0)$.

In fact U_Σ is the projective plane, and each of U_σ,U_κ and U_τ are the affine covers. To see this we need to understand the affine covers of the projective plane. The first affine cover is the set $\{(1 : y : z) : y, z \in \mathbb{C}\}$. If $y \neq 0$ then the point $(1 : y : z)$ can be identified with the point $(\frac{1}{y} : 1 : \frac{z}{y})$ in $\mathbb{C}^2_2$. In affine coordinates, we are identifying the point (y, z) in the first cover with the point $(\frac{1}{y}, \frac{z}{y})$ in the second. Observe that this is the same map as $\Psi_{\sigma,\tau}$. Similarly, if $z \neq 0$ then the point $(1 : y : z)$ can be identified with the point $(\frac{1}{z} : \frac{y}{z} : 1)$ in $\mathbb{C}^2_3$. In affine coordinates we are mapping (y, z) to $(\frac{1}{z}, \frac{y}{z})$. Observe that this is the same map as $\Psi_{\sigma,\kappa}$. We similarly find that the gluing map $\Psi_{\tau,\kappa}$ corresponds to the identification $(x : 1 : z) \to (\frac{x}{z} : \frac{1}{z} : 1)$.

Thus U_Σ is the projective plane. $\square$

Let Δ and Σ be fans in lattices N and N' respectively. We extend Lemma 4.2.1 to define a morphism between U_Δ and U_Σ:

Lemma 4.3.9. *[4, p 22],[6, §VI.6.1] Suppose $\phi : N' \to N$ is a homomorphism of lattices, such that for each cone $\sigma \in \Sigma$ there is some cone $\delta \in \Delta$ such that $\phi(\sigma) \subseteq \delta$. Then ϕ determines a morphism $\Xi : U_\Sigma \to U_\Delta$.*

Definition 4.3.10. The map ϕ is denoted a *homomorphism of fans*. The morphism $\Xi : U_\Sigma \to U_\Delta$ is a *toric morphism*.

We now prove Lemma 4.3.9.

Proof. Let χ be the algebra construction map of Δ and χ_2 the algebra construction map of Σ.

Fix $\sigma \in \Sigma$. By Proposition 4.2.1 ϕ induces a morphism $\Xi_\delta : U_\sigma \to U_\delta$ for every cone δ such that $\phi(\sigma) \subseteq \delta$. We firstly prove that these morphisms respect the gluing maps of U_Δ. That is, we need to prove that for all $x \in U_\sigma$, $\Xi_{\delta_1}(x)$ and $\Xi_{\delta_2}(x)$ correspond to the same point under the gluing maps of U_Δ.

Let δ be the intersection of every face in Δ containing $\phi(\sigma)$, and let $\psi : A_\delta \to A_\sigma$ be the induced algebra homomorphism. From Proposition 4.2.1, $\psi(\chi^w) = \chi_2^{\phi^*(w)}$, where $w \in S_\sigma$. The induced morphism is $\Xi_\delta : U_\sigma \to U_\delta,\ m \to \psi^{-1}(m)$. Suppose κ is another face containing $\psi(\sigma)$, with induced algebra homomorphism $\xi : A_\kappa \to A_\sigma,\ \xi(\chi^w) = \chi_2^{\phi^*(w)}$. Observe that ξ is of the same form as ψ, except it is restricted to elements of the form χ^w where $w \in S_\kappa$. Since $S_\kappa \subseteq S_\delta$, we find $\xi = \psi|_{A_\kappa}$. Thus the morphism induced by ξ is $\Xi_\kappa : U_\sigma \to U_\kappa,\ m \to \xi^{-1}(m) = \psi^{-1}(m) \cap A_\kappa$. But we see by Proposition 4.2.4 that $\psi^{-1}(m)$ and $\psi^{-1}(m) \cap A_\kappa$ correspond to the same point in U_Δ via the inclusion $U_\delta \hookrightarrow U_\kappa$. Hence $\Xi_\delta(m) = \Xi_\kappa(m)$. Thus the individual morphisms $U_\sigma \to U_\delta$ respect the gluing maps of Δ, and therefore patch together to give a well-defined morphism $\Xi_\sigma : U_\sigma \to U_\Delta$.

Secondly we need to check that these morphisms respect the gluing maps of U_Σ. That is, if γ and σ are in Σ and $x \in U_\gamma \cap U_\sigma$, we need to check $\Xi_\gamma(x) = \Xi_\sigma(x)$. Firstly, we assume $\gamma \prec \sigma$. Let $\delta \in \Delta$ be such that $\phi(\sigma) \subseteq \delta$. Note that $\phi(\gamma) \subseteq \phi(\sigma) \subseteq \delta$. We find by Proposition 4.2.1 that the algebra homomorphism corresponding to σ is $\psi_\sigma : A_\delta \to A_\sigma,\ \psi_\sigma(\chi^w) = \chi_2^{\phi^*(w)}$ (where $w \in S_\delta$). Similarly the homomorphism corresponding to γ is $\psi_\gamma : A_\delta \to A_\gamma,\ \psi_\gamma(\chi^w) = \chi_2^{\phi^*(w)}$ (where $w \in S_\delta$). Observe that these maps are identical under the natural inclusion of Proposition 4.2.1: $\psi_\sigma(\chi^w) = \chi_2^{\phi^*(w)} \in A_\sigma \hookrightarrow A_\gamma$. Thus if m is a maximal ideal in A_γ then $\Xi_\gamma(m) = \psi_\gamma^{-1}(m) = \psi_\sigma^{-1}(m) = \psi_\sigma^{-1}(m \cap A_\sigma)$ (as the image of ψ_γ is contained in A_σ). Since $m = m \cap A_\sigma$ under the gluing map $U_\gamma \hookrightarrow U_\sigma$, we have proved $\Xi_\gamma(m) = \Xi_\sigma(m)$, as required. Now suppose γ is not a face of σ. Let $x \in \sigma \cap \gamma$. Then $\Xi_\gamma(x) = \Xi_{\sigma\cap\gamma}(x) = \Xi_\sigma(x)$, as required.

We conclude that the individual induced morphisms $U_\sigma \to U_\delta$ respect the gluing maps of both U_Σ and U_Δ. Thus they patch together to form a morphism $U_\Sigma \to U_\Delta$. $\square$

Chapter 5

The Torus Action

Let Σ be an n-dimensional fan. In this chapter we define an action of the torus $(\mathbb{C}^*)^n$ on U_Σ. The tours action sheds a lot of insight on the structure of U_Σ. The orbits of U_Σ are embedded torii, and there is essentially a 1-1 correspondence between the orbits and cones in the fan. In subsequent chapters we will use orbits to describe the resolution of the singularities of U_Σ.

On a historical note, it was the theory of algebraic groups (varieties which act on themselves as a group) which gave rise to the study of toric varieties in the 1970's.[1] In some respects the embedded torus and its associated action on U_Σ is the defining feature of Toric Varieties.

We predominantly adopt the notation of Fulton [4], but also refer to material in Ewald [6] and Oda [11].

5.1 The Torus Action

Let Σ be a fan in the vector space $V = \mathbb{R}^n$, with lattice N. Let $\sigma \in \Sigma$. In Section 4.2 we saw that $U_{\{0\}}$ is the torus $T_N = (\mathbb{C}^*)^n$. Now T_N is an Abelian group with multiplication $(s_1, \ldots, s_n).(t_1, \ldots, t_n) = (s_1 t_1, \ldots, s_n t_n)$. However since $U_{\{0\}} = T_N$ we can also interpret this multiplication as T_N *acting* on $U_{\{0\}}$. For an introduction to the theory behind the *group action on a set* refer to [15, §2.7].

Let $(x_1, \ldots, x_n)$ be the standard coordinates of $U_{\{0\}}$. Refer to Example 4.2.6 for more details on $U_{\{0\}}$.

Definition 5.1.1. The *Action* of T_N on $U_{\{0\}}$ is

$$(t_1, \ldots, t_n).(x_1, \ldots, x_n) = (t_1 x_1, \ldots, t_n x_n)$$

Note that $(t_1 x_1, \ldots, t_n x_n) \in U_{\{0\}}$ because the only constraint on the coordinates is

[1] For more historical details refer to the introductions of [6] and [11].

$t_i x_i \neq 0$, and this is true as t_i and x_i are always nonzero. Notice also that the orbit of this action is all of $U_{\{0\}}$.

Under the SH realisation of $U_{\{0\}}$, $T_N = Hom_{sg}(M, \mathbb{C})$. Thus

Lemma 5.1.2. *In terms of the SH realisation of the spectrum, the torus action is $(T_N \times U_{\{0\}}) \to U_{\{0\}}$: $(t, y) \to t.y$, where $(t.y)(u) = t(u)y(u)$.*

Proof. Let $y \in Hom_{sg}(M, \mathbb{C})$, with corresponding coordinates $(y_1, \ldots, y_n)$. By definition, $(t.y)$ has coordinates $(t_1 y_1, \ldots, t_n y_n)$. The corresponding semigroup homomorphism $t.y$ is defined such that $(t.y)(a_1 e_1^* + \ldots + a_n e_n^*) = (t_1 y_1)^{a_1} \ldots (t_n y_n)^{a_n} = t(a_1 e_1^* + \ldots + a_n e_n^*).y(a_1 e_1^* + \ldots + a_n e_n^*)$. □

In the construction of the toric variety U_Σ we embed $U_{\{0\}}$ in every other variety U_σ ($\sigma \in \Sigma$). By Proposition 4.2.4 the embedding is $x \to x|_{S_\sigma}$, where $x \in Hom_{sg}(M, \mathbb{C})$. If $t \in T_N$, it follows that under this embedding $t.x \to (t.x)|_{S_\sigma} = t|_{S_\sigma}.x|_{S_\sigma}$. (Note that by $x|_{S_\sigma}$ we mean x restricted to S_σ.) This naturally suggests an extension of the torus action to all of U_σ, such that the torus action is compatible with the gluing map.

Definition 5.1.3. The torus T_N acts on U_σ via $(t, x) \to t|_{S_\sigma}.x$ ($t \in T_N, x \in U_\sigma$).

We determine the torus action on U_σ in coordinates. Let $\{w_1, \ldots, w_j\}$ be generators of σ. Let the corresponding coordinates be $(x_1, \ldots, x_j)$. If $t \in Hom_{sg}(M, \mathbb{C})$ then

Lemma 5.1.4. *[6, §VI.2.8] The torus action in coordinates is $t.(y_1, \ldots, y_j) = (t(w_1)y_1, \ldots, t(w_j)y_j)$.*

We check that the torus action is compatible with the gluing map:

Lemma 5.1.5. *Suppose $\tau \prec \sigma \in \Sigma$.*

i. Suppose $t \in T_N, x \in U_\tau$. Then $\Psi_{\tau,\sigma}(t.x) = t.\Psi_{\tau,\sigma}(x)$ and $\Psi_{\sigma,\tau}(t.y) = t.\Psi_{\sigma,\tau}(y)$ (where $y \in \Psi_{\tau,\sigma}(U_\tau)$).

ii. More generally, let σ and σ' be cones in Σ. If $x \in U_\sigma$ and $\Psi_{\sigma,\sigma'}$ is defined at x, then $\Psi_{\sigma,\sigma'}(t.x) = t.\Psi_{\sigma,\sigma'}(x)$.

Proof. i. $\Psi_{\tau,\sigma}(t.x) = (t.x)|_{S_\sigma} = (t|_{S_\sigma}).(x|_{S_\sigma}) = t.\Psi_{\tau,\sigma}(x)$. Similarly $\Psi_{\sigma,\tau}(t.y)$ is the natural extension of $t.y$ to U_σ, so $\Psi_{\sigma,\tau}(t.y) = t.y = t.\Psi_{\sigma,\tau}(y)$.

ii. $\Psi_{\sigma,\sigma'}$ decomposes into $\Psi_{\sigma\cap\sigma',\sigma'} \circ \Psi_{\sigma,\sigma\cap\sigma'}$. The result now follows by (1).

□

This leads us to a definition of the torus action on U_Σ. Let $x \in U_\Sigma$. Let σ and σ' be any two cones such that $x \in U_\sigma$ and $x \in U_{\sigma'}$. Then by Lemma 5.1.5 $\Psi_{\sigma,\sigma'}(t.x) = t.\Psi_{\sigma,\sigma'}(x)$. So the point obtained by acting t on x in U_σ is the same as the point obtained by acting t on x in $U_{\sigma'}$ (up to isomorphism). Thus we can make the following definition:

Definition 5.1.6. Suppose $t \in T_N$ and $x \in U_\Sigma$. If $x \in U_\delta$ for some $\delta \in \Sigma$, define $t.x$ to be $t|_{S_\delta}.x$.

5.2 The Spectrum of a Quotient Fan

Our aim is to identify the orbits of the torus action on U_Σ. It turns out that the orbit closures are embedded toric varieties. These embedded toric varieties derive from the fan obtained by quotienting Σ by a face $\tau \in \Sigma$. Hence we turn to the spectra of quotient fans.

Let $\sigma \in \Sigma$. Suppose $\tau \prec \sigma$ is a k-dimensional face and let W be the smallest subspace of V containing τ. In Section 3.3 we saw that we can project σ onto $W^\perp$ to obtain $\overline{\sigma} = \{proj_W(v) : v \in \sigma\}$. Note that $W^\perp$ is a subspace of V (not V^*), but $\tau^\perp$ is a subspace of $(W^\perp)^*$. We now extend our definition of the quotient of a cone to the quotient of a fan.

Definition 5.2.1. Denote the set of all cones in Σ containing τ, $Star(\tau)$. The quotient of Σ by τ is $\overline{\Sigma} = \{\overline{\sigma} : \sigma \in Star(\tau)\}$.

Lemma 5.2.2. *[4, p 52],[6, §VI.5.4.3] $\overline{\Sigma}$ is a fan in $W^\perp$. For each cone $\overline{\sigma}$, $\overline{\sigma}^\vee = \sigma^\vee \cap \tau^\perp$. That is, $\overline{\sigma}^\vee$ is a face of $\sigma^\vee$ and equals $(\tau)^*$.*

Proof. Let $\overline{\sigma} \in \overline{\Sigma}$. The fact that the faces of $\overline{\sigma}$ also belong to $\overline{\Sigma}$ follows from Proposition 3.3.5: the faces of $\overline{\sigma}$ are $\{\overline{\alpha} : \alpha \prec \sigma\}$. Let $\overline{\kappa} \in \overline{\Sigma}$. We need to check that $\overline{\sigma} \cap \overline{\kappa}$ is a face of both $\overline{\sigma}$ and $\overline{\kappa}$. By the definition of a fan $\sigma \cap \kappa$ is a face of each of σ and κ. It follows by Proposition 3.3.5 that $\overline{\sigma} \cap \overline{\kappa}$ is a face of each of $\overline{\sigma}$ and $\overline{\kappa}$.

The second assertion follows from Proposition 3.3.5: $\overline{\sigma}^\vee = \sigma^\vee \cap (W^\perp)^* = \sigma^\vee \cap \tau^\perp$. By Proposition 3.5.2, $\overline{\sigma}^\vee$ is a cone and equals $(\tau)^*$. $\square$

We define lattices in $W^\perp$ and $(W^\perp)^*$. Let $N(\tau) = N \cap W^\perp$ and $M(\tau) = M \cap (W^\perp)^*$ be its dual.

Remark 5.2.3. It is important to be aware of the fact that we are considering $\overline{\sigma}$ as a cone in $W^\perp$, not in V. If we were considering $\overline{\sigma}$ as a cone in V, then the dual would be V^* (not $(W^\perp)^*$) and $W \subseteq \overline{\sigma}^\vee$. $\square$

Definition 5.2.4. We introduce new notation to differentiate between the spectra obtained through quotienting σ by different faces.

$$U_\sigma(\tau) := U_{\overline{\sigma}} \qquad V(\tau) := U_{\overline{\Sigma}}$$

.

We will see that $V(\tau)$ can be emedded in U_Σ, and that $V(\tau)$ is the closure of an orbit of the torus action. But to see this we firstly need to show that $U_\sigma(\tau)$ can be embedded as a closed subset of U_σ.

By Lemma 5.2.2, $S_{\overline{\sigma}} = \sigma^\vee \cap \tau^\perp \cap M(\tau) = \sigma^\vee \cap \tau^\perp \cap M$. This is the face τ^* of $\sigma^\vee$. Let $\{w_1, \ldots, w_j\}$ be generators of $\sigma^\vee \cap \tau^\perp \cap M$, and let $\{w_{j+1}, \ldots, w_k\}$ additional vectors not in $\sigma^\vee \cap \tau$ such that $\{w_1, \ldots, w_k\}$ generate $\sigma^\vee \cap M$ (it is possible to find these generators thanks to Lemma 3.2.4). Let $(x_1, \ldots, x_k)$ be the corresponding coordinates of U_σ.

We define a morphism ϕ which embeds $U_\sigma(\tau)$ as a closed subset of U_σ. In terms of semigroups, ϕ is a map $Hom_{sg}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C}) \to Hom_{sg}(\sigma^\vee \cap M, \mathbb{C})$. We define ϕ to be extension by zero. That is, if $y \in Hom_{sg}(\sigma^\vee \cap \tau^\perp \cap M, \mathbb{C})$ and $a \in \sigma^\vee \cap \tau^\perp \cap M$ then define $\phi(y)(a) = y(a)$; for all other $a \in \sigma^\vee \cap M$ define $\phi(y)(a) = 0$.

Lemma 5.2.5. *The embedding $\phi : U_\sigma(\tau) \hookrightarrow U_\sigma$ is well-defined. In coordinates ϕ maps $(y_1, \ldots, y_j) \to (y_1, \ldots, y_j, 0, \ldots, 0)$. ϕ is a closed embedding.*

Proof. Suppose $y_i \neq 0$ for some $i > j$. Then $\phi(w_i) = y_i \neq 0$, which contradicts the definition. Thus the corresponding coordinate map is $(y_1, \ldots, y_j) \to (y_1, \ldots, y_j, 0, \ldots, 0)$. We check that these are valid coordinates. Suppose $c \in \sigma^\vee \cap M$ and that $c = a_1 w_1 + \ldots + a_k w_k = b_1 w_1 + \ldots + b_k w_k$. By Corollary 4.1.11 it suffices to check that

$$y_1^{a_1} \ldots y_k^{a_k} = y_1^{b_1} \ldots y_k^{b_k} \tag{5.1}$$

Suppose $c \in \sigma^\vee \cap \tau^\perp \cap M$. Now $\sigma^\vee \cap \tau^\perp$ is a face, so by Lemma 3.2.7 $w_i \leq c \Rightarrow w_i \in \sigma^\vee \cap \tau^\perp$ (Note $\leq$ is the partial ordering of Section 3.2). This means if $i > j$ then $a_i = b_i = 0$. Thus $c = a_1 w_1 + \ldots + a_j w_j = b_1 w_1 + \ldots + b_j w_j$. But since y is a well-defined homomomorphism on $\sigma^\vee \cap \tau^\perp \cap M$, $y(c) = y_1^{a_1} \ldots y_j^{a_j} = y_1^{b_1} \ldots y_j^{b_j}$ and (5.1) is true.

Suppose $c \notin \sigma^\vee \cap \tau^\perp$ but $c \in \sigma^\vee \cap M$. Then $\exists h, i > j$ such that $a_i \neq 0$ and $b_h \neq 0$ (because otherwise $c \in \sigma^\vee \cap \tau^\perp$). Since $y_i = y_h = 0$, both monomials in (5.1) are 0. Thus $(y_1, \ldots, y_j, 0 \ldots, 0)$ are valid coordinates and $\phi(y) \in U_\sigma$.

ϕ is clearly a morphism of affine varieties. In fact ϕ is an isomorphism between $U_\sigma(\tau)$ and $U_\sigma \cap V(x_{j+1}x_{j+2}\dots x_k)$. It suffices to prove the existence of an inverse $\psi : U_\sigma \cap V(x_{j+1}\dots x_k) \to U_\sigma$. Define $\psi(y_1,\dots,y_j,0,\dots,0) = (y_1,\dots,y_j)$. We check these are valid coordinates. Suppose $a_1w_1 + \ldots + a_jw_j = b_1w_1 + \ldots + b_jw_j$. Then, since $(y_1,\dots,y_j,0,\dots,0) \in U_\sigma$, by Corollary 4.1.11 $y_1^{a_1}\dots y_j^{a_j} = y_1^{b_1}\dots y_j^{b_j}$. But this is the condition necessary and sufficient for $(y_1,\dots,y_j) \in U_\sigma(\tau)$. □

Suppose $\tau \prec \gamma \prec \sigma \in \Sigma$. Let $\Psi_{\gamma,\sigma} : U_\gamma \hookrightarrow U_\sigma$ be the gluing map. Since $\overline{\gamma} \prec \overline{\sigma} \in \overline{\Sigma}$, we can also find a gluing map $\Psi_{\overline{\gamma},\overline{\sigma}} : U_\gamma(\tau) \to U_\sigma(\tau)$. It turns out that the closed embedding of Lemma 5.2.5 commutes with these gluing maps. Let ϕ_γ be the closed embedding $U_\gamma(\tau) \hookrightarrow U_\gamma$ and let ϕ_σ be the closed embedding $U_\sigma(\tau) \hookrightarrow U_\sigma$.

Lemma 5.2.6. *The embedding maps commute:* $\phi_\sigma \circ \Psi_{\overline{\gamma},\overline{\sigma}} = \Psi_{\gamma,\sigma} \circ \phi_\gamma$

Proof. We prove this using the SH realisation of the spectrum. Let $x \in Hom_{sg}(S_{\overline{\gamma}}, \mathbb{C})$. Then $\phi_\gamma(x)$ is the semigroup homomorphism such that $\phi_\gamma(x)(a) = x(a)$ if $a \in \gamma^\vee \cap \tau^\perp \cap M$ and $\phi_\gamma(x)(a) = 0$ otherwise. $\Psi_{\gamma,\sigma}(\phi_\gamma(x))$ is the restriction of $\phi_\gamma(x)$ to $\sigma^\vee \cap M$. Since $\gamma^\vee \cap \tau^\perp \cap M \cap \sigma^\vee \cap M = \sigma^\vee \cap \tau^\perp \cap M$, we find that

$$\Psi_{\gamma,\sigma}(\phi_\gamma(x))(a) = \begin{cases} a & \text{if } a \in \sigma^\vee \cap \tau^\perp \cap M \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

Conversely $\Psi_{\overline{\gamma},\overline{\sigma}}(x)$ is the restriction of x to $\overline{\sigma}^\vee \cap M(\tau) = \sigma^\vee \cap \tau^\perp \cap M$. $\phi_\sigma(\Psi_{\overline{\gamma},\overline{\sigma}}(x))$ is the extension of $\Psi_{\overline{\gamma},\overline{\sigma}}(x)$ by zero: that is it equals x on $\sigma^\vee \cap \tau^\perp \cap M$ and is zero everywhere else. But this is the same function as (5.2). □

In diagram form, we have proved

$$\begin{array}{ccc} U_\gamma(\tau) & \longrightarrow^{\Psi_{\overline{\gamma},\overline{\sigma}}} & U_\sigma(\tau) \\ \phi_\gamma \downarrow & & \downarrow \phi_\sigma \\ U_\gamma & \longrightarrow^{\Psi_{\gamma,\sigma}} & U_\sigma \end{array} \tag{5.3}$$

By Lemma 5.2.5 we can identify $U_\gamma(\tau)$ with a closed subset of U_γ (by the map ϕ_γ). Similarly we can identify $U_\sigma(\tau)$ with a closed subset of U_σ (by the map ϕ_σ). Under this identification it can be seen from (5.3) that, the map $\Psi_{\gamma,\sigma} : U_\gamma \to U_\sigma$ restricts to $\Psi_{\overline{\gamma},\overline{\sigma}} : U_\gamma(\tau) \to U_\sigma(\tau)$. In other words, $\Psi_{\overline{\gamma},\overline{\sigma}} = \Psi_{\gamma,\sigma}|_{U_\gamma(\tau)}$. Now general gluing maps $\Psi_{\sigma,\kappa}$ are by definition the composition of two such open embeddings: $\Psi_{\sigma,\kappa} = \Psi_{\sigma\cap\kappa,\kappa} \circ \Psi_{\sigma,\sigma\cap\kappa}$. Thus this result holds for general cones in Σ as well. Suppose $\tau \prec \sigma$ and $\tau \prec \kappa$,

Lemma 5.2.7. *The gluing maps are consistent:* $\Psi_{\overline{\sigma},\overline{\kappa}} = \Psi_{\sigma,\kappa}|_{U_\sigma(\tau)}$. *Hence we can embed* $V(\tau)$ *as the closed subset* $\bigcup_{\sigma \succ \tau} U_\sigma(\tau) \subseteq U_\Sigma$.

Proof. Suppose $x \in U_\sigma(\tau)$ and $y \in U_\kappa(\tau)$. By the definition of U_Σ we identify x and y in if and only if there exists a gluing map such that $\Psi_{\sigma,\kappa}(x) = y$. But by Lemma 5.2.6 this is equivalent to $\Psi_{\overline{\sigma},\overline{\kappa}}(x) = y$. □

Lastly we prove that the embedding of Corollary 5.2.7 is order reversing.

Lemma 5.2.8. *Suppose* $\tau \prec \kappa \in \Sigma$. *Then* $U_\sigma(\kappa)$ *is a closed subset of* $U_\sigma(\tau)$. *Hence under the embeddings* $V(\kappa) \hookrightarrow U_\Sigma$ *and* $V(\tau) \hookrightarrow U_\Sigma$, $V(\kappa)$ *is a closed subset of* $V(\tau)$.

Proof. We choose generators $\{w_1, \ldots, w_j, \ldots, w_k, \ldots, w_l\}$ such that: $\{w_1, \ldots, w_j\}$ generate $\sigma^\vee \cap \kappa^\perp \cap M$, $\{w_1, \ldots, w_k\}$ generate $\sigma^\vee \cap \tau^\perp \cap M$ and $\{w_1, \ldots, w_l\}$ generate $\sigma^\vee \cap M$. Assume also that if $i > j$ then $w_i \notin \sigma^\vee \cap \kappa^\perp \cap M$ and if $i > k$ then $w_i \notin \sigma^\vee \cap \tau^\perp \cap M$. Let $y \in Hom_{sg}(\sigma^\vee \cap \kappa^\perp \cap M)$ have coordinates $(y_1, \ldots, y_j)$. By Lemma 5.2.5 the coordinates of the embedded y are given by the $l-tuple$ $(y_1, \ldots, y_j, 0, \ldots, 0)$. Thus it is evident that $U_\sigma(\kappa)$ embeds as the closed set $U_\sigma(\tau) \cap (\bigcap_{i=j+1}^{k} V(y_i))$.

These embeddings patch together to give a closed embedding of $V(\kappa)$ in $V(\tau)$. □

Remark 5.2.9. Equivalently we could have embedded $U_\sigma(\kappa)$ into $U_\sigma(\tau)$ and then embedded $U_\sigma(\tau)$ into U_σ. These are merely two different ways of extending $y \in Hom_{sg}(\sigma^\vee \cap \kappa^\perp \cap M, \mathbb{C})$ by zero to all of $\sigma^\vee \cap M$. □

5.3 The Orbits of the Torus Action

Thus far we have identified one orbit of the torus action on U_Σ: the embedded torus $U_{\{0\}}$. Since this orbit is dense it must contain every other orbit in its closure. For this reason we denote it the *big* orbit, following [6]. In fact it turns out that every orbit is an embedded torus.

Definition 5.3.1. [4, p 52] Let O_τ be the torus corresponding to $N(\tau)$. That is, $O_\tau = U_\tau(\tau) = Hom_{sg}(\tau^\perp \cap M, \mathbb{C})$.

As the notation suggests, we will see that O_τ is an orbit of the torus action.

Definition 5.3.2. The *distinguished point* of O_τ is the semigroup homomorphism

$$y_\tau(a) = \begin{cases} 1 & \text{if } a \in \tau^\perp \\ 0 & \text{otherwise} \end{cases}$$

Lemma 5.3.3. *O_τ is an orbit of the torus action on U_Σ.*

Proof. Choose a basis $\{f_1, \ldots, f_j, \ldots, f_n\}$ for M such that $\{f_1, \ldots, f_j\}$ is a basis for $\tau^\perp \cap M$. Let the corresponding coordinates be $(x_1, \ldots, x_n)$. By Lemma 5.2.5, $\forall i > j \ x_i = 0$.

Let y_τ be the distinguished point of O_τ and let y be an arbitrary point in O_τ. We find $t \in T_N$ such that $t.y_\tau = y$, which implies that O_τ belongs to the orbit containing y_τ. We can specify t such that $\forall i \leq j,\ t(f_i) = y(f_i)$. This is possible because $\{f_i\}_{i=1}^{j}$ is a linearly independent set, so there are no restrictions on $t(f_i)$ other than $t(f_i) \neq 0$ for t to be a well-defined semigroup homomorphism.

Furthermore O_τ is closed under the torus action. This can be seen from the fact that if $t \in T_N$, $(t.y)(a) = t(a)y(a) = 0$ if $a \notin \tau^\perp \cap M$ because $y(a) = 0$. Thus $(t.y) \in O_\tau$. □

Corollary 5.3.4. *$V(\tau)$ is the closure of O_τ.*

Proof. Since the closed embedding $V(\tau) \hookrightarrow U_\Sigma$ preserves the Zariski Topology, the result follows from the fact that O_τ is dense in $V(\tau)$ (Lemma 4.3.5). □

Thus O_τ is the big orbit of $V(\tau)$.

Proposition 5.3.5. *[4, p 54] We describe U_Σ in terms of orbits O_τ and embedded toric varieties $V(\tau)$:*

i. $U_\sigma = \coprod_{\tau \prec \sigma} O_\tau$

ii. $U_\Sigma = \coprod_{\delta \in \Sigma} O_\delta$

iii. $O_\tau = V(\tau) - \bigcup_{\tau \prec \gamma, \tau \neq \gamma} O_\gamma$

iv. $V(\tau) = \coprod_{\gamma \succ \tau} O_\gamma$

Proof. (i): We follow [4, p 54]. Let $y \in U_\sigma$ be a semigroup homomorphism. If y is nonzero everywhere then by Proposition 4.2.4 y belongs to the embedded torus T_N. If otherwise, let α be the set $y^{-1}(\mathbb{C}^*)$. Note that if $a + b \in \alpha$ then $y(a)y(b) \neq 0$ and

$a, b \in \alpha$. It follows by Lemma 3.4.6 that the convex hull of α, which we denote γ, is a face of $\sigma^\vee$. Therefore $y \in O_{\gamma^*}$.

(ii): We find $U_\Sigma = \bigcup_{\kappa \in \Sigma} U_\kappa = \bigcup_{\kappa \in \Sigma} (\coprod_{\tau \prec \kappa} O_\tau)$. Since the orbits $\{O_\tau : \tau \in \Sigma\}$ are all disjoint, we find $U_\Sigma = \coprod_{\delta \in \Sigma} O_\delta$.

(iii): Since $V(\tau)$ is a toric variety, we can assume $\tau = \{0\}$ and it suffices for us to show

$$T_N = U_\Delta - \bigcup_{\gamma \neq 0} V(\gamma) \tag{5.4}$$

We intersect both sides of this equation with U_σ, to obtain $T_N = U_\sigma - \bigcup_{\gamma \neq 0} V(\gamma) \cap U_\sigma$. Since $U_\Sigma = \bigcup_{\sigma \in \Sigma} U_\sigma$, if we can prove $T_N = U_\sigma - \bigcup_{\gamma \neq 0} V(\gamma) \cap U_\sigma$ then (5.4) must be true. Using (i), this is equivalent to proving $T_N = \coprod_{\tau \prec \sigma} O_\tau - \bigcup_{\gamma \neq 0} V(\gamma) \cap U_\sigma$. Now $T_N \cap V(\gamma) = \emptyset$ if $\gamma \neq \{0\}$. The reason is, using the SH realisation of the spectrum, if $x \in V(\gamma)$ then x is zero outside of $\gamma^\perp$, but members of T_N are nonzero everywhere. Hence we only need to prove $\bigcup_{\gamma \neq 0} V(\gamma) \cap U_\sigma = \coprod_{\tau \prec \sigma, \tau \neq \{0\}} O_\tau$. But $(\supseteq)$ follows from the fact $O_\tau \subseteq V(\tau)$. The other inclusion $(\subseteq)$ follows from the fact that $V(\gamma) \cap U_\sigma \subseteq U_\sigma - T_N$.

(iv): Refer to [4, p 50] for a proof. □

Some remarks are called for.

Remark 5.3.6. One might think that if $\kappa^\perp = \sigma^\perp$ then $O_\kappa = O_\sigma$. After all, we defined $O_\kappa = Hom_{sg}(\kappa^\perp \cap M, \mathbb{C})$ and $O_\sigma = Hom_{sg}(\sigma^\perp \cap M, \mathbb{C})$. However we only identify points in U_σ and U_κ if there is a gluing map between them. □

Example 5.3.7. We compute the orbits of Example 4.3.7. The big orbit is $O_\tau = \mathbb{C}^*$. The other two orbits are the points y_σ and y_κ. □

Example 5.3.8. We compute the orbits of Example 4.3.8. We write them in homogeneous coordinates. $T_N = \{(x : y : z) : x, y, z \neq 0\}$. Note that $T_N \simeq (\mathbb{C}^*)^2$. There are 3 1-dimensional orbits: $O_{\sigma \cap \kappa}, O_{\sigma \cap \tau}$ and $O_{\kappa \cap \tau}$. These are the sets $\{(0 : y : z) : y, z \neq 0\}$, $\{(x : 0 : z) : x, z \neq 0\}$ and $\{(x : y : 0) : x, z \neq 0\}$. Finally there are three orbits of degree 0 (i.e. points): y_σ, y_κ and y_τ. These are the points $(1 : 0 : 0)$, $(0 : 1 : 0)$ and $(0 : 0 : 1)$. □

Let Σ be a fan in the vector space V_Σ with lattice N_Σ, and let Δ be a fan in the vector space V_Δ with lattice N_Δ. Let $\phi : N_\Sigma \to N_\Delta$ be a fan homomorphism. By Lemma 4.3.9 ϕ induces a toric morphism $\Xi : U_\Sigma \to U_\Delta$. We find that the torus action is preserved.

Theorem 5.3.9. *Let T_Σ be the big torus in U_Σ. Let $y \in U_\Delta$ and $t \in T_\Sigma$. Then $\Xi(t.y) = \Xi(t).\Xi(y)$.*

Proof. Let $\phi^* : M_\Delta \to M_\Sigma$ be the dual of ϕ. Let $\sigma \in \Sigma$ contain y and let $\delta \supseteq \phi(\sigma)$. By Lemma 4.2.1, $\Xi(t.y) = (t.y) \circ (\phi^*|_{S_\delta}) = (t \circ \phi^*)|_{S_\delta}.(y \circ \phi^*)|_{S_\delta} = \Xi(t).\Xi(y)$. □

Theorem 5.3.9 implies that Ξ preserves a lot of the orbit structure. Let M_δ and M_σ be the dual lattices.

Corollary 5.3.10. *[4, p 56] Let $\sigma \in \Sigma$ and let δ be the smallest cone of Δ containing $\phi(\sigma)$. Then*

i. $\Xi(y_\sigma) = y_\delta$

ii. $\Xi(O_\sigma) \subseteq O_\delta$.

Proof.

i. $\phi(\sigma)$ contains a point v in the relative interior of δ because otherwise δ would not be the smallest face containing $\phi(\sigma)$. By Lemma 3.4.5 $\delta^\vee \cap v^\perp = \delta^\perp$. Let $\phi^* : \delta^\vee \to \sigma^\vee$ be the dual. Now $\phi^*(\delta^\perp) \subseteq \sigma^\perp$: because if $a \in \delta^\perp$ and $b \in \sigma$ then $\phi^*(a)(b) = a(\phi(b)) = 0$ (as $\phi(b) \in \delta$). Also $\phi^*(\delta^\vee - \delta^\perp) \subseteq \sigma^\vee - \sigma^\perp$. This is true because if $a \in \delta^\vee - \delta^\perp$ and $b \in \sigma$ is such that $\phi(b) = v$ then $\phi^*(a)(b) = a(v)$. As $a \in \delta^\vee - \delta^\perp = \delta^\vee - \delta^\vee \cap v^\perp$, we find $a \notin v^\perp$ and $a(v) \neq 0$. Recall $\Xi(y_\sigma)(c) = y_\sigma \circ \phi^*(c)$ $(c \in M_\delta)$. Thus if $c \in M_\delta \cap \delta^\perp$, $\phi^*(c) \in \sigma^\perp$ and $\Xi(y_\sigma)(c) = 1$. Conversely if $c \notin M_\delta \cap \delta^\perp$ then $\phi^*(c) \in M_\sigma - \sigma^\perp$ and $\Xi(y_\sigma)(c) = 0$. Thus $\Xi(y_\sigma) = y_\delta$.

ii. Follows immediately from (i).

□

Chapter 6

Singularities in Toric Varieties

In this chapter we draw together the strands of our argument. In the first section we look at the singularities of toric varieties, and observe that U_σ is nonsingular if and only if the generators of σ form part of a basis for N. In the second and third sections we prove that we can refine Δ such that every cone is generated by part of a basis. We will see that the toric morphism associated with this refinement is a resolution of singularities. Finally in the last section we analyse the induced toric morphism using the torus action of the previous chapter.

6.1 Singular Cones

Suppose that σ spans V. In this section we determine necessary and sufficient conditions for U_σ to be nonsingular. This will be our first step towards the resolution of toric singularities.

By Proposition 3.5.2 $\sigma^\perp = \{0\}$ and so $\sigma^\vee$ is strongly convex. Let $(w_1, \ldots, w_j)$ be generators of S_σ, with corresponding coordinates $(x_1, \ldots, x_j)$. It follows that the distinguished point y_σ has coordinates $(0, \ldots, 0)$.

Lemma 6.1.1. *If the generators of σ span V then U_σ is nonsingular if and only if $\{w_i\}_{i=1}^j$ is a basis for N. In this case $U_\sigma \simeq \mathbb{C}^n$.*

Proof. Firstly note that if the generating vectors are a basis for N then we can apply an automorphism to transform them into the standard basis vectors, in which case it is clear $U_\sigma \simeq \mathbb{C}^n$ (and hence U_σ is nonsingular). For the converse, we prove that if y_σ is to be nonsingular it is necessary that σ is generated by a basis for N. By Proposition 2.3.2 the dimension of U_σ equals the dimension of the dense open subset T_N, which is n. Let m be the maximal ideal in $\mathbb{C}[x_1, \ldots, x_j]/I_\sigma$ corresponding to y_σ. Observe that y_σ is not a singularity if and only if $dim(\frac{m}{m^2}) = n$.

Now m is the ideal generated by $(x_1 - 0, \ldots, x_j - 0) = (x_1, \ldots, x_j)$. The corresponding maximal ideal in A_σ is the ideal generated by $(\chi^{w_1}, \ldots, \chi^{w_j})$. Thus by Definition 2.3.3, y_σ is singular precisely when the dimension of $\frac{m}{m^2}$ is greater than n.

The space $\frac{m}{m^2}$ has as a basis all the elements χ^u which do not equal the multiplication of two elements in m. That is, $\frac{m}{m^2}$ has as a basis $\{\chi^u : \ u \text{ is irreducible}\}$. Thus by Proposition 3.7.6 the dimension of $\frac{m}{m^2}$ equals the size of the minimal generating set of S_σ.

From the same proposition we see that every vector in the generating set of $\sigma^\vee$ must be in the unique minimal generating set of S_σ. There are at least n generators of $\sigma^\vee$, as $\sigma^\vee$ spans V. Thus the size of the unique minimal generating set of S_σ is n if and only if the generators of $\sigma^\vee$ also generate S_σ. Since we have assumed σ spans V, this is equivalent to saying the generating vectors of $\sigma^\vee$ are a basis for M.

Now if the generators of $\sigma^\vee$ are a basis for M we can apply a lattice automorphism to map $\sigma^\vee$ to the cone generated by $\{e_i^*\}_{i=1}^n$. The corresponding automorphism of N must map σ to the dual of the cone generated by the vectors $\{e_i^*\}_{i=1}^n$; which is the cone generated by the vectors $\{e_i\}_{i=1}^n$. It follows that σ is generated by a basis for N. □

We extend this Lemma to arbitrary cones.

Proposition 6.1.2. *[4, p 29] If σ is a cone then U_σ is nonsingular if and only if σ is generated by part of a basis for the lattice N, in which case*

$$U_\sigma \simeq \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}, k = dim(\sigma)$$

Proof. We partially follow [4, p 29]. Let σ be a cone of dimension k generated by $\{v_1, \ldots, v_s\}$. Let $W = \mathbb{R}.\sigma$.

We split $N = N_\sigma \oplus N''$ and $\sigma = \sigma' \oplus \{0\}$, where σ' is a cone in N_σ. Let the corresponding splitting of duals be $M' \oplus M''$. Thus as $M'' \subseteq \sigma^\perp$,

$$S_\sigma = ((\sigma')^\vee \cap M') \oplus M'' \tag{6.1}$$

We prove $U_\sigma \simeq U_{\sigma'} \times U_{\{0\}}$, using the SH realisation of the spectra. Suppose $\xi_1 \in Hom_{SG}(S_{\sigma'}, \mathbb{C})$ and $\xi_2 \in Hom_{SG}(M'', \mathbb{C})$. Then we can define $\xi \in Hom_{Sg}(S_\sigma, \mathbb{C})$ as follows. If $u = u' + u''$ is the unique decomposition of $u \in S_\sigma$ then $\xi(u) = \xi_1(u')\xi_2(u'')$. Conversely given $\psi \in Hom_{Sg}(S_\sigma, \mathbb{C})$ we can restrict ψ to get homomorphisms in $Hom_{Sg}(S_{\sigma'}, \mathbb{C})$ and $Hom_{Sg}(M'', \mathbb{C})$. As $U_{M''} = T_{N''}$ we have $U_\sigma \simeq U_{\sigma'} \times T_{N''} \simeq U_{\sigma'} \times (\mathbb{C}^*)^{n-k}$. Since $(\mathbb{C}^*)^{n-k}$ is nonsingular, it can be seen that $U_{\sigma'}$ is nonsingular if and only if U_σ is nonsingular. In turn by Lemma 6.1.1 $U_{\sigma'}$ is nonsingular if and only if the generators of σ' constitute a basis of W. $\square$

Thus we can determine whether a toric variety is nonsingular merely by observing the cones that generate it. We think of a cone as nonsingular if its associated spectrum is nonsingular. More formally,

Definition 6.1.3. A cone is *nonsingular* if it is generated by part of a basis for the lattice. A fan is nonsingular if all of its cones are nonsingular.

Example 6.1.4. The cone σ in Example 4.1.16 and Figure 3.1 is singular, as its generating vectors $2e_1 - e_2$ and e_2 do not form a basis for the lattice. We saw that a coordinate realisation of U_σ is $\{(x, y, z) : xz = y^2\}$. This surface has a singularity at $(0, 0, 0)$ as expected. $\square$

Let Δ and Σ be fans over a lattice N.

Definition 6.1.5. We say Σ is a *refinement* of Δ if each cone in Δ is a union of cones in Σ.

Let Σ be a refinement of Δ. Let ϕ be the identity homomorphism on N. Then ϕ is also a homomorphism of fans $\Sigma \to \Delta$ since by definition each cone in Σ is contained in a cone in Δ.

Proposition 6.1.6. *If Σ and Δ are strongly convex then the induced toric morphism $\Xi : U_\Sigma \to U_\Delta$ is birational. If Σ is nonsingular and Δ is singular, then Ξ is a resolution of the singularities of Δ.*

Proof. T_N is an open dense subset of both U_Σ and U_Δ. Since ϕ is an isomorphism and fixes $\{0\}$, it follows from Corollary 4.2.3 that ϕ induces an isomorphism on the embedded torii. That is, $\Xi|_{T_N}$ is an isomorphism. $\square$

Thus the problem of resolving the singularities of U_Δ has been reduced to combinatorics. To resolve the singularities of U_Δ all we must do is find a nonsingular refinement of Δ. In fact a nonsingular refinement always exists. Proving this requires an understanding of the *multiplicity* of a cone.

6.2 The Multiplicity of a Cone

Let σ be a singular cone generated by $\{v_1, \ldots, v_s\}$.

In order to prove that a nonsingular refinement of σ exists we need a measure of how far removed σ is from being a nonsingular cone. Consider the minimal generating set S of σ. Intuitively, it seems that the larger S is, the further it is removed from being a basis and the more "singular" σ is. The measure we use is the multiplicity of σ; which is in fact equal to the cardinality of S.

Definition 6.2.1. We denote the lattice generated by $\{v_1, \dots v_s\}$, G_σ. That is, $G_\sigma = \mathbb{Z}v_1 + \dots \mathbb{Z}v_s$.

Notice that $G_\sigma = \{Bz | z \in \mathbb{Z}^n\}$.

Consider the quotient $\frac{N_\sigma}{G_\sigma}$; we claim it is a finite Abelian group of index less than or equal to the size of the extraneous set K. If a is any vertex in N_σ then by definition $a = \sum_{i=1}^{s} \lambda_i v_i$. We can add or subtract members of the set $\{v_i\}$ from a until $0 \leq \lambda_i < 1$. Then a belongs to the extraneous set K, which by Corollary 3.7.3 is finite. Thus we can make the following definition:

Definition 6.2.2. The *multiplicity* of σ is the index of the lattice N_σ relative to G_σ. That is, $mult(\sigma) = [N_\sigma : G_\sigma]$.

Now suppose σ is simplicial; that is, its generating vectors are linearly independent. This assumption considerably aids us in determining $mult(\sigma)$. But firstly we need to draw on some standard results from algebra. Consult [15, pp 264-7,688-691] for more details.

Theorem 6.2.3. *[15, pp 264-7]* Fundamental Theorem of Finite Abelian Groups. *Every finite abelian group G is a direct sum of cyclic groups:*

$$G = S(c_1) \oplus \dots \oplus S(c_t)$$

$S(c_i)$ is a cyclic group of order c_i and $c_1|c_2| \dots |c_t$. The $\{c_i\}$ are the invariant factors of G.

Since $\frac{N_\sigma}{G_\sigma}$ is a finite Abelian group, we reach a corollary:

Corollary 6.2.4. *Let $\{a_1, \dots, a_k\}$ be a basis of N_σ. Let v_i' be the coordinate vector of v_i according to this new basis, and A be the matrix $(v_1' : \dots : v_k')$. Then $mult(\sigma) = |det(A)|$.*

Proof. We apply Gaussian row and column operations on A until it is in the *Smith Normal Form* (see [15, p 688]). This means $\exists B$ such that $B = QAP$ where $Q, P \in GL(\mathbb{Z}^k)$ and B is a diagonal matrix with the invariant factors as entries. Now as $det(P) = \pm 1$ and $det(Q) = \pm 1$, $|det(A)| = |det(B)|$. Further $det(B) = c_1 c_2 \dots c_k$. It is clear from Theorem 6.2.3 that $\left|\frac{N_\sigma}{G_\sigma}\right| = c_1 c_2 \dots c_k = mult(\sigma)$. ☐

This means that the determinant of the generating matrix of σ can be used to test whether σ is nonsingular or not. We summarize the results of this section.

Proposition 6.2.5. *Suppose σ is a nonsimplical cone. Suppose $\{a_i\}_{i=1}^k$ is a basis of $\mathbb{R}.\sigma$, and $\{v_i\}_{i=1}^k$ are the generators of σ with respect to this basis. The following are equivalent.*

i. *σ is nonsingular.*

ii. *The extraneous set K equals $\{0\}$.*

iii. *$mult(\sigma) = 1$.*

iv. *The determinant of the matrix $(v'_1 : \ldots : v'_k)$ equals ± 1.*

Proof. (iii) $\Leftrightarrow$ (iv) follows from Corollary 6.2.4.

(iii) $\Leftrightarrow$ (i): If $mult(\sigma) = 1$ then $G_\sigma = N_\sigma$, which means that the generators of σ form a basis for N_σ. They can therefore be extended to a basis for V. Conversely, if σ is nonsingular then its generators form a basis for N_σ and $mult(\sigma) = 1$.

(iii) $\Rightarrow$ (ii): Suppose there exists $v = \sum_i \lambda_i v_i \in K - \{0\}$, where $\{v_i\}$ are generators of σ. Since σ is simplicial this expression is unique. Thus $v \notin G_\sigma$ and $mult(\sigma) \neq 0$.

(ii) $\Rightarrow$ (i): If $K = \{0\}$ then there are no minimal generators of $\sigma \cap N$ in K, and therefore $\sigma \cap N$ is generated by the generating vectors of σ. $\square$

Lemma 6.2.6. *If σ is nonsingular then all of its faces are nonsingular.*

Proof. If σ is nonsingular then the generators S form part of a basis. Since a face τ of σ is generated by a subset of S, it is clear that τ is generated by part of a basis as well. $\square$

6.3 The Subdivision of a Fan

Let Δ be a fan in N. In this section we prove that there is a nonsingular refinement of Δ.

Let $w \in N \cap \Delta$. We *subdivide* Δ through w to obtain a new fan Σ as follows. Let $\delta \in \Delta$ be a cone containing w. For every proper face $\tau \prec \delta$ not containing w, we create a new cone τ' generated by w and τ, and remove δ. That is, if the generators of τ are $\{v_1, \ldots, v_m\}$ then τ' is generated by $\{v_1, \ldots, v_m, w\}$.

Lemma 6.3.1. *The subdivision of Δ through w is another fan Σ. Σ is a refinement of Δ.*

Proof. For any cone $\alpha \in \Delta$, let α' denote the corresponding subdivided cone. Let the subdivided fan be Σ. Suppose τ and κ are faces of respectively δ_1 and δ_2, and that the subdivision is made through $w \in \delta_1 \cap \delta_2$. We show that $\tau' \cap \kappa'$ is the cone $(\tau \cap \kappa)' \in \Sigma$. One inclusion $(\tau \cap \kappa)' \subseteq \tau' \cap \kappa'$ is obvious. For the other inclusion, suppose z is a point in $\tau' \cap \kappa'$: $z = x + \lambda w = y + \mu w$ ($x \in \tau$, $y \in \kappa$). We prove $z \in (\tau \cap \kappa)'$. There are several possibilities. If $\lambda = \mu$, then $x = y$ and $z \in (\tau \cap \kappa)'$. Otherwise assume without loss of generality that $\lambda > \mu$. We find $x + (\lambda - \mu)w = y \in \kappa$. If $y = 0$ then as δ_1 is strongly convex $x = 0$ and $(\lambda - \mu)w = 0 \Rightarrow \lambda = \mu$ - a contradiction. Thus $y \neq 0$ and we find by Lemma 3.2.7 that $x, w \in \delta_1 \cap \delta_2$. As κ is a face of δ_2, the same Lemma implies that $x, w \in \kappa$. Hence $x \in \kappa \cap \tau$. Therefore $z = x + \lambda w \in (\tau \cap \kappa)'$. Thus $\tau' \cap \kappa' \subseteq (\tau \cap \kappa)'$.

Finally we check the subdivision is a refinement. We do this inductively. Suppose δ_0 is the unique cone containing w in its relative interior. Let x be an arbitrary point in δ_0 and let l be the line through x parallel to w. l must intersect the topological boundary of δ exactly once, as $(x + \lambda w) \in \delta_0$ for all $\lambda > 0$. Denote this point y, and denote the facet it is in κ (by Lemma 3.4.2 the boundary of δ is the union of its facets). Then as x equals y plus a multiple of w, it is evident that $x \in \kappa'$. Thus δ_0 is a union of cones in Σ.

Now suppose δ_p is a cone containing w in one of its facets δ_{p-1}. Thanks to Lemma 3.3.6 we can find a chain $\delta_0 \prec \delta_1 \prec \ldots \prec \delta_p$, where δ_i is a facet of δ_{i+1}. To prove that δ_p is a union of cones in Σ, we can assume that δ_{p-1} is a union of cones in the refinement Σ, because then the result will follow by induction.

Let x be any point in δ that isn't in γ. Once again we find the line l line through x parallel to w intersects a facet κ at a point y. x must belong to the subdivided cone κ' because x equals y plus a multiple of w. □

Definition 6.3.2. Let $[v_1, \ldots, v_r]$ represent the refinement obtained by subdividing Δ through v_1, then subdividing the resulting fan through v_2 etc. The *size* of the subdivision is r.

Recall that $\sigma \in \Delta$ is nonsimplicial if its generating vectors are not linearly independent. We firstly refine Δ so that every cone in Δ is simplicial. The fact that this is possible can be proved inductively. Suppose δ is a nonsimplicial cone of dimension n with simplicial faces. We subdivide δ through a point w in its relative interior. If $\tau \prec \delta$ is of dimension k, then τ' will be of dimension $k + 1$ as w does not belong to the subspace spanned by τ. Since by assumption τ is generated by

k vectors, τ' is generated by $k+1$ vectors and therefore is simplicial. Since one-dimensional faces are obviously simplicial, we can subdivde all nonsimplicial cones of dimension two until they are simplicial, then all cones of dimension three, and continue until every cone in Δ is simplicial.

We reach the crux of our argument:

Theorem 6.3.3. *There exists a refinement Σ of Δ such that every cone in Σ is generated by part of a basis. Thus we can always resolve the singularities of Δ through a toric morphism induced by a refinement.*

Proof. This is suggested as an exercise in [4, p 48].

By the above we can subdivide Δ until it is simplicial; so we assume Δ is simplicial. Let Ω_Δ be the set of all maximal cones in Δ (that is, the set of all cones which are not the face of another cone in Δ). Our aim is to find a subdivision Σ of Δ so that all the cones in Ω_Σ are nonsingular, because then it will follow by Lemma 6.2.6 that Σ is nonsingular. Let δ be the cone in Ω_Δ of highest multiplicity. Let the generators of δ be $\{v_1, \dots, v_k\}$, where $dim(\delta) = k$. If $mult(\delta) = 1$ then Δ is nonsingular. Otherwise by Proposition 6.2.5 the extraneous set must contain at least one nonzero vector $v = \sum_{i=1} \lambda_i v_i$ $(0 \leq \lambda_i < 1)$. We subdivide Δ through v to obtain a fan Δ'.

Firstly we observe the k-dimensional cones which replace δ. Let γ_h be the facet generated by $\{v_1, \dots, \hat{v_h}, \dots, v_k\}$ (where $\hat{v_h}$ means "omit v_h"). If $\lambda_h \neq 0$ then $v \notin \gamma_h$, and our subdivision forms a new k-dimensional cone δ_h generated by $\{v_1, \dots, \hat{v_h}, \dots, v_k, v\}$. Choose a basis for N_σ and let v'_i be the coordinate vector of v_i with respect to this basis. It can be seen that the coordinate vector of v with respect to this basis is $v' = \sum_i \lambda_i v'_i$. By Corollary 6.2.4

$$\begin{aligned} mult(\delta_h) &= det(v'_1 : \dots : \hat{v'_h} : \dots : v'_k : v') \\ &= det(v'_1 : \dots : \hat{v'_h} : \dots : v'_k : \sum_j \lambda_j v'_j) \\ &= \sum_j \lambda_j det(v'_1 : \dots : \hat{v'_h} : \dots : v'_k : v'_j) \\ &= \lambda_h det(v'_1 : \dots : v'_k) = \lambda_h mult(\delta) \qquad (6.2) \end{aligned}$$

where (6.2) follows by noting that the vectors $\{v'_1, \dots, \hat{v'_h}, \dots, v'_k, v'_j\}$ are linearly independent precisely when $j = h$. Since $\lambda_h < 1$ we have $mult(\delta_h) < mult(\delta)$. Therefore each of the new k-dimensional cones is of multiplicity strictly less than

δ. If $\beta \in \Omega_\Delta$ is another cone containing w, with dimension l, we likewise find that the new l-dimensional cones are of multiplicity strictly less than $mult(\beta)$.

Let Ω' be the set of all maximal cones in Δ'. Every new cone in Ω' is contained in a cone formed by connecting v to a facet of β, for some $\beta \in \Omega$ containing v. It follows that if δ' is the cone in Ω' of maximum multiplicity then $mult(\delta') < mult(\delta)$. Since $mult(\delta)$ is finite, we can repeat this process a finite number of times until we obtain a subdivision Σ such that the maximal cones of Σ are all nonsingular. It then follows by Lemma 6.2.6 that Σ is nonsingular.

Proposition 6.1.6 implies that the induced toric morphism is a resolution of singularities. □

Recall Definition 3.7.4: a point v is irreducible in Δ if it is irreducible in a cone in Δ to which it belongs. From Lemma 3.5.3 we know that the generators of a cone are uniquely determined: hence we can make the following definition.

Definition 6.3.4. A point v is irreducible* in δ if v is irreducible in δ but not a generating vector of δ. We say v is irreducible* in Δ if v is irreducible* in every cone in Δ to which it belongs.

Lemma 6.3.5. *If $v \in \delta$ but v is not a generating vector of δ then v is not a generating vector of any cone in Δ. Thus v is irreducible* in Δ if and only if v is irreducible* in any cone in Δ to which it belongs.*

Proof. Regarding the first assertion, suppose for a contradiction that v is a generator of $\delta_2 \in \Delta$. Then $v \in \delta_2 \cap \delta$, and by Proposition 3.2.4 v is a generator of $\delta_2 \cap \delta$. Since $\delta_2 \cap \delta$ is a face of δ, we find by the same proposition that v is a generator of δ, which is a contradiction.

The second assertion follows from the first one and Lemma 3.7.5: v is irreducible in Δ if and only if v is irreducible in any cone in δ. □

Lemma 6.3.6. *A cone δ is nonsingular if and only if δ contains no irreducible* points.*

Proof. If δ contains no irreducible* points then Proposition 3.7.6 implies that the extraneous set K is empty. It follows from Corollary 6.2.5 that δ is nonsingular. The other implication is clear. □

Definition 6.3.4 helps us find a minimal bound on the size of the nonsingular refinement.

Lemma 6.3.7. *Suppose $[v_1, \ldots, v_r]$ is a nonsingular refinement of Δ. If v is irreducible* in Δ then $v = v_i$ for some $i < r$. Thus the size of a nonsingular refinement of Δ is greater than or equal to the number of irreducible* elements of Δ.*

Proof. Suppose we subdivide Δ through w to obtain a refinement Σ. Suppose $v \in \sigma$ for some cone $\sigma \in \Sigma$. Then $\sigma \subseteq \delta_2$ for some $\delta_2 \in \Delta$. From Lemma 3.7.5 we know v is irreducible in δ_2. v is obviously irreducible in any subset of δ_2; in particular v is irreducible in σ. Thus in any subdivision the irreducible points remain irreducible.

Proposition 6.2.5 implies that a nonsingular refinement cannot contain any irreducible points other than the generating vectors of cones. Hence v must be a generating vector of some cone in the refinement. This can only mean that a subdivision was made through v. □

6.4 The Toric Morphism Associated with a Refinement

Let Δ be a fan over a lattice N. Suppose Σ is a (not necessarily nonsingular) refinement of Δ. Let $\phi : N \to N$ be the identity homomorphism that maps $\Sigma \hookrightarrow \Delta$. From Lemma 4.3.9 we know that ϕ induces a toric morphism $\Xi : U_\Sigma \to U_\Delta$. We explore this toric morphism in more detail.

In Proposition 5.3.5 we found that

$$U_\Sigma = \coprod_{\sigma \in \Sigma} O_\sigma \qquad U_\Delta = \coprod_{\delta \in \Delta} O_\delta \qquad V(\tau) = \coprod_{\gamma \succ \tau} O_\gamma \tag{6.3}$$

We describe Ξ in terms of this orbit decomposition. The following is stated as an exercise in [4, p 56].

Proposition 6.4.1. *Suppose $\delta \in \Delta$ is the smallest cone in Δ containing $\phi(\sigma)$. Then $\Xi(V(\sigma)) \subseteq V(\delta)$. If $\phi(\sigma) = \delta$ (that is, if the refinement does not alter σ) then $\Xi : U_\sigma \to U_\delta$ is the identity.*

Proof. Suppose $O_\gamma \in V(\sigma)$. Then from (6.3) we see that $\sigma \prec \gamma$. Since ϕ is the identity map, $\phi(\sigma) \subseteq \phi(\gamma)$. Let κ be the smallest cone in Δ containing $\phi(\gamma)$. Then $\phi(\sigma) \subseteq \kappa$. Since δ is the smallest cone in Δ containing $\phi(\sigma)$, we find that $\delta \subseteq \kappa$. Since the intersection of two cones in a fan is another cone, we find $\delta \prec \kappa$

and therefore $O_\kappa \subseteq V(\delta)$. We know from Corollary 5.3.10 that $\Xi(O_\gamma) \subseteq O_\kappa$. Since $O_\kappa \subseteq V(\delta)$, this means $\Xi(O_\gamma) \subseteq V(\delta)$. Since γ is arbitrary, we find $\Xi(V(\sigma)) \subseteq V(\delta)$.

The second assertion follows from the fact that the dual of ϕ, $\phi^* : M \to M$ is the identity. Suppose $x \in U_\sigma$ is a semigroup homomorphism. By Corollary 4.2.3, the induced morphism maps $x \to x \circ \phi^* = x$. □

The following Lemma demonstrates that we can decompose general toric morphisms into the composition of toric morphisms induced by a single subdivision. Suppose Σ is a refinement of Δ, with induced toric morphism $\Xi_1 : U_\Sigma \to U_\Delta$. Similarly suppose Σ_2 is a refinement of Σ, with induced toric morphism $\Xi_2 : U_{\Sigma_2} \to U_\Sigma$. Observe that Σ_2 is also a refinement of Δ. We let $\Xi : U_{\Sigma_2} \to U_\Delta$ be the induced toric morphism.

Lemma 6.4.2. *The induced toric morphisms naturally compose:* $\Xi = \Xi_2 \circ \Xi_1$.

Proof. Let γ be a cone in Σ_2. Thanks to Corollary 5.3.10 we only need to prove $\Xi(y_\gamma) = \Xi_2(\Xi_1(y_\gamma))$. Now $\Xi_1(y_\gamma)$ is y_{γ_2} where γ_2 is the smallest cone in Σ containing γ. Hence $\Xi_2(\Xi_1(y_\gamma))$ is γ_3, where γ_3 is the smallest cone in Δ containing γ_2. But γ_3 must also be the smallest cone in Δ containing γ, so $\Xi(y_\gamma) = y_{\gamma_3}$. □

Now suppose Σ is the refinement obtained by subdividing Δ through a point w. This assumption allows us to write Ξ more explicitly. We know from Corollary 5.3.10 that Ξ preserves the torus action and maps orbits to orbits. Thus to understand Ξ we only need to understand its action on individual orbits. Let τ be the cone $\mathbb{R}_{\geq 0} w$, and let δ be the unique cone containing w in its relative interior (this exists by Corollary 3.4.3).

Proposition 6.4.3. Ξ *is an isomorphism everywhere except* $V(\tau)$, *but is not bijective anywhere on* $V(\tau)$. *In particular,*

i. $\Xi : U_\Sigma - V(\tau) \to U_\Delta - V(\delta)$ *is an isomorphism.*

ii. Suppose $\gamma \in Star(\tau)$ *and* κ *is a facet of* γ *not containing* τ. *If we denote the cone in the refinement generated by* κ *and* w *as* κ', *then* $\Xi(t.y_{\kappa'}) = t.y_\gamma$.

Proof. (i) Using the identities of (6.3) we observe $U_\Sigma - V(\tau) = \coprod_{\sigma \in \Sigma, \tau \not\prec \sigma} O_\sigma$. Now $\tau \not\prec \sigma$ means that σ is a cone in Δ. By Proposition 6.4.1 Ξ is an isomorphism on U_σ, and since $O_\sigma \subseteq U_\sigma$, Ξ is an isomorphism on O_σ.

(ii) From Corollary 5.3.10, $\Xi(y_{\kappa'}) = y_\gamma$. Hence $\Xi(t.y_{\kappa'}) = \Xi(t).\Xi(y_{\kappa'}) = t.y_{\kappa'}$. □

We can write Ξ even more explicitly if we make some further assumptions.

Corollary 6.4.4. *Suppose w is in the relative interior of a maximal cone $\delta \in \Delta$, where $dim(\delta) = dim(V)$. If $\tau = \mathbb{R}_{\geq 0} w$ then*

i. $\Xi^{-1}(y_\delta) = V(\tau)$.

ii. $\Xi : (U_\Sigma - V(\tau)) \to (U_\Delta - y_\delta)$ *is an isomorphism.*

Proof. Since the only face containing w is δ, $Star(\tau)$ is contained in δ. Since $\delta^\perp = \{0\}$, $O_\delta = y_\delta$. It follows from Proposition 6.4.3 that $\Xi(V(\tau)) = y_\delta$. $\square$

We denote $V(\tau)$ the *exceptional divisor*.

Chapter 7

Some Examples of the Resolution of Toric Singularities

We look at some applications of our theory. In the case of 2-dimensional cones the resolution of singularities is very elegant. As we will see, things are a lot more complicated in higher dimensions.

7.1 The Resolution of Singularities of Two-Dimensional Cones

We can say a lot about the resolution of singularities of toric varieties produced by 2-dimensional cones. Our treatment will initially parallel the treatments of Fulton [4, pp 45-50], Oda [11, §1.6,1.7], Reid [13] and Voight [17]. However we will finish by describing the toric morphisms induced by the subdivisions in terms of the torus action.

Let σ be an arbitrary 2-dimensional cone generated by $u = (u_1, u_2)$ and $v = (v_1, v_2)$, where $gcd(u_1, u_2) = 1$ and $gcd(v_1, v_2) = 1$. Suppose σ is singular. If $M = (u : v)$ is the generating matrix then it is possible to perform row operations on M until it is in the form

$$M' = \begin{pmatrix} 0 & m \\ 1 & -k \end{pmatrix} \qquad \text{where } m > k \geq 0 \tag{7.1}$$

The first column of M' is $\binom{0}{1}$ because $gcd(u_1, u_2) = 1$. We can also assume $gcd(m, k) = 1$ because $gcd(v_1, v_2) = 1$. Note that $m \neq 1$ because if $m = 1$ then σ would be nonsingular. We say that σ is in *canonical form* (note that the row operations correspond to a change of basis, so we haven't essentially changed σ).

We make a series of subdivisions. Our first subdivision is through $v_1 = (1, 0)$ (see Figure 7.1). This leaves us with two cones: κ_1 (generated by $\{e_1, e_2\}$) and σ_1 (generated by $\{e_1, me_1 - ke_2\}$).

Figure 7.1: The first subdivision.

Figure 7.2: The automorphisms which put σ_1 into canonical form.

κ_1 is clearly nonsingular. We apply a change of basis to σ_1 so that its generating matrix is in canonical form. We rotate σ_1 anticlockwise by $\frac{\pi}{2}$, to obtain the second cone in Figure 7.2, with generating matrix M_1'. Let $a_1 = \left\lceil \frac{m}{k} \right\rceil$, $k_1 = a_1 k - m$ and $m_1 = k$. We then perform the row operation $Row(2) = Row(2) - a_1 Row(1)$, to obtain the third cone in Figure 7.2 with generating matrix M_2''.

$$M_1 = \begin{pmatrix} 1 & m \\ 0 & -k \end{pmatrix} \to M_1' = \begin{pmatrix} 0 & k \\ 1 & m \end{pmatrix} \to M_2'' = \begin{pmatrix} 0 & m_1 \\ 1 & -k_1 \end{pmatrix}$$

Note that $\left\lceil \frac{m}{k} \right\rceil k$ is the smallest multiple of k greater than m, and therefore $k_1 = a_1 \left\lceil \frac{m}{k} \right\rceil k - m < k = m_1$. Thus M_2'' is in canonical form. Observe that $mult(\sigma_1) = m_1$.

This process can be repeated: we make a subdivision through the point $(1,0)$ (according to the new basis) in σ_1 to obtain a nonsingular cone κ_2 above the line and a cone σ_2 below the line. We then apply a change of basis A_2 to σ_2 so that σ_2 is in canonical form, and continue in this way. Generally, the matrix defining the change of basis from σ_i to σ_{i+1} is

$$A_i = \begin{pmatrix} 1 & 0 \\ -a_1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & a_i \end{pmatrix} \tag{7.2}$$

Because $m_{i+1} = k_i < m_i$ we eventually find an m_r such that $m_r = 1$. The corresponding cone σ_r must have multiplicity 1, and therefore is nonsingular. When this happens we denote the cone κ_{r+1} and terminate the process. Since each cone κ_i is nonsingular the set $\Delta := \{\gamma : \gamma \prec \kappa_i\}$ is a nonsingular refinement of σ.

We denote the initial vertex e_2 as v_0 and the final vertex $me_1 - ke_2$ as v_{r+1}.

The subdivision has very interesting combinatorial properties.

Lemma 7.1.1.

i. The set $\{v_0, \ldots, v_{r+1}\}$ is a minimal generating set for σ.

ii. The added vertices satisfy $a_i v_i = v_{i-1} + v_{i+1}$.

Proof. i. From Proposition 3.7.6 we need to prove that the points $\{v_i\}_{i=1}^{r+1}$ are exactly the irreducible members of σ. We do this inductively. Consider the original cone σ. It is clear that the vertex $v_1 = (1,0)$ is irreducible. Suppose $w \in \sigma_1$. We claim that w is irreducible in σ_1 if and only if it is irreducible in

σ. The "if" statement is obvious. For the "only if" implication, suppose w is irreducible in σ_1 but not irreducible in σ. Then there exist points $x = \binom{x_1}{x_2}$ and $y = \binom{y_1}{y_2}$ in σ such that $\binom{x_1}{x_2} + \binom{y_1}{y_2} = \binom{w_1}{w_2}$, where $x \notin \sigma_1$ (so $x_2 > 0$). Hence $\binom{x_1}{0} + \binom{y_1}{x_2+y_2} = \binom{w_1}{w_2}$. Since $\binom{x_1}{0} \in \sigma_1$ and w is irreducible in σ_1, we find $\binom{y_1}{x_2+y_2} \notin \sigma_1$. Therefore $w_2 = x_2 + y_2 > 0$, which contradicts $w \in \sigma_1$. Thus w is irreducible in σ_1 if and only if it is irreducible in σ. Thus the minimal generating set of σ is the union of the minimal generating set of σ_1 and $\{v_0\}$. We similarly find the minimal generating set of σ_1 is the union of the minimal generating set of σ_2 with $\{v_1\}$. Continuing inductively we find the minimal generating set of σ is $\{v_i\}_{i=1}^{r+1}$.

ii. Consider a cone σ_{i-1}. According to the basis of σ_{i-1}, the coordinates of v_{i-1} are $\binom{0}{1}$ and the coordinates of v_i are $\binom{1}{0}$. v_{i+1} has coordinates $\binom{1}{0}$ according to the basis of σ_i. If v'_{i+1} are the coordinates of v_{i+1} according to the basis of σ_{i-1}, then $v_{i+1} = A_i v'_{i+1}$. A straightforward calculation reveals $v'_{i+1} = \binom{-a_i}{1}$. Thus $a_i v_i = v_{i-1} + v_{i+1}$. Because the automorphisms preserve this equation, we find that $a_i v_i = v_{i-1} + v_{i+1}$ in the original coordinates of σ.

□

Remark 7.1.2. In Lemma 6.3.7 we saw that any nonsingular refinement must involve a subdivision through each irreducible* element. However Lemma 7.1.1 imforms us that the subdivision $[v_1, \ldots, v_r]$ is only through the irreducible* elements. Furthermore performing this subdivision in a different order, for example $[v_2, v_1, v_3, \ldots, v_r]$ clearly produces the same refinement. Thus $[v_1, \ldots, v_r]$ is the unique refinement of minimal size. In higher dimensions it is much harder to find a refinement of minimal size, let alone determine whether it is unique or not. There are two reasons for this.

i. In Lemma 7.1.1 we proved that performing a subdivision through the irreducible point $(1, 0)$ does not create any more irreducible points. It immediately follows from this that we only needed to subdivide through the irreducible* points. However we cannot make the same assumption in higher dimensions.

ii. In 2 dimensions changing the order of the refinement does not affect the refinement itself. However in higher dimensions this is normally not the case. As an example, suppose σ is the cone in three dimensions generated by $\{e_3, e_3 + 3e_1, e_3 + 3e_2\}$. Let $v_1 = e_1 + e_2 + e_3$ and $v_2 = 2e_1 + e_2 + e_3$. It can be observed that the refinement $[v_1, v_2]$ is different from the refinement

Figure 7.3: On the left is the Newton Polygon of the cone generated by e_2 and $5e_1 - 3e_2$, and on the right is its nonsingular refinement.

$[v_2, v_1]$. In the refinement $[v_1, v_2]$, v_1 is in the same cone as $e_3 + 3e_1$, but in the refinement $[v_2, v_1]$, v_1 is not in the same cone as $e_3 + 3e_1$.

□

Remark 7.1.3. The *Newton Polygon* of σ is the convex hull of the nonzero lattice points within σ. See Figure 7.3. In fact the vertices of the Newton Polygon of σ are precisely the points $\{v_i\}_{i=0}^{r+1}$. This makes sense because if v_{i-1} and v_i are consecutive points, then since $\{v_1, v_2\}$ is a basis for $\mathbb{Z}^2$, there can't be any lattice points in the triangle with vertices $\{0, v_1, v_2\}$. Refer to [17] or [12] for more details. □

Definition 7.1.4. The *Hirzebruch-Jung continued fraction* of $\frac{m}{k}$ is

$$\frac{m}{k} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\cdots - \cfrac{1}{b_q}}}$$

Where $b_i = \left\lceil \frac{m_{i-1}}{k_{i-1}} \right\rceil$, $k_i = b_i k_{i-1} - m_{i-1}$, $m_i = k_{i-1}$, $m_0 = m$ and $k_0 = k$. We represent this as $[b_1, \ldots, b_q]$.

It is clear from the above discussion that

Lemma 7.1.5. *The numbers $[a_1, \ldots, a_r]$ are the coefficients of the Hirzebruch-Jung continued fraction expansion of $\frac{m}{k}$.*

Remark 7.1.6. This allows us to state a straightforward algorithm to produce a nonsingular refinement of σ.

i. Write σ in canonical form.
ii. Let $[a_1, \ldots, a_r]$ be the coefficients of the Hirzebruch-Jung continued fraction expansion of $\frac{m}{k}$.
iii. Let $v_0 = \binom{0}{1}$ and $v_1 = \binom{1}{0}$. We then use the recurrence relation $v_{i+1} = a_i v_i - v_{i-1}$ of Lemma 7.1.1 to calculate the rest of the refinement.

□

Finally we explore the toric morphisms induced by this subdivision. Let τ_i be the cone $\mathbb{R}_{\geq 0} v_i$. Thus we can write the refinement Δ as $\{\kappa_i\}_{i=1}^{r+1} \cup \{\tau_i\}_{i=0}^{r+1} \cup$ {The zero cone $\{0\}$}. By Proposition 5.3.5 we can decompose Δ as the disjoint union of T_N, $\coprod_{i=0}^{r+1} O_{\tau_i}$, and $\coprod_{i=1}^{r+1} y_{\kappa_i}$. These are 2-dimensional, 1-dimensional and 0-dimensional (i.e. a point) torii respectively. We denote the intermediate refinement $[v_1, \ldots, v_k]$ as $\Delta_k = \{\kappa_i\}_{i=1}^{k} \cup \{\tau_i\}_{i=0}^{k} \cup \{\sigma_k, \tau_{r+1}\}$.

Let $\Xi_k : U_{\Delta_k} \to U_{\Delta_{k-1}}$ be the toric morphism induced by the refinement $\Delta_{k-1} \to \Delta_k$. Since this refinement corresponds to subdividing through v_k, we find by Corollary 6.4.4 that the exceptional divisor is $V(\tau_k)$. The quotient of κ_i is $\overline{\kappa_i} = \mathbb{R}_{\geq 0} e_2$ and the quotient of σ_i is $\overline{\sigma_i} = \mathbb{R}_{\geq 0}(-e_2)$. The quotient of τ_k is obviously $\overline{\tau_k} = \{0\}$. We saw in Example 4.3.7 what the toric variety $V(\tau_k)$ of this fan is: it is the projective line $\mathbb{P}^1$. The orbits are $O_{\tau_k} \simeq \mathbb{C}^*$ and the points y_{κ_k} and y_{σ_k}. After noting that $\tau_k \in \sigma_{k-1}$, we can describe Ξ_k:

$$\Xi_k : \begin{cases} V(\tau_k) \to y_{\sigma_{k-1}} \\ \text{identity otherwise} \end{cases} \tag{7.3}$$

Remark 7.1.7. Ξ_k^{-1} is an instance of a type of map known as a *blow-up*. The blow-up is the main tool in the resolution of singularities of an algebraic variety. A full account of the blow-up requires an understanding of projective varieties (the analogue of affine varieties in projective space) so we only offer a brief sketch. The blowup ϕ of $\mathbb{C}^n$ at a point P is a birational map, $\phi : X \to \mathbb{C}^n$, where X is a particular closed subset of $\mathbb{C}^n \times \mathbb{P}^{n-1}$. (The definitions of birational maps and closed subsets of projective space are analogous to the definitions in affine space). The blow-up is such that $\phi : X - \phi^{-1}(P) \to \mathbb{C} - P$ is an isomorphism, and $\phi^{-1}(P) \simeq \mathbb{P}^{n-1}$. The blowup essentially replaces P with the set of all lines through P. Refer to [8, pp 28-9]. □

7.2 The Resolution of Singularities of Higher Dimensional Cones

We make some comments on the resolution of singularities of higher dimensional cones. Let Δ be a singular simplicial fan in $V = \mathbb{R}^n$. From Theorem 6.3.3 we know that a nonsingular refinement Σ of Δ exists. We wish to produce a refinement involving as few subdivisions as possible. Given that a refinement must involve a subdivision through every irreducible* point, it seems preferable that we subdivide through irreducible* points first. We can't prove this is a minimal refinement (refer to the discussion in Remark 7.1.2). However Remark 7.2.1 below gives us reason to hope that subdividing through an irreducible* point is more likely to lead to a refinement of minimal size than subdividing through a point that isn't irreducible*.

Remark 7.2.1. Suppose we wish to make a subdivision through a point w that is not irreducible*. Let $\delta \in \Delta$ be a cone containing w, with generators $\{v_1, \ldots, v_s\}$. As

Δ is simplicial, w can be uniquely written as $\sum_{i=1}^{s} t_i v_i$. We denote the subdivision through w, Σ_w.

From the proof of Theorem 6.3.3 we know that the largest new cones in δ are formed by joining w to a facet of δ. We denote such a cone, a facetcone. There is a facetcone for every $t_i \neq 0$; we denote it $\sigma_{w,i}$.

Let u be an irreducible* point such that $u = \sum_{i=1}^{s} u_i v_i \leq w$ (such an irreducible* point must exist by Proposition 3.7.6). Again, the largest new cones resulting from the subdivision of δ through u are the join of u to a facet of δ. There is one such facetcone for every $u_i \neq 0$; we denote such a cone $\sigma_{u,i}$. The multiplicity of $\sigma_{u,i}$ is $u_i mult(\delta)$. Now $u \leq w \Rightarrow u_i \leq t_i$. This means that $u_i \neq 0 \Rightarrow t_i \neq 0$, and therefore there must be at least as many new facetcones resulting from the subdivision through w compared to the subdivision through u. Furthermore even if $u_i > 0$, $mult(\sigma_{w,i}) = t_i mult(\delta) \geq mult(\sigma_{u,i}) = u_i mult(\delta)$. That is, the multiplicities of the $\sigma_{w,i}$ facetcones are greater than or equal to the multiplicities of their counterparts.

Thus it seems that subdividing through an irreducible* point is more likely to lead to nonsingularity, although it needs to be stressed that we haven't proved this. $\square$

Proposition 7.2.2. *We can find a refinement* $[w_1, \ldots, w_p]$ *of* Δ *such that if* Δ_k *is the intermediate refinement* $[w_1, \ldots, w_k]$ *then* w_{k+1} *is irreducible* in* Δ_k.

Proof. If there are no irreducible* elements in Δ then Δ must be nonsingular. Otherwise we can find an irreducible* point to subdivide through. $\square$

7.2.1 The Toric Morphism of a Higher Dimensional Fan

We finish with some comments on the toric morphism. The toric morphism induced by a higher dimensional refinement cannot be described as elegantly as in the 2-dimensional case.

Let $\Xi : U_\Sigma \to U_\Delta$ be the toric morphism induced by the refinement.

Generally, we cannot say much beyond Proposition 6.4.3. Let p be the point through which the subdivision is made, and $\tau = \mathbb{R}_{\geq 0} v$. Let δ be the cone containing p in its relative interior. We find by Proposition 6.4.3 that

i. $\Xi : U_\Sigma - V(\tau) \to U_\Delta - V(\delta)$ is an isomorphism.

ii. Suppose $\gamma \in Star(\tau)$ and κ is a facet of γ not containing τ. If we denote the cone in the refinement generated by κ and w as κ', then $\Xi(t.y_{\kappa'}) = t.y_\gamma$.

We make some comments as to why things are much more complicated in higher dimensions.

Remark 7.2.3. In the 2-dimensional case the point through which the subdivision is made is always in the relative interior. However we cannot assume this in higher dimensions. This means the cones added by the subdivision won't all lie in the one face. Furthermore if δ is the unique cone containing the point through which the subdivision was made in its relative interior, then δ is not necessarily maximal (unlike the 2-dimensional case). Hence O_δ is not necessarily a point. $\square$

Chapter 8

Concluding Remarks

The resolution of the singularities of toric varieties is a remarkably straightforward process compared to other varieties. We have reduced the problem of resolving the singularities of toric varieties to the problem of finding a nonsingular refinement of a cone.

Not only this, our study of the torus action revealed that the combinatorics of fans sheds a lot of insight onto the structure of toric varieties. We were able to avoid a lot of more sophisticated algebraic geometry, such as projective varieties and the blowup map, thanks to the fact that the induced toric morphism can be understood by studying its effect on the orbits of the torus action.

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