THE WEYL ALGEBRAS

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Chapter 1

Introduction

An important result in single-variable calculus is the so-called product rule. That is, for two polynomials (or more generally, functions) $f(x), g(x) : \mathbb{R} \to \mathbb{R}$:

$$\frac{\delta}{\delta x}(fg) = (\frac{\delta}{\delta x}f)g + f(\frac{\delta}{\delta x}g)$$

It turns out that this formula, which is firmly rooted in calculus has very interesting algebraic properties. If $k[x]$ denotes the ring of polynomials in one variable over a characteristic 0 field k, differentiation (in the variable x) can be considered as a map $\delta : k[x] \to k[x]$. It is relatively straighforward to verify that the map δ is in fact a k-linear vector space endomorphism of $k[x]$. Similarly, we can define another k-linear endomorphism X by left multiplication by x ie. $X(f) = xf$. Consider the expression $(\delta \cdot X)f(x)$. Expanding this gives:

$$(\delta \cdot X)f(x) = \delta(xf(x))$$

applying the product rule gives

$$\begin{aligned}(\delta \cdot X)f(x) &= \delta(x)f(x) + x\delta f(x) \\ &= f(x) + (X \cdot \delta)f(x)\end{aligned}$$

noting the common factor of $f(x)$ gives us the relation (this time in the ring of k-linear endomorphisms of $k[x]$):

$$\delta \cdot X = X \cdot \delta + 1$$

where 1 is the identity map. This is the defining relation of the *first Weyl algebra* which can be viewed as the ring of differential operators on $k[x]$ with polynomial coefficients. There also exist higher order Weyl algebras related to the polynomial ring in n variables.

The Weyl algebras arise in a number of contexts, notably as a quotient of the universal enveloping algebra of certain finite-dimensional Lie algebras (arising from the *Heisenberg group*) which have links to quantum mechanics.

The second chapter of this paper covers some basic results on the Weyl algebras, culminating in the proof that they are simple domains. The third chapter covers gradings, filtrations and the concept of an associated graded algebra. The fourth chapter introduces the concept of the Gelfand-Kirillov dimension which is a useful invariant of finitely-generated associative algebras. The final chapter is an exposition of a proof published in [1] that characterises the automorphisms of the first Weyl algebra.

Chapter 2

Basic Results

In the following, k will always be a field of characteristic 0 and all ideals are two-sided unless specifically stated otherwise.

Definition 2.1. Let D be a (not neccessarily commutative) domain. Define $A(D)$ as the non-commutative algebra over D on the two generators p, q with defining relation

$$qp - pq = 1 \tag{2.1}$$

ie.

$$A(D) = \frac{D < p, q >}{(qp - pq - 1)}$$

For a field k of characteristic 0, define the first Weyl algebra over k, denoted by A_1 to be $A(k)$. Define the n$^{\text{th}}$ Weyl algebra for $n > 1$ by $A_n = A(A_{n-1})$ (note that this definition assumes that A_{n-1} is a domain, this is proved later). For convenience assume $A_0 = k$. Note that for $n > 1$ there are extra (implicit) relations: $q_i p_j - p_j q_i = 0$ for $i \neq j$ ie. the generators of different index commute.

Definition 2.2. Define linear maps $X, \delta : k[x] \to k[x]$ by $X(f) = xf$ and $\delta(f) = \frac{\delta f}{\delta x}$ ie. formal differentiation. X and δ generate a sub-algebra of the ring of k-linear endomorphisms of $k[x]$. Applying Leibniz' rule for the differentiation of a product gives $\delta \cdot X = X \cdot \delta + 1$. Call this algebra A_1'.

For $n > 1$ and $1 \leq i \leq n$ define linear maps $X_i, \delta_i : k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ by $X_i(f) = x_i f$ and $\delta_i(f) = \frac{\delta f}{\delta x_i}$ ie. formal partial differentiation with respect to x_i.

Once again, differenting the product yields the relations $\delta_i X_j = X_j \delta_i + 1$ if $i = j$ or $\delta_i X_j = X_j \delta_i$ if $i \neq j$. Call this algebra A'_n. Expressed as a quotient:

$$A'_n = \frac{k < X_1, \ldots, X_n, \delta_1, \ldots, \delta_n >}{(\delta_i X_j - X_i \delta_j - \Delta_{ij})}$$

where

$$\Delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$$

Lemma 2.3. *For any domain D, every $x \in A(D)$ can be expressed as $\sum a_{ij} p^i q^j$ for some finite set $\{(i,j) \in \mathbb{N} \times \mathbb{N}\}$ and $a_{ij} \in D$.*

Proof. Since p, q generate A_n over D, every $x \in A_n$ can be expressed as some finite sum

$$\sum_i b_i p^{r(i,1)} q^{s(i,1)} \ldots p^{r(i,n_i)} q^{s(i,n_i)}$$

where $b_i \in D, n_i \in \mathbb{Z}^+$ and the leading or trailing coefficent ($r_{(i,1)}$ and $s_{(i,n_i)}$ respectively) may be 0. Note that p and q both commute with elements of the base domain D.

For a monomial product term M, define $\#_p(M)$ to be the number of p terms appearing in M. Define $\#_q(M)$ similarly. Let $I(M)$ be the number of 'inversions' in the term M. That is, the sum over every q term in M of the number of p terms which occur to the right. For example:

$$I(p^m q^n) = I(\lambda \in k) = 0$$
$$I(qp) = 1$$
$$I(q^2 p) = I(qp^2) = 2$$
$$I(q^2 p^2) = I(qp^4) = 4$$

Define $I\left(\sum_i M_i\right)$ to be $\max_i \left(I(M_i)\right)$.

Let $R = \sum_i M_i$ be a represention of x in the form described above. If $I(R) > 0$, then for at least one monomial term M_i we must have $I(M_i) > 0$. Thus the monomial M_i must contain at least one factor of the form qp ie. $M_i = AqpB$ where A may be in k and B may be 1. Pick one such term and apply the identity $qp = pq + 1$ to give:

$$M_i' = b_i ApqB + b_i AB$$

calculating gives:

$$\begin{aligned} I(M_i') &= \max(I(ApqB), I(AB)) \\ &= \max(I(m_i) - 1, I(m_i) - (\#_q(A) + \#_p(B)) - 1) \end{aligned}$$

clearly therefore, $I(M_i') = I(M_i) - 1$.

Inductively therefore, the sequence of manipulations $M_i \to M_i'$ must terminate in some M_i^* with $I(M_i^*) = 0$. Applying this to each term of R gives a representation in the required form. □

Corollary 2.3.1. *Any $x \in A_n$ can be expressed as*

$$\sum a_{i_1 \ldots i_n j_1 \ldots j_n} p_1^{i_1} \ldots p_n^{i_n} q_1^{j_1} \ldots q_n^{j_n}$$

Proof. Since k is a domain, the result is true for $n = 1$. A_n is defined recursively as $A(A_{n-1})$. Assuming that A_{n-1} is a domain (again, this is proved shortly) and that the result holds for $n-1$, the result follows by induction on n since the generators of different index commute. □

Lemma 2.4. *Every $x \in A_n'$ can be expressed as $\sum a X_1^{i_1} \ldots X_n^{i_n} \delta_1^{i_1} \ldots \delta_n^{i_n}$ for some finite set $\{(i,j) \in \mathbb{N} \times \mathbb{N}\}$.*

Proof. By writing A_n' recursively as $A_{n-1}' < X_n, \delta_n >$ and using the relation $\delta_n X_n = X_n \delta_n + 1$, the result follows as for (2.3.1). □

Lemma 2.5. *The k-linear map $\phi : A_n \to A_n'$ defined by $\phi(p_i) = X_i$ and $\phi(q_i) = \delta_i$ is an algebra homomorphism.*

Proof. By the universal property, it suffices to check the images of the defining relations $q_ip_i - p_iq_i - 1$ and $q_ip_j - p_jq_i$ for $i \neq j$:

$$\phi(q_ip_i - p_iq_i - 1) = \delta_iX_i - X_i\delta_i - 1 = 0$$
$$\phi(q_ip_j - p_jq_i) = \delta_iX_j - X_j\delta_i = 0$$

□

Lemma 2.6. *For any element of A'_n, the representation given in lemma 2.4 is unique.*

Proof. Suppose that an element $x \in A'_n$ has two distinct representations

$$\sum a_{i_1\ldots i_nj_1\ldots j_n}X_1^{i_1}\ldots X_n^{i_n}\delta_1^{j_1}\ldots\delta_n^{j_n}$$
$$=\sum b_{i_1\ldots i_nj_1\ldots j_n}X_1^{i_1}\ldots X_n^{i_n}\delta_1^{j_1}\ldots\delta_n^{j_n}$$

Cancel all equal terms in the above sums to give two differential operators (A, B) with coefficients $a_{i_1\ldots i_nj_1\ldots j_n}$ and $b_{i_1\ldots i_nj_1\ldots j_n}$ such that $a_{i_1\ldots i_nj_1\ldots j_n} \neq b_{i_1\ldots i_nj_1\ldots j_n}$ for all $i_1\ldots i_nj_1\ldots j_n$. For each $1 \leq k \leq n$, pick j_k^* to be minimal with respect to the property that $a_{i_1\ldots i_nj_1{}^*\ldots j_k{}^*j_{k+1}\ldots j_n}$ and $b_{i_1\ldots i_nj_1{}^*\ldots j_k{}^*j_{k+1}\ldots j_n}$ appear as coefficients of A and B respectively, for some $i_1\ldots i_n,\ j_{k+1}\ldots j_n$. Let $p = x_1^{j_1^*}\ldots x_n^{j_n^*} \in k[x_1,\ldots,x_n]$. Apply the operators A and B to p.

$$Ap = \sum a_{i_1\ldots i_nj_1\ldots j_n}X_1^{i_1}\ldots X_n^{i_n}\delta_1^{j_1}\ldots\delta_n^{j_n}x_1^{j_1^*}\ldots x_n^{j_n^*}$$
$$Bp = \sum b_{i_1\ldots i_nj_1\ldots j_n}X_1^{i_1}\ldots X_n^{i_n}\delta_1^{j_1}\ldots\delta_n^{j_n}x_1^{j_1^*}\ldots x_n^{j_n^*}$$

Consider a single term of the above sums:

$$t_a = a_{i_1\ldots i_nj_1\ldots j_n}X_1^{i_1}\ldots X_n^{i_n}\delta_1^{j_1}\ldots\delta_n^{j_n}x_1^{j_1^*}\ldots x_n^{j_n^*}$$
$$t_b = b_{i_1\ldots i_nj_1\ldots j_n}X_1^{i_1}\ldots X_n^{i_n}\delta_1^{j_1}\ldots\delta_n^{j_n}x_1^{j_1^*}\ldots x_n^{j_n^*}$$

If for all $0 \le k \le n$, $j_k = j_k{}^*$, then

$$t_a = a_{i_1 \ldots i_n j_1 \ldots j_n} (j_1{}^*! \ldots j_n{}^*!)(x_1{}^{i_1} \ldots x_n{}^{i_n})$$
$$t_b = b_{i_1 \ldots i_n j_1 \ldots j_n} (j_1{}^*! \ldots j_n{}^*!)(x_1{}^{i_1} \ldots x_n{}^{i_n})$$

Suppose that the j_k and $j_k{}^*$ differ for some set of indices. Let l be the smallest such index. By the choice of the $j_k{}^*$, we must have $j_l > j_l{}^*$. Since t contains a factor $\delta_l{}^{j_l} x_l{}^{j_l{}^*}$, $t = 0$. Therefore,

$$Ap = \sum a_{i_1 \ldots i_n j_1{}^* \ldots j_n{}^*} (j_1{}^*! \ldots j_n{}^*!)(x_1{}^{i_1} \ldots x_n{}^{i_n})$$
$$Bp = \sum b_{i_1 \ldots i_n j_1{}^* \ldots j_n{}^*} (j_1{}^*! \ldots j_n{}^*!)(x_1{}^{i_1} \ldots x_n{}^{i_n})$$

Since the above are simply polynomials in $x_1, \ldots, x_n$ and $Ap = Bp$, we can equate coefficients which implies that

$$a_{i_1 \ldots i_n j_1{}^* \ldots j_n{}^*} = b_{i_1 \ldots i_n j_1{}^* \ldots j_n{}^*}$$

which is a contradiction.

□

Corollary 2.6.1. *For any element of A_n, the representation given in lemma 2.3 is unique.*

Proof. This follows from considering the homomorphism ϕ defined above. If $x \in A_n$ has distinct representations

$$\sum a_{i_1 \ldots i_n j_1 \ldots j_n} p_1{}^{i_1} \ldots p_n{}^{i_n} q_1{}^{i_1} \ldots q_n{}^{i_n}$$

and

$$\sum b_{i_1 \ldots i_n j_1 \ldots j_n} p_1{}^{i_1} \ldots p_n{}^{i_n} q_1{}^{i_1} \ldots q_n{}^{i_n}$$

then

$$\phi\left(\sum a_{i_1\ldots i_n j_1\ldots j_n} {p_1}^{i_1}\ldots {p_n}^{i_n} {q_1}^{i_1}\ldots {q_n}^{i_n}\right)$$
$$= \sum a_{i_1\ldots i_n j_1\ldots j_n} {X_1}^{i_1}\ldots {X_n}^{i_n} {\delta_1}^{i_1}\ldots {\delta_n}^{i_n}$$

and

$$\phi\left(\sum b_{i_1\ldots i_n j_1\ldots j_n} {p_1}^{i_1}\ldots {p_n}^{i_n} {q_1}^{i_1}\ldots {q_n}^{i_n}\right)$$
$$= \sum b_{i_1\ldots i_n j_1\ldots j_n} {X_1}^{i_1}\ldots {X_n}^{i_n} {\delta_1}^{i_1}\ldots {\delta_n}^{i_n}$$

are distinct representations of $\phi(x) \in A'_n$, a contradiction. □

Lemma 2.7. $A_n \simeq A'_n$

Proof. Take the homomorphism ϕ as above. By Lemma 2.4, any $x' \in A'_n$ can be expressed in the form

$$\sum a_{i_1\ldots i_n j_1\ldots j_n} {X_1}^{i_1}\ldots {X_n}^{i_n} {\delta_1}^{j_1}\ldots {\delta_n}^{j_n}$$

Let

$$x = \sum a_{i_1\ldots i_n j_1\ldots j_n} {p_1}^{i_1}\ldots {p_n}^{i_n} {q_1}^{j_1}\ldots {q_n}^{j_n}$$

Clearly, $\phi(x) = x'$. Therefore ϕ is surjective.

Take $y \in A_n$ and suppose $\phi(y) = 0$. By lemma 2.3, write y as

$$\sum a_{i_1\ldots i_n j_1\ldots j_n} {p_1}^{i_1}\ldots {p_n}^{i_n} {q_1}^{j_1}\ldots {q_n}^{j_n}$$

The image, $\phi(y)$, is therefore

$$\sum a_{i_1\ldots i_n j_1\ldots j_n} {X_1}^{i_1}\ldots {X_n}^{i_n} {\delta_1}^{j_1}\ldots {\delta_n}^{j_n}$$

Since the representation of $\phi(y)$ is unique, we can equate coefficients which implies that all of the $a_{i_1\ldots i_n j_1\ldots j_n}$ are zero. Thus ϕ is injective. □

Lemma 2.8. *For $i, j \in \mathbb{Z}^+$ and p, q the generators for $A(D)$:*

$$qp^iq^j = p^iq^{j+1} + ip^{i-1}q^j$$

Proof. Reduce the expression step by step as in Lemma (2.3):

$$\begin{aligned} qp^iq^j &= (qp)p^{i-1}q^j \\ &= pqp^{i-1}q^j + p^{i-1}q^j \\ &= p(qp)p^{i-2}q^j + p^{i-1}q^j \\ &= p^2qp^{i-2}q^2 + 2p^{i-1}q^j \\ &= \dots \\ &= p^iq^{j+1} + ip^{i-1}q^j \end{aligned}$$

At each step, the left-hand q is moved past one of the p factors to the right, adding the term $p^{i-1}q^j$. Recursively applying this step i times gives the required result. □

Lemma 2.9. *For $i, j, l, m \in \mathbb{Z}^+$ and p, q as above:*

$$(p^iq^j)(p^lq^m) = \sum_{r=0}^{j} r! \begin{pmatrix} j \\ r \end{pmatrix} \begin{pmatrix} l \\ r \end{pmatrix} p^{i+l-r}q^{j+m-r} \tag{2.2}$$

Proof. Consider the product $q^jp^lq^m$. By the above lemma,

$$\begin{aligned} q^jp^lq^m &= q^{j-1}(p^lq^{m+1} + lp^{l-1}q^m) \\ &= q^{j-1}p^lq^{m+1} + lq^{j-1}p^{l-1}q^m \end{aligned}$$

Let

$$\begin{aligned} \alpha(q^jp^lq^m) &= q^{j-1}p^lq^{m+1} \qquad \text{and} \\ \beta(q^jp^lq^m) &= lq^{j-1}p^{l-1}q^m \end{aligned}$$

and note that both α and β both reduce the degree of q^j by one, and that if $l = 0$ then $\beta(q^j p^l q^m) = 0$. The above relation then becomes

$$q^j p^l q^m = \alpha(q^j p^l q^m) + \beta(q^j p^l q^m)$$

inductively, this must terminate after j steps giving 2^j (not neccessarily distinct) terms of a form similar to $\underbrace{\alpha\beta\ldots\alpha}_{j}(q^j p^l q^m)$.

Note that $\alpha\beta(q^j p^l q^m) = \beta\alpha(q^j p^l q^m)$. By commuting these operations, we can write the original expression in the form

$$\begin{aligned} q^j p^l q^m &= \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} \alpha^{j-r}\beta^{r}(q^j p^l q^m) \\ &= \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} \frac{l!}{(l-r)!} p^{l-r} q^{m+(j-r)} \\ &= \sum_{r=0}^{j} r! \begin{pmatrix} j \\ r \end{pmatrix} \begin{pmatrix} l \\ r \end{pmatrix} p^{l-r} q^{j+m-r} \end{aligned}$$

and thus,

$$\begin{aligned} (p^i q^j)(p^l q^m) &= p^i (q^j p^l q^m) \\ &= p^i \left(\sum_{r=0}^{j} r! \begin{pmatrix} j \\ r \end{pmatrix} \begin{pmatrix} l \\ r \end{pmatrix} p^{l-r} q^{j+m-r} \right) \\ &= \sum_{r=0}^{j} r! \begin{pmatrix} j \\ r \end{pmatrix} \begin{pmatrix} l \\ r \end{pmatrix} p^{i+l-r} q^{j+m-r} \end{aligned}$$

□

Definition 2.10. Take $x \in A(D)$ such that $x = \sum a_{i,j} p^i q^j$. Define the degree of x, denoted $\deg(x)$ to be

$$\max_{i,j | a_{i,j} \neq 0} (i + j)$$

Define $\deg(0) = -\infty$. Define the leading terms of x to be the terms $a_{i,j}p^iq^j$ such that $i + j = \deg(x)$ ie. the terms of maximal degree.

Lemma 2.11. *Take* $x, y \in A(D)$*. Then* $\deg(xy) = \deg(x) + \deg(y)$.

Proof. Let $x = \sum_{i,j} a_{i,j}p^iq^j$ and let $y = \sum_{l,m} b_{l,m}p^lq^m$. Expanding the product xy gives:

$$xy = \sum_{i,j}\sum_{l,m} a_{i,j}b_{l,m}p^iq^jp^lq^m$$

Applying lemma (2.9) gives:

$$xy = \sum_{i,j}\sum_{l,m} a_{i,j}b_{l,m}\left(\sum_{r=0}^{j} r!\begin{pmatrix} j \\ r \end{pmatrix}\begin{pmatrix} l \\ r \end{pmatrix} p^{i+l-r}q^{j+m-r}\right)$$

The leading terms of this sum are those of the form:

$$a_{i,j}b_{l,m}p^{i+l}q^{j+m} \qquad\qquad i + j = \deg(x), l + m = \deg(y)$$

$\deg(xy)$ is therefore $i + j + l + m = \deg(x) + \deg(y)$. □

Corollary 2.11.1. A_n *is a domain.*

Proof. If $x, y \in A(D)$ are non-zero, then $\deg(x), \deg(y) \geq 0$. Thus $\deg(xy) = \deg(x) + \deg(y) \geq 0$. Therefore $xy \neq 0$ and hence $A(D)$ is a domain.

Applying the above inductively with k as the base gives the result. □

Definition 2.12. Let R be a ring. A map $\delta : R \to R$ is called a *derivation* if it satisfies $\delta(ab) = \delta(a)b + a\delta(b)$.

Example 2.13. For a ring R, define ad c by $[c, -]$. That is, $(\text{ad } c)(x) = cx - xc$. Observe that

$$\begin{aligned}(\text{ad } c)(x)y + x(\text{ad } c)(y) &= (cx - xc)y + x(cy - yc)\\ &= cxy - xcy + xcy - xyc\\ &= cxy - xyc\\ &= (\text{ad } c)(xy)\end{aligned}$$

Thus, ad c is derivation on R.

Definition 2.14. A derivation δ is called an *inner* derivation if there exists a $c \in R$ such that for all $a \in R$, $\delta(a) = ca - ac$. Note that if a is in the centre of R, then $\delta(a) = 0$.

An ideal $\mathfrak{I} \subset R$ is called a δ-ideal if $\delta(\mathfrak{I}) \subseteq \mathfrak{I}$. The ring R is called δ-simple if the only δ-ideals of R are (0) and R.

Definition 2.15. For a ring R and a derivation δ on R, define $R[x; \delta]$ to be the set of left polynomials $\sum a_i x^i$ with the relation $ax = xa + \delta(a)$ for $a \in R$ extending (by repeated application) to give a multiplication on $R[x; \delta]$. The proof of the fact that this is an associative multiplication which gives a ring structure on $R[x; \delta]$ is referenced in [3].

Lemma 2.16. *(Paraphrased from [3]) For a $\mathbb{Q}$-algebra R, if R is δ-simple for a non-inner derivation δ, then $R[x; \delta]$ is simple.*

Proof. To prove this, we prove a (logically equivalent) partial converse. That is, assume that R is δ-simple, but that $R[x; \delta]$ is not simple ie. with some ideal $I \neq (0), R$ and show that δ must be inner.

Let n be the minimum degree among non-zero (left) polynomials in I, and let $\mathfrak{U}$ be the set of leading coefficients a of polynomials $f \in I$ of degree n, together with 0. Observing that

$$xf - fx = \delta(a)x^n + \dots$$

shows that $\delta(\mathfrak{U}) \subseteq \mathfrak{U}$. Combined with the fact that I is an ideal (and hence $ab, a+b \in \mathfrak{U}$ for any $a, b \in \mathfrak{U}$) this shows that $\mathfrak{U}$ is a non-zero δ-ideal of R. Thus since R is δ-simple, $1 \in \mathfrak{U}$ ie. there exists an element $g = x^n + dx^{n-1} + \ldots$ in I. For any $b \in R$, it follows inductively that $x^n b = bx^n + n\delta(b)x^{n-1} + \ldots$ thus

$$bg - gb = (bd - db - n\delta(b))x^{n-1} + (\text{terms of lower degree})$$

Since $bg - gb \in I$ (and $\mathbb{Q} \subseteq R$),

$$\delta(b) = b\left(\frac{d}{n}\right) - \left(\frac{d}{n}\right)b$$

for every $b \in R$. Thus δ is an inner derivation. □

Lemma 2.17. *For δ defined as $\frac{\delta}{\delta x}$, $A_n = A(A_{n-1}) \simeq A_{n-1}[x][y;\delta]$ where x, y commute with all elements of A_{n-1}.*

Proof. Since every element of A_{n-1} commutes with x, the defining multiplicative relation on $A_{n-1}[x][y;\delta]$ is simply:

$$yx = xy + \delta(y)$$
$$yx = xy + 1$$

Define a k-linear map $\phi : A_n \to A_{n-1}[x][y;\delta]$ by $\psi(p) = x$ and $\psi(q) = y$. The image $\psi(qp - pq - 1)$ is simply $yx - xy - 1 = 0$. Thus ψ is an homomorphism.

Elements of $A_{n-1}[x][y;\delta]$ are (by definition) of the form:

$$\sum p_j(x)y^j$$

where $p_j \in k[x]$. Splitting each term of the sum into monomial components gives an expression of the form:

$$\sum a_{ij}x^iy^j$$

Whence it is clear that

$$\psi\left(\sum a_{ij}p^iq^j\right) = \sum a_{ij}x^iy^j$$

and thus ψ is surjective. Since (as A_{n-1} is a domain) the expressions of the form

$$\sum p_j(x)y^j$$

are unique, $\psi(\alpha) = 0$ implies $\alpha = 0$, whence ψ is injective. □

Lemma 2.18. *Define δ as in lemma 2.17 ie. $\delta = \frac{\delta}{\delta x}$. For a simple ring R, $R[x]$ is δ-simple, and δ is a non-inner derivation on $R[x]$.*

Proof. The map δ is defined as formal differentiation on $R[x]$. Leibniz's rule for the differentiation of a product says that $\delta(ab) = \delta(a)b + a\delta(b)$ and therefore δ is a derivation. The element x is (by definition) central in $R[x]$, but $\delta(x) = 1$ and therefore δ is not inner.

Take any δ-ideal $\mathfrak{I} \lhd R[x]$ (other than (0)). Take $i \in \mathfrak{I}$ and let $d = \deg(i)$. Thus $i = a_dx^d + \ldots + a_0$, with $a_d \neq 0$ and hence $\delta(i) = da_dx^{d-1} + \ldots + a_1$. Since $\mathfrak{I}$ is a δ-ideal, $\delta(i) \in \mathfrak{I}$ and $\deg(\delta(i)) = d - 1$. Hence by reduction, $\mathfrak{I}$ contains an element $i_0 = d!a_d \neq 0$ with $\deg(i_0) = 0$ ie. $i_0 \in R$. Thus, since R is simple, $1 \in \mathfrak{I}$ ie. $\mathfrak{I} = R[x]$. Therefore, $R[x]$ is δ-simple □

Theorem 2.19. *A_n is simple.*

Proof. Define X and δ as in definition 2.2. Lemma 2.18 implies that $A_0[x] = k[x]$ is δ-simple and δ is a non-inner derivation on $A_0[x]$. Applying lemma (2.16) shows that $A_1 \simeq A_0[x][y;\delta]$ is simple.

Assume that A_{n-1} is simple for $n \geq 2$. Then by 2.18, $A_{n-1}[x]$ is δ-simple and δ is an inner derivation on $A_{n-1}[x]$. Thus $A_n \simeq A_{n-1}[x][y;\delta]$ is simple by 2.16. □

Remark. As the kernel of a homomorphism is a two-sided ideal, this implies that all endomorphisms of A_n are either injective, or zero. Whether all endomorphisms are also surjective is an important open question, which if proved, would imply an important result in multi-variable calculus known as the *Jacobian conjecture*.

Chapter 3

Gradings and Filtrations

Definition 3.1. A k-algebra A is $\mathbb{N}$-*graded* or simply *graded*, if there exists a set of subspaces $\mathrm{gr}(A, i)$ such that:

$$\mathrm{gr}(A, i).\mathrm{gr}(A, j) \subseteq \mathrm{gr}(A, i + j)$$

and

$$A = \bigoplus_{i \in \mathbb{N}} \mathrm{gr}(A, i)$$

Elements of the nth graded piece $\mathrm{gr}(A, n)$ are called homogeneous of degree n.

Example 3.2. The above statement about homogeneous elements suggests a relation to the idea of an homogeneous polynomial. Take the polynomial algebra $P = k[x_1, \ldots, x_n]$ and let $\mathrm{gr}(P, i)$ be the vector space generated over k by the monomials of degree i in the variables $x_1, \ldots, x_n$ for $i \geq 0$ ie. homogeneous polynomials of degree i. Clearly $\mathrm{gr}(P, i).\mathrm{gr}(P, j) = \mathrm{gr}(P, i + j)$ for $0 \leq i, j$ and $P = \bigoplus_i \mathrm{gr}(P, i)$. This is called the grading by degree and the ith graded component is commonly denoted $k[x_1, \ldots, x_n]_i$.

Take $k[x, y]$ graded by degree. Take two elements a, b and consider their decomposition into homogeneous components $a = \sum_{i \geq 0} a_i$ and $b = \sum_{j \geq 0} b_j$ with $a_i \in k[x, y]_i$ and $b_j \in k[x, y]_j$. The product ab can therefore be decomposed as a sum of the products of homogeneous terms:

$$ab = \sum_{i,j \geq 0} a_i b_j$$

Note that the (i, j)th term of the above sum has degree $i + j$. This gives a homogeneous decomposition of the product:

$$ab = \sum_{n \geq 0} \sum_{i+j=n} a_i b_j$$

Definition 3.3. A k-algebra A is $\mathbb{N}$-*filtered* or simply *filtered*, if there exists a set of subspaces $\mathrm{fp}(A, i)$ such that:

$$\mathrm{fp}(A, i) \subseteq \mathrm{fp}(A, i+1), \ \forall i \in \mathbb{N}$$
$$1 \in \mathrm{fp}(A, 0)$$
$$\mathrm{fp}(A, i).\mathrm{fp}(A, j) \subseteq \mathrm{fp}(A, i+j)$$
$$A = \bigcup_{i \in \mathbb{N}} \mathrm{fp}(A, i)$$

Example 3.4. Drawing on Lemma (2.3), any element of the algebra A_1 can be expressed as $x = \sum_i a_i p^{m_i} q^{n_i}$. Defining $\deg(x) = \max_i\{m_i + n_i\}$ suggests a filtration of A_1 by degree. Define the nth filtered part $\mathrm{fp}(A_1, n)$ by $\{x \in A_1 : \deg(x) \leq n\}$ ie. not neccessarily homogeneous elements of degree $\leq n$. Clearly $A_1 = \bigcup_i \mathrm{fp}(A_1, i)$ and $1 \in \mathrm{fp}(A_1, 0)$. Lemma (2.11) implies that $\mathrm{fp}(A_1, i).\mathrm{fp}(A_1, j) \subseteq \mathrm{fp}(A_1, i+j)$.

Definition 3.5. Given a filtered algebra A with filtered pieces $\mathrm{fp}(A, i)$, define the associated graded algebra $\mathrm{gr}(A)$ corresponding to this filtration to be:

$$\mathrm{gr}(A) = \bigoplus_i \frac{\mathrm{fp}(A, i)}{\mathrm{fp}(A, i-1)}$$

defining $\mathrm{fp}(A, -1) = 0$ and with multiplication defined on (left)-cosets by:

$$[x + \mathrm{fp}(A, i-1)] \cdot [y + \mathrm{fp}(A, j-1)] = [xy + \mathrm{fp}(A, i+j-1)]$$

and extended component-wise to multiplication on $\mathrm{gr}(A)$.

Lemma 3.6. *For a filtered algebra A, gr(A) is an algebra.*

Proof.

- The coset $[1+\mathrm{fp}(A,-1)]$ contains only the element 1, which acts as the multiplicative identity since $[x+\mathrm{fp}(A,i-1)]\cdot[1+\mathrm{fp}(A,-1)] = [x+\mathrm{fp}(A,i+0-1)]$.
- Likewise, the coset $[0+\mathrm{fp}(A,-1)]$ contains only the element 0, which acts as the additive identity.
- We need to check that the multiplication is well defined. Take the cosets $\mathcal{X} = [x+\mathrm{fp}(A,m-1)]$ and $\mathcal{Y} = [y+\mathrm{fp}(A,n-1)]$ and take the elements $a \in \mathcal{X}$ and $b \in \mathcal{Y}$. Note that $x \in \mathrm{fp}(A,m)$ and $y \in \mathrm{fp}(A,n)$ and write the elements as:

  $$a = x + a' \qquad a' \in \mathrm{fp}(A,m-1)$$
  $$b = y + b' \qquad b' \in \mathrm{fp}(A,n-1)$$

  and form the product:

  $$ab = xy + xb' + a'y + a'b'$$

  Applying the rule for multiplication of filtered pieces gives:

  $$xy \in \mathrm{fp}(A,m+n)$$
  $$xb' \in \mathrm{fp}(A,m+n-1)$$
  $$a'y \in \mathrm{fp}(A,m+n-1)$$
  $$a'b' \in \mathrm{fp}(A,m+n-2)$$

  Thus, $ab \in [xy+\mathrm{fp}(A,m+n-1)]$. Thus multiplication in $\mathrm{gr}(A)$ is well-defined.
- The other ring axioms follow easily from the definition.

□

Lemma 3.7. *The associated graded algebra of A_1 is $k[p,q]$, the commutative polynomial ring on the variables p and q.*

Proof. Take the filtration by degree defined in Example (3.4). Let $\mathrm{gr}(A)$ be the associated graded algebra corresponding to this filtration. Take two homogeneous elements of $\mathrm{gr}(A)$,

$$x = [a_0 p^0 q^i + \cdots + a_i p^i q^0 + \mathrm{fp}(A, i-1)]$$

and

$$y = [b_0 p^0 q^j + \cdots + b_i p^j q^0 + \mathrm{fp}(A, j-1)]$$

Where as usual, some of the coefficients a or b may be 0. Take the product xy:

$$xy = [(a_0 p^0 q^i + \cdots + a_i p^i q^0)(b_0 p^0 q^j + \cdots + b_i p^j q^0) + \mathrm{fp}(A, i+j-1)]$$

Consider a single term of the product:

$$(a_m p^m q^{i-m})(b_n p^n q^{j-n}) = a_m b_n p^m q^{i-m} p^n q^{j-n}$$

Reducing this expression as in 2.3 gives a representation of this term as

$$a_m b_n p^{m+n} q^{i+j-m-n} + (\text{terms of lower degree})$$

Thus the product can be expressed as

$$\sum_{m,n} a_m b_n p^{m+n} q^{i+j-m-n} + (\text{terms of lower degree})$$

Note that all terms in the left hand sum have degree $i+j$. Therefore as a member of the coset $xy + \mathrm{fp}(A, i+j-1)$ this is simply

$$\left[\sum_{m,n} a_m b_n p^{m+n} q^{i+j-m-n} + \mathrm{fp}(A, i+j-1)\right]$$

Thus multiplication of homogeneous elements behaves exactly as in $k[p,q]$ graded by degree. Since $\mathrm{gr}(A)$ is the direct sum of these homogeneous components, the multiplication extended to the whole ring is also identical to that in $k[p,q]$.

This allows us to define a homomorphism $\phi : \mathrm{gr}(A) \to k[p,q]$ defined on the graded components of $\mathrm{gr}(A)$ by mapping $[x + \mathrm{fp}(A, i-1)]$ to the leading terms of the unique representation of x considered as polynomials in $k[p,q]_i$ and extended component-wise to $\mathrm{gr}(A)$. This is clearly an homomorphism as shown above. It is also easy to show that it is bijective. □

Definition 3.8. A filtered algebra A for which the associated graded algebra $\mathrm{gr}(A)$ is commutative (eg. A_n) is called an *almost-commutative* algebra.

Proposition 3.9. *Any finitely-generated almost-commutative algebra A is both left and right noetherian.*

Proof. See [2] proposition 7.1. □

Corollary 3.9.1. *A_n is both left and right noetherian.*

Remark. It has been proved that an algebra (over k) is almost commutative if and only if it is a homomorphic image of the universal enveloping algebra of some finite dimensional Lie algebra over k.

Chapter 4

Gelfand-Kirillov Dimension

Let A be a finitely generated k-algebra. Let V be a finite-dimensional generating subspace for A. That is, if

$$\text{fp}(A, n) = k + V + V^2 + \ldots + V_n$$

then:

$$A = \bigcup_{n \geq 0} \text{fp}(A, n)$$

Note that this is a filtration of A with filtered pieces $\text{fp}(A, n)$.

Define the function $d_V(n) : \mathbb{N} \to \mathbb{R}$ by

$$d_V(n) = \dim_k(\text{fp}(A, n))$$

Note that if the algebra A is finite dimensional, then there exists some N such that for all $n > N$, $\text{fp}(A, n) = \text{fp}(A, N)$ ie. the function $d_V(n)$ becomes stationary.

Note that the function $d_V(n)$ depends on A and the choice of generating subspace V. It would be nice to remove this dependence to get an invariant of the algebra A. This is done with the following construction.

The idea here is to form an equivalence relation on the functions $d_V(n)$ by comparing their asymptotic growth rate. This turns out to be exactly the right definition to avoid the dependence on the choice of generating subspace.

Definition 4.1. Let Φ denote the set of eventually non-decreasing positive valued functions $f : \mathbb{N} \to \mathbb{R}$ ie. those for which there exists an $n_0 \in \mathbb{N}$ such that for all n, $f(n) \geq 0$ and for all $n \geq n_0$

$$f(n+1) \geq f(n)$$

Define a relation $\leq^*$ on Φ by setting $f \leq^* g$ iff there exist $c \in \mathbb{R}$, $m \in \mathbb{N}$ such that for all n sufficiently large,

$$f(n) \leq cg(mn)$$

Lemma 4.2. *The relation $\leq^*$ is a preorder relation on the set Φ.*

Proof.

- (reflexive) Taking $c = m = 1$ gives $f \leq^* f$.
- (transitive) Take $f, g, h \in \Phi$ and assume that $f \leq^* g$ and $g \leq^* h$ ie.

$$f(n) \leq c_0 g(m_0 n) \qquad \forall n > n_0$$
$$g(n) \leq c_1 h(m_1 n) \qquad \forall n > n_1$$

Let $n_2 = \max(n_0, n_1)$. Then since f and g are non-decreasing,

$$f(n) \leq c_0 c_1 h(m_0 m_1 n) \qquad \forall n > n_2$$

Therefore $f \leq^* h$.

□

Define an equivalence relation on Φ by $f \sim g$ iff $f \leq^* g$ and $g \leq^* f$. Denote the partial order induced on the quotient $\Phi/\sim$ by $\leq$. For an $f \in \Phi$, the equivalence class $\mathcal{G}(f) \in \Phi/\sim$ is called the *growth* of f.

Lemma 4.3. *[2] Let A be a finitely generated k-algebra with finite dimensional generating subspaces V and W. If $d_V(n)$ and $d_W(n)$ denote the dimensions of $\sum_{i=0}^{n} V^i$ and $\sum_{i=0}^{n} W^i$, respectively, then $\mathcal{G}(d_V) = \mathcal{G}(d_W)$.*

Proof. Since

$$A = \bigcup_{n=0}^{\infty}(V^0 + \ldots + V^n) = \bigcup_{n=0}^{\infty}(W^0 + \ldots + W^n)$$

there exist positive integers s and t such that

$$W \subseteq \sum_{i=0}^{s} V^i \text{ and } V \subseteq \sum_{i=0}^{t} W^i$$

Thus $d_W(n) \leq d_V(sn)$ and $d_V(n) \leq d_W(tn)$, whence $d_V \sim d_W$. □

Thus the *growth* of an algebra A, defined to be $\mathcal{G}(d_V)$, is independent of the choice of generating subspace V.

Example 4.4. Let $A = k[p, q]$, the commutative polynomial ring in two variables. Take the generating subspace $V = kp + kq$. Take the basis $B_1 = \{p, q\}$ for V. Let B_n be the corresponding basis for V^n (formed as the product $B_1 \cdot B_{n-1}$). Assume that the basis B_{n-1} consists of all monomials of degree $n - 1$ ie.

$$B_{n-1} = \left\{p^{n-1}q^0, \ldots, p^0q^{n-1}\right\}$$

Then calculating B_n simply gives

$$B_n = \left\{p^nq^0, \ldots, p^1q^{n-1}, p^{n-1}q^1, \ldots, p^0q^n\right\} = \left\{p^nq^0, \ldots, p^0q^n\right\}$$

Thus, inductively, V^n is the space spanned by all (commutative) monomials of degree n. The set $\bigcup B_n$ is linearly independent and hence

$$\begin{aligned} d_V(n) &= \dim_k \sum_{i=0}^{n} V^n \\ &= \sum_{i=0}^{n} \dim_k V^n \\ &= \frac{1}{2}(n+1)(n+2) \end{aligned}$$

Thus $\mathcal{G}(d_V) = \mathcal{G}(n^2)$ ie. the polynomial algebra in 2 variables has quadratic growth. This example extends simply to show that the polynomial algebra in m variables has degree m polynomial growth.

It will be interesting at this point to try to calculate the growth of the first Weyl algebra A_1. To do so we will need the following lemma:

Lemma 4.5. *Let $M_n \subset A_1$ be the subspace spanned by the monomials $p^i q^j$ with $i + j = n$. Let $V = M_1$ ie. $kp + kq$, then for $n \geq 2$,*

$$V^n = \bigoplus_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} M_{n-2i}$$

Proof. As above, let B_n be a basis for V_n with $B_1 = \{p, q\}$. Let $S_n = B_1 \cdot B_{n-1}$. Calculating for $n = 2$ gives:

$$\begin{aligned} S_2 &= B_1 \cdot B_1 \\ &= \{p^2, pq, qp, q^2\} \\ &= \{p^2, pq, pq + 1, q^2\} \end{aligned}$$

This reduces to the basis:

$$B_2 = \{p^2, pq, q^2, 1\}$$

Which proves the result for $n = 2$. Assume that the result holds for $n - 1$. That is,

$$B_{n-1} = \bigcup_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \{p^j q^{((n-1)-2i)-j} : 0 \leq j \leq (n-1) - 2i\}$$

Calculating the product gives

$$S_n = \bigcup_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\begin{array}{c} \{p^{j+1}q^{((n-1)-2i)-j} : 0 \leq j \leq (n-1)-2i\} \quad \cup \\ \{qp^j q^{((n-1)-2i)-j} : 0 \leq j \leq (n-1)-2i\} \end{array} \right)$$

Applying Lemma (2.8) to the appropriate terms in the above expression (those qp^lq^m with $l > 0$) gives

$$S_n = \bigcup_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\begin{array}{r} \{p^{j+1}q^{((n-1)-2i)-j} : 0 \leq j \leq (n-1)-2i\} \quad \cup \\ \{p^j q^{(n-2i)-j} + jp^{j-1}q^{((n-1)-2i)-j} : 1 \leq j \leq (n-1)-2i\} \quad \cup \\ \{q^{n-2i}\} \quad \end{array} \right)$$

For a given i, the first and third terms in the above expression contain all monomials of degree $n-2i$. The elements of the second term are simply all monomials of degree $n-2i-2$ added to some monomial of degree $n-2i$. They are therefore linearly dependent on the other elements of S_n and can be ignored except in the case where n is even and $i = \lfloor \frac{n-1}{2} \rfloor$. In this case, it is easily verified that the only element of the second term in $pq + 1$ which, since pq is in the span when $i = \lfloor \frac{n-1}{2} \rfloor - 1$, adds 1 to the span of S_n.

Thus inductively,

$$V^n = \text{span}(S_n) = \bigoplus_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} M_{n-2i}$$

Note that for all $n >= 0$, $V^n \subset V^{n+1}$. □

Example 4.6. The subspace $V = kp + kq$ is clearly a generating subspace for A_1. Define

$$\text{fp}(A_1, n) = \bigcup_{i=0}^{n} V^i$$

Since V^n contains all monomials of degree n, $\mathrm{fp}(A_1, n)$ is just the span of all monomials of degree $\leq n$.

$$\mathrm{fp}(A_1, n) = \bigoplus_{i=0}^{n} M_i$$

The dimension $\dim(M_i)$ is simply $i + 1$. Therefore $d_V(n)$ is simply

$$\begin{aligned} d_V(n) &= \dim(\mathrm{fp}(A_1, n)) \\ &= \dim\left(\bigoplus_{i=0}^{n} M_i\right) \\ &= \sum_{i=0}^{n} (i+1) \\ &= \frac{1}{2} n(n+1) + n \end{aligned}$$

Thus $\mathcal{G}(d_V(n)) = \mathcal{G}(n^2)$ ie. the growth of A_1 is quadratic just as for $k[p, q]$, its associated graded algebra. This is not a coincidence, and is an example of a more general result.

Example 4.7. Let $F_n = k < x, y >$ be the free algebra on 2 variables. Let $V = kx + ky$ be a generating subspace for F_n. Define a filtration of F_n by

$$F_n = \bigcup_{i \geq 0} \mathrm{fp}(F_n, i)$$

where

$$\mathrm{fp}(F_n, i) = \sum_{j=0}^{i} V^j$$

It is not hard to see that V^j has dimension 2^j and that the above sum is direct. Thus

$$\begin{aligned} d_V(n) &= \sum_{i=0}^{n} 2^i \\ &= 2^{n+1} - 1 \end{aligned}$$

Therefore $\mathcal{G}(d_V) = \mathcal{G}(2^n)$ ie. exponential growth.

The growth is an important invariant of an algebra. It can, however, be somewhat unwieldy to calculate in practice. Looking at the definition of the equivalence relation on growth functions, what we are really interested in is the asymptotic growth of the algebra. This is formalised in the definition of the Gelfand-Kirillov dimension.

Definition 4.8. The *Gelfand-Kirillov dimension* of a k-algebra A is

$$\operatorname{GKdim}(A) = \sup_V \overline{\lim} \log_n d_V(n)$$

Where $\overline{\lim}$ denotes the limit superior and the supremum $\sup_V$ is taken over all finitely-generated subspaces of A.

Lemma 4.9. *[2] Take $f, g \in \Phi$ (two eventually non-decreasing functions $\mathbb{N} \to \mathbb{R}$). The following hold:*

- $\overline{\lim} \log_n f(n) = \inf\{\rho \in \mathbb{R} : \mathcal{G}(f) \leq \mathcal{G}(n^\rho)\}$
- *If $\mathcal{G}(f) = \mathcal{G}(g)$ then $\overline{\lim} \log_n f(n) = \overline{\lim} \log_n g(n)$*

Proof. Let r denote $\overline{\lim} \log_n f(n)$ and s denote $\inf\{\rho \in \mathbb{R} : \mathcal{G}(f) \leq \mathcal{G}(n^\rho)\}$. □

The first part of the preceeding lemma shows that for an algebra with polynomial growth, say $\mathcal{G}(n^a)$, then the Gelfand-Kirillov dimension of the algebra is the polynomial degree, a. It also shows that for any algebra with super-polynomial growth eg. the free algebra with exponential growth, the Gelfand-Kirillov dimension is infinite.

Note that in a previous lemma, we showed that for two generating subspaces V and W, $\mathcal{G}(d_V(n)) = \mathcal{G}(d_W(n))$. Thus for any generating subspace V, we can drop the supremum and write

$$\text{GKdim}(A) = \overline{\lim} \log_n d_V(n)$$

The previous lemma also shows that the Gelfand-Kirillov dimension gives an equivalence which is no finer than that given by the growth.

Remark. It has been proven (see [2] Chapter 2) that the range of possible values for the Gelfand-Kirillov dimension of an algebra is $\{0\} \cup \{1\} \cup [2, \infty)$.

Remark. The Gelfand-Kirillov dimension can also be defined in a natural way for modules over an algebra. The study of modules over the Weyl algebras is a particularly interesting application. It turns out that the minimum Gelfand-Kirillov dimension for a module over the n^{th} Weyl algebra A_n is $2n$. Modules of this minimal dimension are known as *holonomic* modules and are linked with holonomic systems of linear differential equations.

Proposition 4.10. *If A is an almost-commutative algebra with associated graded algebra gr(A) then GKdim(A) = GKdim(gr(A)).*

Proof. See [2] proposition 6.6. □

Chapter 5

Automorphisms of A_1

In this chapter, we show that all automorphisms of the first Weyl algebra, A_1, are generated by a set of automorphisms $\Phi_{n,\lambda}, \Phi'_{n,\lambda}$ which are defined shortly. This proof originally appears in the paper [1] by Dixmier.

At this point, we need to define a number of concepts which are used in the argument that follows.

Definition 5.1. If $f = \sum a_{ij} x^i y^j \in k[x, y]$, denote by $E(f)$ the set of pairs (i, j) such that $a_{ij} \neq 0$. If ρ, σ are real numbers, define

$$v_{\rho,\sigma}(f) = \sup_{(i,j)\in E(f)} (\rho i + \sigma j)$$

(for convenience define $v_{\rho,\sigma}(0) = -\infty$). Denote by $E(f, \rho, \sigma)$ the set of pairs $(i, j) \in E(f)$ such that $\rho i + \sigma j = v_{\rho,\sigma}(f)$. If $f \neq 0$, we have $E(f, \rho, \sigma) \neq \emptyset$. If $E(f) = E(f, \rho, \sigma)$, we say that f is (ρ, σ)-homogeneous of (ρ, σ)-degree $v_{\rho,\sigma}(f)$.

Remark. The above gives a grading of $k[x, y]$ by (ρ, σ)-degree. This fact is not required and the proof is omitted, but it does demonstrate the fact that there exist more general gradings than those considered earlier.

Definition 5.2. Suppose $\mathfrak{a} = \sum a_{ij} p^i q^j \in A_1$, and σ, ρ are real numbers. By analogy with the previous definition (5.1), define $E(\mathfrak{a})$, $v_{\rho,\sigma}(\mathfrak{a})$ and $E(\mathfrak{a}, \rho, \sigma)$. The polynomial

$$\sum_{(i,j)\in E(\mathfrak{a},\rho,\sigma)} a_{ij} x^i y^j \in k[x, y]$$

is called the (ρ, σ)-associated polynomial of $\mathfrak{a}$.

Remark. The previous definition is an example of an associated graded algebra of A_1 corresponding to a more general filtration than that previously considered.

Lemma 5.3. *Let $f \in k[x, y]$ be a (ρ, σ)-homogeneous polynomial of (ρ, σ)-degree v. Then,*

- $\rho x \frac{\delta f}{\delta x} + \sigma y \frac{\delta f}{\delta y} = vf$.
- *If ρ and σ are linearly independent over $\mathbb{Q}$, f is a monomial.*

Proof. Let $g = x^i y^j$ with $\rho i + \sigma j = v$. We have

$$\begin{aligned} \rho x \frac{\delta g}{\delta x} + \sigma y \frac{\delta g}{\delta y} &= \rho x i x^{i-1} y^j + \sigma y j x^i y^{j-1} \\ &= (\rho i + \sigma j) g \end{aligned}$$

and hence the first result.

If $(i, j) \in E(f)$ and $(i', j') \in E(f)$, we have $\rho i + \sigma j = \rho i' + \sigma j' = v$, thus $\rho(i - i') = \sigma(j' - j)$. Since ρ and σ are linearly independent, we must have $i = i'$ and $j = j'$, whence f is a monomial. □

Lemma 5.4. *Let $f, g \in k[x, y]$ be (ρ, σ)-homogeneous polynomials of (ρ, σ) degrees v, w. Then the following hold,*

1.

$$\sigma y \left(\frac{\delta f}{\delta x} \frac{\delta g}{\delta y} - \frac{\delta f}{\delta y} \frac{\delta g}{\delta x} \right) = wg \frac{\delta f}{\delta x} - vf \frac{\delta g}{\delta x}$$

Moreover, if both v and w are integers,

$$\sigma y \left(\frac{\delta f}{\delta x} \frac{\delta g}{\delta y} - \frac{\delta f}{\delta y} \frac{\delta g}{\delta x} \right) = f^{-w+1} g^{v+1} \frac{\delta}{\delta x} (g^{-v} f^w)$$

2.

$$-\rho x\left(\frac{\delta f}{\delta x}\frac{\delta g}{\delta y}-\frac{\delta f}{\delta y}\frac{\delta g}{\delta x}\right)=wg\frac{\delta f}{\delta y}-vf\frac{\delta g}{\delta y}$$

Moreover, if both v and w are integers,

$$-\rho x\left(\frac{\delta f}{\delta x}\frac{\delta g}{\delta y}-\frac{\delta f}{\delta y}\frac{\delta g}{\delta x}\right)=f^{-w+1}g^{v+1}\frac{\delta}{\delta y}(g^{-v}f^{w})$$

Proof. Applying lemma 5.3 part 1,

$$\begin{aligned}\sigma y\left(\frac{\delta f}{\delta x}\frac{\delta g}{\delta y}-\frac{\delta f}{\delta y}\frac{\delta g}{\delta x}\right)&=\frac{\delta f}{\delta x}\left(wg-\rho x\frac{\delta g}{\delta x}\right)-\left(vf-\rho x\frac{\delta f}{\delta x}\right)\frac{\delta g}{\delta x}\\&=wg\frac{\delta f}{\delta x}-vf\frac{\delta g}{\delta x}\end{aligned}$$

Suppose that v and w are integers. Then

$$\frac{\delta}{\delta x}(g^{-v}f^{w})=g^{-v-1}f^{w-1}\left(-v\frac{\delta g}{\delta x}f+gw\frac{\delta f}{\delta x}\right)$$

thus, taking into account the preceeding,

$$\sigma y\left(\frac{\delta f}{\delta x}\frac{\delta g}{\delta y}-\frac{\delta f}{\delta y}\frac{\delta g}{\delta x}\right)=f^{-w+1}g^{v+1}\frac{\delta}{\delta x}(g^{-v}f^{w})$$

□

Lemma 5.5. *Take i,j,l,m integers ≥ 0. Then*

$$\begin{aligned}(p^iq^j)(p^lq^m)&=p^{i+l}q^{j+m}+jlp^{i+l-1}q^{j+m-1}+\frac{1}{2!}j(j-1)l(l-1)p^{i+l-2}q^{j+m-2}\\&+\frac{1}{3!}j(j-1)(j-2)l(l-1)(l-2)p^{i+l-3}q^{j+m-3}+\dots\end{aligned}$$

Proof. This follows by expanding the expression derived in 2.9. □

Lemma 5.6. *Take* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A_1$ *with* $\mathfrak{c} = \mathfrak{a}\mathfrak{b}$*. Suppose*

$$\mathfrak{a} = \sum a_{ij} p^i q^j \qquad \mathfrak{b} = \sum b_{ij} p^i q^j \qquad \mathfrak{c} = \sum c_{ij} p^i q^j$$

Let

$$f = \sum a_{ij} x^i y^j \qquad g = \sum b_{ij} x^i y^j \qquad h = \sum c_{ij} x^i y^j$$

Then

$$h = fg + \frac{\delta f}{\delta y}\frac{\delta g}{\delta x} + \frac{1}{2!}\frac{\delta^2 f}{\delta y^2}\frac{\delta^2 g}{\delta x^2} + \frac{1}{3!}\frac{\delta^3 f}{\delta y^3}\frac{\delta^3 g}{\delta x^3} + \dots$$

Proof. It suffices to prove for $\mathfrak{a} = p^i q^j$, $\mathfrak{b} = p^l q^m$, which follows from lemma 5.5. □

Lemma 5.7. *Take* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A_1$ *such that* $\mathfrak{c} = [\mathfrak{a}, \mathfrak{b}]$*. Suppose*

$$\mathfrak{a} = \sum a_{ij} p^i q^j \qquad \mathfrak{b} = \sum b_{ij} p^i q^j \qquad \mathfrak{c} = \sum c_{ij} p^i q^j$$

and let

$$f = \sum a_{ij} x^i y^j \qquad g = \sum b_{ij} x^i y^j \qquad h = \sum c_{ij} x^i y^j$$

then

$$h = \frac{\delta f}{\delta x}\frac{\delta g}{\delta y} - \frac{\delta f}{\delta y}\frac{\delta g}{\delta x} + \frac{1}{2!}\left(\frac{\delta^2 f}{\delta x^2}\frac{\delta^2 g}{\delta y^2} - \frac{\delta^2 f}{\delta y^2}\frac{\delta^2 g}{\delta x^2}\right) + \frac{1}{3!}\left(\frac{\delta^3 f}{\delta x^3}\frac{\delta^3 g}{\delta y^3} - \frac{\delta^3 f}{\delta y^3}\frac{\delta^3 g}{\delta x^3}\right) + \dots$$

Proof. This follows directly from lemma 5.6 □

Lemma 5.8. *Take* $\mathfrak{a}$ *and* $\mathfrak{b}$ *non-zero elements of* A_1*, and* ρ, σ *real numbers such that* $\rho + \sigma > 0$*. Let* $v = v_{\rho,\sigma}(\mathfrak{a})$ *and* $w = v_{\rho,\sigma}(\mathfrak{b})$*. Let* f_1 *and* g_1 *be the* (ρ, σ)*-associated polynomials of* $\mathfrak{a}$ *and* $\mathfrak{b}$*.*

(i) There exists a pair (t, u) *of elements of* A_1*, possessing the following properties:*

(a) $[x, y] = t + u$

(b) $E(t) = E(t, \rho, \sigma)$ *and* $v_{\rho,\sigma}(t) = v + w - (\rho + \sigma)$

(c) $v_{\rho,\sigma}(u) < v + w - (\rho + \sigma)$

(ii) The following conditions are equivalent:

(ii 1) $t = 0$

(ii 2) $\frac{\delta f_1}{\delta x}\frac{\delta g_1}{\delta y} - \frac{\delta f_1}{\delta y}\frac{\delta g_1}{\delta x} = 0$

(ii 3) If v *and* w *are integers,* g_1^v *is a multiple of* f_1^w

(iii) If $t \neq 0$*, the* (ρ, σ)*-associated polynomial of* $[x, y]$ *is* $\frac{\delta f_1}{\delta x}\frac{\delta g_1}{\delta y} - \frac{\delta f_1}{\delta y}\frac{\delta g_1}{\delta x}$.

Proof. Introduce the notation of lemma 5.7. Then h is the sum of $\frac{\delta f_1}{\delta x}\frac{\delta g_1}{\delta y} - \frac{\delta f_1}{\delta y}\frac{\delta g_1}{\delta x}$, which is (ρ, σ) homogeneous of (ρ, σ) degree $v + w - (\rho + \sigma)$, and a polynomial h^* such that

$$v_{\rho,\sigma}(h^*) < v + w - (\rho + \sigma)$$

This proves (i), (iii) and the equivalence (ii 1)⇔(ii 2). If v and w are integers, the equivalence (ii 2)⇔(ii 3) follows from lemma 5.4. □

Definition 5.9. For a field k, denote the algebraic closure of k by $\overline{k}$. For an algebra A over k, denote the algebra $A \otimes_k \overline{k}$ by $\overline{A}$.

Let A be an algebra over k, and take $\mathfrak{a} \in A$. For all $y \in \overline{A}$, define $V_y = \sum_{n \geq 0} k(\mathrm{ad}\, \mathfrak{a})^n y$. Denote by $F(\mathfrak{a}; A)$, or $F(\mathfrak{a})$, the set of $y \in A$ such that $\dim V_y < \infty$ ie. the set of elements for which the subspace V_y is finite dimensional. We have $F(x; \overline{A}) = F(x; A) \otimes_k \overline{k}$.

If $\lambda \in \overline{k}$, denote by $F(\mathfrak{a}, \lambda; \overline{A})$ the set of $y \in F(\mathfrak{a}; \overline{A})$ such that $(\mathrm{ad}\, \mathfrak{a} - \lambda)^n y$ is zero for n sufficiently large. If $\lambda \in k$, define $F(x, \lambda; A) = F(x, \lambda; \overline{A}) \cap A$, such that $F(x, \lambda; \overline{A}) = F(x, \lambda; A) \otimes_k \overline{k}$.

Denote by $N(\mathfrak{a}; A)$ or $N(\mathfrak{a})$ the set $F(\mathfrak{a}, 0; A)$. This is the set of $y \in A$ such that $(\mathrm{ad}\, \mathfrak{a})|_V$ is nilpotent. For $n = 0, 1, 2, \ldots$, denote by $N(\mathfrak{a}, n; A)$ or $N(\mathfrak{a}, n)$ the kernel of $(\mathrm{ad}\, \mathfrak{a})^{n+1}$. Note that $N(\mathfrak{a}, 0) = C(\mathfrak{a})$, the set of elements which commute with $\mathfrak{a}$.

For $\lambda \in \overline{k}$, denote by $D(\mathfrak{a}, \lambda; \overline{A})$ the set of $y \in \overline{A}$ such that $(\operatorname{ad} \mathfrak{a})y = \lambda y$. Define

$$D(x; \overline{A}) = \bigoplus_{\lambda \in \overline{k}} D(x, \lambda; \overline{A})$$

and define $D(x) = D(x; A) = D(x; \overline{A}) \cap A$.

Lemma 5.10. *Take any $\mathfrak{a} \in A_1$, and Φ an automorphism of A_1. Then $C(\Phi(\mathfrak{a})) = \Phi(C(\mathfrak{a}))$, $N(\Phi(\mathfrak{a})) = \Phi(N(\mathfrak{a}))$ and $D(\Phi(\mathfrak{a})) = \Phi(D(\mathfrak{a}))$.*

Proof. Suppose that $\mathfrak{b} \in C(\mathfrak{a})$. That is,

$$\mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a} = 0$$

consider $\Phi(\mathfrak{a})\Phi(\mathfrak{b}) - \Phi(\mathfrak{b})\Phi(\mathfrak{a})$.

$$\begin{aligned}\Phi(\mathfrak{a})\Phi(\mathfrak{b}) - \Phi(\mathfrak{b})\Phi(\mathfrak{a}) &= \Phi(\mathfrak{a}\mathfrak{b}) - \Phi(\mathfrak{b}\mathfrak{a}) \\ &= \Phi(\mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a}) \\ &= \Phi(0) = 0\end{aligned}$$

Thus $\Phi(C(\mathfrak{a})) \subseteq C(\Phi(\mathfrak{a}))$. Applying the above calculation in the other direction shows that $C(\Phi(\mathfrak{a})) \subseteq \Phi(C(\mathfrak{a}))$. Thus $\Phi(C(\mathfrak{a})) = C(\Phi(\mathfrak{a}))$.

Suppose that $\mathfrak{b} \in D(\mathfrak{a}, \lambda)$ that is,

$$\mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a} = \lambda\mathfrak{b}$$

consider $\Phi(\mathfrak{a})\Phi(\mathfrak{b}) - \Phi(\mathfrak{b})\Phi(\mathfrak{a})$.

$$\begin{aligned}\Phi(\mathfrak{a})\Phi(\mathfrak{b}) - \Phi(\mathfrak{b})\Phi(\mathfrak{a}) &= \Phi(\mathfrak{a}\mathfrak{b} - \mathfrak{b}\mathfrak{a}) \\ &= \Phi(\lambda\mathfrak{b}) \\ &= \lambda\Phi(\mathfrak{b})\end{aligned}$$

Thus $\Phi(D(\mathfrak{a}, \lambda)) \in D(\Phi(\mathfrak{a}, \lambda))$. Once again, applying the calculation in the other direction gives the reverse inclusion. Therefore $D(\Phi(\mathfrak{a})) = \Phi(D(\mathfrak{a}))$.

It is not hard to see that

$$\Phi((\text{ad } \mathfrak{a})^n \mathfrak{b}) = (\text{ad } \Phi(\mathfrak{a}))^n \Phi(\mathfrak{b})$$

and thus reasoning as for $C(\mathfrak{a})$, $N(\Phi(\mathfrak{a})) = \Phi(N(\mathfrak{a}))$. □

Corollary 5.10.1. *If equality holds between any of the $N(\mathfrak{a})$, $D(\mathfrak{a})$ and $C(\mathfrak{a})$, then the same equality holds between the $N(\Phi(\mathfrak{a}))$, $D(\Phi(\mathfrak{a}))$ and $C(\Phi(\mathfrak{a}))$. Likewise if any of the $N(\mathfrak{a})$, $D(\mathfrak{a})$ and $C(\mathfrak{a})$ are equal to any Φ-invariant set (eg. 0 or A_1), then the same equality holds for the $N(\Phi(\mathfrak{a}))$, $D(\Phi(\mathfrak{a}))$ and $C(\Phi(\mathfrak{a}))$.*

Lemma 5.11. $C(p) = k[p]$

Proof. Clearly $k[p] \subseteq C(p)$. Take any $\mathfrak{a} \in C(p)$, and let $\mathfrak{b} = [q, \mathfrak{a}]$, then

$$\begin{aligned}
[\mathfrak{b}, p] &= [q\mathfrak{a} - \mathfrak{a}q, p] \\
&= (q\mathfrak{a} - \mathfrak{a}q)p - p(q\mathfrak{a} - \mathfrak{a}q) \\
&= q\mathfrak{a}p - \mathfrak{a}qp - pq\mathfrak{a} + p\mathfrak{a}q \\
&= qp\mathfrak{a} - \mathfrak{a}qp - pq\mathfrak{a} + \mathfrak{a}pq && \text{since } \mathfrak{a} \text{ commutes with } p \\
&= pq\mathfrak{a} + \mathfrak{a} - \mathfrak{a}pq - \mathfrak{a} - pq\mathfrak{a} + \mathfrak{a}pq \\
&= 0
\end{aligned}$$

and hence $[q, \mathfrak{a}] \in C(p)$. Suppose there exists some $\mathfrak{c} \in C(p)$ with

$$\mathfrak{c} = \sum a_{ij} p^i q^j$$

with at least one j non-zero. Note that if p commutes with both $\mathfrak{a}$ and $\mathfrak{b}$, then p commutes with $\mathfrak{a} + \mathfrak{b}$. Using this, and the fact that $k[p] \subseteq C(p)$, cancel all terms for which $j = 0$ to get some $\mathfrak{c}' \in C(p)$ for which every term contains a positive power of q. Calculate $[q, \mathfrak{c}']$:

$$\begin{aligned}
[q, \mathfrak{c}'] &= q\mathfrak{c}' - \mathfrak{c}'q \\
&= \sum a_{ij}(qp^iq^j - p^iq^{j+1}) && \text{since } [-,-] \text{ is bilinear} \\
&= \sum a_{ij}(p^iq^{j+1} + ip^{i-1}q^j - p^iq^{j+1}) && \text{by lemma 2.8} \\
&= \sum a_{ij}ip^{i-1}q^j
\end{aligned}$$

Note that we have decreased the degree in p by one in each term, and every term for which $i = 0$ becomes zero. Pick i_0 to be maximal with respect to the property that $a_{i_0,j}$ is non-zero for some j. And apply the above operation i_0 times to get another element $\mathfrak{c}'' \in C(p)$. By the choice of i_0,

$$\mathfrak{c}'' = \sum_j a_{i_0j}q^j$$

since terms with a lower power of p will vanish (note that the terms of this sum are not neccessarily unique). Equating coefficients of $[p, \mathfrak{c}'']$ implies that $[p, q^j] = 0$, which is a contradiction. Therefore $\mathfrak{c} \notin C(p)$ and $C(p) = k[p]$. □

Lemma 5.12. *Let A be an algebra over k. Take $\lambda \in k$ and $\mathfrak{a}, \mathfrak{b} \in A$ such that $(ad\ \mathfrak{a} - \lambda)^2\mathfrak{b} = 0$. Then*

1. $(ad\ \mathfrak{a} - n\lambda)^n\mathfrak{b}^n = n!((ad\ \mathfrak{a} - \lambda)\mathfrak{b})^n$ *for* $n = 1, 2, 3, \dots$
2. $(ad\ \mathfrak{a} - n\lambda)^{n+1}\mathfrak{b}^n = 0$

Proof. The assumption $(\text{ad}\ \mathfrak{a} - \lambda)^2\mathfrak{b} = 0$ can be written as

$$(\text{ad}\ \mathfrak{a} - \lambda)\mathfrak{b} \in D(\mathfrak{a}, \lambda)$$

Since $D(\mathfrak{a}, \lambda; A).D(\mathfrak{a}, \mu; A) \subset D(\mathfrak{a}, \lambda + \mu; A)$,

$$((\text{ad}\ \mathfrak{a} - \lambda)\mathfrak{b})^n \in D(\mathfrak{a}, n\lambda) \qquad \text{for } n = 1, 2, 3, \dots \tag{5.1}$$

Equality 1 is clear for $n = 1$. Assume it holds for n. Then

$$\begin{aligned}
(\text{ad } \mathfrak{a} - (n+1)\lambda)^{n+1}\mathfrak{b}^{n+1} &= (\text{ad } \mathfrak{a} - (n+1)\lambda)^{n+1}(\mathfrak{b}^n \cdot \mathfrak{b}) \\
&= ((\text{ad } \mathfrak{a} - n\lambda)^{n+1}\mathfrak{b}^n)\mathfrak{b} \\
&+ (n+1)((\text{ad } \mathfrak{a} - n\lambda)^n\mathfrak{b}^n)((\text{ad } \mathfrak{a} - \lambda)\mathfrak{b}) \\
&+ \frac{1}{2}(n+1)n((\text{ad } \mathfrak{a} - n\lambda)^{n-1}\mathfrak{b}^n)((\text{ad } \mathfrak{a} - \lambda)^2\mathfrak{b}) \\
&+ \dots \\
&= ((\text{ad } \mathfrak{a} - n\lambda)^{n+1}\mathfrak{b}^n)\mathfrak{b} \\
&+ (n+1)((\text{ad } \mathfrak{a} - n\lambda)^n\mathfrak{b}^n)((\text{ad } \mathfrak{a} - \lambda)\mathfrak{b})
\end{aligned}$$

Applying equation 5.1 and induction,

$$\begin{aligned}
(\text{ad } \mathfrak{a} - n\lambda)^{n+1}\mathfrak{b}^n &= (\text{ad } \mathfrak{a} - n\lambda)(n!((\text{ad } \mathfrak{a} - \lambda)\mathfrak{b})^n) \\
&= 0
\end{aligned}$$

thus

$$\begin{aligned}
(\text{ad } \mathfrak{a} - (n+1)\lambda)^{n+1}\mathfrak{b}^{n+1} &= (n+1)((\text{ad } \mathfrak{a} - n\lambda)^n\mathfrak{b}^n)((\text{ad } \mathfrak{a} - \lambda)\mathfrak{b}) \\
&= (n+1)n!((\text{ad } \mathfrak{a} - \lambda)\mathfrak{b})^n((\text{ad } \mathfrak{a} - \lambda)\mathfrak{b}) \\
&= (n+1)!((\text{ad } \mathfrak{a} - \lambda)\mathfrak{b})^{n+1}
\end{aligned}$$

and thus we have shown equality 1. Equality 2 is follows from equality 1 and equation 5.1. □

Lemma 5.13. *Take $\mathfrak{a} \in A_1$. Consider $F(\mathfrak{a})$ as a right $C(\mathfrak{a})$ module, and suppose that it is finitely generated over $C(\mathfrak{a})$, then $F(\mathfrak{a}) = C(\mathfrak{a})$.*

Proof. Suppose that $N(\mathfrak{a}) \neq C(\mathfrak{a})$. Let $(\mathfrak{b}_1, \ldots, \mathfrak{b}_r)$ be a set of generators for $N(\mathfrak{a})$ as a $C(\mathfrak{a})$ module. There exists an integer $n > 0$ such that

$$(\text{ad } \mathfrak{a})^n \mathfrak{b}_1 = \ldots = (\text{ad } \mathfrak{a})^n \mathfrak{b}_r = 0$$

and thus $(\text{ad } \mathfrak{a})^n(N(\mathfrak{a})) = 0$. Or, there exists a $\mathfrak{b} \in N(\mathfrak{a})$ such that

$$(\text{ad } \mathfrak{a})\mathfrak{b} \neq 0, \ (\text{ad } \mathfrak{a})^2\mathfrak{b} = 0$$

Applying lemma 5.12, $(\text{ad } \mathfrak{a})^n \mathfrak{b}^n \neq 0$, a contradiction.

Suppose that $D(\mathfrak{a}) \neq C(\mathfrak{a})$. Then $D(\mathfrak{a}\lambda) \neq 0$ implies that $D(\mathfrak{a}, n\lambda) \neq 0$ for all integers $n > 0$. Therefore $D(\mathfrak{a})$ is an infinite direct sum of non-zero $C(\alpha)$ modules, which is a contradiction. □

Lemma 5.14. *Take ρ, σ integers > 0. Take $\mathfrak{a} \in A_1$, $\mathfrak{b} \in F(\mathfrak{a})$, $v = v_{\rho,\sigma}(\mathfrak{a})$, $w = v_{\rho,\sigma}(\mathfrak{b})$, and let f and g be the (ρ, σ)-associated polynomials of $\mathfrak{a}$ and $\mathfrak{b}$ respectively. Suppose that $v > \rho + \sigma$ and that f is not a monomial. Then one of the following is true:*

(a) f^w is a multiple of g^v

(b) $\sigma > \rho$, σ is a multiple of ρ, and $f(x, y)$ is of the form $\lambda x^\alpha (x^{\frac{\sigma}{\rho}} + \mu y)^\beta$, for $\lambda, \mu \in k$, α, β integers ≥ 0

(c) $\rho > \sigma$, ρ is a multiple of σ, and $f(x, y)$ is of the form $\lambda y^\alpha (y^{\frac{\sigma}{\rho}} + \mu x)^\beta$, for $\lambda, \mu \in k$, α, β integers ≥ 0

(d) $\rho = \sigma$, and $f(x, y)$ is of the form $\lambda(\mu x + \nu y)^\alpha (\mu' x + \nu' y)^\beta$, for $\lambda, \mu, \nu, \mu', \nu' \in k$, α, β integers ≥ 0

Proof. Let $\mathfrak{b}_n = (\text{ad } \mathfrak{a})^n \mathfrak{b}$, for $n = 0, 1, 2, \ldots$. Then $v_{\rho,\sigma}(\mathfrak{b}_0) = w$. It is impossible to have $v_{\rho,\sigma}(\mathfrak{b}_n) = w + n(v - \rho - \sigma)$ for all n (because $v - \rho - \sigma > 0$ and $\mathfrak{b} \in F(\mathfrak{a})$).

Thus there exists an $n \geq 0$ such that

$$v_{\rho,\sigma}(\mathfrak{b}_m) = w + m(v - \rho + \sigma) \qquad \text{for } m \leq n \text{ and}$$
$$v_{\rho,\sigma}(\mathfrak{b}_{n+1}) < w + (n+1)(v - \rho - \sigma)$$

Let h be the (ρ, σ)-associated polynomial of $\mathfrak{b}_n$. Let $v_{\rho,\sigma}(\mathfrak{b}_n) = t$. Applying lemma 5.8, f^t is a multiple of h^v. If $n = 0$, we have $\mathfrak{b}_n = \mathfrak{b}$, $h = g$ and $t = w$ ie. case (a). Suppose from now on that $n > 0$. Consider $\mathfrak{b}_{n-1}$. Let l be the (ρ, σ)-associated polynomial of $\mathfrak{b}_{n-1}$. We have $v_{\rho,\sigma}(\mathfrak{b}_n) - v_{\rho,\sigma}(\mathfrak{b}_{n-1}) = v - \rho - \sigma$, thus

$$v_{\rho,\sigma}(\mathfrak{b}_{n-1}) = t - v + \rho + \sigma$$

Thus

$$\begin{aligned} \sigma y h &= \sigma y \left(\frac{\delta f}{\delta x}\frac{\delta l}{\delta y} - \frac{\delta f}{\delta y}\frac{\delta l}{\delta x} \right) && \text{lemma 5.8 part 3} \\ &= f^{-t+v-\rho-\sigma+1} l^{v+1} \frac{\delta}{\delta x}(l^{-v} f^{l-v+\rho+\sigma}) && \text{lemma 5.4 part 1} \end{aligned}$$

thus, since f^t is a multiple of h^v,

$$\frac{\delta}{\delta x}\left(\left(\frac{h}{fl}\right)^v f^{\rho+\sigma} \right) = \sigma y \left(\frac{h}{fl}\right)^{v+1} f^{\rho+\sigma} \tag{5.2}$$

By utilising lemma 5.4 part 2 instead of part 1, it follows similarly that

$$\frac{\delta}{\delta y}\left(\left(\frac{h}{fl}\right)^v f^{\rho+\sigma} \right) = -\rho x \left(\frac{h}{fl}\right)^{v+1} f^{\rho+\sigma} \tag{5.3}$$

Consider f,h and l as polynomials in x with coefficients in $k(y)$. Take $\mu \in \overline{k(y)}$. If μ is a zero of $\frac{h}{fl}$ of order $\nu > 0$ and a zero of f of order $\nu' \geq 0$, the relation 5.2 shows that $v\nu + (\rho + \sigma)\nu' - 1 = (v + 1)\nu + (\rho + \sigma)\nu'$, which is impossible. Thus $\frac{h}{dl}$ is non-zero on $\overline{k(y)}$. Therefore, $\frac{fl}{h} \in k(y)[x]$. Applying 5.3, we see similarly that $\frac{fl}{h} \in k(x)[y]$. Thus there exists a non-zero polynomial $m \in k[x, y]$ such that $fl = hm$. Since f, h and l are (ρ, σ) homogeneous of (ρ, σ) degrees v, t and $t - v + \rho + \sigma$, m is (ρ, σ)-homogeneous and

$$v_{\rho,\sigma}(m) = v + (t - v + \rho + \sigma) - t = \rho + \sigma$$

The relations 5.2 and 5.3 can now be written as

$$\frac{\delta}{\delta x}\left(\frac{f^{\rho+\sigma}}{m^v}\right) = \sigma y \frac{f^{\rho+\sigma}}{m^{v+1}} \tag{5.4}$$

$$\frac{\delta}{\delta y}\left(\frac{f^{\rho+\sigma}}{m^v}\right) = \rho x \frac{f^{\rho+\sigma}}{m^{v+1}} \tag{5.5}$$

Consider f and m as elements of $k(y)[x]$ (or $k(x)[y]$). Applying relations 5.4 and 5.5, all zeroes of f in $\overline{k(y)}$ (or $\overline{k(x)}$) are zeroes of m in $\overline{k(y)}$ (or $\overline{k(x)}$, respectively).

If m is a monomial, then we have shown that f is a monomial, contrary to assumption. Thus $E(m)$ contains at least two elements. And yet, if $(i, j) \in E(m)$, we have $\rho i + \sigma j = \rho + \sigma$. If $i > 0$ and $j > 0$, it follows that $(i, j) = (1, 1)$. As $E(m)$ is not simply $\{(1, 1)\}$ by the preceeding argument, $E(m)$ contains an element of the form $(i, 0)$, which is neccessarily of the form $\left(\frac{\rho+\sigma}{\rho}, 0\right)$, or an element of the form $(0, j)$ which is neccessarily of the form $\left(0, \frac{\rho+\sigma}{\sigma}\right)$. We thus have one of the following cases:

First case $\rho < \sigma$, σ is a multiple of ρ, $E(m) = \left\{(1, 1), (1 + \frac{\rho}{\sigma}, 0)\right\}$, and

$$m(x, y) = \mu x^{1+\frac{\sigma}{\rho}} + \nu xy$$

with $\mu, \nu \in k$, $\mu, \nu \neq 0$.

Second case $\rho > \sigma$, ρ is a multiple of σ, $E(m) = \left\{(1, 1), (0, 1 + \frac{\rho}{\sigma})\right\}$, and

$$m(x, y) = \mu y^{1+\frac{\rho}{\sigma}} + \nu xy$$

Third case $\rho = \sigma$, $E(m) \subset \{(2,0),(1,1),(0,2)\}$, and

$$m(x,y) = \mu x^2 + \nu xy + \zeta y^2$$

with at least two of μ, ν, ζ non-zero.

Consider the first case. Then $m \in k(x)[y]$ has one zero in $\overline{k(x)}$. Thus $f \in k(x)[y]$ has, for its only zero in $\overline{k(x)}$, the zero of m. Thus there exists an integer $\beta \geq 0$ and an element $\tau(x) \in k(x)$ such that $f = \tau(x)(\nu y + \mu x^{\frac{\sigma}{\rho}})^{\beta}$. Since $v \neq 0$, we have $\tau(x) \in k[x]$. For the other part, all zeroes of $f \in k(y)[x]$ in $\overline{k(y)}$ are zeroes of $m \in k(y)[x]$. This proves that the only zero of $\tau(x)$ is at $x = 0$, and thus $\tau(x)$ is a monomial. This places us in case (b). We see similarly that in the second case places us in case (c).

Consider now the third case. If $\zeta = 0$, we have $\mu, \nu \neq 0$. Reasoning as for the first case, we have (d). Similarly if $\mu = 0$. Suppose that $\mu, \zeta \neq 0$. If $m(x,y) = \mu(x + \eta y)(x + \theta y)$ with $\eta, \theta \in k$, $f \in k(y)[x]$ has its sole zeros at $-\eta y$, and $-\theta y$ in $\overline{k(y)}$, thus

$$f = \tau(y)(x + \eta y)^{\alpha}(x + \theta y)^{\beta}$$

with $\tau(y) \in k(y)$ and α, β integers ≥ 0. Clearly, $\tau(y) \in k[y]$ and, exchanging x and y, we see that $\tau(y) \in k$, and we have established (d). Finally, suppose that

$$m(x,y) = \mu(x + \eta y)(x + \theta y)$$

with $\eta, \theta \in \overline{k} \setminus k$, η and θ conjugate over k. We have $f = \tau(y)(x + \eta y)^{\alpha}(x + \theta y)^{\beta}$, this time with $\alpha = \beta$ and also $\tau(y) \in k$. But then f is a multiple of a power of m, whence

$$v_{\rho,\sigma}(f^{\rho+\sigma}) = (\rho + \sigma)v = v_{\rho,\sigma}(m^v)$$

we have $\frac{f^{\rho,\sigma}}{m^v} \in k$, which is a contradiction following 5.4 and 5.5. □

Proposition 5.15. *Take ρ, σ integers > 0, $\mathfrak{a} \in A_1$, $v = v_{\rho,\sigma}(\mathfrak{a})$, let f be the (ρ, σ)-associated polynomial of $\mathfrak{a}$. Suppose that*

1. *$v > \rho + \sigma$*
2. *f is not a monomial*
3. *None of the cases (b), (c), or (d) in lemma 5.14 hold*

   Then $F(\mathfrak{a}) = C(\mathfrak{a})$.

Proof. Let Λ be the set of integers λ for which there exists a $\mathfrak{b} \in F(\mathfrak{a})$ with $v_{\rho,\sigma}(\mathfrak{b}) = \lambda$. Since $F(\mathfrak{a})$ is closed under addition, $\Lambda + \Lambda \subset \Lambda$, and in particular $\{0, v, 2v, \ldots\} \subset \Lambda$. Let Λ' be the image in $\mathbb{N}/v\mathbb{N}$ of Λ under the map $n \mapsto n + v\mathbb{N}$. In every (coset) element of Λ', choose the smallest element. Denote these elements by $\lambda_0 = 0, \lambda_1, \ldots, \lambda_r$ where $r \leq v$. The elements of Λ are then of the following form:

$$\begin{array}{lllll} 0, & v, & 2v, & 3v, & \ldots \\ \lambda_1, & \lambda_1 + v, & \lambda_1 + 2v, & \lambda_1 + 3v, & \ldots \\ \vdots & & & & \\ \lambda_r, & \lambda_r + v, & \lambda_r + 2v, & \lambda_r + 3v, & \ldots \end{array}$$

Let $\mathfrak{b}_i$ be an element of $F(\mathfrak{a})$ such that $v_{\rho,\sigma}(\mathfrak{b}_i) = \lambda_i$. Take $\mathfrak{b} \in F(\mathfrak{a})$. We will show by induction that $\mathfrak{b} \in k[\mathfrak{a}]\mathfrak{b}_0 + k[\mathfrak{a}]\mathfrak{b}_1 + \ldots + k[\mathfrak{a}]\mathfrak{b}_r$. It is obvious for $v_{\rho,\sigma}(\mathfrak{b}) = 0$. Suppose that it holds for $v_{\rho,\sigma}(\mathfrak{b}) < n$ and consider the case where $v_{\rho,\sigma}(\mathfrak{b}) = n > 0$. Then there exists an $i \in \{0, 1, \ldots, r\}$ and an integer $s \geq 0$ such that $v_{\rho,\sigma}(\mathfrak{a}^s \mathfrak{b}_k) = n$. Let g and h be the (ρ, σ)-associated polynomials of $\mathfrak{b}$ and $\mathfrak{a}^s \mathfrak{b}_i$. Applying lemma 5.14, g^v and h^v are scalar multiples of f^n and thus of each other. Thus there exists a $\lambda \in k$ such that $v_{\rho,\sigma}(\mathfrak{b} - \lambda \mathfrak{a}^s \mathfrak{b}_i) < n$. We have therefore $\mathfrak{b} - \lambda \mathfrak{a}^s \mathfrak{b}_i \in F(\mathfrak{a})$, and the result follows by induction on $\mathfrak{b} - \lambda \mathfrak{a}^s \mathfrak{b}_i$.

Thus we have $F(\mathfrak{a}) = \sum_i k[\mathfrak{a}]\mathfrak{b}_i$ and the result follows from lemma 5.13. □

Definition 5.16. For $\lambda \in k$ and $n \in \mathbb{N}$, define the k-linear maps $\Phi_{n,\lambda}, \Phi'_{n,\lambda} : A_1 \to A_1$ by

$$\Phi_{n,\lambda}(p) = p \qquad \Phi_{n,\lambda}(q) = q + \lambda p^n$$
$$\Phi'_{n,\lambda}(p) = p + \lambda q^n \qquad \Phi'_{n,\lambda}(q) = q$$

Lemma 5.17. *For all $\lambda \in k$ and $n \in \mathbb{N}$, $\Phi_{n,\lambda}$ and $\Phi'_{n,\lambda}$ are k-linear automorphisms of A_1.*

Proof. The image of $qp - pq - 1$ under $\Phi_{n,\lambda}$ is

$$\begin{aligned}(q + \lambda p^n)p - p(q + \lambda p^n) - 1 &= qp + \lambda p^{n+1} - pq - \lambda p^{n+1} - 1\\ &= pq + 1 + \lambda p^{n+1} - pq - \lambda p^{n+1} - 1\\ &= 0\end{aligned}$$

thus $\Phi_{n,\lambda}$ is an homomorphism. Similarly for $\Phi'_{n,\lambda}$. As mentioned at the end of chapter 2, the fact that A_1 is simple implies that any non-zero endomorphism of A_1 is injective. This follows as the kernel is an ideal of A_1 and is therefore either 0 or all of A_1. Thus both $\Phi_{n,\lambda}$ and $\Phi'_{n,\lambda}$ are injective. Let $q_0 = -\lambda p^n + q$ and let $p_0 = p - \lambda q^n$. We have,

$$\Phi_{n,\lambda}(p) = p \qquad \Phi_{n,\lambda}(q_0) = -\lambda p^n + q + \lambda p^n = q$$
$$\Phi'_{n,\lambda}(p_0) = p + \lambda q^n - \lambda q^n = p \qquad \Phi'_{n,\lambda}(q) = q$$

Thus p and q are in the images of both $\Phi_{n,\lambda}$ and $\Phi'_{n,\lambda}$ and they are thus both surjective. □

Definition 5.18. Let G denote the group of automorphisms of A_1 generated by the $\Phi_{n,\lambda}, \Phi'_{n,\lambda}$ for all n, λ.

Let V be the vector space $kp+kq$. The group $SL(V)$ consists of maps $\theta : V \to V$ of the form $vp+wq \mapsto (av+cw)p+(bv+dw)q$ with $ad-bc=1$. Since the map θ is defined for p and q, we can extend it to a multiplicative k-linear map $\theta' : A_1 \to A_1$ defined by

$$\theta'(p) = ap + bq \qquad \theta'(q) = cp + dq \qquad ad - bc = 1$$

Note that $\theta'|_V = \theta$. Calculating,

$$\begin{aligned}
\theta(qp - pq - 1) &= (cp + dq)(ap + bq) - (ap + bq)(cp + dq) - 1 \\
&= acp^2 + bcpq + adqp + bdq^2 - acp^2 - adpq - bcqp - bdq^2 - 1 \\
&= bcpq + adpq + ad - adpq - bcpq - bc - 1 \\
&= (ad - bc) - 1 \\
&= 0
\end{aligned}$$

Thus θ' is an homomorphism. Let $p_0 = dp - bq$ and $q_0 = -cp + aq$. It is simple to calculate that $\theta'(p_0) = p$ and $\theta'(q_0) = q$. Thus both $p, q \in \operatorname{im}(\theta')$ and therefore θ' is surjective. As in the previous lemma, the fact that A_1 is simple implies that θ' is injective.

Definition 5.19. As above, let V be the vector space $kp + kq$. As we have just shown, all elements of $SL(V)$ extend to give an automorphism of A_1. We obtain thus a group G' of automorphisms of A_1. It is not hard to see that the restrictions $\Phi_{1,\lambda}|_V$ and $\Phi'_{1,\lambda}|_V$ generate the group $SL(V)$. Thus $G' \subset G$. Denote by $\overline{G'}$ the analogous group acting on $\overline{A_1}$. In particular, denote by Ψ the element of G' such that $\Psi(p) = q$, $\Psi(q) = -p$, this is known as the *Fourier transform*.

Lemma 5.20. *Take* $\mathfrak{a} = \alpha p^2 + \beta pq + \gamma q^2 \in A_1$ *with* $\alpha, \beta, \gamma \in k$. *There exist* $\Phi, \Theta \in \overline{G'}$ *such that*

$$\Phi(\mathfrak{a}) = \theta pq + \zeta$$

$$\Theta(\mathfrak{a}) = \begin{cases} \alpha' p^2 + \zeta', & \text{if } \beta^2 - \alpha\gamma = 0 \\ \alpha' p^2 + \gamma' q^2 + \zeta', & \text{if } \beta^2 - \alpha\gamma \neq 0 \end{cases}$$

for $\alpha', \gamma', \zeta' \in k$ *and* $\theta, \zeta \in \overline{k}$.

Proof. Noting that if $\Phi(p) = ap + bq$ and $\Phi(q) = cp + dq$,

$$\begin{aligned}
\Phi(\mathfrak{a}) &= \alpha(ap+bq)^2 + \beta(ap+bq)(cp+dq) + \gamma(cp+dq)^2 \\
&= \alpha a^2 p^2 + \alpha abpq + \alpha abqp + \alpha b^2 q^2 \\
&+ \beta acp^2 + \beta adpq + \beta bcqp + \beta bdq^2 \\
&+ \gamma c^2 p^2 + \gamma cdpq + \gamma cdqp + \gamma d^2 q^2 \\
&= (\alpha a^2 + \beta ac + \gamma c^2)p^2 + (2\alpha ab + \beta ad + \beta bc + 2\gamma cd)pq \\
&+ (\alpha b^2 + \beta bd + \gamma d^2)q^2 + \alpha ab + \beta bc + \gamma cd
\end{aligned}$$

which is simply the action of $\Phi|_V$ on $\mathfrak{a}$ considered as a quadratic form in $k[p,q]$ plus a scalar term. The result thus follows from the analogous results for real quadratic forms on two variables. □

Lemma 5.21. *For* $\mathfrak{a} \in k[p]$, $N(\mathfrak{a}) = A_1$.

Proof. First note that for $\mathfrak{b}, \mathfrak{c} \in A_1$, if $(\text{ad } \mathfrak{a})^m \mathfrak{b} = 0$ and $(\text{ad } \mathfrak{a})^n \mathfrak{c} = 0$ then if $o = \max(m,n)$, $(\text{ad } \mathfrak{a})^o(\mathfrak{b} + \mathfrak{c}) = (\text{ad } \mathfrak{a})^o \mathfrak{b} + (\text{ad } \mathfrak{a})^o \mathfrak{c} = 0$ and $(\text{ad } \mathfrak{a})^o(\mathfrak{b}\mathfrak{c}) = ((\text{ad } \mathfrak{a})^o \mathfrak{b})\mathfrak{c} + \mathfrak{b}((\text{ad } \mathfrak{a})^o \mathfrak{c}) = 0$ ie. $N(\mathfrak{a})$ is both multiplicatively and additively closed.

We have $p \in N(\mathfrak{a})$ and $[\mathfrak{a}, q] \in k[p]$, whence $q \in N(\mathfrak{a})$ and thus $A_1 \subseteq N(\mathfrak{a})$. □

Lemma 5.22. *Take* $\mathfrak{a} = \lambda p^2 + \mu q^2 + \nu$ *with* $\lambda, \mu, \nu \in k$, $\lambda \neq 0, \mu \neq 0$. *Then* $D(\mathfrak{a}) = A_1$.

Proof. By lemma 5.20, there exists a $\Phi \in \overline{G'}$ such that $\Phi(\mathfrak{a}) = \zeta pq + \theta$ with $\zeta, \theta \in \overline{k}$. Note that $[pq, p] = -p$ and $[pq, q] = -q$, and hence $[pq, p^i q^j] = (m - l)p^i q^j$. Thus, for any $\mathfrak{b} \in A_1$ with $\mathfrak{b} = \sum b_{ij} p^i q^j$, $[\Phi(\mathfrak{a}), \mathfrak{b}] = \sum (m - l)\zeta b_{ij} p^i q^j$. The individual terms of $\mathfrak{b}$ are therefore in the components $D(\Phi(\mathfrak{a}), (m - l)\zeta b_{ij}; \overline{A_1})$ and hence $\mathfrak{b} \in D(\Phi(\mathfrak{a}); \overline{A_1})$. But, since $\mathfrak{b}$ was chosen to lie within A_1, $\mathfrak{b} \in D(\Phi(\mathfrak{a}); \overline{A_1}) \cap A_1 = D(\Phi(\mathfrak{a}); A_1)$. Thus by corollary 5.10.1, $D(\mathfrak{a}) = D(\mathfrak{a}; A_1) = A_1$. □

Lemma 5.23. *Let $\mathfrak{a}$ be an element of A_1 of the form*

$$a_{00} + a_{10}p + a_{20}p^2 + \ldots + a_{r0}p^r + a_{01}q + a_{11}pq, \qquad a_{ij} \in k$$

Then there exists a $\Phi \in G$ such that $\Phi(\mathfrak{a})$ is of the form

$$b_{00} + b_{10}p + b_{01}q + b_{11}pq, \qquad b_{ij} \in k$$

Proof. The result is trivial if $r \leq 1$. Suppose the result holds for $r - 1$. If $a_{11} \neq 0$, we can scale such that $a_{11} = 1$. We have the following:

$$\begin{aligned} \Phi_{r-1,-a_{r0}}(\mathfrak{a}) &= a_{00} + a_{10}p + \ldots + a_{r0}p^r + a_{01}(q - a_{r0}p^{r-1}) + p(q - a_{r0}p^{r-1}) \\ &= a_{00} + a_{10}p + \ldots + a_{r-2,0}p^{r-2} + a_{r-1,0}p^{r-1} + a_{01}q + pq \end{aligned}$$

and, noting that the maximum degree in p has fallen by one, the result follows by induction. If $a_{11} = 0$ and $a_{01} \neq 0$, we can scale such that $a_{01} = 1$. We have:

$$\begin{aligned} \Phi_{r,-a_{r0}}(\mathfrak{a}) &= a_{00} + a_{10}p + \ldots + a_{r0}p^r + q - a_{r0}p^r \\ &= a_{00} + a_{10}p + \ldots + a_{r-1,0}p^{r-1} + q \end{aligned}$$

and again the result follows by induction. □

Lemma 5.24. *Let $\mathfrak{a}$ be an element of A_1, of the form*

$$\alpha p^2 + 2\beta pq + \gamma q^2 + \delta p + \epsilon q + \zeta \qquad (\alpha, \beta, \ldots, \zeta \in k)$$

- *If* $\beta^2 - \alpha\gamma = 0$*, there exists a* $\Phi \in G$ *such that* $\Phi(\mathfrak{a}) \in k[p]$
- *If* $\beta^2 - \alpha\gamma \neq 0$*, there exists a* $\Phi \in G$ *and* $\lambda, \mu, \nu \in k$ *with* $\lambda \neq 0$*,* $\mu \neq 0$ *such that* $\Phi(\mathfrak{a}) = \lambda p^2 + \mu q^2 + \nu$

Proof. If $\beta^2 - \alpha\gamma = 0$, then by lemma 5.20 there exists a $\Phi_1 \in G'$ such that

$$\Phi_1(\mathfrak{a}) = \alpha' p^2 + \delta' p + \epsilon' q + \zeta'$$

If $\epsilon' = 0$, the result is proved. If $\epsilon' \neq 0$, can scale such that $\epsilon' = 1$. We have

$$\begin{aligned} \Phi_{2,-\alpha'}(\Phi_1(\mathfrak{a})) &= \alpha' p^2 + \delta' p + q - \alpha' p^2 + \zeta' \\ &= \delta' p + q + \zeta' \end{aligned}$$

and it suffices to apply some element of G'.

If $\beta^2 - \alpha\gamma \neq 0$, then again by lemma 5.20 there exists a $\Phi_1 \in G'$ such that

$$\Phi_1(\mathfrak{a}) = \alpha' p^2 + \gamma' q^2 + \delta' p + \epsilon' q + \zeta'$$

with $\alpha', \gamma' \neq 0$. We have

$$\begin{aligned} y &= \Phi_{0,-\frac{1}{2}\epsilon'\gamma'^{-1}}(\Phi_1(\mathfrak{a})) \\ &= \alpha' p^2 + \delta' p + \zeta' + \gamma'(q - \frac{1}{2}\epsilon'\gamma'^{-1})^2 + \epsilon'(q - \frac{1}{2}\epsilon'\gamma'^{-1}) \\ &= \alpha' p^2 + \delta' p + \zeta' + \gamma' q^2 + \zeta_1 \end{aligned}$$

Likewise, there exists a $\Phi_2 \in G'$ such that $\Phi_2(y) = \alpha' p^2 + \gamma' q^2 + \zeta_2$. □

Lemma 5.25. *Let* $\mathfrak{a} = \sum a_{ij} p^i q^j \in A_1$*. Let* r *be the smallest integer* ≥ 0 *such that* $a_{i0} = 0$ *for* $i > r$*. Let* s *be the smallest integer* ≥ 0 *such that* $a_{0j} = 0$ *for* $j > s$*. Suppose that there exist integers* $i_1, j_1 \geq 0$ *such that* $a_{i_1,j_1} \neq 0$*,* $(i_1, j_1) \neq (1,1)$ *and* $si_1 + rj_1 > rs$*. Then* $F(\mathfrak{a}) \neq A_1$*.*

Proof. If $i_1 = 0$, we have $rj_1 > rs$, or $j_1 > s$, which contradicts the definition of s. Thus $i_1 > 0$ and similarly $j_1 > 0$.

Take real numbers $\rho, \sigma > 0$ (relatively irrational) such that

$$\sigma i_1 + \rho j_1 > \rho s$$
$$\sigma i_1 + \rho j_1 > r\sigma$$

(eg. take $\rho = r + \delta$, $\sigma = s + \epsilon$ for $0 < \delta s, r\epsilon < 1$). By the definition of $v_{\sigma,\rho}(\mathfrak{a})$, there exist integers $i_2, j_2 \geq 0$ such that $a_{i_2 j_2} \neq 0$, $\sigma i_2 + \rho j_2 = v_{\sigma,\rho}(\mathfrak{a})$. Thus, by the maximality of $v_{\sigma,\rho}(\mathfrak{a})$,

$$\sigma i_2 + \rho j_2 > \sigma i_1 + \rho j_1 > \rho s$$
$$\sigma i_2 + \rho j_2 > \sigma i_1 + \rho j_1 > r\sigma$$

This implies that $i_2, j_2 > 0$ (otherwise if, say, $i_2 = 0$ then $j_2 > j_1$, which contradicts the choice of j_1). If $i_2 = j_2 = 1$, we have

$$\begin{aligned} \sigma + \rho &\leq \sigma i_1 + \rho j_1 \\ &\leq \sigma i_2 + \rho j_2 \\ &= \sigma + \rho \end{aligned}$$

hence $i_1 = i_2$, $j_1 = j_2$ and $(i_1, j_1) = (1, 1)$, contrary to assumption. Thus $i_2 > 1$, or $j_2 > 1$. By applying the second part of lemma 5.3, the (σ, ρ)-associated polynomial of $\mathfrak{a}$ is $a_{i_2 j_2} x^{i_2} y^{j_2}$.

Suppose that $i_2 \leq j_2$. For $n = 0, 1, 2, \ldots$, let $\mathfrak{b}_n = (\text{ad } \mathfrak{a})^n q$. We aim to show by induction on n that the (σ, ρ)-associated polynomial of $\mathfrak{b}_n$ is

$$b_n x^{n(i_2-1)} y^{1+n(j_2-1)}$$

with $b_n \in k, b_n \neq 0$. The result is obvious for $n = 0$. Suppose that the result holds for n. Applying lemma 5.8, the (σ, ρ)-associated polynomial of $\mathfrak{b}_{n+1} = [\mathfrak{a}, \mathfrak{b}_n]$ is

$$(i_2 + nj_2 - ni_2)a_{i_2 j_2} b_n x^{i_2+n(i_2-1)-1} y^{j_2+1+n(j_2-1)-1}$$

which proves the assertion for $n+1$. We also have

$$v_{\sigma,\rho}(\mathfrak{b}_n) = \sigma n(i_2 - 1) + \rho(1 + n(j_2 - 1))$$

Since $i_2 > 1$ or $j_2 > 1$, we see that $v_{\sigma,\rho}$ tends to infinity with n. Thus $q \notin F(\mathfrak{a})$ and $F(\mathfrak{a}) \neq A_1$.

If $i_2 \geq j_2$, we have $p \notin F(\Psi(\mathfrak{a}))$, by the preceeding, thus $F(\Psi(\mathfrak{a})) \neq A_1$ and $F(\mathfrak{a}) \neq A_1$ by corollary 5.10.1. □

Lemma 5.26. *Take $\mathfrak{a} \in A_1$ with $F(\mathfrak{a}) = A_1$. Then there exists a $\Phi \in G$ such that $\Phi(\mathfrak{a})$ posesses one of the following properties: either $\Phi(\mathfrak{a}) \in k[p]$ or $\Phi(\mathfrak{a})$ is of the form $\lambda p^2 + \mu q^2 + \nu$ with $\lambda, \mu, \nu \in k$, $\lambda, \mu \neq 0$.*

Proof. Introduce integers r, s as in the previous lemma. We argue inductively on $r + s$. If $r \leq 2$ and $s \leq 2$, the previous lemma shows that $v_{1,1} \leq 2$, and we apply lemma 5.24. Suppose therefore that $r > 2$ or $s > 2$ and that the result holds for $r + s < n$. Consider the case $r + s = n$.

Using the automorphism Ψ (to swap p and q if neccessary), we suppose that $r \geq s$. If $s \leq 1$, $\mathfrak{a}$ is, by lemma (5.25), of the form

$$a_{00} + a_{10}p + a_{20}p^2 + \ldots + a_{r0}p^r + a_{01}q + a_{11}pq$$

and it suffices to apply lemmas 5.23 and 5.24. Suppose from now on, therefore, that $r \geq s \geq 2$ and $r > 2$, whence $r + s < rs$. If $(i, j) \in E(\mathfrak{a})$, lemma 5.25 shows that either $si + rj \leq rs$, or $i = j = 1$, in which case $si + rj = s + r < rs$. Thus $v_{s,r}(\mathfrak{a}) = rs$ and the (s, r)-associated polynomial of $\mathfrak{a}$ is of the form

$$f(x, y) = a_{r0}x^r + \ldots + a_{0s}y^s \qquad \text{with } a_{r0} \neq 0, a_{0s} \neq 0 \tag{5.6}$$

Since $F(\mathfrak{a}) = A_1 \neq C(\mathfrak{a})$, applying proposition 5.15 with $\rho = s$, $\sigma = r$ implies that one of cases b, c or d of lemma 5.14 holds. Since $r \geq s$, we are in either case b or case d.

Suppose that we are in case b. Then r is a multiple of s and applying theorem 5.6 , f is a scalar multiple of $(x^{\frac{r}{s}} + \mu y)^s$ with $\mu \in k, \mu \neq 0$. Suppose that

$$\mathfrak{a} = (p^{\frac{r}{s}} + \mu q)^s + \sum_{(i,j) \in E} a_{ij} p^i q^j$$

with $si + rj < rs$ for $(i, j) \in E$. Then

$$\begin{aligned} \mathfrak{b} &= \Phi_{\frac{r}{s}, \frac{-1}{\mu}}(\mathfrak{a}) \\ &= \mu^s q^s + \sum_{(i,j) \in E} a_{ij} p^i (q - \mu^{-1} p^{\frac{r}{s}})^j \end{aligned}$$

We have

$$v_{s,r}(q - \mu^{-1} p^{\frac{r}{s}}) = r \qquad \text{and} \qquad v_{s,r}(p) = s$$

thus

$$v_{s,r}\left(\sum_{(i,j) \in E} a_{ij} p^i (q - \mu^{-1} p^{\frac{r}{s}})^j \right) < rs$$

If we denote as r_1 and s_1 the analogous integers to r and s, but related to $\mathfrak{b}$, we see that $s_1 = s$ and $r_1 < r$. Inductively, there exists a $\Phi \in G$ such that $\Phi(\mathfrak{b})$ posesses one of the required properties. Thus $\Phi(\mathfrak{b}) = (\Phi \circ \Phi_{\frac{r}{s}, \frac{-1}{\mu}})(\mathfrak{a})$, and the lemma is proved in this case.

Suppose that we are in case d of lemma 5.14. Then $r = s$ and f is a multiple of $(x+\mu y)^\alpha (x+\nu y)^{r-\alpha}$ with $\mu, \nu \in k$ and α an integer such that $0 \leq \alpha \leq r$. Multiplying $\mathfrak{a}$ by a scalar if neccessary, suppose that

$$\mathfrak{a} = (p+\mu q)^\alpha (p+\nu q)^{r-\alpha} + \sum_{(i,j)\in E} a_{ij} p^i q^j$$

with $i + j < r$ for $(i,j) \in E$. If neccessary, exchange μ and ν, and assume that $\alpha > 0$. Then

$$\begin{aligned} \mathfrak{b} &= \Phi_{1,\frac{-1}{\mu}}(\mathfrak{a}) \\ &= \mu^\alpha q^\alpha ((1-\nu\mu^{-1})p + \nu q)^{r-\alpha} + \sum_{(i,j)\in E} a_{ij} p^i (q - \mu^{-1} p)^j \end{aligned}$$

If we again denote as r_1 and r_1 the integers analogous to the r and s but relative to $\mathfrak{b}$, we see that $s_1 = s = r$ and $r_1 < r$. Induction therefore will terminate in case b. □

Lemma 5.27. *Take* $\mathfrak{a} \in A_1$

- *If* $N(\mathfrak{a}) = A_1$*, then there exists a* $\Phi \in G$ *such that* $\Phi(\mathfrak{a}) \in k[p]$
- *If, in addition,* $C(\mathfrak{a}) = k[\mathfrak{a}]$*, then there exists a* $\Phi \in G$ *such that* $\Phi(\mathfrak{a}) = p$

Proof. Since $N(\mathfrak{a}) \subseteq F(\mathfrak{a})$, lemma 5.26 implies that there exists a $\Phi \in G$ such that either $\Phi(\mathfrak{a}) \in k[p]$ or $\Phi(\mathfrak{a}) = \lambda p^2 + \mu q^2 + \nu$. But since $N(\mathfrak{a}) \nsubseteq D(\mathfrak{a})$, $D(\mathfrak{a}) \neq A_1$ and the contrapositive of lemma 5.22 implies that $\Phi(\mathfrak{a}) \neq \lambda p^2 + \mu q^2 + \nu$.

Now suppose that $N(\mathfrak{a}) = A_1$ and $C(\mathfrak{a}) = k[\mathfrak{a}]$. Thanks to the first result, we assume $\mathfrak{a} \in k[p]$. Thus $p \in C(\mathfrak{a}) = k[\mathfrak{a}]$, and hence $\mathfrak{a} \in k.1 + k.p$, and the result follows. □

Theorem 5.28. *The group of* k*-linear automorphisms of* A_1 *is generated by the automorphisms* $\Phi_{n,\lambda}$ *and* $\Phi'_{n,\lambda}$.

Proof. We take Φ to be an automorphism of A_1 and prove that $\Phi \in G$. We have $N(\Phi(p)) = A_1$ (lemma 5.21), and $C(\Phi(p)) = k[\Phi(p)]$ (5.11). Applying lemma 5.27, we reduce to the case where $\Phi(p) = p$. Then

$$[p, \Phi(q) - q] = \Phi([p, q]) - [p, q] = 1 - 1 = 0$$

Thus $\Phi(q) \in q + k[p]$, and Φ is a product of automorphisms $\Phi_{n,\lambda}$. □

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